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A functor-valued extension of knot quandles

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Abstract. For an oriented knot K, we construct a functor from the category of pointed quandles to the category of quandles in three different ways. This functor-valued invariant of a knot is an extension of the knot quandle. We also extend the quandle cocycle invariants of knots by using these quandle-valued invariants, and study their properties.

1. Introduction.

A *quandle* is a set Q with a binary operation $*$ which satisfies some axioms, and the pointed quandle is a pair (Q, h) consisting of a quandle Q and its element h. For an oriented knot K, one can associate a quandle Q_K called the *knot quandle*. The knot quandle distinguishes all knots up to orientation [**7**], [**8**]. Moreover, using the homology theories of quandles, the knot quandle provides a knot invariant called a *quandle cocycle invariant* [**4**], [**5**].

The aim of this paper is to extend the knot quandle as a functor. For each oriented knot K in S^3, we construct the *quandle invariant functor* I_K, which is a functor from $\mathscr{PQ}$, the category of pointed quandles, to $\mathscr{Q}$, the category of quandles. Thus, for each pointed quandle (Q, h) we obtain a quandle-valued invariant $I_K(Q, h)$ of a knot K. The classical knot quandle Q_K appears as $I_K(T_1)$, the quandle-valued invariant corresponding to the trivial 1-quandle T_1. We also construct an extension of quandle cocycle invariants by using the quandle-valued invariant of knots.

Our results are summarized as follows.

Theorem 1. *Let K be an oriented knot. Then there exists a functor $I_K : \mathscr{PQ} \to \mathscr{Q}$ from the category of pointed quandles to the category of quandles, having the following properties.*

1. *For the trivial 1-quandle T_1, $I_K(T_1)$ is the knot quandle Q_K.*

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2. $H_1^Q(I_K(Q,h);\boldsymbol{Z}) \cong H_1^Q(Q;\boldsymbol{Z})$.
3. *If (Q,h) is a finite pointed quandle, then there exists a characteristic homology class $[K]_{Q,h} \in H_2^Q(I_K(Q,h);\boldsymbol{Z})$ which vanishes if and only if K is unknot.*
4. *For the dual knot $-K^*$, which is the mirror of K with the opposite orientation, $I_K = I_{-K^*}$ and $[-K^*]_{Q,h} = -[K]_{Q,h}$ hold.*

Using the characteristic class $[K]_{Q,h}$, we define a cocycle invariant as in the classical knot quandles. The explicit construction of cocycle invariants will be given in Section 6, in which section we also study fundamental properties of the extension of cocycle invariants.

We construct the quandle invariant functor by three different ways. The first method is algebraic. We use a representation of the braid groups derived from a pointed quandle. Such a representation can be seen as a generalization of the Artin type representation of the braid groups introduced in [**3**]. The second method is combinatorial and uses knot diagrams. We define the quandle-valued invariants by giving a presentation, which extends the Wirtinger presentation of the knot quandle. This point of view is useful when we extend the quandle cocycle invariants. The last method is geometric. We construct a topological pair of space whose positive fundamental quandle coincides with the quandle-valued invariant of knots. Such a spatial realization of the quandle invariant is obtained by gluing a topological space along the knot.

Our quandle-valued invariants are generalizations of the group-valued invariant defined by Crisp-Paris [**3**] and Wada [**9**]. Indeed, many arguments in this paper is a direct generalization of the arguments in [**3**]. A new aspect which did not appear in [**3**] is the homology theory, which is used to define a cocycle invariant and makes our invariant more useful.

Finally, we remark that although we restrict our attention to knots, but the construction of the quandle invariant functors and the characteristic classes are valid for oriented links as well with some modifications. Thus our results are directly extended for link cases.

2. Quandles, racks and braids.

A quandle is a set Q with a binary operation $*$ which satisfies the following three axioms.

Q1: $a * a = a$ for all $a \in Q$.
Q2: For all $a, b \in Q$ there exists a unique element $a \overline{*} b \in Q$ such that $a = (a \overline{*} b) * b = (a * b) \overline{*} b$.
Q3: $(a * b) * c = (a * c) * (b * c)$ for all $a, b, c \in Q$.

If $(Q, *)$ does not satisfy the axiom **Q1** but satisfy both **Q2** and **Q3**, then it is called a *rack*.

EXAMPLE 1. We present examples of quandle which will be used in later.

- Let X be a finite set of cardinal n, and consider an operation $*$ defined by $x * y = x$ for all $x, y \in X$. Then $(X, *)$ is a quandle. We call $(X, *)$ the *trivial n-quandle* and denote by T_n.
- A group G can be considered as a quandle by the operation $*$ defined by the conjugation $x * y = y^{-1}xy$. We call this quandle *the conjugacy quandle* associated to the group G and denote by Q_G. Conversely, for a quandle Q one can obtain the *associated group* $\mathrm{Ass}(Q)$ defined by the presentation

$$\mathrm{Ass}(Q) = \langle q \in Q \mid p * q = q^{-1}pq \rangle.$$

We call a pair (Q, h) consisting of a quandle Q and its element $h \in Q$ a *pointed quandle*. A map between two quandles $\tau : (Q, *_Q) \to (P, *_P)$ is called a *(quandle) morphism* if τ preserves the operation $*$, that is, $\tau(a *_Q b) = \tau(a) *_P \tau(b)$ holds for all $a, b \in Q$. We denote by $\mathrm{Aut}(Q)$ the group of automorphisms of Q. A morphism between pointed quandles (Q, h) and (P, i) is, by definition, a quandle morphism f which satisfies the condition $f(h) = i$. We denote the category of pointed quandles and the category of quandles by $\mathscr{P}\mathscr{Q}$ and $\mathscr{Q}$ respectively.

As in the group case, the notions of the free quandles, free products, and the presentation of quandles are defined in a similar way.

3. Representations of the braid group associated to pointed quandles.

Let B_n be the braid group of n-strands and $\sigma_1, \ldots, \sigma_{n-1}$ be the standard generators of B_n. The closure of a braid β, an oriented link in S^3, is denoted by $\widehat{\beta}$. In this section we define a representation of the braid group $\rho_{Q,h} : B_n \to \mathrm{Aut}(Q^{*n})$ for a pointed quandle (Q, h).

Let $Q^{*n} = Q_1 * Q_2 * \cdots * Q_n$ be the free product of n-copies of the quandle Q. For $q \in Q$, we denote by q_i the element in $Q_i \subset Q^{*n}$ which corresponds to q. For each integer $k = 1, 2, \ldots, n-1$, let τ_k be an automorphism of Q^{*n} defined by

$$\tau_k : \begin{cases} q_k \mapsto q_{k+1} \mathbin{\overline{*}} h_k & \\ q_{k+1} \mapsto q_k * h_k & \\ q_i \mapsto q_i & (i \neq k, k+1) \end{cases}$$

The following proposition is easily confirmed by a direct calculation.

PROPOSITION 1. *The map $\rho_{Q,h} : B_n \to \mathrm{Aut}(Q^{*n})$ defined by $\rho_{Q,h}(\sigma_i) = \tau_i$ is a group homomorphism.*

We call this representation the *representation associated to a pointed quandle* (Q, h). By considering the associated group of Q, we also have a representation $\rho'_{Q,h} : B_n \to \mathrm{Aut}(\mathrm{Ass}(Q^{*n}))$, given by

$$\rho'_{Q,h}(\sigma_i) : \begin{cases} q_k \mapsto h_k^{-1} q_{k+1} h_k \\ q_{k+1} \mapsto h_k q_k h_k^{-1} \\ q_i \mapsto q_i & (i \neq k, k+1) \end{cases}$$

This representation is called the *Artin type representation* of B_n associated to the pair $(\mathrm{Ass}(Q), h)$ defined in [**3**].

