J. Math. Soc. Japan
Vol. 64, No. 3 (2012) pp. 969–984
doi: 10.2969/jmsj/06430969

A theorem of Hadamard-Cartan type for Kähler magnetic fields

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(Received Feb. 3, 2011)

Abstract. We study the global behavior of trajectories for Kähler magnetic fields on a connected complete Kähler manifold M of negative curvature. Concerning these trajectories we show that theorems of Hadamard-Cartan type and of Hopf-Rinow type hold: If sectional curvatures of M are not greater than c (< 0) and the strength of a Kähler magnetic field is not greater than $\sqrt{|c|}$, then every magnetic exponential map is a covering map. Hence arbitrary distinct points on M can be joined by a minimizing trajectory for this magnetic field.

1. Introduction.

The aim of this paper is to study some global properties on trajectories for Kähler magnetic fields, especially on a Kähler manifold of negative curvature. Generally, a closed 2-form on a Riemannian manifold is said to be a *magnetic field* because it is considered as a generalization of static magnetic fields on a Euclidean 3-space (cf. [**9**]). When a magnetic field is given, we can consider a dynamical system on the unit tangent bundle which is a perturbation of the geodesic flow. For manifolds of negative curvature, it is well-known that their geodesic flows are hyperbolic. In his paper [**5**], Gouda, having the structure stability theorem on hyperbolic flows in his mind, studied the relationship between hyperbolicity of magnetic fields and curvature conditions of base manifolds.

To treat arbitrary magnetic fields on a given manifold and to investigate their properties are very interesting study, but the author thinks that there is another angle in the study of magnetic fields. When we study Riemannian manifolds, it is needless to say that geodesics play quite an important role. There is an interaction between shapes of Riemannian manifolds and properties of their geodesics. From this point of view the author hopes the following: If one considers magnetic fields corresponding to some geometric structure on base manifolds, then there is

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2010 *Mathematics Subject Classification.* Primary 53C22; Secondary 53B35.

Key Words and Phrases. Kähler magnetic fields, trajectories, trajectory-spheres, trajectory-harps, comparison theorem, theorem of Hopf-Rinow, theorem of Hadamard-Cartan.

The author is partially supported by Grant-in-Aid for Scientific Research (C) (No. 20540071), Japan Society for the Promotion of Sciences.

some interaction between properties of this structure and properties of trajectories, which are motions of charged particles under magnetic fields.

In this paper we take Kähler manifolds. On a Kähler manifold M with complex structure J and Riemannian metric $\langle\ ,\ \rangle$, we have natural closed 2-forms which are constant multiples of the Kähler form $\boldsymbol{B}_J$. They are typical examples of uniform magnetic fields, and are called *Kähler magnetic fields*. Here, uniform means that the strength of a magnetic field does not depend on the choice of unit tangent vectors of a base manifold (see [**2**] for the definition). A smooth curve γ parameterized by its arclength is said to be a trajectory for a Kähler magnetic field $\boldsymbol{B}_\kappa = \kappa \boldsymbol{B}_J$ if it satisfies $\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$. When we study Riemannian geometry, the theorem of Hopf-Rinow which shows the equivalence of completeness and geodesical completeness is one of basic results (see for example [**4**], [**8**]). As a corollary of this theorem we have an important property of geodesics: For arbitrary distinct points p, q on a connected complete Riemannian manifold there is a minimizing geodesic joining them. We consider this property for trajectories for Kähler magnetic fields.

THEOREM 1. *Let $\boldsymbol{B}_\kappa$ be a Kähler magnetic field on a connected complete Kähler manifold M whose sectional curvatures satisfy* $\mathrm{Riem}^M \le c < 0$. *If $|\kappa| \le \sqrt{|c|}$, for arbitrary distinct points $p, q \in M$, there is a minimizing trajectory for $\boldsymbol{B}_\kappa$ which goes from p to q. In particular, when M is simply connected, there exists a unique trajectory for $\boldsymbol{B}_\kappa$ of p to q.*

On a manifold of nonpositive curvature we have one more important result on geodesics which is called the theorem of Hadamard-Cartan: Every exponential map on this manifold is a covering map. We can generalize this in the following manner. We define magnetic exponential maps from trajectories for magnetic fields by just the same way as for exponential maps.

THEOREM 2. *Let M be a connected complete Kähler manifold whose sectional curvatures satisfy* $\mathrm{Riem}^M \le c < 0$. *If $|\kappa| \le \sqrt{|c|}$, then every magnetic exponential map for $\boldsymbol{B}_\kappa$ is a covering map. In particular, when M is simply connected, every magnetic exponential map for $\boldsymbol{B}_\kappa$ is bijective.*

Our idea is based on comparison theorems on Jacobi fields. In Section 2, we study magnetic exponential maps on complex space forms. After discussing a comparison theorem on trajectory-harps, which are constructed by trajectories and geodesics joining their points, in Section 3, we show our results.

2. Trajectory-spheres on a complex space form.

Let M be a complete Kähler manifold. We take a Kähler magnetic field $\boldsymbol{B}_\kappa$ on M. One can easily see that every trajectory is defined on $(-\infty, \infty)$ (see [**7**]). For a unit tangent vector $v \in UM$, we denote by γ_v a trajectory for $\boldsymbol{B}_\kappa$ with initial vector v. Given a point $p \in M$ we define a *magnetic exponential map* $\boldsymbol{B}_\kappa \exp_p : T_pM \to M$ of the tangent space T_pM at $p \in M$ by

$$\boldsymbol{B}_\kappa \exp_p(w) = \begin{cases} \gamma_{w/\|w\|}(\|w\|), & \text{if } w \neq 0_p, \\ p, & \text{if } w = 0_p. \end{cases}$$

Clearly, for a trivial magnetic field $\boldsymbol{B}_0$, it is an exponential map $\exp_p : T_pM \to M$. In [**2**] we studied its differential by investigating magnetic Jacobi fields (see Section 4). In this paper we study some properties of themselves.

For a positive r we put $\boldsymbol{B}_\kappa S_p(r) = \{\boldsymbol{B}_\kappa \exp_p(rv) \mid v \in U_pM\}$ and call it a *trajectory-sphere* of radius r centered at p. Trivially, when $\kappa = 0$, it is a geodesic sphere $S_p(r)$ of radius r. We are interested in the difference between geodesic spheres and trajectory-spheres. It is well-known that geodesics emanating from the center of a geodesic sphere cross this geodesic sphere orthogonally. We are also interested in how trajectories cross trajectory-spheres. In this section we study these points on a complex space form, which is one of a complex projective space $\boldsymbol{C}P^n$, a complex Euclidean space $\boldsymbol{C}^n$ and a complex hyperbolic space $\boldsymbol{C}H^n$. When γ is a trajectory for $\boldsymbol{B}_\kappa$, we call its restriction $\gamma|_{[a,b]}$ onto a finite interval $[a, b]$ a trajectory-segment, and call its restriction onto $(-\infty, 0]$ or on $[0, \infty)$ a trajectory half line.

On a complex Euclidean space $\boldsymbol{C}^n$, a trajectory γ for $\boldsymbol{B}_\kappa$ is a circle of radius $1/|\kappa|$ in the sense of Euclidean geometry, hence is closed of length $2\pi/|\kappa|$. Every trajectory-sphere coincides with some geodesic sphere.

