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Locally o-minimal structures

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Abstract. In this paper we study (strongly) locally o-minimal structures. We first give a characterization of the strong local o-minimality. We also investigate locally o-minimal expansions of $(\boldsymbol{R}, +, <)$.

1. Introduction.

Toffalori and Vozoris [**8**] introduced the notion of local o-minimality and that of strong local o-minimality, by weakening the definition of o-minimality. A typical example of locally o-minimal structure is $(\boldsymbol{R}, +, <, \sin)$, which is not o-minimal (see [**8**, Theorem 2.7]). They systematically investigated the notions, and, among many others, showed that any weakly o-minimal structure is locally o-minimal.

In this paper we first give a characterization of the strong local o-minimality. This characterization shows that a strongly locally o-minimal structure really resembles an o-minimal structure if it is viewed locally. In [**4**], [**9**], [**7**], several generalizations of the cell decomposition theorem were studied in the weakly o-minimal context. In this paper, using the characterization, we show that the local version of cell decomposition holds for strongly locally o-minimal structures.

We then introduce the notion of simple products of two structures. This notion is already implicit in [**8**], and in the present paper we give an explicit definition. Using the method of taking simple products, a number of structures are shown to be (strongly) locally o-minimal. For example, in Section 4, we show that any structure of the form $(\boldsymbol{R}, +, <, P)$ with $P \subset \boldsymbol{Z}$ is locally o-minimal. Conversely, we also show that any locally o-minimal structure expanding $(\boldsymbol{R}, +, <, \boldsymbol{Z})$ can be written as a simple product of $\boldsymbol{Z}$ and an o-minimal structure.

We only assume the reader's familiarity with a few basic model theoretic notions. In Section 2, we recall some definitions and results on (local) o-minimality. The notion of local structures is introduced here. For a structure M and its subset

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A, the local structure A_{def} is defined roughly as the set A with M-definable subsets. A_{def} is an important tool in our characterization.

In Section 3, we give a characterization of strong local o-minimality, using local structures (see Theorem 9). The local monotonicity theorem and the local cell decomposition theorem (for strongly locally o-minimal structures) are easily obtained from our characterization. In this section, we also introduce the notion of uniform local o-minimality, and study the relation between this notion and (strong) local o-minimality. Several examples will be given.

Section 4 is the section for simple products. Let M and N be two structures. If the product $M \times N$ is simple in our sense then every definable subset of $M \times N$ has the form $A \times B$, where $A \subset M$ is M-definable and $B \subset N$ is N-definable. Simple products play important roles in constructing locally o-minimal structures (see Theorem 19). As an application, we can show the following:

- Let $\boldsymbol{R}^*$ be a nonstandard real closed field elementarily extending $\boldsymbol{R}$. Then $(\boldsymbol{R}^*, +, <, P)$ is locally o-minimal, where P is a unary predicate whose interpretation is $\boldsymbol{R}$.

In Section 5, we concentrate on expansions of the additive structure $(\boldsymbol{R}, +, <)$. For an expansion M of $(\boldsymbol{R}, +, \cdot, <)$ we easily have that M is locally o-minimal if and only if M is o-minimal. So the restriction to additive structures seems natural. The main result (Theorem 25) of this section is the following:

- Let M be a locally o-minimal expansion of $(\boldsymbol{R}, +, <, \boldsymbol{Z})$. Then M is expressed as a simple product of $\boldsymbol{Z}$ and $I = [0, 1)_{\text{def}}$.

General references on o-minimal structures are [**1**], [**2**], [**5**], see also [**6**].

2. Preliminaries.

Our notations are standard. L denotes a language. $M, N, \ldots$ are used to denote L-structures. The universe of M is also denoted by M. $A, B, \ldots$ are used to denote subsets of some structures. We use $x, y, \ldots$ for variables. Formulas are denoted by $\varphi, \psi, \ldots$. We simply say that A is definable in M (or M-definable) if it is definable in M using parameters from M. So, if $A \subset M^n$ is definable, then there is an L-formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and parameters $b_1, \ldots, b_m \in M$ such that $A = \varphi(x_1, \ldots, x_n, b_1, \ldots, b_m)^M$ (the set of all tuples satisfying $\varphi(x_1, \ldots, x_n, b_1, \ldots, b_m)$). A family $\mathscr{F}$, consisting of M-definable sets, will be called uniformly definable if $\mathscr{F}$ has the form $\mathscr{F} = \{\varphi(x_1, \ldots, x_n, b_1, \ldots, b_m)^M : b_1, \ldots, b_m \in M\}$.

Definition 1. Let M be an L-structure and A a subset of M.

1. For $n \in \omega$, $Def^n(A, M)$ is the set of all subsets of M^n of the form $A^n \cap D$, where D is an M-definable subset of M^n. $Def(A, M) = \bigcup_{n\in\omega} Def^n(A, M)$.
2. We simply write $Def(M)$ for $Def(M, M)$ (i.e. the set of all M-definable sets).

DEFINITION 2. Let $A \subset M$. We prepare an n-ary predicate symbol P_X for each $X \in Def^n(A, M)$, and let L_A be the language $\{P_X : X \in Def(A, M)\}$. The local structure A_{def} of A is the following L_A-structure:

- The universe of A_{def} is A;
- The interpretation of P_X in A_{def} is X, for all $X \in Def(A, M)$.