EXAMPLE 2. Let F_n be the free group of rank n generated by $\{x_1, x_2, \ldots, x_n\}$, which is the fundamental group of the n-punctured disc $D_n = D^2 - \{n \text{ points}\}$. It is known that the braid group B_n is identified with the relative mapping class group $MCG(D_n, \partial D_n)$, the group of isotopy classes of homeomorphisms of D_n which fixes ∂D_n pointwise [**2**].

The action of the braid groups on D_n induces the representation $\Phi : B_n \to \mathrm{Aut}(\pi_1(D_n)) = \mathrm{Aut}(F_n)$, explicitly written as

$$\Phi(\sigma_k) : \begin{cases} x_k \mapsto x_k^{-1} x_{k+1} x_k \\ x_{k+1} \mapsto x_k \\ x_i \mapsto x_i & (i \neq k, k+1). \end{cases}$$

The representation Φ is identical with the associated group representation $\rho'_{T_1,q}$. It is classically known that both $\rho_{T_1,q}$ and $\rho'_{T_1,q}$ are faithful [**2**].

First we show that the representation $\rho_{Q,h}$ is faithful.

PROPOSITION 2. *For a pointed quandle (Q, h), the representation $\rho_{Q,h} : B_n \to \mathrm{Aut}(Q^{*n})$ is faithful.*

PROOF. Let us consider the subquandle $T = (\{h\}, *)$, which is isomorphic to the trivial 1-quandle T_1, and consider the subquandle $T^{*n} \subset Q^{*n}$. Since $\tau_i|_{T^{*n}} = T^{*n}$, by considering the restriction, we obtain a map $\rho_{Q,h}|_{T^{*n}} : B_n \to \mathrm{Aut}(T^{*n})$ defined by $\rho_{Q,h}|_{T^{*n}}(\beta) = \rho_{Q,h}(\beta)|_{T^{*n}}$. By definition $\rho_{Q,h}|_{T^{*n}}$ coincides with the representation $\rho_{T_1,h}$, so $\rho_{Q,h}|_{T^{*n}}$ is faithful. Hence we conclude $\rho_{Q,h}$ is faithful. $\square$

Now using the representation associated to a pointed quandle (Q,h), we construct the quandle invariant functor. For an n-braid β, we define a *quandle invariant* $I_\beta(Q,h)$ as the quotient of Q^{*n} by the set of relations $\{[\rho_{Q,h}(\beta)](q) = q \mid q \in Q^{*n}\}$. For a pointed quandle morphism $f : (Q,h) \to (R,i)$, we define a morphism between quandle invariants $I_\beta(f) : I_\beta(Q,h) \to I_\beta(R,i)$ by $[I_\beta(f)](q_i) = [f(q)]_i$. This defines a functor $I_\beta : \mathscr{PQ} \to \mathscr{Q}$.

THEOREM 2. *The functor I_β is a knot invariant.*

PROOF. Recall that the Markov theorem (see [**2**], for example) states that the closures of two braids α, β represent the same oriented link if and only if α is converted to β by applying following two operations.

Conjugation: $\alpha \to \gamma^{-1}\alpha\gamma$ where $\alpha, \gamma \in B_n$.
(De)Stabilization: $\alpha \leftrightarrow \alpha\sigma_n^{\pm 1}$ where $\alpha \in B_n$.

First we show the invariance under the conjugation. Since $\rho_{Q,h}(\gamma)$ is an automorphism of Q^{*n}, the set of relations $\{[\rho_{Q,h}(\alpha)](q) = q\}$ is equivalent to the set of relations $\{[\rho_{Q,h}(\alpha\gamma)](q) = [\rho_{Q,h}(\gamma)](q)\}$. Hence $I_\alpha(Q,h)$ and $I_{\gamma^{-1}\alpha\gamma}(Q,h)$ are isomorphic as a quandle.

The isomorphism τ_γ between $I_\alpha(Q,h)$ and $I_{\gamma^{-1}\alpha\gamma}(Q,h)$ is given by $\tau_\gamma(q_i) = [\rho_{Q,h}(\gamma)](q_i)$. Thus, the following diagram commutes for any pointed quandle morphisms $f : (Q,h) \to (R,i)$.

$$\begin{tikzcd}
I_\alpha(Q,h) \arrow[r, "I_\alpha(f)"] \arrow[d, "\tau_\gamma"] & I_\alpha(R,i) \arrow[d, "\tau_\gamma"'] \\
I_{\gamma^{-1}\alpha\gamma}(Q,h) \arrow[r, "I_{\gamma^{-1}\alpha\gamma}(f)"] & I_{\gamma^{-1}\alpha\gamma}(R,i)
\end{tikzcd}$$

Therefore I_β is invariant as a functor under the conjugations.

Next we show the invariance under the positive stabilization. First observe that $[\rho_{Q,h}(\alpha)](q_{n+1}) = q_{n+1}$. From the relation $[\rho_{Q,h}(\alpha\sigma_n)](q_{n+1}) = [\rho_{Q,h}(\alpha)](q_n \mathbin{\overline{*}} h_n) = q_{n+1}$, we obtain the equation

$$\begin{aligned}
[\rho_{Q,h}(\alpha\sigma_n)](q_n) &= [\rho_{Q,h}(\alpha)](q_{n+1} * h_n) = q_{n+1} * [\rho_{Q,h}(\alpha)](h_n) \\
&= [\rho_{Q,h}(\alpha)](q_n \mathbin{\overline{*}} h_n) * [\rho_{Q,h}(\alpha)](h_n) \\
&= [\rho_{Q,h}(\alpha)](q_n).
\end{aligned}$$

Thus, the relation $[\rho_{Q,h}(\alpha\sigma_n)](q_n) = q_n$ implies $[\rho_{Q,h}(\alpha)](q_n) = q_n$. Similarly, the relation $[\rho_{Q,h}(\alpha)](q_n) = q_n$ implies $[\rho_{Q,h}(\alpha\sigma_n)](q_n) = q_n$. Thus, there is a natural isomorphism $\tau_+ : I_{\alpha\sigma_n}(Q,h) \to I_\alpha(Q,h)$, which is defined by $\tau_+(q_i) = q_i$ $(i = 1,\ldots,n)$ and $\tau_+(q_{n+1}) = q_n$. Now the following diagram commutes for each pointed quandle morphism $f : (Q,h) \to (R,i)$, hence the functor I_β is invariant under the positive stabilization.

$$\begin{tikzcd}
I_{\alpha\sigma_n}(Q,h) \arrow[r, "I_{\alpha\sigma_n}(f)"] \arrow[d, "\tau_+"] & I_{\alpha\sigma_n}(R,i) \arrow[d, "\tau_+"'] \\
I_\alpha(Q,h) \arrow[r, "I_\alpha(f)"] & I_\alpha(R,i)
\end{tikzcd}$$

The invariance under the negative stabilization is similar. □

Now we obtain the first definition of quandle invariant functor.

