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Continuous limit of the difference second Painlevé equation and its asymptotic solutions

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Abstract. The discrete second Painlevé equation $\mathrm{dP_{II}}$ is mapped to the second Painlevé equation $\mathrm{P_{II}}$ by its continuous limit, and then, as shown by Kajiwara et al., a rational solution of $\mathrm{dP_{II}}$ also reduces to that of $\mathrm{P_{II}}$. In this paper, regarding $\mathrm{dP_{II}}$ as a difference equation, we present a certain asymptotic solution that reduces to a triply-truncated solution of $\mathrm{P_{II}}$ in this continuous limit. In a special case our solution corresponds to a rational one of $\mathrm{dP_{II}}$. Furthermore we show the existence of families of solutions having sequential limits to truncated solutions of $\mathrm{P_{II}}$.

1. Introduction.

The non-autonomous mapping

$$\mathrm{dP_{II}} \qquad y_{n+1} + y_{n-1} = \frac{(a_0 n + 2)y_n + a_1}{1 - y_n^2}$$

with $a_0, a_1 \in \boldsymbol{C}$ is known as the discrete second Painlevé equation [**2**], [**6**], [**8**], [**9**]. If we put

$$i\varepsilon n = x, \quad y_n = i\varepsilon v(x), \quad a_0 = -i\varepsilon^3, \quad a_1 = -i\varepsilon^3\alpha,$$

this becomes

$$v(x + i\varepsilon) + v(x - i\varepsilon) = \frac{(2 - \varepsilon^2 x)v(x) - \varepsilon^2\alpha}{1 + \varepsilon^2 v(x)^2}, \tag{1.1}$$

which may be regarded as a difference equation with respect to the variable x. Equation (1.1) is also written in the form

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$$(i\varepsilon)^{-2}\big(v(x+i\varepsilon)+v(x-i\varepsilon)-2v(x)\big)=\frac{2v(x)^3+xv(x)+\alpha}{1+\varepsilon^2 v(x)^2}, \tag{1.2}$$

and the limit $\varepsilon \to 0$ yields the second Painlevé equation

$$\mathrm{P_{II}} \qquad v''=2v^3+xv+\alpha$$

$('=d/dx)$ [**7**]. If $a_1/a_0 \in \boldsymbol{Z}$ or $\alpha \in \boldsymbol{Z}$, then $\mathrm{dP_{II}}$ or (1.1) admits a rational solution (see [**4**], [**10**]). Kajiwara et al. [**4**] proved that this rational solution reduces to that of $\mathrm{P_{II}}$ in the limit $\varepsilon \to 0$. In general, however, few results are known about the relation between a solution of a difference equation and that of the resultant differential equation of its continuous limit, or about the behaviour of a solution in the process of the limit.

The purpose of this paper is to show that this continuous limit of (1.1) to $\mathrm{P_{II}}$ may be analytically justified also for certain kinds of asymptotic solutions of (1.1) including the rational solution mentioned above as a special case. Our main results are stated in Section 2. In Theorem 2.2, for each $\alpha \in \boldsymbol{C}$, we present an asymptotic solution of (1.1) that reduces to a triply-truncated solution of $\mathrm{P_{II}}$ as $\varepsilon \to 0$. Furthermore, in Theorem 2.4 we describe one-parameter families of solutions having sequential limits to truncated solutions of $\mathrm{P_{II}}$.

The solution in Theorem 2.2 is asymptotic to a formal power series given in Theorem 2.1, which is proved in Section 3. In Section 5, we derive a nonlinear difference equation (cf. (5.3)) equivalent to (1.1) by using some lemmas given in Section 4. In Section 6, for this equivalent equation, we show the existence of a solution asymptotic to zero, from which our desired solution in Theorem 2.2 immediately follows. The equivalent equation may be regarded as a nonlinear perturbation of an associated linear difference equation (cf. (6.1)). In our argument we have to handle these equations in a domain where the value of $|\varepsilon^2 x|$ ranges from 0 to $+\infty$, and we need some appropriate expressions of solutions of this associated linear equation uniformly valid in this domain. Such uniform expressions are given in Corollary 6.3. In Section 7 we prove Theorem 2.4 by constructing the families of solutions in strip domains. In showing their relation to truncated solutions of $\mathrm{P_{II}}$, we use the fact that the associated linear difference equation is mapped to a differential equation of Airy type by its continuous limit. The final section is devoted to the proof of Proposition 6.2 for a system of linear difference equations. We find a fundamental matrix solution of it, which yields the uniform expressions in Corollary 6.3. The key to constructing it is the Euler-Maclaurin summation formula (cf. Lemma 8.7).