Proposition 1. *For a Kähler magnetic field $\boldsymbol{B}_\kappa$ on $\boldsymbol{C}^n$, we have the following for an arbitrary point $p \in \boldsymbol{C}^n$:*

(1) *For $0 \leq r \leq 2\pi/|\kappa|$, we have $\boldsymbol{B}_\kappa S_p(r) = S_p(\ell_\kappa(r; 0))$ with $\ell_\kappa(r; 0) = (2/|\kappa|)\sin(|\kappa|r/2)$;*
(2) *Every trajectory for $\boldsymbol{B}_\kappa$ emanating from p and the outward unit normal of $\boldsymbol{B}_\kappa S_p(r)$ make the angle $\theta_\kappa(r; 0) = |\kappa|r/2$ when $0 < r < 2\pi/|\kappa|$.*

Proof. We take a trajectory γ for $\boldsymbol{B}_\kappa$ with $\gamma(0) = p$. Since $\gamma([0, 2\pi/|\kappa|])$ is a circle, when $0 < r \leq \pi/|\kappa|$, the geodesic segment joining $\gamma(0)$ and $\gamma(r)$ is a subtense for the circular arc $\gamma([0, r])$. Considering a triangle of vertices $p, \gamma(r)$ and the center of the circle γ and a sector of circular arc $\gamma([0, r])$, we find that the distance between p and $\gamma(r)$ is $(2/|\kappa|)\sin(|\kappa|r/2)$ and that $\theta_\kappa(r;0)$ coincides with the half of the angle $|\kappa|r$ of the sector. As we can consider similarly for the case $\pi/|\kappa| < r \leq 2\pi/|\kappa|$, we get the conclusion. $\square$

This example shows that the corollary of the theorem of Hopf-Rinow does not hold in general: On $\boldsymbol{C}^n$, if the distance between two points is greater than $2/|\kappa|$, then they can not be joined by trajectory-segments for $\boldsymbol{B}_\kappa$. Clearly, if the distance between two points is not greater than $2/|\kappa|$, then they can be joined by a trajectory-segment for $\boldsymbol{B}_\kappa$.

By the expressions of $\ell_\kappa(r;0)$ and $\theta_\kappa(r;0)$ for $0 < r < \pi/|\kappa|$, we find

1) $\ell_\kappa(r;0) = \ell_{-\kappa}(r;0)$, $\theta_\kappa(r;0) = \theta_{-\kappa}(r;0)$,
2) $\kappa \mapsto \ell_\kappa(r;0)$ is monotone decreasing on $0 \leq \kappa < 1/r$;
3) $\kappa \mapsto \delta_\kappa(r;0) = \cos\theta_\kappa(r;0)$ is monotone decreasing on $0 \leq \kappa < 1/r$.

Generally, when γ is a trajectory for $\boldsymbol{B}_\kappa$, then a curve $\tilde{\gamma}$ which is given by reversing the parameter of γ, that is $\tilde{\gamma}(t) = \gamma(r-t)$, is a trajectory for $\boldsymbol{B}_{-\kappa}$. If a geodesic σ satisfies $\sigma(0) = \gamma(0)$ and $\sigma(\ell) = \gamma(r)$, we see the angle between $\dot{\gamma}(0)$ and $\dot{\sigma}(0)$ coincides with the angle between $\dot{\tilde{\gamma}}(0)$ and $\dot{\tilde{\sigma}}(0)$, where $\tilde{\sigma}$ is given by $\tilde{\sigma}(t) = \sigma(\ell - t)$. Thus $\theta_\kappa(r;0)$ also shows the angle between $\dot{\gamma}(0)$ and $\dot{\sigma}(0)$. On $\boldsymbol{C}^n$ the following property is clear.

PROPOSITION 2. *If the distance between two distinct points $p, q \in \boldsymbol{C}^n$ is not greater than $2/|\kappa|$, there is a unique trajectory for $\boldsymbol{B}_\kappa$ which goes from p to q.*

On a complex projective space $\boldsymbol{C}P^n(c)$ of constant holomorphic sectional curvature c, a trajectory γ for $\boldsymbol{B}_\kappa$ is a "small" circle on a totally geodesic $\boldsymbol{C}P^1(c) = S^2$ (see [**1**]). It is closed of length $2\pi/\sqrt{\kappa^2+c}$. On $\boldsymbol{C}P^n$ also every trajectory-sphere coincides with some geodesic sphere.

PROPOSITION 3. *For a Kähler magnetic field $\boldsymbol{B}_\kappa$ on $\boldsymbol{C}P^n(c)$, we have the following for an arbitrary point $p \in \boldsymbol{C}P^n$ and an arbitrary r with $0 < r < 2\pi/\sqrt{\kappa^2+c}$:*

(1) *$\boldsymbol{B}_\kappa S_p(r) = S_p(\ell_\kappa(r;c))$ with $\ell_\kappa(r;c)$ which satisfies*

$$\sqrt{\kappa^2+c}\,\sin\big(\sqrt{c}\,\ell_\kappa(r;c)/2\big) = \sqrt{c}\,\sin\big(\sqrt{\kappa^2+c}\,r/2\big);$$

(2) *Every trajectory for $\boldsymbol{B}_\kappa$ emanating from p and the outward unit normal of a trajectory-sphere $\boldsymbol{B}_\kappa S_p(r)$ make the angle $\cos^{-1}\delta_\kappa(r;c)$, where*

$$\delta_\kappa(r;c) = \frac{\sqrt{\kappa^2+c}\cos\bigl(\sqrt{\kappa^2+c}\,r/2\bigr)}{\sqrt{\kappa^2+c\cos^2\bigl(\sqrt{\kappa^2+c}\,r/2\bigr)}}.$$

On the interval $\bigl[0,\pi/\sqrt{\kappa^2+c}\,\bigr]$, the function $r\mapsto \ell_\kappa(r;c)$ is monotone increasing and the function $r\mapsto\delta_\kappa(r;c)$ is monotone decreasing.

PROOF. We first study the case $c=4$. For a trajectory γ for $\boldsymbol{B}_\kappa$, we consider its horizontal lift $\hat\gamma$ with respect to the Hopf fibration $\varpi : S^{2n+1}(1)\bigl(\subset \boldsymbol{C}^{n+1}\bigr)\to \boldsymbol{C}P^n(4)$. It is of the form

$$\hat\gamma(t) = e^{\sqrt{-1}\kappa t/2}\Bigl\{\cos\frac{1}{2}\sqrt{\kappa^2+4}\,t\;z+(\kappa^2+4)^{-1/2}\sin\frac{1}{2}\sqrt{\kappa^2+4}\,t\bigl(-\sqrt{-1}\kappa z+2v\bigr)\Bigr\},$$

where $z\in S^{2n+1}\subset \boldsymbol{C}^{n+1}$ satisfies $\varpi(z)=\gamma(0)$ and a horizontal vector $(z,v)\in T_zS^{2n+1}$ with respect to ϖ satisfies $d\varpi((z,v))=\dot\gamma(0)$. A horizontal lift $\hat\sigma$ of the minimizing geodesic σ joining $\gamma(0)$ and $\gamma(r)$ is of the form $\hat\sigma(t)=\cos t\,z+\sin t\,u$ with some $u=e^{\sqrt{-1}\theta}v$, because γ lies on some totally geodesic $\boldsymbol{C}P^1$. As we have $\gamma(r)=\sigma(\ell_\kappa(r;c))$, there is a real number ϕ with $\hat\gamma(r)=e^{\sqrt{-1}\phi}\hat\sigma(\ell_\kappa(r;c))$. We take the Hermitian products of both sides on $\boldsymbol{C}^{n+1}$ with v and with z. As the Hermitian product $\langle\!\langle v,z\rangle\!\rangle$ of z and v satisfies $\langle\!\langle v,z\rangle\!\rangle=0$, we have