DEFINITION 1. Let K be an oriented knot represented as a closed braid $\widehat{\alpha}$. The *quandle invariant functor* I_K is a functor $I_\alpha : \mathscr{PQ} \to \mathscr{Q}$. For a pointed quandle (Q,h), we call a quandle $I_K(Q,h)$ *the quandle invariant* associated to (Q,h).

By definition, it is easy to see the quandle invariant functor I_K has the following properties.

PROPOSITION 3. *Let K be an oriented knot.*

1. *If $\tau : (Q,h) \to (R,i)$ is a surjective morphism of pointed quandles, then $I_K(\tau)$: $I_K(Q,h) \to I_K(R,i)$ is also surjective.*
2. *For each pointed quandle (Q,h), $I_K(Q,h) = I_{-K^*}(Q,h)$, where $-K^*$ is the dual of K.*

PROOF. The assertion 1 is obvious from the definition of $I_K(Q,h)$. Let $K = \widehat{\beta}$. Then, $-K^* = \widehat{\beta^{-1}}$, so the relation $[\rho_{Q,h}(\beta)](q) = q$ is equivalent to the relation $[\rho_{Q,h}(\beta^{-1})](q) = q$, hence $I_K(Q,h)$ is isomorphic to $I_{-K^*}(Q,h)$. □

4. Diagrammatic description of quandle invariants.

We give an alternative definition of the quandle invariant functor by using a knot diagram. This construction is more combinatorial and is useful to study the (co)homology of the quandle invariants.

Let D be an oriented knot diagram. That is, D is an image of a projection of a knot on the plane so that is has only transverse double points. At each double point we assign the "over and under" information by breaking the underpassing segment. We call a connected component of diagram D a *large arc* and let $\mathscr{A}(D) = \{A_1, A_2, \dots, A_m\}$ be the set of large arcs of D. Each large arc A_i is decomposed to the subarcs $a_{i,1}, a_{i,2}, \dots, a_{i,k_i}$ by removing the points on A_i which correspond to the double points of the projection. We call these subarcs *small arcs* of D, and denote the set of small arcs by $\mathscr{S}\mathscr{A}(D)$. For a small arc a, we denote the large arc containing a as its subarc by the corresponding large letter A.

Let (Q, h) be a pointed quandle and $Q^{*m} = Q_A * Q_B * \cdots$ be the free product of m-copies of Q, where $m = \sharp\mathscr{A}(D)$. Each copy of Q is labeled by the large arc of D. For each $q \in Q$ we denote by q_A the element of $Q_A \subset Q^{*m}$ which corresponds to q. For $q \in Q$, we first define the map $c_q : \mathscr{S}\mathscr{A}(D) \to Q^{*m}$ by the following manner.

1. For a small arc a which contains the starting point of the large arc A, we define $c_q(a) = q_A$.
2. Let x be a crossing point of D and put a, a', b, c as in Figure 1. Assume that we have defined the value $c_q(a)$. Then, we define $c_q(a')$ by

$$\begin{cases} c_q(a') = c_q(a) \,\overline{*}\, h_A \text{ if the crossing } x \text{ is positive.} \\ c_q(a') = c_q(a) * h_A \text{ if the crossing } x \text{ is negative.} \end{cases}$$

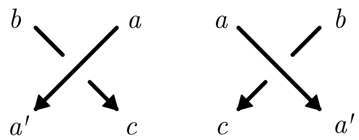

Figure 1. Labeling of small arcs around the crossing point.

Using the map c_q, we associate a relation $R(x; q)$ for each crossing point x and $q \in Q$ by the rule

$$R(x;q) : \begin{cases} c_q(c) = c_q(b) * h_A \text{ if the crossing } x \text{ is positive.} \\ c_q(c) = c_q(b) \,\overline{*}\, h_A \text{ if the crossing } x \text{ is negative.} \end{cases}$$

Now we define the quandle invariant $I_D(Q, h)$ by the presentation

$$I_D(Q,h) = \langle q_A \ \ (A \in \mathscr{A}(D),\ q \in Q) \mid R(x;q) \ (x : \text{crossings of } D,\ q \in Q) \rangle.$$

As in the definition using braid representation, I_D is a functor $I_D : \mathscr{P}\mathscr{Q} \to \mathscr{P}$, by defining $[I_D(f)](q_A) = f(q)_A$ for a pointed quandle morphism $f : (Q, h) \to (R, i)$.

Proposition 4. *The functor I_D is a knot invariant, and it coincides with the quandle invariant functor I_K defined in the previous section.*

Proof. We only prove an invariance of the quandle invariant $I_D(Q,h)$ for each pointed quandle (Q,h). Invariance as a functor is routine. By definition, it is easy to confirm that if a diagram D is a closed braid diagram $\widehat{\beta}$, then $I_D(Q,h) = I_\beta(Q,h)$. Thus, we only need to show that $I_D(Q,h)$ is a link invariant.

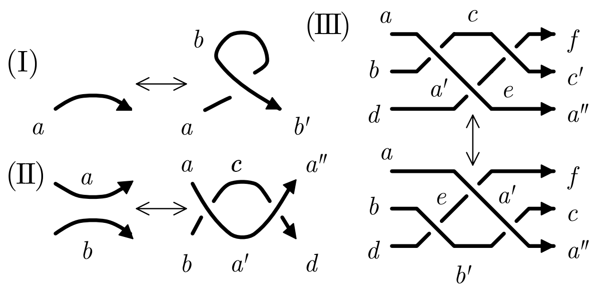

Figure 2. Reidemeister move invariance.

4.1. Invariance of Reidemeister move I.
Let x be the newly-added crossing generated by the Reidemeister move I. We consider the case x is a positive crossing. The negative crossing case is proved in a similar way. Put a, b, b' as in Figure 2 (I). Then the relation $R(x;q)$ is $c_q(b) = c_q(a) * h_B$, hence $c_q(b') = c_q(b) \overline{*} h_B = c_q(a)$. Thus this move does not change the quandle invariant.

4.2. Invariance of Reidemeister move II.
We consider the Reidemeister move II depicted in Figure 2 (II). Other cases are proved in a similar way. Take small arcs $\{a, a', a'', b, c, d\}$ as in Figure 2 (II). Then the relations at these two crossings are given by

$$\begin{cases} c_q(c) = c_q(b) * h_A \\ c_q(d) = c_q(c) \overline{*} h_A. \end{cases}$$

Hence $c_q(b) = c_q(d)$ and the contributions of the quandle Q_C to $I_D(Q,h)$ vanish. Hence these two diagrams define the same quandle.

4.3. Invariance of Reidemeister move III.
We consider the Reidemeister move III depicted in Figure 2 (III). Other cases are proved in a similar way. Take small arcs $\{a, a', a'', b, b', c, c', d, e, f\}$ as in Figure 2 (III). First of all, in the diagram above, three crossings provide the relations

$$\begin{cases} c_q(c) = c_q(b) * h_A \\ c_q(e) = c_q(d) * h_A \\ c_q(f) = c_q(e) * h_C. \end{cases}$$

Thus, $c_q(f) = (c_q(d) * h_A) * h_C$. By putting $q = h$, we obtain $h_C = h_B * h_A$, so $c_q(f) = (c_q(d) * h_A) * (h_B * h_A) = (c_q(d) * h_B) * h_A$. Similarly, we get $c_q(c') = (c_q(b) * h_A) \mathbin{\overline{*}} (h_B * h_A) = (c_q(b) \mathbin{\overline{*}} h_B) * h_A$.