REMARK 3. In general, $Def(A_{\text{def}})$ and $Def(A, M)$ are not equal. However, if A is a definable subset of M, then we have $Def(A_{\text{def}}) = Def(A, M)$.

From now on, we assume that M has the form $(M, <, \dots)$ and that $<^M$ is a dense linear ordering, unless otherwise stated. An open interval of M is a set of the form (a, b), where $a \in M \cup \{-\infty\}$ and $b \in M \cup \{\infty\}$. Recall that M is said to be o-minimal if every definable subset of M is a finite union of points and open intervals in M. The notion of local o-minimality and that of strongly local o-minimality were defined in [**8**]. The ordered pair of a and b is usually denoted by $\langle a, b\rangle$, avoiding confusion of pairs and intervals.

DEFINITION 4. 1. M is called locally o-minimal if for any definable set $A \subset M$ and $a \in M$ there is an open interval $I \ni a$ such that $I \cap A$ is a finite union of intervals and points.
2. M is strongly locally o-minimal, if for any $a \in M$ there is an open interval $I \ni a$ such that whenever A is a definable subset of M then $I \cap A$ is a finite union of intervals and points.
3. M is uniformly locally o-minimal if for any $\varphi(x, y_1, \dots, y_n) \in L$ and $a \in M$ there is an open interval $I \ni a$ such that $I \cap \varphi(x, b_1, \dots, b_n)^M$ is a finite union of intervals and points for any $b_1, \dots, b_n \in M$.

The following facts are proved in [**8**, Corollaries 2.5 and 3.9].

FACT 5. 1. Local o-minimality is preserved under elementary equivalence.
2. Strong local o-minimality is not preserved under elementary equivalence.

Several examples are given below.

EXAMPLE 6. Let $L = \{<\} \cup \{P_i : i \in \omega\}$, where P_i is an unary predicate. Let $M = (\boldsymbol{Q}, <^M, P_0^M, P_1^M, \dots)$ be the structure defined by $P_i^M = \{a \in M : a < 2^{-i}\sqrt{2}\}$. Then M is uniformly locally o-minimal, but it is not strongly locally o-minimal.

If we assume the saturation, we can show the following:

Proposition 7. *Let M be a uniformly locally o-minimal structure. Suppose that M is ω-saturated. Then M is strongly locally o-minimal.*

Proof. Let $a \in M$. Choose an L-formula $\varphi(x, y)$ arbitrarily. By the uniformity of M, there is an open interval $I \ni a$ and numbers $n_b \in \omega$ ($b \in M$) such that $I \cap \varphi(x, b)^M$ is a union of n_b many intervals and points. We may assume that each n_b is chosen minimum. By the saturation of M, n_b's are uniformly bounded, say by $n_\varphi \in \omega$. (Otherwise, by saturation, there would be $b \in M$ such that $I \cap \varphi(x, b)^M$ cannot be expressed as a finite union of intervals and points.) Let $\theta_\varphi(u, v)$ be the formula saying that for any z the set of $x \in (u, v)$ with $\varphi(x, z)$ is a union of n_φ many intervals and points. Then the following set

$$\Gamma(u, v) = \{u < a < v\} \cup \{\theta_\varphi(u, v) : \varphi \in L\}$$

is finitely satisfiable in M. So, by saturation, there are $c, d \in M$ realizing the set Γ. The open interval $I = (c, d)$ witnesses the strong local o-minimality. $\square$

Example 8. We show that there is an ω-saturated locally o-minimal structure that is not uniformly locally o-minimal. For each non-negative $q \in \boldsymbol{Q}$, we prepare a binary predicate $P_q(x, y)$. $L = \{<, P_q\}_{q \in \boldsymbol{Q}^+}$ is our language. We define an L-structure $M = (\boldsymbol{Q}, <^M, P_q^M)_{q \in \boldsymbol{Q}^+}$ by the following:

- $<^M$ is the standard ordering on $\boldsymbol{Q}$;
- $P_q(a, b) \iff a + \sqrt{2} \cdot q \le b$ (in $\boldsymbol{R}$).

$T = \mathrm{Th}_L(M)$ admits elimination of quantifiers. For showing this, let M^* be an ω-saturated model of T. For $r \in \boldsymbol{R}^+ \cup \{\infty\}$, let $\Gamma_r(x, y)$ be the following set of quantifier-free formulas.

$$\{x < y\} \cup \left\{P_q(x, y) : q \in \boldsymbol{Q}^+, \sqrt{2}q \le r\right\} \cup \left\{\neg P_q(x, y) : q \in \boldsymbol{Q}^+, r < \sqrt{2}q\right\}.$$