$$\begin{cases} e^{\sqrt{-1}(\phi+\theta-(\kappa r/2))}\sin\ell_\kappa(r;4) = \dfrac{2}{\sqrt{\kappa^2+4}}\sin\dfrac{1}{2}\sqrt{\kappa^2+4}\,r, \\[2ex] e^{\sqrt{-1}(\phi-(\kappa r/2))}\cos\ell_\kappa(r;4) = \cos\dfrac{1}{2}\sqrt{\kappa^2+4}\,r - \dfrac{\sqrt{-1}\kappa}{\sqrt{\kappa^2+4}}\sin\dfrac{1}{2}\sqrt{\kappa^2+4}\,r. \end{cases}$$

Taking the absolute values of both sides of the first equality, we get $\sin\ell_\kappa(r;4)=\bigl(2/\sqrt{\kappa^2+4}\bigr)\sin(\sqrt{\kappa^2+4}\,r/2)$. Since the curve $t\mapsto\gamma(r-t)$ is a trajectory for $\boldsymbol{B}_{-\kappa}$, we have $\delta_\kappa(r;4)=\cos\theta$. We hence obtain the relations in the case $c=4$.

In order to study general cases, we consider a homothetical change of metrics. If we change the metric $\langle\ ,\ \rangle$ on a Kähler manifold homothetically to the metric $\lambda^2\langle\ ,\ \rangle$ with some positive λ, for a trajectory γ for $\boldsymbol{B}_\kappa$ with respect to the original metric, the curve $\tilde\gamma(t)=\gamma(t/\lambda)$ is a trajectory for $\boldsymbol{B}_{\kappa/\lambda}$ with respect to the new metric. Since sectional curvatures change λ^{-2}-times of the original sectional curvatures, we obtain the conclusion. $\square$

As was used in the above proof, we have $\ell_\kappa(r;c)=\ell_{-\kappa}(r;c)$ and $\delta_\kappa(r;c)=\delta_{-\kappa}(r;c)$ for trajectories on $\boldsymbol{C}P^n(c)$. The proof also shows the following.

PROPOSITION 4.

(1) *If the distance between two distinct points $p, q \in \boldsymbol{C}P^n(c)$ is not greater than $(2/\sqrt{c}) \sin^{-1} \sqrt{c/(\kappa^2 + c)}$, then there is a unique trajectory for $\boldsymbol{B}_\kappa$ which goes from p to q.*
(2) *If the distance between p, q is longer than $(2/\sqrt{c}) \sin^{-1} \sqrt{c/(\kappa^2 + c)}$, then there are no trajectory-segments joining these points.*

On a complex hyperbolic space $\boldsymbol{C}H^n(c)$ of constant holomorphic sectional curvature c, every trajectory for a Kähler magnetic field is a curve without self-intersections and lies on some totally geodesic $\boldsymbol{C}H^1(c) = H^2$ (see [**1**]). Features of trajectories depend on strengths of Kähler magnetic fields. When $|\kappa| > \sqrt{|c|}$, a trajectory for $\boldsymbol{B}_\kappa$ is closed of length $2\pi/\sqrt{\kappa^2 + c}$, and when $|\kappa| \leq \sqrt{|c|}$, it is unbounded. On $\boldsymbol{C}H^n$ also every trajectory-sphere coincides with some geodesic sphere.

PROPOSITION 5. *For a Kähler magnetic field $\boldsymbol{B}_\kappa$ on $\boldsymbol{C}H^n(c)$, we have the following at an arbitrary point $p \in \boldsymbol{C}H^n$.*

(1) *$\boldsymbol{B}_\kappa S_p(r) = S_p(\ell_\kappa(r; c))$ with $\ell_\kappa(r; c)$ which satisfies the following relation*;

$$\begin{cases} \sqrt{|c| - \kappa^2} \sinh\big(\sqrt{|c|}\,\ell_\kappa(r; c)/2\big) = \sqrt{|c|} \sinh\big(\sqrt{|c| - \kappa^2}\, r/2\big), & \text{if } |\kappa| < \sqrt{|c|}, \\ 2 \sinh\big(\sqrt{|c|}\,\ell_\kappa(r; c)/2\big) = \sqrt{|c|}\, r, & \text{if } \kappa = \pm\sqrt{|c|}, \\ \sqrt{\kappa^2 + c} \sinh\big(\sqrt{|c|}\,\ell_\kappa(r; c)/2\big) = \sqrt{|c|} \sin\big(\sqrt{\kappa^2 + c}\, r/2\big), & \text{if } |\kappa| > \sqrt{|c|}. \end{cases}$$

(2) *Every trajectory for $\boldsymbol{B}_\kappa$ emanating from p and the outward unit normal of a trajectory-sphere $\boldsymbol{B}_\kappa S_p(r)$ make the angle $\cos^{-1} \delta_\kappa(r; c)$, where*

$$\delta_\kappa(r; c) = \begin{cases} \dfrac{\sqrt{|c| - \kappa^2} \cosh\big(\sqrt{|c| - \kappa^2}\, r/2\big)}{\sqrt{|c| \cosh^2\big(\sqrt{|c| - \kappa^2}\, r/2\big) - \kappa^2}}, & \text{if } |\kappa| < \sqrt{|c|}, \\ \dfrac{2}{\sqrt{|c| r^2 + 4}}, & \text{if } \kappa = \pm\sqrt{|c|}, \\ \dfrac{\sqrt{\kappa^2 + c} \cos\big(\sqrt{\kappa^2 + c}\, r/2\big)}{\sqrt{\kappa^2 + c \cos^2\big(\sqrt{\kappa^2 + c}\, r/2\big)}}, & \text{if } |\kappa| > \sqrt{c}. \end{cases}$$

Here, in the case $|\kappa| > \sqrt{|c|}$ we only consider r with $0 \leq r \leq 2\pi/\sqrt{\kappa^2 + c}$.