On the other hand, from the diagram below, three crossings provide the relations

$$\begin{cases} c_q(e) = c_q(d) * h_B \\ c_q(f) = c_q(e) * h_A \\ c_q(c) = c_q(b') * h_A \end{cases}$$

so $c_q(c) = (c_q(b) \mathbin{\overline{*}} h_B) * h_A$ and $c_q(f) = (c_q(d) * h_B) * h_A$. Thus, the map c_q takes the same value on each small arc and these two diagrams defines the same quandle. □

From this diagrammatic definition, it is quite easy to check our quandle invariant functor is indeed an extension of the classical knot quandle.

Proof of Theorem 1.1. Let us take the trivial 1-quandle $T_1 = \{q\}$. Then the relation $R(x)$ at the crossing x is $q_C = q_B * q_A$, which is a relation in the classical Wirtinger presentation of the knot quandles in [**7**], [**8**]. □

Next we show that quandle invariants naturally contain the knot quandle as a subquandle.

Proposition 5. *There are a natural injection of the knot quandle $\iota : Q_K \to I_K(Q, h)$ and a natural surjection to the knot quandle $p : I_K(Q, h) \to Q_K$. Moreover, $p \circ \iota = id$.*

Proof. This proposition follows from Theorem 1.1 and the fact that the trivial 1-quandle is the initial and the final object of the category $\mathscr{P}\mathscr{Q}$. More precisely, let $i : T_1 \to (Q, h)$ be the natural inclusion and $\pi : (Q, h) \to T_1$ be the natural surjection. Then $\iota = i_*$ and $p = \pi_*$. □

We remark that from the presentation of $I_K(Q, h)$, for each long arc A of a knot diagram D the natural map $Q_A \to I_D(Q, h)$ is an injection. So we may also regard Q as a subquandle of $I_D(Q, h)$ by specifying an arc A of the diagram D. In the knot theory view point, this corresponds to a base point $* \in K$, thus this is equivalent to consider the corresponding long knot.

5. Homology and cohomology of quandle invariants.

In this section, we study the homology and cohomology groups of the quandle invariant $I_K(Q,h)$. For a quandle $(X,*)$, let $C_n^R(X)$ be the free abelian group generated by n-tuples of elements of X and $C_n^D(X)$ be the subgroup of $C_n^R(X)$ generated by n-tuples $(x_1,x_2,\ldots,x_n)$ of X with $x_i = x_{i+1}$ for some i. Let $\partial_n : C_n^R \to C_{n-1}^R$ be a homomorphism defined by

$$\partial_n(x_1,x_2,\ldots,x_n) = \sum_{i=1}^{n}(-1)^i\big[(x_1,x_2,\ldots,\widehat{x_i},\ldots,x_n) \\ - (x_1 * x_i, x_2 * x_i, \ldots, x_{i-1} * x_i, x_{i+1},\ldots,x_n)\big]$$

Then both $(C_*^R(X),\partial_*)$ and $(C_*^D(X),\partial_*)$ are chain complexes. Let $C_*^Q(X)$ be the quotient complex $(C_*^R(X),\partial_*)/(C_*^D(X),\partial_*)$. For an abelian group G, the G-coefficient n-th *quandle homology* and *cohomology groups* are defined by

$$H_n^Q(X;G) = H_n\big(C_*^Q(X)\otimes G\big), \quad H_Q^n(X;G) = H^n\big(\mathrm{Hom}(C_*^Q(X),G)\big)$$

respectively. The quandle (co)homology is an invariant of quandles, so the (co)homology groups of quandle invariant $I_K(Q,h)$ also define knot invariants.

Remark 1. The above definition of the quandle (co)homology is the simplest one. There is a general theory of quandle (co)homologies, including twisted coefficients [**1**], [**4**]. Many results in this section also remains true for such generalized homology theories. In particular, we can also extend the “generalized” cocycle invariants defined in [**4**], which is an extension of classical cocycle invariants. We mainly restrict the classical (abelian coefficient) case for the sake of simplicity.

First we observe that the quandle invariant contains all information of the homology and cohomology of knot quandles.

Lemma 1. *Let K be an oriented knot and (Q,h) be a pointed quandle. Let $\iota : Q_K \hookrightarrow I_K(Q,h)$ and $p : I_K(Q,h) \to Q_K$ be the natural maps in Proposition* 5. *Then for any coefficient group G, ι induces an injection of the homology groups $\iota_* : H_*^Q(Q_K;G) \hookrightarrow H_*^Q(I_K(Q,h);G)$. Similarly, p induces an injection of the cohomology groups.*

Proof. Since $p \circ \iota = id$, this is clear. □

Now we determine the first quandle homology group $H_1^Q(I_K(Q,h),\boldsymbol{Z})$.

Proof of Theorem 1.2. Let D be an oriented knot diagram which represents K. For $q \in Q$ and large arcs $A, B \in \mathscr{A}(D)$, we consider two elements $q_A, q_B \in I_K(Q,h)$. Since K is a knot, by definition of $I_K(Q,h)$ we may find a sequence of elements $\{h_i\}_{i=1,\dots,m}$ in $I_K(Q,h)$ such that

$$q_B = \big((\cdots(q_A *' h_1) *' h_2) *' \cdots\big) *' h_m$$

holds, where $*'$ represents either $*$ or $\overline{*}$. Thus q_A and q_B represents the same 1st homology class in $I_K(Q,h)$. Hence the map $Q \to I_K(Q,h)$ defined by $q \mapsto q_A$ induces an isomorphism between the 1st quandle homologies. □

Next we proceed to study the 2nd homology group. For an oriented knot K, Eisermann showed that if K is not an unknot, then $H_2^Q(Q_K;\boldsymbol{Z}) = \boldsymbol{Z}$ and $H_2^Q(Q_K;\boldsymbol{Z})$ is generated by the *orientation class* $[K]$. He also showed that if K is unknot, then $H_2^Q(Q_K;\boldsymbol{Z}) = 0$ [**6**].

We extend the orientation class for the quandle invariant $I_K(Q,h)$. *From now on, we assume that the quandle Q is finite.*

Let D be a knot diagram. Let us define a 2-chain $(D) \in C_2^Q(I_K(Q,h);\boldsymbol{Z})$ by

$$(D) = \sum_{q\in Q}\sum_{x} \varepsilon(x)\cdot\{(c_q(a),h_B) - (c_q(b),h_B)\}.$$

where x runs all crossing points of D and $\varepsilon(x)$ denotes the sign of the crossing x. At each crossing x, we take small arcs a, b, b', c as in Figure 3.

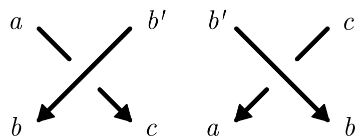

Figure 3. Definition of 2-chain (D).

Lemma 2. *The 2-chain (D) is a cycle and its representing homology class $[D] \in H_2^Q(I_K(Q,h);\boldsymbol{Z})$ is a knot invariant.*

Proof. First we show (D) is a 2-cycle. The boundary of (D) is given by

$$\begin{aligned}\partial(D) &= \sum_{x}\sum_{q} \partial\varepsilon(x)\{(c_q(a),h_B) - (c_q(b),h_B)\}\\ &= \sum_{x}\sum_{q} \varepsilon(x)(c_q(a) * h_B - c_q(a)) - \varepsilon(x)(c_q(b) - c_q(b) * h_B)\end{aligned}$$

$$= \sum_x \sum_q \varepsilon(x)(c_q(c) - c_q(a)) - \sum_q \varepsilon(x)(c_q(b) - c_q(b) * h_B)$$
$$= \sum_x \sum_q \varepsilon(x)(c_q(c) - c_q(a)) = \sum_x \sum_q (q_C - q_A).$$

Hence each long arc A contributes $\partial(D)$ by $\sum_q q_A$ at its initial point and by $-\sum_q q_A$ at its end point. Thus these two contributions cancel each other, so $\partial(D) = 0$.