Intuitively speaking, $\Gamma_r(x, y)$ asserts that the distance of two points $x < y$ is r. Let $A = \{a_1 < \cdots < a_n\}$ and $B = \{b_1 < \cdots < b_n\}$ be two finite subsets of M^*. We will write $A \simeq B$ if we have

$$M^* \models \Gamma_r(a_i, a_j) \iff M^* \models \Gamma_r(b_i, b_j),$$

for all $i, j \le n$ and $r \in \boldsymbol{R}^+ \cup \{\infty\}$. Let $c \in M^*$ be any element. We want to find an element $d \in M^*$ with $Ac \simeq Bd$. To simplify our argument, we treat the case when c is bigger than A. Choose $r_1, \ldots, r_n$ such that $\Gamma_{r_i}(a_i, c)$ holds ($i = 1, \ldots, n$). Let us consider the following set $\Delta(x)$:

$$\bigcup_{1\le i\le n} \Gamma_{r_i}(b_i, x).$$

Since M^* is ω-saturated, we can find $d \in M^*$ such that $\Gamma_{r_n}(b_n, d)$. Then this d automatically satisfies $\Delta(x)$. Now we have $Ac \simeq Bd$. The above argument shows that T admits elimination of quantifiers. From the elimination of quantifiers, we see that M is locally o-minimal.

Now we show that M is not uniformly locally o-minimal. Let (b, c) be a small interval containing a. Notice that the following sentence is a member of T:

$$\forall x \forall x' \big(x < x' \to \exists y (P_1(x, y) \wedge \neg P_1(x', y))\big).$$

So we can choose $q \in M^*$ such that $P_1(b, q) \wedge \neg P_1(c, q)$. Then the set X defined by $P_1(x, q)$ divides (b, c) into two convexes C_1 and C_2. Neither C_1 nor C_2 are intervals.

3. Strong local o-minimality.

The following theorem is easy but important.

Theorem 9. *The following two conditions are equivalent*:

1. *M is strongly locally o-minimal.*
2. *For any finite subset $\{a_1, \dots, a_n\}$ of M, there are left-open and right-closed intervals I_i with $a_i \in (I_i)^\circ$ such that, by putting $I = \bigcup_{1\le i\le n} I_i$, I_{def} is o-minimal. (I° is the interior of I.)*

Proof. $1 \to 2$: Choose any $a_1, \dots, a_n \in M$. Then, by the strong local o-minimality, there are intervals $I_i = (b_i, c_i]$ with $a_i \in (I_i)^\circ$ $(i = 1, \dots, n)$ such that, for any definable set $X \subset M$, $X \cap I_i$ is a finite union of points and open intervals in M $(i = 1, \dots, n)$. We may assume that $a_1 < \cdots < a_n$ and $I_1 < \cdots < I_n$.

Let $I = \bigcup I_i$ and choose any $Y \in Def^1(I_{\mathrm{def}})$. Then Y is a definable subset of M and $Y = Y \cap I = \bigcup_i (Y \cap I_i)$. By the item 1, there are d_{ik}'s and e_{ik}'s such that

$$Y \cap I_i = (d_{i1}, e_{i1}) \cup \cdots \cup (d_{im_i}, e_{im_i}) \cup \{\text{finite points}\}.$$

Hence Y is a finite union of convex sets. Using the fact that $<^M$ is dense, we may assume that $d_{i2}, \dots, d_{im_i}$, $e_{i1}, \dots, e_{im_i} \in I_i$. The point d_{i1} need not be an element in I_i. However, even if $d_{i1} \notin I_i$, in I, $Y \cap I_i$ can be written as

$$Y \cap I_i = (-\infty, e_{i1}) \cup \cdots \cup (d_{im_i}, e_{im_i}) \cup \{\text{finite points}\} \quad (\text{if } i = 1),$$
$$Y \cap I_i = (c_{i-1}, e_{i1}) \cup \cdots \cup (d_{im_i}, e_{im_i}) \cup \{\text{finite points}\} \quad (\text{if } i > 1).$$

So, in I_{def}, Y is expressed as a finite union of intervals and points in I.

$2 \to 1$: Assume 2. Let $\{a\}$ be a singleton set in M. Choose an interval $I' = (b, c]$ witnessing the condition in 2. Notice that $I = (b, c)$ also satisfies the required condition in 2, i.e., I_{def} is o-minimal. Let $X \in Def^1(M)$. Then we have $X \cap I \in Def^1(I_{\text{def}})$. By the o-minimality, we have

$$X \cap I = I_1 \cup \cdots \cup I_m \cup \{\text{finite points}\},$$

for some open intervals in the sense of I_{def}. Notice that each I_i is an interval in M. So $X \cap I$ is a finite union of intervals and points in M. Thus we are done. □

The following definition is taken from [**8**].

Definition 10. We say that a definable unary (possibly partial) function f has *local monotonicity* if, for every point $a \in M$, there exists some open interval I containing a such that $\operatorname{dom} f \cap I$ can be broken up into a finite union of points and open intervals, on each of which f is constant, strictly increasing, or strictly decreasing. We say that M has local monotonicity if every definable unary function f of M has local monotonicity.

In [**8**], it was shown that a strongly locally o-minimal structure satisfies local monotonicity. In o-minimal case, we can add the local continuity in the monotonicity theorem. As we will see later, this is not the case of local o-minimality. However, by Theorem 9, we can prove the following:

Proposition 11. *Let M be strongly locally o-minimal. Let D be a definable set of M and $f : D \to M$ a definable function. Then, for any $a \in D$, there are open intervals $I \subset M$ containing a and $J \subset M$ containing $f(a)$ such that, by putting $f^* = f \cap (I \times J)$, the domain of f^* can be broken up into a finite union of points and open intervals, on each of which f^* is constant, strictly increasing and continuous, or strictly decreasing and continuous.*

The following example shows that the replacement of f by f^* in the above proposition is necessary.