When $|\kappa| \leq \sqrt{|c|}$ the function $r \mapsto \ell_\kappa(r; c)$ is monotone increasing and satisfies $\lim_{r \to \infty} \ell_\kappa(r; c) = \infty$, and the function $r \mapsto \delta_\kappa(r; c)$ is monotone decreasing *and satisfies* $\lim_{r\to\infty}\delta_\kappa(r;c)=\sqrt{1-(\kappa^2/|c|)}$. *When* $|\kappa|>\sqrt{|c|}$*, on the interval* $\big[0,\pi/\sqrt{\kappa^2+c}\,\big]$*, the function* $r\mapsto\ell_\kappa(r;c)$ *is monotone increasing and the function* $r\mapsto\delta_\kappa(r;c)$ *is monotone decreasing.*

Proof. When $c=-4$, we consider a fibration $\varpi: H_1^{2n+1}\ (\subset \boldsymbol{C}_1^{n+1})\to \boldsymbol{C}H^n(-4)$ of an anti-de Sitter space H_1^{2n+1}. For a trajectory γ for $\boldsymbol{B}_\kappa$, its horizontal lift $\hat\gamma$ with respect to this fibration is given as

$$\hat\gamma(t)=\begin{cases} e^{\sqrt{-1}\kappa t/2}\Big\{\cosh\dfrac{1}{2}\sqrt{4-\kappa^2}\,t\,z \\ \qquad\qquad +(4-\kappa^2)^{-1/2}\sinh\dfrac{1}{2}\sqrt{4-\kappa^2}\,t\,\big(-\sqrt{-1}\kappa z+2v\big)\Big\}, & \text{if } |\kappa|<2,\\ e^{\pm\sqrt{-1}t}\big\{\big(1\mp\sqrt{-1}t\big)z+tv\big\}, & \text{if } \kappa=\pm 2,\\ e^{\sqrt{-1}\kappa t/2}\Big\{\cos\dfrac{1}{2}\sqrt{\kappa^2-4}\,t\,z \\ \qquad\qquad +(\kappa^2-4)^{-1/2}\sin\dfrac{1}{2}\sqrt{\kappa^2-4}\,t\,\big(-\sqrt{-1}\kappa z+2v\big)\Big\}, & \text{if } |\kappa|>2,\end{cases}$$

where $z\in H_1^{2n+1}\subset\boldsymbol{C}^{n+1}$ satisfies $\varpi(z)=\gamma(0)$ and a horizontal vector $(z,v)\in T_zH_1^{2n+1}$ with respect to ϖ satisfies $d\varpi((z,v))=\dot\gamma(0)$. Along the same lines as in the proof of Proposition 3, we get the conclusion. □

For trajectories on $\boldsymbol{C}H^n(c)$ we also have $\ell_\kappa(r;c)=\ell_{-\kappa}(r;c)$ and $\delta_\kappa(r;c)=\delta_{-\kappa}(r;c)$. We should also note that the assertion of Theorem 1 holds on a complex hyperbolic space by the proof of Proposition 5.

Proposition 6 (cf. [**1**]). *Let* $p,q\in\boldsymbol{C}H^n(c)$ *be distinct points.*

(1) *When* $|\kappa|\le\sqrt{|c|}$*, there is a unique trajectory for* $\boldsymbol{B}_\kappa$ *which goes from* p *to* q*.*
(2) *When* $|\kappa|>\sqrt{|c|}$*, if the distance between* p *and* q *is not greater than* $(2/\sqrt{|c|})\sinh^{-1}\sqrt{|c|/(\kappa^2+c)}$*, then there is a unique trajectory-segment for* $\boldsymbol{B}_\kappa$ *which goes from* p *to* q*.*
(3) *When* $|\kappa|>\sqrt{|c|}$ *and if the distance between* p,q *is greater than* $(2/\sqrt{|c|})\sinh^{-1}\sqrt{|c|/(\kappa^2+c)}$*, then they can not be joined by trajectory-segments for* $\boldsymbol{B}_\kappa$*.*

3. A comparison theorem on trajectory-harps.

We now consider trajectories on general Kähler manifolds. Let M be a complete Kähler manifold and $\gamma:[0,T]\to M$ be a trajectory-segment or a trajectory half line for a Kähler magnetic field $\boldsymbol{B}_\kappa$ on M. This means that when $0 < T < \infty$ it is a trajectory-segment and when $T = \infty$ the curve $\gamma : [0, \infty) \to M$ is a trajectory half line. We suppose $\gamma\big([0,T]\big)$ lies on a geodesic ball centered at $\gamma(0)$ and of radius of injectivity at $\gamma(0)$. We also suppose $\gamma(t) \neq \gamma(0)$ for all $t \in (0,T)$. For this trajectory-segment or trajectory half line, we define a variation $\alpha_\gamma : [0,T) \times \boldsymbol{R} \to M$ of geodesics as follows:

i) $\alpha_\gamma(t,0) = \gamma(0)$,
ii) when $t = 0$, the curve $s \mapsto \alpha_\gamma(0,s)$ is the geodesic of initial vector $\dot\gamma(0)$,
iii) when $t \neq 0$, the curve $s \mapsto \alpha_\gamma(t,s)$ is the geodesic of unit speed joining $\gamma(0)$ and $\gamma(t)$.

We shall call this variation the *trajectory-harp* associated with γ. We denote by $\ell_\gamma(t)$ the distance $d(\gamma(0),\gamma(t))$ between $\gamma(0)$ and $\gamma(t)$ and call it the *string-length* at t. As γ is parameterized by its arclength, it is clear that it satisfies $\ell_\gamma(t) \le t$ for all $0 \le t \le T$. We define the *string-cosine* $\delta_\gamma(t)$ at t by $\delta_\gamma(t) = \langle \dot\gamma(t), (\partial\alpha/\partial s)(t, \ell_\gamma(t))\rangle$. When we consider a trajectory-segment or a trajectory half line $\gamma : \big[0, 2\pi/\sqrt{\kappa^2+c}\big) \to \boldsymbol{C}M^n(c)$ for $\boldsymbol{B}_\kappa$ on a complex space form $\boldsymbol{C}M^n(c)$ of constant holomorphic sectional curvature c, we have $\ell_\gamma(t) = \ell_\kappa(t;c)$ and $\delta_\gamma(t) = \delta_\kappa(t;c)$ by the study of the previous section. Here, we regard $2\pi/\sqrt{\kappa^2+c}$ infinity when $\kappa^2 + c \le 0$. From now on we use this convention without noticing.

LEMMA 1. *For a trajectory-segment or a trajectory half line γ for $\boldsymbol{B}_\kappa$ on a Kähler manifold, its string-length and string-cosine satisfy the following properties*:

(1) $\ell'_\gamma(t) = \delta_\gamma(t)$;
(2) $\ell_\gamma(0) = 0$, $\delta_\gamma(0) = 1$, $\lim_{t\downarrow 0} \delta'_\gamma(t) = 0$, $\lim_{t\downarrow 0}\delta''_\gamma(t) = -\kappa^2/4$.

PROOF. (1) We set $\hat\alpha(t,u) = \alpha\big(t, \ell_\gamma(t)u\big)$. As we have $\ell_\gamma(t)^2 = \int_0^1 \|(\partial\hat\alpha/\partial u)(t,u)\|^2 du$ and $u \mapsto \hat\alpha(t,u)$ is a geodesic, we find

$$\begin{aligned} 2\ell'_\gamma(t)\ell_\gamma(t) &= \int_0^1 2\Big\langle \nabla_{\partial/\partial t}\frac{\partial\hat\alpha}{\partial u}, \frac{\partial\hat\alpha}{\partial u}\Big\rangle du = 2\int_0^1 \Big\langle \nabla_{\partial/\partial u}\frac{\partial\hat\alpha}{\partial t}, \frac{\partial\hat\alpha}{\partial u}\Big\rangle du \\ &= 2\int_0^1 \frac{d}{du}\Big\langle \frac{\partial\hat\alpha}{\partial t}, \frac{\partial\hat\alpha}{\partial u}\Big\rangle du = 2\Big\langle \frac{\partial\hat\alpha}{\partial t}(t,1), \frac{\partial\hat\alpha}{\partial u}(t,1)\Big\rangle. \end{aligned}$$

Since $\hat\alpha(t,1) = \gamma(t)$, we get the conclusion.