We show the homology class $[(D)]$ is a knot invariant.

First we check the invariance of Reidemeister move I. Let x be the newly-added crossing generated by Reidemeister move I. We consider the case x is a positive crossing. The negative case is similar. Take small arcs a, b, b' around x as in the Figure 2 (I). Then the contribution of the newly-added crossing to the cycle (D) is

$$\sum_q (c_q(a), h_B) - (c_q(b'), h_B) = \sum_q (c_q(a), h_B) - (c_q(a), h_B) = 0,$$

hence the cycle (D) itself is invariant under the Reidemeister move I.

Next we consider the Reidemeister move depicted in Figure 2 (II). Other cases are similar. Take small arcs a, a', a'', b, c, d as in Figure 2 (II). Then the newly-added two crossings contribute the cycle (D) by

$$\sum_q (c_q(b), h_A) - (c_q(a'), h_A) - (c_q(d), h_A) + (c_q(a''), h_A)$$
$$= \sum_q -(c_q(a'), h_A) + (c_q(a''), h_A) = 0$$

hence the cycle (D) itself is invariant under the Reidemeister move II.

Finally we show the invariance under the Reidemeister move III. We consider the Reidemeister move depicted in Figure 2 (III). Other cases are similar. Take small arcs $\{a, a', a'', b, b', c, c', d, e, f\}$ as in Figure 2 (III). Then the three crossings in the diagram above contribute the cycle (D) by

$$\sum_q \big\{(c_q(b), h_A) - (c_q(a'), h_A) + (c_q(d), h_A) - (c_q(a''), h_A) + (c_q(e), h_C) - (c_q(c'), h_C)\big\}$$

and the three crossings in the diagram below contribute the cycle (D) by

$$\sum_q \{(c_q(d), h_B) - (c_q(b'), h_B) + (c_q(e), h_A) \\ - (c_q(a'), h_A) + (c_q(b'), h_A) - (c_q(a''), h_A)\}.$$

Their difference is a boundary of the 3-chain

$$\sum_q \{(q_D, h_B, h_A) - (q_B, h_B, h_A)\}.$$

Thus the homology class does not change under the Reidemeister move III. □

The (Q, h)-*fundamental class* (or orientation class) of a knot K is, by definition, $[(D)] \in H_2^Q(I_K(Q, h); \boldsymbol{Z})$ where D is a knot diagram representing K. From Lemma 2, this is independent of the choice of a diagram D. We denote this homology class by $[K]_{Q,h}$. For $Q = T_1$, the definition of the T_1-fundamental class agrees with the orientation class $[K]$ defined by Eisermann [**6**].

Proof of Theorem 1.3 and 4. Let $p_* : H_2^Q(I_K(Q, h); \boldsymbol{Z}) \to H_2^Q(K_Q; \boldsymbol{Z})$ be the map in Proposition 5. From the definition of $[K]_{Q,h}$, $p_*([K]_{Q,h}) = \sharp Q \cdot [K]$. Since $[K] = 0$ if and only if K is unknot [**6**], we conclude $[K]_{Q,h} = 0$ if and only if K is an unknot. The assertion 4 follows from the definition of the cycle (D). □

Remark 2. The isomorphism class of the quandle invariant $I_K(Q, h)$ only depends on the quandle homology class $[h] \in H_1^Q(Q; \boldsymbol{Z})$, because if h and h' represent the same homology class, then the set of the relations for $I_K(Q, h)$ and $I_K(Q, h')$ coincide. In particular, if $H_1^Q(Q; \boldsymbol{Z}) = \boldsymbol{Z}$, then the isomorphism class of $I_K(Q, h)$ is independent of a choice of $h \in Q$. However, we need to fix a point $h \in Q$ to realize I_K as a functor. The situation is similar to the fundamental group, since the isomorphism class of fundamental group is independent of the choice of base point if the underlying space is path-connected. Thus in our theory, an element h plays a very similar role to the base point in the theory fundamental group. This is why we call a pair (Q, h) a *pointed* quandle.

We remark that for unknot K, the quandle invariant $I_K(Q, h)$ is isomorphic to Q, so its 2nd homology does not always vanish whereas $H_2^Q(K_Q; \boldsymbol{Z}) = 0$.

The (Q, h)-fundamental class is decomposed as the sum of the *partial fundamental class* as follows. For an element $q \in Q$, we denote by $[q]$ the h-orbit of q. That is, $[q]$ is a subset of Q defined by $[q] = \{(\cdots(q * h) * h)\cdots * h) * h, (\cdots(q \mathbin{\overline{*}} h) \mathbin{\overline{*}} h)\cdots \mathbin{\overline{*}} h) \mathbin{\overline{*}} h\}$. The quandle Q is decomposed as a disjoint union of h-orbits as $Q = [h] \coprod [q_1] \coprod \cdots \coprod [q_k]$. For a knot diagram D and $q \in Q$, define

$$(D)_{[q]} = \sum_{q \in [q]} \sum_{x} \varepsilon(x) \cdot (c_q(a_x), c_q(b_x)).$$

where x runs all crossings of D. As in the proof of Lemma 2, $(D)_{[q]}$ is a cycle and its homology class is also independent of a choice of a diagram D. Thus the homology class $[(D)_{[q]}]$ also defines a knot invariant $[K]_{Q,h;[q]}$. We call this homology class the *partial* (Q,h)*-fundamental class* of K relative to $[q]$. From the proof of Lemma 2 and the proof of Theorem 1.3, we observe the partial (Q,h)-fundamental class has the following property.

Corollary 1. *Let K be an oriented knot and (Q,h) be a finite pointed quandle. We denote the h-orbit decomposition of Q by $Q = [q_0] \coprod \cdots \coprod [q_k]$. Then,*

1. $[K]_{Q,h} = \sum_{i=0}^{k} [K]_{Q,h;[q_i]}$.
2. *For each $q \in Q$, $[K]_{Q,h;[q]}$ is trivial if and only if K is unknot.*

We close this section by giving questions about the (co)homology of quandle invariants. As we have seen, the 1st homology group of quandle invariants contains no information of K. We would like to ask this phenomenon always occurs for all degrees.

Question 1.
1. Are the (co)homology group of $I_K(Q,h)$ always determined by $H_*^Q(Q_K; \boldsymbol{Z})$ and $H_*^Q(Q; \boldsymbol{Z})$?
2. Are the betti numbers of $I_K(Q,h)$ always determined by the betti numbers of Q_K and Q?

6. Quandle cocycle invariant via quandle invariant $I_K(Q,h)$.

In this section we extend the quandle cocycle invariants by using the quandle invariant $I_K(Q,h)$ and study their properties. As in the previous section, we always assume that every pointed quandle (Q,h) is finite.