Example 12. Let M be any o-minimal structure and let $a \in M$. Let $f : \{a\} \times M \to M^2$ be the function defined by $\langle a, b\rangle \mapsto \langle b, a\rangle$. Then $N = (M^2, <_{lex}, f)$ is an M-definable structure (in eq-sense), where $<_{lex}$ is the lexicographic ordering on M^2. So, N is strongly locally o-minimal. However, f is discontinuous at any point.

As in the o-minimal setting, we can define cells and cell decompositions of definable sets in the locally o-minimal setting, see [**3**]. We have the following proposition by Theorem 9:

Proposition 13. *Assume that $M = (M, <, \dots)$ is a strongly locally o-minimal structure. Let $a \in M^n$. Then, the following results hold.*

1. *Let $X_1, \dots, X_m$ be definable subsets of M^n. Then there is an open box $B \ni a$ and a finite decomposition $\mathscr{P}$ of B into cells partitioning $X_1 \cap B, \dots, X_m \cap B$.*
2. *Let $X \subset M^n$ be a definable set and $f : X \to M$ a definable function. Then there is an open box $B \ni \langle a, f(a)\rangle$ such that for the restriction $f^* = f \cap B$, the domain of f^* admits a finite decomposition $\mathscr{P}$ into cells so that for any $Y \in \mathscr{P}$, $f^*|Y$ is continuous.*
3. *Let $X \subset M^{n+1}$ be a definable set and $b \in M$. Suppose that $X_c = \{d \in M : \langle c, d\rangle \in X\}$ is finite for any $c \in M^n$. Then, there is an open box $B \ni a$, an open interval $I \ni b$ and $K \in \omega$ such that $|X_c \cap I| \leq K$ for all $c \in B$.*

4. Simple products.

Let L_1, L_2 and L be languages. For simplicity, we assume that these languages are relational. Under this assumption, a binary function will be treated as a ternary relation. Let M_i be an L_i-structure ($i = 1, 2$).

Definition 14. 1. Let $A \subset {M_1}^n$ and $B \subset {M_2}^n$. Then $A * B$ is the subset of N^n, $N = M_1 \times M_2$, defined by:

$$A * B := \big\{\langle\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle\rangle \in N^n : \langle a_1, \dots, a_n\rangle \in A \ , \langle b_1, \dots, b_n\rangle \in B\big\}.$$

2. Let N be an L-structure whose universe is the product $M_1 \times M_2$. We say that N is a simple product of M_1 and M_2 if for any $P(x_1, \dots, x_n) \in L$ there are M_1-definable sets $A_1, \dots, A_k \subset {M_1}^n$ and M_2-definable sets $B_1, \dots, B_l \subset {M_2}^n$ such that P^N is a boolean combination of the following sets
   - $A_i * {M_2}^n \quad (i = 1, \dots, k)$,
   - ${M_1}^n * B_i \quad (i = 1, \dots, l)$.

Many important structures can be expressed using simple products.

Example 15. 1. Let M_1 and M_2 be two ordered sets. The lexicographic order $<^N$ on the product $N = M_1 \times M_2$ can be expressed as

$$<^N = \big[(<^{M_1}) * M_2\big] \cup \big[((=^{M_1}) * M_2) \cap (M_1 * (<^{M_2}))\big].$$

So $(N, <^N)$ is a simple product.

2. Let M_1 and M_2 be two groups. The product group of M_1 and M_2 is a simple product.
3. Let $I = ([0,1), <, +)$ be the additive group of reals modulo 1. Let $N = \boldsymbol{Z} \times I$ be the simple product defined by:

$$+^N = (P * Q) \cup (P' * Q'),$$

where $P = \{\langle m,n,k\rangle \in \boldsymbol{Z}^3 : m+n = k\}$, $P' = \{\langle m,n,k\rangle \in \boldsymbol{Z}^3 : m+n+1 = k\}$, $Q = \{\langle a,b,c\rangle \in [0,1)^3 : a, b < a +^I b = c\}$ and $Q' = \{\langle a,b,c\rangle \in [0,1)^3 : c = a +^I b < a, b\}$. Then N is isomorphic to $\boldsymbol{R} = (\boldsymbol{R}, +, <)$ by the mapping $\langle n, a\rangle \mapsto n + a$.

Remark 16. Let N be the simple product of M_1 and M_2. Let A be a definable subset of ${M_1}^n$. Then the complement of $A * {M_2}^n$ in N^n can be written as $({M_1}^n \setminus A) * {M_2}^n$. So, in the definition of simple products, we can replace "boolean combination" by "positive boolean combination".