(2) The first two equalities are trivial. For the third equality, by the definition of the string-cosine, we get

$$\begin{aligned}\delta_\gamma'(t) &= \Big\langle \nabla_{\dot\gamma}\dot\gamma(t), \frac{\partial\alpha}{\partial s}(t,\ell_\gamma(t))\Big\rangle + \Big\langle \dot\gamma(t), \Big(\nabla_{\partial/\partial t}\frac{\partial\alpha}{\partial s}\Big)(t,\ell_\gamma(t))\Big\rangle \\ &\quad + \ell_\gamma'(t)\Big\langle \dot\gamma(t), \Big(\nabla_{\partial/\partial s}\frac{\partial\alpha}{\partial s}\Big)(t,\ell_\gamma(t))\Big\rangle \\ &= \kappa\Big\langle J\dot\gamma(t), \frac{\partial\alpha}{\partial s}(t,\ell_\gamma(t))\Big\rangle + \Big\langle \dot\gamma(t), \Big(\nabla_{\partial/\partial t}\frac{\partial\alpha}{\partial s}\Big)(t,\ell_\gamma(t))\Big\rangle.\end{aligned}$$

Since $\|\partial\alpha/\partial s\| \equiv 1$, we have $\langle \nabla_{\partial/\partial t}\partial\alpha/\partial s, \partial\alpha/\partial s\rangle \equiv 0$. As we have $\dot\gamma(0) = (\partial\alpha/\partial s)(0,0)$, we get the third equality.

To get the fourth equality, we continue our calculation.

$$\begin{aligned}\delta_\gamma''(t) &= -\kappa^2\Big\langle \dot\gamma(t), \frac{\partial\alpha}{\partial s}(t,\ell_\gamma(t))\Big\rangle + 2\kappa\Big\langle J\dot\gamma(t), \Big(\nabla_{\partial/\partial t}\frac{\partial\alpha}{\partial s}\Big)(t,\ell_\gamma(t))\Big\rangle \\ &\quad + \Big\langle \dot\gamma(t), \Big(\nabla_{\partial/\partial t}\nabla_{\partial/\partial t}\frac{\partial\alpha}{\partial s}\Big)(t,\ell_\gamma(t))\Big\rangle \\ &\quad + \ell_\gamma'(t)\Big\langle \dot\gamma(t), \Big(\nabla_{\partial/\partial s}\nabla_{\partial/\partial t}\frac{\partial\alpha}{\partial s}\Big)(t,\ell_\gamma(t))\Big\rangle.\end{aligned}$$

Since $\gamma(t) = \alpha(t,\ell_\gamma(t))$, we see $\dot\gamma(t) = (\partial\alpha/\partial t)(t,\ell_\gamma(t)) + \ell_\gamma'(t)(\partial\alpha/\partial s)(t,\ell_\gamma(t))$, hence we have

$$\begin{aligned}\kappa J\dot\gamma(t) &= \nabla_{\dot\gamma}\dot\gamma(t) \\ &= \Big(\nabla_{\partial/\partial t}\frac{\partial\alpha}{\partial t}\Big)(t,\ell_\gamma(t)) + 2\ell_\gamma'(t)\Big(\nabla_{\partial/\partial t}\frac{\partial\alpha}{\partial s}\Big)(t,\ell_\gamma(t)) + \ell_\gamma''(t)\frac{\partial\alpha}{\partial s}(t,\ell_\gamma(t)).\end{aligned}$$

We therefore obtain

$$\kappa J\dot\gamma(0) = 2\lim_{t\downarrow 0}\Big(\nabla_{\partial/\partial t}\frac{\partial\alpha}{\partial s}\Big)(t,\ell_\gamma(t)),$$

because $\lim_{t\downarrow 0}\ell_\gamma'(t) = \delta_\gamma(0) = 1$ and $\lim_{t\downarrow 0}\ell_\gamma''(t) = \lim_{t\downarrow 0}\delta_\gamma'(t) = 0$. As we have $\big\langle \nabla_{\partial/\partial t}\partial\alpha/\partial s, \partial\alpha/\partial s\big\rangle(t,s) \equiv 0$, we see

$$0 = \frac{d}{dt}\Big\langle \nabla_{\partial/\partial t}\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial s}\Big\rangle = \Big\|\nabla_{\partial/\partial t}\frac{\partial\alpha}{\partial s}\Big\|^2 + \Big\langle \nabla_{\partial/\partial t}\nabla_{\partial/\partial t}\frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial s}\Big\rangle.$$

Thus we find

$$\lim_{t\downarrow 0}\delta_\gamma''(t) = -\kappa^2+\kappa^2-\frac{\kappa^2}{4}+\lim_{t\downarrow 0}\Big\langle \dot\gamma(t), R\Big(\frac{\partial\alpha}{\partial s}(t,\ell_\gamma(t)), \frac{\partial\alpha}{\partial t}(t,\ell_\gamma(t))\Big)\frac{\partial\alpha}{\partial s}(t,\ell_\gamma(t))\Big\rangle$$
$$= -\frac{\kappa^2}{4}.$$

We get the conclusion. □

We now prepare a comparison theorem on trajectory-harps. Concerning the string-length function for trajectory-harps on a complex space form $\boldsymbol{C}M^n(c)$, Examples 1, 2 and 3 show that $\ell_\kappa(\,\cdot\,;c) : \big[0,\pi/\sqrt{\kappa^2+c}\,\big] \to [0,\infty)$ is monotone increasing, because $\ell_\kappa'(t;c) = \delta_\kappa(t;c) > 0$ if $t \neq \pi/\sqrt{\kappa^2+c}$. We denote by $\ell \mapsto \tau_\kappa(\ell;c)$ the inverse function of the function $t \mapsto \ell_\kappa(t;c)$. We hence have $\tau_\kappa(\cdot\,;c) : \big[0,\ell_\kappa(\pi/\sqrt{\kappa^2+c};c)\big] \to \big[0,\pi/\sqrt{\kappa^2+c}\,\big]$.

THEOREM 3. *Let M be a complete Kähler manifold whose sectional curvatures satisfy* $\mathrm{Riem}^M \leq c$ *for some constant c, and let $\gamma : [0,T] \to M$ be a trajectory-segment or a trajectory half line for a Kähler magnetic field $\boldsymbol{B}_\kappa$ on M with $0 < T \leq \pi/\sqrt{\kappa^2+c}$ which lies on the ball centered at $\gamma(0)$ and of injectivity radius at $\gamma(0)$. We then have the following*:

(1) $\gamma(t) \neq \gamma(0)$ *for each* $t \in [0,T]$;
(2) *The string-length of the trajectory-harp associated with γ is monotone increasing and satisfies $\ell_\gamma(t) \geq \ell_\kappa(t;c)$ for $0 \leq t \leq T$*;
(3) *The string-cosine of the trajectory-harp associated with γ satisfies $\delta_\gamma(t) \geq \delta_\kappa(\tau_\kappa(\ell_\gamma(t);c);c)$ for $0 \leq t \leq T_*$. Here, we set T_* to be the minimum positive t_* $(\leq T)$ with $\ell_\gamma(t_*) = \ell_\kappa\big(\pi/\sqrt{\kappa^2+c};c\big)$ if such t_* exists, and set $T_* = T$ if $\ell_\gamma(t) < \ell_\kappa\big(\pi/\sqrt{\kappa^2+c};c\big)$ for all $0 \leq t \leq T$.*

PROOF. We take positive $\hat\kappa$ so that $|\kappa| < \hat\kappa$. By Lemma 1 we see there is a positive ϵ which satisfies $\delta_\gamma(t) > \delta_{\hat\kappa}(t;c)$ and $\delta_{\hat\kappa}'(t;c) < \delta_\gamma'(t) < 0$ for $0 < t < \epsilon$.