Throughout this paper, for functions $\phi_1(\varepsilon,x)$ and $\phi_2(\varepsilon,x)$ not necessarily real-valued, we often write $\phi_1(\varepsilon,x) \ll \phi_2(\varepsilon,x)$ or $\phi_2(\varepsilon,x) \gg \phi_1(\varepsilon,x)$ if $\phi_1(\varepsilon,x) = O(\phi_2(\varepsilon,x))$, that is, $\phi_1(\varepsilon,x)/\phi_2(\varepsilon,x)=O(1)$.

2. Main results.

Let us begin with a formal solution of (1.1).

THEOREM 2.1. *Equation* (1.1) *possesses a formal solution of the form*

$$\hat{v}(\varepsilon, x) := \sum_{j \geq 1} c_j(\varepsilon) x^{-j}. \tag{2.1}$$

The coefficients $c_j(\varepsilon)$ are polynomials in ε^2 and α, which are uniquely determined, and the first eleven of them are listed as follows:

$$\begin{aligned}
c_1(\varepsilon) &= -\alpha, \quad c_2(\varepsilon) = c_3(\varepsilon) = 0, \\
c_4(\varepsilon) &= 2\alpha(\alpha^2 - 1), \quad c_5(\varepsilon) = c_6(\varepsilon) = 0, \\
c_7(\varepsilon) &= -4\alpha(\alpha^2 - 1)(3\alpha^2 - 10), \quad c_8(\varepsilon) = 0, \\
c_9(\varepsilon) &= -4\alpha(\alpha^2 - 1)(12\alpha^2 - 35)\varepsilon^2, \\
c_{10}(\varepsilon) &= 8\alpha(\alpha^2 - 1)(12\alpha^4 - 117\alpha^2 + 280), \\
c_{11}(\varepsilon) &= -4\alpha(\alpha^2 - 1)(37\alpha^2 - 84)\varepsilon^4.
\end{aligned}$$

Moreover, $\hat{v}(\varepsilon, x)$ has the properties:

(1) *the series*

$$\hat{v}_0(x) := \hat{v}(0, x) = \sum_{j \geq 1} c_j(0) x^{-j}$$

is a formal solution of $\mathrm{P_{II}}$;

(2) *for each $\varepsilon \in \boldsymbol{C}$, the series $\hat{v}(\varepsilon, x)$ converges if and only if $\alpha \in \boldsymbol{Z}$.*

The second Painlevé equation $\mathrm{P_{II}}$ possesses a solution $v_{\mathrm{II}}(x)$ admitting the asymptotic representation

$$v_{\mathrm{II}}(x) \sim \hat{v}_0(x) = \sum_{j \geq 1} c_j(0) x^{-j} \tag{2.2}$$

as $x \to \infty$ through any closed sector contained in $|\arg x - \pi| < 2\pi/3$, which is one of the triply-truncated solutions of $\mathrm{P_{II}}$ (see [**3**]). This is the unique solution that satisfies (2.2) in a sector $|\arg x - \pi| < \theta_0$ with $\pi/3 < \theta_0 < 2\pi/3$.

Formal series (2.1) is decomposed into two parts:

$$\hat{v}(\varepsilon, x) = \hat{v}_0(x) + \varepsilon^2 \hat{v}_*(\varepsilon, x) \tag{2.3}$$

with

$$\hat{v}_*(\varepsilon, x) := \sum_{j \ge 1} c_j^*(\varepsilon) x^{-j}, \quad c_j^*(\varepsilon) := \varepsilon^{-2}(c_j(\varepsilon) - c_j(0)) \in \boldsymbol{C}[\varepsilon^2].$$

For positive numbers ε_0 and r, we define the interval $E(\varepsilon_0) \subset \boldsymbol{R}$ and the half-plane $H(r) \subset \boldsymbol{C}$ by

$$E(\varepsilon_0): \ 0 < \varepsilon < \varepsilon_0, \qquad H(r): \ \operatorname{Re} x < -r.$$

For $x \in H(r)$ we take $\arg x$ so that $|\arg x - \pi| < \pi/2$. Our solution admitting $\hat{v}(\varepsilon, x)$ as its asymptotic expansion is given by the following:

THEOREM 2.2. *Let ε_0 be a given positive number. Then equation* (1.1) *possesses a solution of the form*

$$v(\varepsilon, x) := v_{\mathrm{II}}(x) + \varepsilon^2 v_*(\varepsilon, x)$$

in the domain $E(\varepsilon_0) \times H(r_0) \subset \boldsymbol{R} \times \boldsymbol{C}$, provided that $r_0 = r_0(\varepsilon_0)$ is sufficiently large. Here

(1) *$v(\varepsilon, x)$ and $v_*(\varepsilon, x)$ are continuous in $\varepsilon \in E(\varepsilon_0)$ and holomorphic in $x \in H(r_0)$;*
(2) *$v_*(\varepsilon, x)$ admits the asymptotic representation*

$$v_*(\varepsilon, x) \sim \hat{v}_*(\varepsilon, x) = \sum_{j \ge 1} c_j^*(\varepsilon) x^{-j}$$

uniformly for $\varepsilon \in E(\varepsilon_0)$ as $x \to \infty$ through any closed sector contained in $|\arg x - \pi| < \pi/2$.

In addition, if a solution $\tilde{v}(\varepsilon, x)$ satisfies $\tilde{v}(\varepsilon, x) = O(x^{-1})$ uniformly for $\varepsilon \in E(\varepsilon_0)$ in $H(r_0)$, then $\tilde{v}(\varepsilon, x) \equiv v(\varepsilon, x)$.

REMARK 2.1. In a half-plane, an asymptotic property of $v_*(\varepsilon, x)$ may be described, at least, in a weaker form. For each integer $N \ge 2$, we have

$$v_*(\varepsilon, x) = \sum_{j=1}^{N} c_j^*(\varepsilon) x^{-j} + O((\operatorname{Re} x)^{-N-1})$$

uniformly for $\varepsilon \in E(\varepsilon_0)$ as $x \to \infty$ through $H(r_0^{(N)})$, where $r_0^{(N)} \geq r_0$, in particular $r_0^{(2)} = r_0$, is a sufficiently large positive number (cf. the argument in Section 6.3).

As an immediate consequence of Theorem 2.2 combined with Remark 2.1 we have the following:

COROLLARY 2.3. *The solution $v(\varepsilon, x)$ satisfies $v(\varepsilon, x) \to v_{\mathrm{II}}(x)$ as $\varepsilon \to 0$ uniformly in the half-plane $H(r_0)$.*

REMARK 2.2. If $\alpha \in \boldsymbol{Z}$, the series $\hat{v}(\varepsilon, x)$ converges to a rational solution of (1.1) (cf. Section 3), and $v_{\mathrm{II}}(x)$ $(= \lim_{\varepsilon \to 0} v(\varepsilon, x))$ is also a rational solution of $\mathrm{P_{II}}$. About rational solutions this corollary agrees with the result of [**4**].

For positive numbers δ $(< \pi/2)$, r and R, let $S_-(\delta, r, R)$ and $S_+(\delta, r, R)$ be the strip domains defined by, respectively,

$$S_{\pm}(\delta, r, R) := \{x \mid -(r+R) < \operatorname{Re} x < -r, \;\; \pm(\arg x - \pi) > \delta\} \subset H(r).$$

THEOREM 2.4. *Let δ $(< \pi/2)$ be a given positive number. Then equation* (1.1) *possesses one-parameter families of solutions $\mathscr{V}_-(\delta)$ and $\mathscr{V}_+(\delta)$ given by*

$$\mathscr{V}_{\pm}(\delta) := \big\{v_{\pm}(\sigma, \varepsilon, x) := v(\varepsilon, x) + V_{\pm}(\sigma, \varepsilon, x) \mid \sigma \in \boldsymbol{C} \setminus \{0\}\big\}$$

with the properties:

(1) *for every $R \geq 1$, $V_{\pm}(\sigma, \varepsilon, x)$ restricted to $E(\varepsilon_{\sigma,R}) \times S_{\pm}(\delta, r_\sigma, R)$ are continuous in $\varepsilon \in E(\varepsilon_{\sigma,R})$ and holomorphic in $x \in S_{\pm}(\delta, r_\sigma, R)$, and admit the expressions*

$$V_{\pm}(\sigma, \varepsilon, x) = \sigma x^{-1} \exp\biggl(\frac{1}{i\varepsilon} \int_{x_0^{\pm}}^{x} \varphi_{\pm}(\varepsilon, t) dt\biggr)(1 + O(x^{-1/2}))$$