Lemma 17. *Suppose that $N = M_1 \times M_2$ is a simple product. Let D be an N-definable subset of N^n. Then there are M_1-definable sets $A_1, \ldots, A_k \subset {M_1}^n$ and M_2-definable sets $B_1, \ldots, B_l \subset {M_2}^n$ such that D is a positivve boolean combination of $A_i * {M_2}^n$ $(i = 1, \ldots, k)$ and ${M_1}^n * B_i$ $(i = 1, \ldots, k)$.*

Proof. For simplicity, we assume D is $\emptyset$-definable. Choose an L-formula $\varphi(x_1, \ldots, x_n)$ defining D. If φ is an atomic formula, the lemma follows from the definition of simple products. Our proof proceeds by induction on the complexity of φ. The case when φ has the form $\psi \wedge \chi$ or $\psi \vee \chi$ is clear. If φ has the form $\neg\psi$, then we can apply Remark 16. So we assume that φ has the form $\exists y \psi$. Further, for simplicity of the notation, we assume $\psi = \psi(x, y)$, where x and y are single variables. By the induction hypothesis, ψ^N has the form

$$\bigcup_{1 \le i \le k} \big(A_i * {M_2}^2\big) \cap \big({M_1}^2 * B_i\big),$$

where all $A_i \subset {M_1}^2$ and all $B_i \subset {M_2}^2$ are definable sets. Then φ^N is the following set.

$$\bigcup_{\langle a,b\rangle \in M_1 \times M_2} \bigcup_{1 \le i \le k} \big\{\langle a_1, b_1\rangle : \langle\langle a_1, b_1\rangle, \langle a, b\rangle\rangle \in (A_i * {M_2}^2) \cap ({M_1}^2 * B_i)\big\}.$$

This set is equal to

$$\bigcup_{1\le i\le k}\ \bigcup_{\langle a,b\rangle\in M_1\times M_2}\{\langle a_1,b_1\rangle : \langle a_1,a\rangle\in A_i, \langle b_1,b\rangle\in B_i\}.$$

Finally, notice that the set $\bigcup_{\langle a,b\rangle\in M_1\times M_2}\{\langle a_1,b_1\rangle : \langle a_1,a\rangle\in A_i, \langle b_1,b\rangle\in B_i\}$ is equal to $\{\langle a_1,b_1\rangle : a_1\in proj(A_i), b_1\in proj(B_i)\}$, where $proj$ is the projection map to the first coordinate. Since $proj(A_i)$ and $proj(B_i)$ are definable sets, the induction step is complete. □

Lemma 18. *Let N be a simple product of M_1 and M_2. For every $a\in M_1$ and every definable set $D\subset N$, the section $D_a=\{b\in M_2 : \langle a,b\rangle\in D\}$ is definable in M_2.*

Proof. We can find definable sets $A_i\subset M_1$ and $B_i\subset M_2$ such that $D=\bigcup_{1\le i\le k}(A_i * {M_2}^n)\cap({M_1}^n * B_i)$. Then D_a can be written as

$$D_a=\bigcup\{B_i : a\in A_i\}.$$

So D_a is a definable subset of M_2. □

Theorem 19. *For $i=1,2$, let $M_i=(M_i,<^{M_i},\dots)$ be an expansion of a linear order. Let $N=(N,<^N,\dots)$ be a simple product of M_1 and M_2, where $<^N$ is given by the lexicographic ordering.*

1. *Suppose that M_2 is a (strongly) locally o-minimal structure without endpoints. Then N is (strongly) locally o-minimal.*
2. *Suppose that M_2 is an o-minimal structure possibly with endpoints. Suppose also that M_1 is a discrete order. Then N is strongly locally o-minimal.*

Proof. We prove 2, since 1 can be proven similarly. We assume that M_1 has the form $(-\infty,m]$, where m is the maximum element. Let $\langle a,b\rangle\in N$ be any point. First assume that $b\in M_2$ is not an endpoint. Let I be an open M_2-interval with $I\ni b$. Then $I'=\{a\}\times I$ is an N-interval containing $\langle a,b\rangle$. Let $D\subset N$ be any definable set. Then $D\cap I'=\{a\}\times(D_a\cap I)$. By Lemma 18, D_a is a definable subset of M_2. So $D_a\cap I$ is a finite union of intervals and points. Hence $D\cap I'$ is a finite union of intervals and points in the sense of N. This shows the o-minimality of $I'_{\rm def}$, and hence we have the strong local o-minimality of N.

Then we treat the case that b is the maximum element m. Choose any $c\in M_2\smallsetminus\{b\}$ and let $I=(\{a\}\times(c,b])\cup(\{a+1\}\times(-\infty,c))$, where $a+1$ is the successor of a in the discrete structure M_1. (If $a+1$ does not exist, we can put $I=\{a\}\times(c,b]$.) As in the previous case, $I_{\rm def}$ is o-minimal, hence N is strongly locally o-minimal. □

Example 20. Let $A \subset \boldsymbol{Z}$ and P a new unary predicate symbol. Then the structure $(\boldsymbol{R}, +, <, P^{\boldsymbol{R}})$ with $P^{\boldsymbol{R}} = A$ is locally o-minimal.