We take the maximal positive $T_{\hat\kappa} \leq \min\{T, \pi/\sqrt{\hat\kappa^2+c}\}$ which satisfies the following two conditions for all $0 \leq t \leq T_{\hat\kappa}$:

i) $\ell_\gamma(t) \leq \ell_{\hat\kappa}\big(\pi/\sqrt{\hat\kappa^2+c};c\big)$,
ii) $\delta_\gamma(t) \geq \delta_{\hat\kappa}(\tau_{\hat\kappa}(\ell_\gamma(t);c);c)$.

We should note that the second condition guarantees that $\gamma(t) \neq \gamma(0)$. We shall show that $\ell_\gamma(T_{\hat\kappa}) = \ell_{\hat\kappa}\big(\pi/\sqrt{\hat\kappa^2+c};c\big)$ if $T_{\hat\kappa} < \min\{T, \pi/\sqrt{\hat\kappa^2+c}\}$. We suppose $\ell_\gamma(T_{\hat\kappa}) < \ell_{\hat\kappa}\big(\pi/\sqrt{\hat\kappa^2+c};c\big)$ and $T_{\hat\kappa} < \min\{T, \pi/\sqrt{\hat\kappa^2+c}\}$. By the maximality of $T_{\hat\kappa}$ we have $\delta_\gamma(T_{\hat\kappa}) = \delta_{\hat\kappa}(\tau_{\hat\kappa}(\ell_\gamma(T_{\hat\kappa});c);c)$. We here estimate the derivative

$$\frac{d\delta_\gamma}{dt}(t) = \kappa\Big\langle J\dot\gamma(t), \frac{\partial\alpha}{\partial s}(t,\ell_\gamma(t))\Big\rangle + \Big\langle \dot\gamma(t), \Big(\nabla_{\partial/\partial t}\frac{\partial\alpha}{\partial s}\Big)(t,\ell_\gamma(t))\Big\rangle.$$

By the definition of δ_γ we see

$$\kappa\Big\langle J\dot\gamma(t), \frac{\partial\alpha}{\partial s}(t,\ell_\gamma(t))\Big\rangle \geq -|\kappa|\sqrt{1-\delta_\gamma^2(t)}.$$

We hence estimate the second term. We put $Z_t(s) = (\partial\alpha/\partial t)(t,s)$, which is a Jacobi field along a geodesic $s \mapsto \alpha(t,s)$ and is orthogonal to $(\partial\alpha/\partial s)(t,s)$. We denote by $\hat\alpha : [0, \pi/\sqrt{\hat\kappa^2+c}\,] \to \boldsymbol{C}M^1(c)$ a trajectory-harp associated with a trajectory $\hat\gamma$ for $\boldsymbol{B}_{\hat\kappa}$ on $\boldsymbol{C}M^1(c) \cong \boldsymbol{R}M^2(c)$ and put $\widehat{Z}_t(s) = (\partial\hat\alpha/\partial t)(t,s)$, which is a Jacobi field on a real space form $\boldsymbol{R}M^2(c)$ of constant sectional curvature c. Since $\dot\gamma(t) = Z_t(\ell_\gamma(t)) + \delta_\gamma(t)(\partial\alpha/\partial s)(t,\ell_\gamma(t))$, we have $\|Z_t(\ell_\gamma(t))\|^2 = 1-\delta_\gamma^2(t)$.

By the comparison theorem on Jacobi fields, we find the following:

$$\begin{aligned}
&\Big\langle \dot\gamma(t), \Big(\nabla_{\partial/\partial t}\frac{\partial\alpha}{\partial s}\Big)(t,\ell_\gamma(t))\Big\rangle \\
&= \big\langle Z_t(\ell_\gamma(t)), (\nabla_{\partial/\partial s}Z_t)(\ell_\gamma(t))\big\rangle \\
&= \|Z_t(\ell_\gamma(t))\|^2 \times \frac{\langle Z_t(\ell_\gamma(t)), (\nabla_{\partial/\partial s}Z_t)(\ell_\gamma(t))\rangle}{\|Z_t(\ell_\gamma(t))\|^2} \\
&\geq \|Z_t(\ell_\gamma(t))\|^2 \times \frac{\langle \widehat{Z}_{\tau_{\hat\kappa}(\ell_\gamma(t);c)}(\ell_\gamma(t)), (\nabla_{\partial/\partial s}\widehat{Z}_{\tau_{\hat\kappa}(\ell_\gamma(t);c)})(\ell_\gamma(t))\rangle}{\|\widehat{Z}_{\tau_{\hat\kappa}(\ell_\gamma(t);c)}(\ell_\gamma(t))\|^2} \\
&= \frac{1-\delta_\gamma^2(t)}{1-\delta_{\hat\kappa}^2(\tau_{\hat\kappa}(\ell_\gamma(t);c);c)}\big\langle \widehat{Z}_{\tau_{\hat\kappa}(\ell_\gamma(t);c)}(\ell_\gamma(t)), (\nabla_{\partial/\partial s}\widehat{Z}_{\tau_{\hat\kappa}(\ell_\gamma(t);c)})(\ell_\gamma(t))\big\rangle.
\end{aligned}$$

As we have $\delta_\gamma(T_{\hat\kappa}) = \delta_{\hat\kappa}(\tau_{\hat\kappa}(\ell_\gamma(T_{\hat\kappa});c);c)$, we obtain

$$\begin{aligned}
\frac{d\delta_\gamma}{dt}(T_{\hat\kappa}) &> -\hat\kappa\sqrt{1-\delta_{\hat\kappa}(\tau_{\hat\kappa}(\ell_\gamma(T_{\hat\kappa});c);c)^2} \\
&\qquad + \big\langle \widehat{Z}_{\tau_{\hat\kappa}(\ell_\gamma(T_{\hat\kappa});c)}(\ell_\gamma(T_{\hat\kappa})), \big(\nabla_{\partial/\partial s}\widehat{Z}_{\tau_{\hat\kappa}(\ell_\gamma(T_{\hat\kappa});c)}\big)(\ell_\gamma(T_{\hat\kappa}))\big\rangle \\
&= \hat\kappa\Big\langle J\dot{\hat\gamma}(\tau_{\hat\kappa}(\ell_\gamma(T_{\hat\kappa});c)), \frac{\partial\hat\alpha}{\partial s}(\tau_{\hat\kappa}(\ell_\gamma(T_{\hat\kappa});c), \ell_\gamma(T_{\hat\kappa}))\Big\rangle \\
&\qquad + \Big\langle \dot{\hat\gamma}(\tau_{\hat\kappa}(\ell_\gamma(T_{\hat\kappa});c)), \Big(\nabla_{\partial/\partial t}\frac{\partial\hat\alpha}{\partial s}\Big)(\tau_{\hat\kappa}(\ell_\gamma(T_{\hat\kappa});c), \ell_\gamma(T_{\hat\kappa}))\Big\rangle \\
&= \frac{d\delta_{\hat\kappa}}{dt}(\tau_{\hat\kappa}(\ell_\gamma(T_{\hat\kappa});c)) = \frac{d}{du}\delta_{\hat\kappa}(\tau_{\hat\kappa}(\ell_\gamma(u);c))\Big|_{u=T_{\hat\kappa}}.
\end{aligned}$$

By the maximality of $T_{\hat\kappa}$ we find it is a contradiction, hence we can conclude that either $T_{\hat\kappa} = \min\{T, \pi/\sqrt{\hat\kappa^2+c}\}$ or $\ell_\gamma(T_{\hat\kappa}) = \ell_{\hat\kappa}\bigl(\pi/\sqrt{\hat\kappa^2+c}; c\bigr)$ holds.