Let D be an oriented knot diagram, and X be a finite quandle. Take a G-coefficient quandle 2-cocycle of X, $\phi : X \times X \to G$. We call a quandle morphism $\rho : I_K(Q,h) \to X$ a (Q,h)*-extended X-coloring.* For the classical knot quandle Q_K, we simply call a quandle morphism $\rho : Q_K \to X$ an *X-coloring.* For $q \in Q$ and a (Q,h)-extended coloring ρ, we denote by ρ_q the map $\rho \circ \overline{c_q} : \mathscr{S}\mathscr{A}(D) \to X$, where $\overline{c_q}$ is the composite of the coloring map $c_q : \mathscr{S}\mathscr{A}(D) \to Q^{*\sharp\mathscr{A}(D)}$ and the projection map $p : Q^{*\sharp\mathscr{A}(D)} \to I_K(Q,h)$.

At each crossing x of D we define a weight $W(x,q;\rho)$ by

$$W(x,q;\rho) = \left\{ \phi(\rho_q(a), \rho(h_B)) \cdot \phi(\rho_q(b), \rho(h_B))^{-1} \right\}^{\varepsilon(x)}.$$

The (Q,h)-*extended quandle cocycle invariant* is defined by the sum of the all weights

$$\Phi_{\phi,(Q,h)}(D) = \sum_{\rho} \prod_{q \in Q} \prod_{x} W(x,q;\rho) \in \boldsymbol{Z}[G]$$

where x runs all crossings of D and ρ runs all Q-extended X-colorings.

THEOREM 3. *The (Q,h)-extended quandle cocycle invariant is equal to the value $\sum_\rho \langle [\phi], \rho_*([D]) \rangle$. Here $\langle\ ,\ \rangle$ represents the pairing of the homology and the cohomology. Thus, $\Phi_{\phi,(Q,h)}(D)$ is a knot invariant and its value depends on the cohomology class $[\phi] \in H^2_Q(X;G)$.*

PROOF. From the definition of $[D]$, we obtain the equality

$$\langle [\phi], \rho_*([D]) \rangle = \sum_{\rho} \left(\prod_{q \in Q} \prod_{x} \left\{ \phi(\rho(c_q(a)), \rho(h_B)) \cdot \phi(\rho(c_q(b)), \rho(h_B))^{-1} \right\}^{\varepsilon(x)} \right).$$

The right hand is the definition of $\Phi_{\phi,(Q,h)}(D)$. $\square$

By definition, for the trivial 1-quandle T_1, the T_1-extended quandle cocycle invariant coincides with the classical quandle cocycle invariant $\Phi_\phi(K)$ defined in [**5**].

From the homological viewpoint of the cocycle invariant, we can decompose the (Q,h)-extended quandle cocycle invariants by using the partial fundamental classes. For a pointed quandle (Q,h) and $q \in Q$, let us define the *partial quandle cocycle invariant* $\Phi_{\phi,(Q,h);[q]}(K)$ by

$$\Phi_{\phi,(Q,h);[q]}(K) = \sum_{\rho} \big\langle [\phi], \rho_*([K]_{Q,h;[q]}) \big\rangle.$$

COROLLARY 2. *The partial quandle cocycle invariant $\Phi_{\phi,(Q,h);[q]}(K)$ is a knot invariant.*

Now let us proceed to study the properties of (Q,h)-extended quandle cocycle invariants. Unfortunately, in many cases (Q,h)-extended quandle cocycle invariants are determined by Q and usual quandle cocycle invariants $\Phi_\phi(K)$ as we shall explain below.

Before stating our results, we review some notions about quandle morphisms. We say a quandle morphism $f : Q \to R$ is *trivial* if $f(Q) = \{r\}$ for some $r \in R$. For each element $q \in Q$, the map $[*q] : Q \to Q$, $x \mapsto x * q$ defines a quandle automorphism of Q. The *inner automorphism group* $\mathrm{Inn}(Q)$ of Q is a subgroup of $\mathrm{Aut}(Q)$ generated by $\{[*q] \mid q \in Q\}$. By the definition of inner automorphisms, each nontrivial inner automorphism has at least one fixed point.

As a first step, we study the relationships between a (Q,h)-extended quandle coloring $\psi : I_K(Q,h) \to X$ and a usual knot coloring $\rho : Q_K \to X$. Let D be an oriented knot diagram of K. We assume that D is a long knot diagram with distinguished arcs A and A' which contains the point of infinity, as shown in Figure 4. Then, as we remarked earlier, $Q = Q_A$ is regarded as a subquandle of $I_K(Q,h)$.

For a (Q,h)-extended coloring $\psi : I_K(Q,h) \to X$, we obtain quandle morphisms $\rho_\psi : Q_K \to X$ and $f_\psi : Q \to X$ by considering the restriction of ψ to Q_K and Q_A respectively. ρ_ψ and f_ψ satisfy $\rho_\psi(h_A) = f_\psi(h)$.

Conversely, let $\rho : Q_K \to X$ and $f : Q \to X$ be quandle morphisms which satisfy $\rho(h_A) = f(h)$. Let us try to construct a (Q,h)-extended coloring by extending ρ and f. First we define $\psi_{\rho,f}(q_A) = f(q)$. Using the defining relations of $I_K(Q,h)$, we can uniquely determine the value $\psi_{\rho,f}(q_B)$ for other long arcs $B \in \mathscr{A}(D)$ as the following way. Let A, B, C be arcs of the diagram D around a crossing point x, as in Figure 4. Assume that we have already defined the value $\psi_{\rho,f}(q_A)$. Let p, n be the number of the positive and negative crossing points contained in A. Then, we define $\psi_{\rho,f}(q_C)$ by

$$\psi_{\rho,f}(q_C) = [*\rho(h_B)]^{\varepsilon(x)} \circ [*\rho(h_A)]^{n-p}(\psi_{\rho,f}(q_A)).$$

For each $x \in X$, let $f_x : Q \to X$ be a trivial quandle morphism defined by $f_x(q) = x$ for all $q \in Q$. Then the above procedure defines an inner automorphism $A_{\rho,D} : X \to X$, which sends $x \in X$ to $\psi_{\rho,f_x}(q_{A'}) \in X$, the color induced on A'. The inner automorphism $A_{\rho,D}$ only depends on the diagram D and ρ.

Observe that this construction of $\psi_{\rho,f}$ defines a well-defined morphism $I_K(Q,h) \to X$ if and only if the colorings on A and A' coincide, that is, $A_{\rho,D}(f(q)) = f(q)$ holds for all $q \in Q$.

Summarizing, we proved the following lemma.

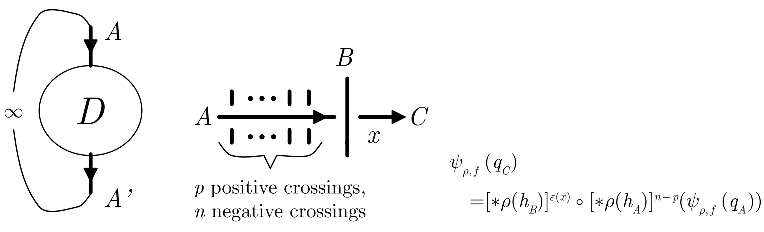

Figure 4. The definition of $\psi_{\rho,f}(q_C)$ and $A_{\rho,D}$.

LEMMA 3. *There is a one-to-one correspondence between the* (Q,h)*-extended* X*-colorings of a knot* K *and the pair* (ρ, f) *consisting of an* X*-coloring* $\rho : Q_K \to X$ *and a quandle morphism* $f : Q \to X$ *which satisfy the two conditions*:

1. $\rho(h_A) = f(h)$.
2. $A_{\rho,D}(f(q)) = f(q)$ *for all* $q \in Q$.