Proof. Let $I = ([0, 1), +, <)$ be the additive group of reals modulo 1. Let P_0 be a unary predicate symbol and ${P_0}^{\boldsymbol{Z}} = A$. There is a simple product $N = \boldsymbol{Z} \times I$ such that $N \cong (\boldsymbol{R}, +, <)$. We give a P-structure on N by

$$P^N = {P_0}^{\boldsymbol{Z}} * \{0\}.$$

Then (N, P^N) is a simple product, hence it is locally o-minimal by Theorem 19. It is easy to see that $(N, P^N) \cong (\boldsymbol{R}, +, <, A)$. □

Example 21. Let $(\boldsymbol{R}^*, +, \cdot, <, \boldsymbol{Z}^*)$ be a saturated elementary extension of $(\boldsymbol{R}, +, \cdot, <, \boldsymbol{Z})$. Let P be a new unary predicate symbol such that $P^{\boldsymbol{R}^*} = \boldsymbol{Q}$. Then $(\boldsymbol{R}^*, +, <, P^{\boldsymbol{R}^*})$ is locally o-minimal. To see this, using the saturation, choose a positive infinitesimal $h \in \boldsymbol{R}^*$ such that $h\boldsymbol{Z}^* = \{hn : n \in \boldsymbol{Z}^*\} \supset \boldsymbol{Q}$. Then, for a similar reason as in the previous example, $(\boldsymbol{R}^*, +, <, \boldsymbol{Q})$ is given by a simple product of $\boldsymbol{Z}^*$ and $[0, h)^*$.

Example 22. Let $\boldsymbol{R}^*$ be a nonstandard real closed field extending $\boldsymbol{R}$. Then $(\boldsymbol{R}^*, +, <, P^{\boldsymbol{R}^*})$ is locally o-minimal, where $P^{\boldsymbol{R}^*} = \boldsymbol{R}$. This is a corollary of the following more general statement:

Let $(G, 0, +, -, <)$ be a divisible ordered abelian group and $G_0 \subset G$ a subgroup. Suppose there is an $h \in G$ such that $nh < |a|$ for all $n \in \boldsymbol{N}$ and $a \in G_0 \backslash \{0\}$. Then $(G, 0, <, +, -, P^G)$ is locally o-minimal, where $P^G = G_0$.

Proof. First notice that every ordered divisible abelian group with the language $L = \{0, +, -, <\}$ has quantifier elimination. Let $H = \{a \in G : \exists n \in \boldsymbol{N}, |a| < nh\}$. Then H is also a divisible ordered abelian group. So H is an o-minimal structure with the language L. Let $G' \supset G_0$ be a maximal divisible subgroup of G such that $G' \cap H = \{0\}$. Then G splits as the direct sum of G' and H. It is easy to check that $(G, 0, +, -, <, G_0)$ is given by a simple product of G' and H. □

5. Locally o-minimal structures on $\boldsymbol{R}$.

As is shown in the last section, the structure $(\boldsymbol{R}, +, <, \boldsymbol{Z})$ is locally o-minimal. On the other hand, for an expansion M of $(\boldsymbol{R}, +, \cdot, <)$, M is locally o-minimal if and only if it is o-minimal. So, in the study of local o-minimality, it may be important to consider structures without multiplication. In this section we show that any locally o-minimal expansion R of $(\boldsymbol{R}, +, <, \boldsymbol{Z})$ is given by a simple product of $\boldsymbol{Z}$ and $I = [0, 1)$.

We start with some basic remarks on local o-minimality.

REMARK 23. 1. For any $a \in \boldsymbol{R}$, the structure $M = (\boldsymbol{R}, +, <, a\boldsymbol{Z})$ is locally o-minimal. If $a \in \boldsymbol{R}$ is an irrational number, then the structure $N = (\boldsymbol{R}, +, <, \boldsymbol{Z}, a\boldsymbol{Z})$ is not locally o-minimal, since 0 is a limit of the set $\{m + x : m \in \boldsymbol{Z}, x \in a\boldsymbol{Z}\}$.

2. Let M be locally o-minimal. Let $K \subset M$ be a (nonempty) compact definable subset of M. Then K_{def} is an o-minimal structure. (K is possibly not dense, but it is a finite union of dense subsets.)

PROOF. Let A be a definable subset of K_{def}. First notice that A is definable in M also. We show that A is a finite union of intervals (in the sense of K_{def}) and points. Let $a \in K_{\mathrm{def}}$. By the local o-minimality, we can choose an open interval $I \subset M$ with $a \in I$ such that $K \cap I$ has one of the following form:

(a) (b, c), $(b, c]$, $[b, c)$,
(b) $\{a\}$,

where $b < c$ and $b \le a \le c$. But, by the closedness of K, the endpoints b and c must belong to K. So $K \cap I$ is an interval (or a point) in K_{def}. Since other cases can be treated similarly, we assume $K \cap I = [b, c]$ and $b < a < c$. Now we consider the set $K \cap I \cap A$. By the local o-minimality of M, there are $b_1, c_1 \in M$ such that $K \cap (b_1, c_1) \cap A$ is a finite union of intervals and points. We may assume that $b < b_1 < a < c_1 < c$. So, by letting $I_a = (b_1, c_1)$, $K \cap I_a \cap A$ is a finite union of intervals in K and points in K. Since $\bigcup_{a \in K} I_a$ is an open covering of K, by compactness of K, there is a finite set $F \subset K$ such that $\bigcup_{a \in F} I_a \supset K$. Then $K \cap A = \bigcup_{a \in F} (K \cap I_a \cap A)$ is a finite union of intervals and points in the sense of K_{def}. □

LEMMA 24. *Let M be a locally o-minimal expansion of $(\boldsymbol{R}, +, <)$ and let $I = [0, 1)$. Suppose that a family $\mathscr{X} \subset Def^n(I, M)$ is at most countable. If $\mathscr{X}$ is uniformly M-definable, then it is finite.*

PROOF. We use the fact that any compact subset of M is o-minimal (see Remark 23). So we know that I_{def} is an o-minimal structure.