We have a monotone decreasing sequence $\{\hat\kappa_j\}_{j=1}^\infty$ with $\hat\kappa_j > |\kappa|$ and $\lim_{j\to\infty}\hat\kappa_j = |\kappa|$ which satisfies one of the following conditions for all j:

1) $T_{\hat\kappa_j} = \min\bigl\{T, \pi/\sqrt{\hat\kappa_j^2+c}\bigr\}$,

2) $T_{\hat\kappa_j} < \min\bigl\{T, \pi/\sqrt{\hat\kappa_j^2+c}\bigr\}$ and $\ell_\gamma(T_{\hat\kappa_j}) = \ell_{\hat\kappa}\bigl(\pi/\sqrt{\hat\kappa_j^2+c}; c\bigr)$.

In the first case it is clear that $\lim_{j\to\infty} T_{\hat\kappa_j} = T$. In the second case, as we have $\ell'_\gamma(t) = \delta_\gamma(t) > 0$ in the interior of $\bigcup_j \bigl[0, T_{\hat\kappa_j}\bigr]$, hence ℓ_γ is monotone increasing on this domain and $T_{\hat\kappa_j}$ is monotone increasing. We have $\lim_{j\to\infty} T_{\hat\kappa_j} = T_*$. Since $\lim_{\hat\kappa\downarrow\kappa}\delta_{\hat\kappa}(\tau_{\hat\kappa}(\ell_\gamma(t); c); c) = \delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c)$ for each t, we obtain $\delta_\gamma(t) \geq \delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c)$ for all $0 \leq t \leq T_*$.

We next compare $\ell_\gamma(t)$ and $\ell_\kappa(t; c)$. For a positive $\hat\kappa$ with $|\kappa| < \hat\kappa$ we take the maximal positive $S_{\hat\kappa} \leq \min\bigl\{T, \pi/\sqrt{\hat\kappa^2+c}\bigr\}$ satisfying $\ell_\gamma(t) \geq \ell_{\hat\kappa}(t; c)$ for all $0 < t \leq S_{\hat\kappa}$. We shall show $S_{\hat\kappa} = \min\bigl\{T, \pi/\sqrt{\hat\kappa^2+c}\bigr\}$. If we suppose $S_{\hat\kappa} < \min\bigl\{T, \pi/\sqrt{\hat\kappa^2+c}\bigr\}$, then $\ell_\gamma(S_{\hat\kappa}) = \ell_{\hat\kappa}(S_{\hat\kappa}; c)$ holds. We hence have

$$\delta_\gamma(S_{\hat\kappa}) \geq \delta_\kappa(\tau_\kappa(\ell_\gamma(S_{\hat\kappa}); c); c) = \delta_\kappa(S_{\hat\kappa}; c) > \delta_{\hat\kappa}(S_{\hat\kappa}; c).$$

The maximality of $S_{\hat\kappa}$ shows that it is a contradiction. By the same argument as for δ_γ we obtain that $\ell_\gamma(t) \geq \ell_\kappa(t; c)$ for all $0 \leq t \leq T$. We get the conclusion. □

A simply connected complete Riemannian manifold is said to be a Hadamard manifold if it is non-positively curved. As a consequence of Theorem 3 we have the following.

COROLLARY 1. *Let M be a Kähler Hadamard manifold whose sectional curvatures satisfy* $\mathrm{Riem}^M \leq c < 0$. *When κ satisfies $|\kappa| \leq \sqrt{|c|}$, for a trajectory γ for $\boldsymbol{B}_\kappa$ on M, its string-length and string-cosine of the associated trajectory-harp satisfy the following properties.*

(1) *Its string-cosine satisfies $\delta_\gamma(t) \geq \delta_\kappa(\tau_\kappa(\ell_\gamma(t); c); c) > \sqrt{1-(\kappa^2/c)}$ for all $t \geq 0$.*

(2) *Its string-length $\ell_\gamma(t)$ is monotone increasing and satisfies $\ell_\gamma(t) \geq \ell_\kappa(t; c)$ for all $t \geq 0$. In particular, it satisfies $\lim_{t\to\infty}\ell_\gamma(t) = \infty$, hence both of the sets $\gamma([0,\infty))$ and $\gamma((-\infty, 0])$ are unbounded.*

4. A theorem of Hadamard-Cartan type.

In the previous section we study the behavior of trajectories. In order to show our main theorems we here briefly recall properties of differentials of magnetic exponential maps given in [**2**]. A vector field Y along a trajectory γ for a Kähler magnetic field $\boldsymbol{B}_\kappa$ on a Kähler manifold is said to be a normal *magnetic Jacobi field* if it satisfies

$$\begin{cases} \nabla_{\dot\gamma}\nabla_{\dot\gamma}Y - \kappa J\big(\nabla_{\dot\gamma}Y\big) + R(Y,\dot\gamma)\dot\gamma = 0, \\ \langle \nabla_{\dot\gamma}Y, \dot\gamma\rangle = 0. \end{cases}$$

Normal magnetic Jacobi fields for $\boldsymbol{B}_\kappa$ correspond to variations of trajectories for $\boldsymbol{B}_\kappa$. As we suppose trajectories have unit speed we need the second condition. Being different from usual Jacobi fields, for magnetic Jacobi fields, there is an interaction between the component which is tangent to a trajectory and the component which is orthogonal to a trajectory. For a magnetic Jacobi field Y along a trajectory γ, let $Y^\sharp(t)$ denote the component of $Y(t)$ orthogonal to $\dot\gamma(t)$.

Theorem 4 ([**3**], cf. [**2**]). *Let M be a Kähler manifold whose sectional curvatures satisfy $\mathrm{Riem}^M \le c < 0$ and γ be a trajectory for $\boldsymbol{B}_\kappa$ on M. If Y is a normal magnetic Jacobi field along γ with $Y^\sharp(0) = 0$, then it satisfies $\|Y^\sharp(t)\| \ge \|\nabla_{\dot\gamma}Y^\sharp(0)\| s_\kappa(t;c)$. Here*

$$s_\kappa(t;c) = \begin{cases} (1/\sqrt{|c|-\kappa^2})\sinh\sqrt{|c|-\kappa^2}\,t, & \kappa^2 < |c|, \\ t, & \kappa = \pm\sqrt{|c|}, \\ (1/\sqrt{\kappa^2+c})\sin\sqrt{\kappa^2+c}\,t, & \kappa^2 > |c|. \end{cases}$$

Corollary 2. *When M is a Kähler manifold whose sectional curvatures satisfy $\mathrm{Riem}^M \le c < 0$, every magnetic exponential map for $\boldsymbol{B}_\kappa$ does not have singular points if $|\kappa| \le \sqrt{|c|}$.*

We now show our main theorems which correspond to theorems of Hopf-Rinow and of Hadamard-Cartan on geodesics.