Now we provide some computations of (Q,h)-extended cocycle invariants.

PROPOSITION 6. *Let* (Q,h) *and* X *be finite* (*pointed*) *quandles and* K *be an oriented knot. Assume that one of the following conditions holds.*

1. *There are no non-trivial quandle morphisms* $Q \to X$.
2. *Each non-trivial inner automorphism of* X *has only one fixed point, and* $A_{\rho,D}$ *is non-trivial for all quandle morphisms* $\rho : Q_K \to X$.

Then

$$\Phi_{\phi,(Q,h)}(K) = P^{\sharp Q}(\Phi_\phi(K))$$

holds where $P^i : \boldsymbol{Z}G \to \boldsymbol{Z}G$ *is a map defined by* $g \mapsto g^i$ *and* $\Phi_\phi(K) = \Phi_{\phi,T_1}(K)$ *is the classical quandle cocycle invariant.*

PROOF. Let D be a long knot diagram of K.

First assume that assumption 1. holds. Since there are no non-trivial quandle morphisms from Q to X, by Lemma 3 there is a one-to-one correspondence between the set of (Q,h)-extended colorings and the usual knot colorings. For a usual knot coloring ρ, we denote by ψ_ρ its corresponding (Q,h)-extended coloring. This coloring map ψ_ρ is defined by $\psi_\rho(q_A) = \rho(h_A)$ for each $A \in \mathscr{A}(D)$. Let us denote by $W'(x,\rho)$ the classical weight $\phi(\rho(h_A), \rho(h_B))^{\varepsilon(x)}$. Then, the (classical) quandle cocycle invariant of K is defined by the sum of classical weights $\Phi_\phi(K) = \sum_\rho \prod_x W'(x;\rho)$.

By definition of ψ_ρ, for a crossing point x of D and an arbitrary element $q \in Q$, the equality $W(x,q;\psi_\rho) = W'(x;\rho)$ holds.

Similarly, if assumption 2. holds, by Lemma 3, there is also a one-to-one correspondence between the set of (Q,h)-extended colorings and the usual knot colorings. Thus in this case the same equality of weights holds.

Thus in either case, we obtain the equality

$$\begin{aligned}\Phi_{\phi,(Q,h)}(K) &= \sum_\rho \prod_q \prod_x W(x,q;\psi_\rho) \\ &= \sum_\rho \prod_x W'(x;\rho)^{\sharp Q} = P^{\sharp Q}(\Phi_\phi(K)). \qquad \square\end{aligned}$$

Example 3. We give some examples where the criterion of Proposition 6 work.

1. Let S_4 be the Alexander quandle $\boldsymbol{Z}_2[T,T^{-1}]/(T^2+T+1)$ and R_3 be the dihedral quandle of order 3. Then, there are no non-trivial quandle morphism $R_3 \to S_4$. Hence for all $h \in R_3$, the (R_3,h)-extended cocycle invariant for a cocycle of S_4 is determined by the usual cocycle invariant.
2. Let K be a knot with less than 7 crossings. Then, the inner automorphisms $A_{D,\rho}$ are non-trivial for all coloring maps $\rho : Q_K \to S_4$. Since each non-trivial inner automorphism of S_4 has exactly one fixed point, we conclude that for such knots, the (S_4,h)-extended cocycle invariant for a cocycle of S_4 is determined by the usual cocycle invariant.

Next we consider the case that Q is a trivial quandle. In this case we can also represent the extended cocycle invariants by the classical cocycle invariants, but the formula is slightly complicated.

Proposition 7. *Let T_m be the trivial m-quandle and ϕ be a 2-cocycle of a finite quandle X. Then there exist integers $\{N_i\}_{i=1,\dots,m}$ such that for every oriented knot K, (T_m,h)-extended quandle cocycle invariants satisfy the equality*

$$\Phi_{\phi,(T_m,h)}(K) = \sum_{i=1}^{m} N_i \cdot P^i(\Phi_\phi(K)).$$

The integers N_i depend on only m and X.

Proof. We assume $m \geq 2$, since $m = 1$ case is trivial. Let D be a long knot diagram of K. For an X-coloring $\rho : Q_K \to X$, let F_ρ be the set of quandle morphisms $f : Q \to X$ such that $\rho(h_A) = f(h)$. Let X' be the image of $\rho(Q_K)$.

Since the knot quandle Q_K is connected (that is, the action of the inner automorphism group of Q_K is transitive), X' is also connected. Thus for all $f \in F_\rho$ and $q \in Q$, if $f(q) \notin X'$, then $f(q) * x' = f(q)$ holds for all $x' \in X'$. Similarly, by the same reason, if $f(q) \in X'$, then $f(q) = f(h)$ holds.

By definition, $A_{\rho,D}$ belongs to the subgroup of $\mathrm{Inn}(X)$ generated by $\{[*x'] \mid x' \in X'\}$. Since ρ is a knot coloring, $A_{\rho,D}(\rho(h_A)) = \rho(h_A)$ always holds. Therefore, from the above observations, the inner automorphism $A_{\rho,D}$ is always trivial when it is restricted to $f(T_m)$. Thus by Lemma 3, for all $f \in F_\rho$, a pair (ρ,f) always defines a (T_m,h)-extended coloring $\psi_{\rho,f}$.

For $q \in Q$, if $f(q) \neq f(h)$, then $\psi_{\rho,f}(c_q(b)) = f(q)$ for each small arc b. Thus in this case $W(x,q;\psi_{\rho,f}) = 1$ holds for each crossing point x. On the other hand, if $f(q) = f(h)$, then $\psi_{\rho,f}(c_q(b)) = \rho(h_B)$ for each small arc b. Thus, in this case

$W(x,q;\psi_{\rho,f})$ is equal to the classical weight $W'(x;\rho)$.

Now we define the integer N_i as follows. Let us take an element $x \in X$ and define $N_i = \sharp\{f : T_m \to X \mid \sharp f^{-1}(x) = i\}$ for $i = 1,\dots,m$. The integers N_i are independent of a choice of x, and only depend on m and X.

Then by using the obtained equality of weights, we obtain the desired equality

$$\begin{aligned}\Phi_{\phi,(T_m,h)}(D) &= \sum_{\rho}\sum_{F_\rho}\prod_{q\in T_m}\prod_{x} W(x,q;\psi_{\rho,f}) = \sum_{\rho}\sum_{i=1}^{m} N_i \prod_{x} W'(x;\rho)^i \\ &= \sum_{i=1}^{m} N_i \left(\sum_{\rho}\prod_{x} W'(x;\rho)^i\right) = \sum_{i=1}^{m} N_i \cdot P^i(\Phi_\phi(D)). \qquad \square\end{aligned}$$

As these examples suggest, in many simple cases extended quandle cocycle invariants are determined by the usual quandle cocycle invariants (and the pointed quandle (Q,h)). In fact, the author cannot find an example of knots whose extended cocycle invariants can distinguish them while the corresponding classical cocycle invariant cannot. Thus, we would like to pose the following question.