with the integrands satisfying

$$\begin{aligned} \varphi_{\pm}(\varepsilon, x) &= (1 + O(x^{-1})) \log \rho_{\pm}(\varepsilon, x), \\ \rho_{\pm}(\varepsilon, x) &:= 1 + \frac{e^{-\pi i}\varepsilon^2 x}{2} \pm \sqrt{e^{-\pi i}\varepsilon^2 x + \frac{\varepsilon^4 x^2}{4}} \end{aligned} \tag{2.4}$$

and with base points $x_0^{\pm} = |x_0^{\pm}| e^{(\pi \pm \delta)i}$ depending only on δ, where $\varepsilon_{\sigma,R} = \varepsilon_{\sigma,R}(\delta)$ (respectively, $r_\sigma = r_\sigma(\delta)$) is a sufficiently small (respectively, large) positive number depending on (σ, δ, R) (respectively, only on (σ, δ)) and the square root is chosen so that $\operatorname{Re}\sqrt{\cdot} > 0$ as $\operatorname{Re}(e^{-\pi i}\varepsilon^2 x) \to +\infty$;

(2) *for each σ there exists a sequence $\{\varepsilon_n \mid n \in \boldsymbol{N}\}$ such that, for every $R \geq 1$, $v_{\pm}(\sigma, \varepsilon_n, x)$ with $\varepsilon_n < \varepsilon_{\sigma,R}$ converge uniformly on every compact set contained in $S_{\pm}(\delta, r_\sigma, R)$ as $\varepsilon_n \to 0$, and the limit functions in the strip domains coincide with truncated solutions of* $\mathrm{P_{II}}$ *admitting the expressions*

$$v_{\mathrm{II}}(x) + \sigma C_{\pm} x^{-1/4} \exp\left(\mp \frac{2}{3} x^{3/2}\right)(1 + O(x^{-1/2})),$$

respectively, as $x \to \infty$ through

$$S_{\pm}(\delta, r_\sigma, +\infty) := \bigcup_{R \geq 1} S_{\pm}(\delta, r_\sigma, R) = \{x \mid \operatorname{Re} x < -r_\sigma, \ \ \pm(\arg x - \pi) > \delta\},$$

where $C_{\pm} \in \boldsymbol{C} \setminus \{0\}$ are some constants independent of σ.

Remark 2.3. The constants r_σ and $\varepsilon_{\sigma,R}$ satisfy $r_\sigma \leq r_{\sigma'}$ and $\varepsilon_{\sigma,R} \geq \varepsilon_{\sigma',R}$ if $|\sigma'| \geq |\sigma|$. Furthermore, for $\delta \neq \delta'$, if $S_{\pm}(\delta, r_\sigma(\delta), R) \cap S_{\pm}(\delta', r_\sigma(\delta'), R') \neq \emptyset$, then solutions $v_{\pm}(\sigma, \varepsilon, x) \in \mathscr{V}_{\pm}(\delta)$ with $\varepsilon < \min\{\varepsilon_{\sigma,R}(\delta), \varepsilon_{\sigma,R'}(\delta')\}$ are continued analytically to $S_{\pm}(\delta', r_\sigma(\delta'), R')$, respectively (see Section 7.2).

Remark 2.4. From the expressions of $\varphi_{\pm}(\varepsilon, x)$ given above, we may derive, for fixed base points $x_0^{\pm}$,

$$\frac{1}{i\varepsilon}\int_{x_0^{\pm}}^{x} \varphi_{\pm}(\varepsilon, t)dt = \frac{i}{\varepsilon^3}\left(\left(1 + \frac{e^{-\pi i}\varepsilon^2 x}{2}\right)\log\left(1 + \frac{e^{-\pi i}\varepsilon^2 x}{2} \pm \sqrt{\frac{\varepsilon^4 x^2}{4} + e^{-\pi i}\varepsilon^2 x}\right)\right.$$
$$\left. \mp \sqrt{\frac{\varepsilon^4 x^2}{4} + e^{-\pi i}\varepsilon^2 x}\right) + O(x^{1/2})$$

uniformly in $E(\varepsilon_0) \times H(r_0)$ by using the facts that the primitive functions of $\log(z \pm \sqrt{z^2-1})$ are $z\log(z \pm \sqrt{z^2-1}) \mp \sqrt{z^2-1}$, respectively, and that $\log \rho_{\pm}(\varepsilon, t) = O(\varepsilon t^{1/2})$. Hence, in $E(\varepsilon_{\sigma,R}) \times S_{\pm}(\delta, r_\sigma, R)$,

$$V_{\pm}(\sigma, \varepsilon, x) = \sigma \exp\left(\mp \frac{2}{3} x^{3/2}(1 + o(1))\right)$$

as $\varepsilon^2 x \to 0$, $x \to \infty$, and

$$V_{\pm}(\sigma, \varepsilon, x) = \sigma \exp\left(\mp \frac{ix}{\varepsilon}\big(\log(\varepsilon^2 x) - 1 - \pi i + o(1)\big)\right)$$

as $\varepsilon^2 x \to \infty$ (see also Proposition 6.1, (3), (4) and Remark 6.1). For each ε, $V_\pm(\sigma,\varepsilon,x)$ decay as $x \to \infty$ through $S_\pm(\delta, r_\sigma, R)$, respectively.