We proceed by induction on n. First let $n = 1$ and let $\mathscr{X}$ be uniformly definable. By the o-minimality of I_{def}, for each $X \in \mathscr{X}$, $\delta(X) = cl(X) - X^\circ$ is finite. So $\Delta = \bigcup_{X \in \mathscr{X}} \delta(X)$ is at most countable. Moreover, by the uniform M-definability, $\Delta \setminus \{1\}$ is an I_{def}-definable set. Again, by the o-minimality of I_{def}, Δ must be finite. From this, we see that $\mathscr{X}$ is a finite set.

Now we consider the case when $\mathscr{X} \subset M^{n+1}$ is a uniformly definable countable family. For $X \in \mathscr{X}$ and $a \in I^n$, let X_a be the section $\{b \in I : \langle a, b\rangle \in X\}$ and let $\delta(X_a) = cl(X_a) - (X_a)^\circ$. As in the case $n = 1$, the set $\Delta_a = \bigcup_{X \in \mathscr{X}} \delta(X_a)$ is a finite set. So $\{\Delta_a : a \in I^n\}$ is a uniformly I_{def}-definable family of finite sets in I. By the uniform finiteness (o-minimality of I_{def}), there is a number k such that, for any $a \in I^n$, $|\Delta_a| \leq k$.

We enumerate $\Delta_a \cup \{0, 1\}$ as $\{d_0(a), d_1(a), \ldots, d_{k+1}(a)\}$ in increasing order. For $F, G \subset \{0, \ldots, k+1\}$, let $J_{a,F,G}$ be the union of all singletons $\{d_i(a)\}$ $(i \in F)$ and open intervals $(d_i(a), d_{i+1}(a))$ $(i \in G)$. Then, for any $X \in \mathscr{X}$ and $a \in I^n$, we can find F, G with $X_a = J_{a,F,G}$. Using this fact, we define definable sets

$$Y_{X,F,G} = \{a \in I^n : X_a = J_{a,F,G}\},$$

and we put $\mathscr{Y} = \{Y_{X,F,G}\}_{X,F,G}$. There are only finitely many $\langle F, G\rangle$'s. So the family $\mathscr{Y}$ (consisting of subsets of I^n) is a uniformly M-definable family. From this, using the induction hypothesis, we know that $\mathscr{Y}$ is a finite family. Now notice that if $Y_{X,F,G} = Y_{X',F,G}$ for all F, G, then $X = X'$. So we know that $\mathscr{X}$ is a finite family. □

Theorem 25. *Let M be a locally o-minimal expansion of $(\boldsymbol{R}, +, <, \boldsymbol{Z})$. Then M is expressed as a simple product of $\boldsymbol{Z}$ and $I = [0, 1)_{\mathrm{def}}$.*

Proof. Let L be the language of M. Let P be an n-ary predicate symbol in L. For each $\eta = \langle \eta(1), \ldots, \eta(n)\rangle \in \boldsymbol{Z}^n$, we define

$$D_\eta = \{\langle d_1, \ldots, d_n\rangle \in I^n : \langle \eta(1) + d_1, \ldots, \eta(n) + d_n\rangle \in P^M\}.$$

Then, using the predicate for $\boldsymbol{Z}$, we can show that $\mathscr{X} = \{D_\eta\}_\eta$ is a uniformly M-definable family. Since $\mathscr{X}$ is at most countable, it must be finite, by Lemma 24. So we can enumerate $\mathscr{X}$ as $X_0, \ldots, X_k$. For $i = 0, \ldots, k$, let $A_i = \{\eta \in \boldsymbol{Z}^n : D_\eta = X_i\}$. Now we regard $\boldsymbol{Z}$ as a $\{A_i : i \leq k\}$-structure. We give a simple structure on $N = \boldsymbol{Z} \times I$ by

$$P^N = A_0 * X_0 \cup \cdots \cup A_k * X_k.$$

Now it is sufficient to show the following.

Claim A. *The natural mapping $\langle m, a\rangle \mapsto m + a$ gives an isomorphism of N and M.*

Suppose that $\langle\langle m_1, a_1\rangle, \ldots, \langle m_n, a_n\rangle\rangle$ is a member of P^N. Then, by the definition of P^N, there is $i \leq k$ such that

$$\langle\langle m_1, a_1\rangle, \dots, \langle m_n, a_n\rangle\rangle \in A_i * X_i.$$

So we have (1) $\langle m_1, \dots, m_n\rangle \in A_i$ and (2) $\langle a_1, \dots, a_n\rangle \in X_i$. From (1) and the definition of A_i, we have $D_{\langle m_1,\dots,m_n\rangle} = X_i$. From this and (2), we have $\langle a_1, \dots, a_n\rangle \in D_{\langle m_1,\dots,m_n\rangle}$. Hence $\langle a_1 + m_1, \dots, a_n + m_n\rangle \in P^M$. The other direction can be shown similarly. □

Theorem 25 shows that, if the given locally o-minimal expansion of $(\boldsymbol{R}, <, +)$ has $\boldsymbol{Z}$ as a definable set, then it can be expressed as a simple product. The next proposition shows that there is a locally o-minimal expansion M having the properties (1) M has an infinite discrete definable set and (2) M cannot be expressed as a simple product of the form $\boldsymbol{Z} \times I$ (see Remark 27 below).