Question 2. Let X be a finite quandle and ϕ be a 2-cocycle of X. For two oriented knots K and K', if their classical quandle cocycle invariants $\Phi_\phi(K)$ and $\Phi_\phi(K')$ are the same, then for each finite pointed quandle (Q,h), are the (Q,h)-extended cocycle invariants $\Phi_{\phi,(Q,h)}(K)$ and $\Phi_{\phi,(Q,h)}(K')$ always the same?

Even if the above question has an affirmative answer, it might be difficult to construct an explicit formula to write the (Q,h)-extended cocycle invariants by the classical cocycle invariants.

7. Spatial realization of quandle invariants.

In this section we describe a spatial realization of the quandle invariant $I_K(Q,h)$ in some special cases. The content of this section is a direct extension of section 3 of [**3**] and proofs are almost the same, so we only sketch the proof. We remark that this approach does not produce the quandle invariant functor, because it is not known that every quandle and quandle morphism admits a spatial realization as the fundamental quandles, unlike the group cases.

First we review the notion of the fundamental quandle introduced by Joyce [**7**] and Matveev [**8**]. A pointed pair of topological space is a triple $(X,A,*)$ consisting of a topological space X, its subspace A and a base point $* \in X\backslash A$. A map of pair of topological spaces is a continuous map $f : (X,A,*) \to (Y,B,*')$ with $f^{-1}(B) = A$ and $f(*) = *'$.

Let $N = \{z \in \boldsymbol{C} \mid |z| \leq 1\} \cup \{z \in \boldsymbol{R} \subset \boldsymbol{C} \mid -5 \leq z \leq -1\}$. We denote by $-\!\!\odot$ the pointed pair of topological space $(N, 0, -5)$. The fundamental quandle $Q(X, A, *)$ of a pointed pair of topological space $(X, A, *)$ is defined as the homotopy classes of the map $f : -\!\!\odot \to (X, A, *)$. The quandle operation $*$ is defined by Figure 5.

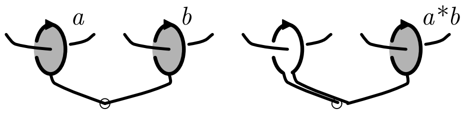

Figure 5. Quandle operation $*$.

Under some conditions, for example, if X is a compact smooth manifold and A is a proper codimension two submanifold, we can define the notion of positive intersections. The *positive fundamental quandle* $Q^+(X, A, *)$ is a subquandle of $Q(X, A, *)$ generated by the map $f : -\!\!\odot \to (X, A, *)$ which positively intersects with A at the point $f(0)$. Geometrically, the knot quandle Q_K of a knot K is defined as the positive fundamental quandle $Q^+(S^3, K, *)$.

Let $K = \widehat{\beta}$ be a knot represented as a closed n-braid. Let Q be a positive fundamental quandle of a pointed topological pair $(X, A, *)$ and $f : -\!\!\odot \to (X, A, *)$ be a map which represents the element $h \in Q$.

Let D be a 2-disc $D = \{z \in \boldsymbol{C} \mid |z| \leq n + 1\}$, $P = \{p_i = (i, 0) \in D \mid i = 1, 2, \dots, n\}$ and $*$ be the base point lying on ∂D. We consider the pointed topological pair $(D, P, *)$. Let $g_i : -\!\!\odot \to (D, P, *)$ be the map defined as in Figure 6 and we denote its image in D by N_i. Now glue n-copies of a pointed topological pair $(X, A, *)$ to $(D, P, *)$ along N_i by the map $f \circ g_i^{-1}$. Let us denote the obtained pointed topological pair by $(Z, S, *)$. Then, the fundamental quandle $Q^+(Z, S, *)$ is isomorphic to Q^{*n}.

Let C_i (resp. $C_{i,i+1}$) be a simple closed curve in D which encloses p_i (resp. p_i and p_{i+1}). We denote the half-Dehn twist along C_i (resp. $C_{i,i+1}$) by τ_i (resp. $\tau_{i,i+1}$). Let $T_i : D\backslash P \to D\backslash P$ be a homomorphism defined by $\tau_i^{-3}\tau_{i+1}^{-1}\tau_{i,i+1}$ (See Figure 6). The homeomorphism T_i can be extended as a homeomorphism of the pointed topological pair $T_i^X : (Z, S, *) \to (Z, S, *)$.

The following lemma is proved by the same way as in the proof of Proposition 3.2 in [**3**].

Lemma 4. *Let (Q, h) be a pointed quandle and $(X, K, *)$ be a pointed topological pair defined as the above. Then the induced homomorphism $\Phi : B_n \to Aut(Q^{*n})$ defined by $\sigma_i \mapsto (T_i^X)_*$ is identical with the associated braid representation $\rho_{Q,h}$.*

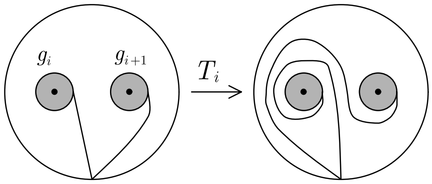

Figure 6. Maps g_i and T_i.

Now the geometric construction goes as follows. Fix a word representative of β, and let $B : (Z, S, *) \to (Z, S, *)$ be a homomorphism which corresponds to β. Then by Lemma 4, the induced map $B_* : Q^{*n} \to Q^{*n}$ is identical with the image of the associated braid representation $\rho_{Q,h}(\beta)$. Let $(M(Z), M(S), *)$ be the mapping torus of B. Then the total space $M(Z)$ has a torus boundary $\partial D \times S^1$. Along this torus boundary, attach a solid torus so that $\{*\} \times S^1$ is identified with $\partial D^2 \times \{\text{point}\}$. We denote the obtained pointed pair of space by $(\Omega, M(S), *)$.

THEOREM 4. *The positive fundamental quandle of the pointed pair of topological space* $(\Omega, M(S), *)$ *is isomorphic to the quandle invariant* $I_K(Q, h)$.

PROOF. Let T_Q be another copy of Q^{*n}. We denote an element of T_Q corresponding to $q \in Q^{*n}$ by t_q. Then the positive fundamental quandle $Q^+(M(Z), M(S), *)$ has a presentation

$$Q^+(M(Z), M(S), *) = \langle q, t_q \mid t_q = [\rho_{Q,h}(\beta)](q) \rangle$$

Geometrically, t_q is represented by a map depicted in Figure 7. Then, gluing a solid torus corresponds to the adding relations $\{q = t_q\}$, hence the positive fundamental quandle of $(\Omega, M(S), *)$ has a presentation

$$Q^+(\Omega, M(S), *) = \langle q \in Q^{*n} \mid q = [\rho_{Q,h}(\beta)](q) \rangle,$$

which is a presentation of the quandle invariant $I_K(Q, h)$. $\square$

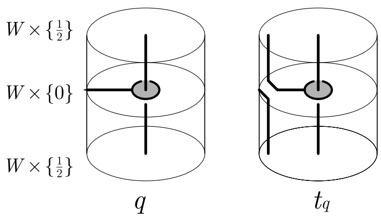

Figure 7. Generators q and t_q of $Q^+(M(Z), M(S), *)$.

This point of view provides a geometrical meaning of quandle invariants in some special cases.

Example 4. Let FQ_n be the free quandle of rank n generated by $\{q_1, \dots, q_n\}$. This quandle is the positive fundamental quandle of the n-punctured disc $(D^2, \{p_1, \dots, p_n\}, *)$. By Theorem 4, for each i, $I_K(FQ_n, q_i)$ is isomorphic to the link quandle of the n-parallelization of the knot K.

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