Proof of Theorem 1 (Existence). By taking the universal covering of M, we may suppose M is a Kähler Hadamard manifold. Given a point $p \in M$ we show the magnetic exponential map $\boldsymbol{B}_\kappa \exp_p : T_pM \to M$ is surjective. Since we do not have magnetic conjugate points for $\boldsymbol{B}_\kappa$ with $|\kappa| \le \sqrt{c}$, we see that the image of $\boldsymbol{B}_\kappa \exp_p$ is an open set. On the other hand, if a sequence of points $q_i = \boldsymbol{B}_\kappa \exp_p(r_i v_i)$ with unit tangent vectors $v_i \in U_pM$ and $r_i > 0$ converges to q $(\ne p)$, Corollary 1 guarantees that $\{r_i\}$ is bounded. As $r_i \ge d(p, q_i)$, taking subsequences we see r_{i_j} converges to some positive r_0 and v_{i_j} to some $v_0 \in U_pM$. Hence we have $q = \boldsymbol{B}_\kappa \exp_p(r_0 v_0)$, and obtain that the image of $\boldsymbol{B}_\kappa \exp_p$ is closed. Thus the connectedness of M guarantees that $\boldsymbol{B}_\kappa \exp_p$ is surjective. Hence, for an arbitrary point $q \in M$, we have a trajectory-segment for $\boldsymbol{B}_\kappa$ from p to q. If there exist trajectory-segments for $\boldsymbol{B}_\kappa$ from p to q more than one, by the same argument as above, we have a trajectory-segment of minimal length. We get the assertion of Theorem 1 except the uniqueness. □

PROOF OF THEOREM 2. We show this theorem by the same argument as for exponential maps. We may suppose M is Hadamard Kähler. Since $\boldsymbol{B}_\kappa \exp_p : T_pM \to M$ does not have singular points, with the induced metric $\langle\ ,\ \rangle_\kappa$ and induced complex structure on T_pM, it is a local holomorphic isometry and sectional curvatures of T_pM are not greater than c. For a unit tangent vector $v \in U_0(T_pM) \cong U_pM$, the curve $t \to tv$ is a trajectory for $\boldsymbol{B}_\kappa$ on T_pM.

We here show the origin $0_p \in T_pM$ and $sv \in T_pM$ is joined by a unique geodesic-segment of unit speed. Since $\boldsymbol{B}_\kappa \exp_p$ is a local isometry, we see for sufficiently small s we can join 0_p and sv by a unique geodesic-segment of unit speed. We put t_* the maximal positive number such that for all $0 < t < t_*$ we have a unique geodesic-segment of unit speed joining 0_p and tv. If we suppose $t_* < \infty$, by taking a trajectory-harp associated with a trajectory γ_v for $\boldsymbol{B}_\kappa$ of initial vector v on M, we see the set $\{\alpha(t,s) \mid 0 \le t \le t_*,\ 0 \le s \le \ell_\gamma(t)\}$ is compact. Thus it is covered by finite open sets each of which gives the local isometric property. Hence there is positive ϵ such that for all $0 < t < t_* + \epsilon$ we have a geodesic-segment of unit speed joining 0_p and tv. If there are two geodesics σ_1, σ_2 of unit speed on T_pM joining 0_p and tv, as M is a Hadamard manifold, we conclude that the two geodesic-segments $\boldsymbol{B}_\kappa \exp_p(\sigma_1)$ and $\boldsymbol{B}_\kappa \exp_p(\sigma_2)$ on M must coincide. The local isometric property at $\boldsymbol{B}_\kappa \exp_p(0_p) = p$ shows σ_1 and σ_2 coincide. We therefore find that $t_* = \infty$.

In order to complete the proof, we show T_pM is complete. Take a Cauchy sequence $\{w_j\} \subset T_pM$ and denote $w_j = r_j v_j$ with nonnegative r_j and $v_j \in U_0(T_pM)$. Since $\{w_j\}$ is a bounded set and the distance between 0 and w_j coincides with the distance between p and $\boldsymbol{B}_\kappa \exp_p(w_j)$, Corollary 1 guarantees that $\{r_j\}$ is bounded. As the unit tangent space $U_0(T_pM)$ is compact, we have a convergent subsequence $\{w_{j_i}\}$, which shows that $\{w_j\}$ converges to some point on T_pM. Thus T_pM is complete. We hence find that $\boldsymbol{B}_\kappa \exp_p : T_pM \to M$ is a covering map. □

When M is a Hadamard Kähler manifold, as $\boldsymbol{B}_\kappa \exp_p$ is bijective, we see there is only one trajectory-segment joining p and an arbitrary q ($\neq p$). This completes the proof of Theorem 1.

We should note that a minimizing trajectory for $\boldsymbol{B}_\kappa$ of p to q ($p \neq q$) does not coincide with that of q to p in general. As we mentioned in Section 2, the image of a trajectory for $\boldsymbol{B}_\kappa$ of q to p coincides with the image of a trajectory for $\boldsymbol{B}_{-\kappa}$ of p to q.

We can generalize Theorem 1 in the following manner. For a point $p \in M$ on a Riemannian manifold M, we denote by $\overline{B_r(0_p)}$ a closed ball in T_pM of radius r centered at the origin 0_p, and by $\overline{B_r(p)}$ a distance closed ball in M of radius r centered at p.

THEOREM 5. *Let M be a complete Kähler manifold whose sectional curvatures satisfy $\mathrm{Riem}^M \leq c$ for some constant c. For an arbitrary positive r with $r < \pi/\sqrt{\kappa^2 + c}$, the image $\boldsymbol{B}_\kappa \exp_p(\overline{B_r(0_p)})$ contains the distance closed ball $\overline{B_{\ell_\kappa(r;c)}(p)}$.*

PROOF. We may suppose M to be simply connected. By the same argument as in the proof of Theorem 1, we see $\boldsymbol{B}_\kappa \exp_p(\overline{B_r(0_p)})$ is a closed subset of M. Therefore, if we suppose $\overline{B_{\ell_\kappa(r;c)}(p)} \setminus \boldsymbol{B}_\kappa \exp_p(\overline{B_r(0_p)}) \neq \emptyset$, we have $B_{\ell_\kappa(r;c)}(p) \setminus \boldsymbol{B}_\kappa \exp_p(\overline{B_r(0_p)})$ is a non-empty open subset. Since $\boldsymbol{B}_\kappa \exp_p(\overline{B_r(0_p)})$ is connected and contains an open neighborhood of p, for sufficiently small positive ϵ $(< \ell_\kappa(r;c))$ we have $q \in S_{\ell_\kappa(r;c)-\epsilon}(p) \cap \partial \boldsymbol{B}_\kappa \exp_p(\overline{B_r(0_p)})$. We denote as $q = \boldsymbol{B}_\kappa \exp_p(t_0 v_0)$ with some $v_0 \in U_pM$ and t_0 satisfying $0 < t_0 \leq r$. As the distance between q and p is smaller than $\ell_\kappa(r;c)$, Theorem 3 shows that $t_0 < r$. Since $\boldsymbol{B}_\kappa \exp_p$ is regular at $t_0 v_0$, we find that some open neighborhood of $\boldsymbol{B}_\kappa \exp_p(t_0 v_0)$ is contained in $\boldsymbol{B}_\kappa \exp_p(\overline{B_r(0_p)})$. It is a contradiction. □

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