Proposition 26. *Let $E = \{e^n : n \in \omega\}$, where e is the base of the natural logarithm. Then the structure $(\boldsymbol{R}, +, <, E)$ is locally o-minimal.*

Proof. Let $(\boldsymbol{R}^*, +, <, E^*)$ be a proper elementary extension of $(\boldsymbol{R}, +, <, E)$ with infinitesimals. Let μ be the monad of 0, i.e. $\mu = \{a \in \boldsymbol{R}^* : |a| < r\ (\forall r \in \boldsymbol{R})\}$. Let $D^* \subset \boldsymbol{R}^*$ be the smallest divisible group containing E^*.

Claim A. $D^* \cap \mu = \{0\}$.

Assume otherwise. We consider $\boldsymbol{R}$ and $\boldsymbol{R}^*$ as $\boldsymbol{Q}$-modules. Then there is an infinitesimal $\varepsilon \in \boldsymbol{R}^* \setminus \{0\}$ and finitely many rationals $q_i \in \boldsymbol{Q}$ and E^*-elements $\alpha_1 < \dots < \alpha_n$ such that $\varepsilon = q_1\alpha_1 + \dots + q_n\alpha_n$. By $(\boldsymbol{R}, +, <, E) \prec (\boldsymbol{R}^*, +, <, E^*)$, for any positive $r \in \boldsymbol{R}$, there are E-elements $a_1 < \dots < a_n$ such that $0 \neq |q_1a_1 + \dots + q_na_n| < r$. We show that this is impossible. For fixed $s_1, \dots, s_n \in \boldsymbol{Q}$, let $A_{s_1\dots s_n} = \{|s_1e^{m_1} + \dots + s_ne^{m_n}| : m_1 < \dots < m_n \in \omega\}$. Then, by induction on n, we can show that for any $s_1, \dots, s_n \in \boldsymbol{Q}$ and positive $r \in \boldsymbol{R}$, there are only finitely many elements $a \in A_{s_1\dots s_n}$ with $a \leq r$. (End of Proof of Claim A)

Using Claim A, choose a maximal divisible group $X \subset \boldsymbol{R}^*$ extending D^* such that $X \cap \mu = \{0\}$. Then we have $\boldsymbol{R}^* = X \oplus \mu$, and X is a representative set of $\boldsymbol{R}^*/\mu$. X has a natural induced order. On $M = X \times \mu$, we can define naturally $+^M$ and $<^M$ so that M becomes a simple product. We also define E^M by

$$\langle a, b\rangle \in E^M \iff a \in E^* \text{ and } b = 0.$$

Then the expanded structure $M = (M, +^M, <^M, E^M)$ is still simple. So M is a locally o-minimal structure, by Theorem 19.

Claim B. *Let $\sigma : \boldsymbol{R}^* \to M$ be the natural mapping defined by $\alpha \mapsto \langle a, b\rangle$, where $a \in X$ and $b \in \mu$ are (unique) elements with $\alpha = a + b$. Then σ is an isomorphism.*

We only need to check $\sigma(E^*) = E^M$. Let $\alpha \in E^*$. Then $\alpha \in X$ and $\sigma(\alpha) = (\alpha, 0)$. So $\sigma(\alpha)$ belongs to E^M. The other inclusion follows similarly. (End of Proof of Claim B)

By Claim B, we see that $(\boldsymbol{R}^*, +, <, E^*)$ is locally o-minimal. Since the local o-minimality is preserved under elementary equivalence (see Fact 5), we have the local o-minimality of $(\boldsymbol{R}, +, <, E)$. □

Remark 27. 1. The structure $(\boldsymbol{R}, +, <, E)$ cannot be expressed as a simple product of the form $\boldsymbol{Z} \times I$. For otherwise, both E and $\boldsymbol{Z}$ are definable in the structure $\boldsymbol{Z} \times I$. Then $(E + \boldsymbol{Z}) \cap I$ is a countable (infinite) definable set having an accumulation point. But this contradicts the local o-minimality of $\boldsymbol{Z} \times I$.

2. Let us say that $E \subset \boldsymbol{R}$ is a good set if for all $n \in \omega$ and for all $q_1, \ldots, q_n \in \boldsymbol{Q} \setminus \{0\}$, the set $\{|q_1 a_1 + \cdots + q_n a_n| : a_i \in E\}$ has a positive infimum. Then, for any good E, we can prove the local o-minimality of $(\boldsymbol{R}, +, <, E)$, exactly by the same argument as above. Moreover, if $P_0, P_1, \ldots$ are relations on E, then the structure $(\boldsymbol{R}, +, <, E, P_0, P_1, \ldots)$ is also locally o-minimal.

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