REMARK 2.5. For each ε such that $i\varepsilon^3 \in \boldsymbol{R} \setminus \{0\}$, equation (1.1) admits asymptotic solutions that can be continued meromorphically to the whole complex plane $\boldsymbol{C}$. Such solutions are easily obtained from the asymptotic solutions given by [**11**, Theorems 2.11 and 2.12]. It is easy to see that, for every $R \geq 1$, the solutions $v_\pm(\sigma,\varepsilon,x)$ with $\varepsilon \in E(\varepsilon_{\sigma,R})$ are continued meromorphically to the strip domain $-(r_\sigma + R) < \operatorname{Re} x < -r_\sigma$. For $v_\pm(\sigma,\varepsilon,x)$ and $v(\varepsilon,x)$ except for rational solutions, however, the possibility of their meromorphic continuation to $\boldsymbol{C}$ is an open problem.

3. Proof of Theorem 2.1.

In this and the subsequent two sections, ε denotes a *complex* parameter. Set $v(x) = \sum_{j\geq 1} c_j(\varepsilon) x^{-j}$ and substitute it into (1.2). Since

$$\begin{aligned} &v(x+i\varepsilon) + v(x-i\varepsilon) - 2v(x) \\ &\quad = \sum_{j\geq 1} c_j(\varepsilon) x^{-j} \big((1 + i\varepsilon x^{-1})^{-j} + (1 - i\varepsilon x^{-1})^{-j} - 2 \big) \\ &\quad = 2\varepsilon^2 \sum_{j\geq 1}\sum_{k\geq 1} \frac{c_j(\varepsilon)(j)_{2k}}{(2k)!} (-1)^k \varepsilon^{2(k-1)} x^{-j-2k}, \end{aligned} \tag{3.1}$$

we have

$$\begin{aligned} \alpha + c_1(\varepsilon) + \sum_{j\geq 1} c_{j+1}(\varepsilon) x^{-j} = -2\Big(\sum_{j\geq 1} c_j(\varepsilon)x^{-j}\Big)^3 - 2\bigg(1 + \varepsilon^2 \Big(\sum_{j\geq 1} c_j(\varepsilon) x^{-j}\Big)^2\bigg) \\ \times \sum_{j\geq 1}\sum_{k\geq 1} \frac{c_j(\varepsilon)(j)_{2k}}{(2k)!} (-1)^k \varepsilon^{2(k-1)} x^{-j-2k} \end{aligned} \tag{3.2}$$

with $(j)_{2k} := j(j+1)\cdots(j+2k-1)$. Comparing the coefficients of x^{-j} on both sides, we have

$$\begin{gathered} c_1(\varepsilon) = -\alpha, \quad c_2(\varepsilon) = c_3(\varepsilon) = 0, \\ c_j(\varepsilon) = \Pi_j(\varepsilon^2; c_1(\varepsilon), \ldots, c_{j-1}(\varepsilon)) \quad (j \geq 4), \end{gathered}$$

where $\Pi_j(\varepsilon^2; c_1, \ldots, c_{j-1})$ are polynomials in $(\varepsilon^2, c_1, \ldots, c_{j-1})$ with integer coefficients. Then $c_j(\varepsilon) \in \boldsymbol{C}[\varepsilon^2]$ $(j \geq 1)$ are recursively determined, and we obtain the formal solution $\hat{v}(\varepsilon, x)$ of (1.1) as in Theorem 2.1.

Equality (3.2) with $\varepsilon = 0$ is written in the form

$$\alpha + x \sum_{j \geq 1} c_j(0) x^{-j} + 2\Big(\sum_{j \geq 1} c_j(0) x^{-j}\Big)^3 = \sum_{j \geq 1} j(j+1) c_j(0) x^{-j-2},$$

which implies that $\hat{v}_0(x)$ is a formal solution of $\mathrm{P_{II}}$.

To prove property (2) of the theorem, suppose that $\hat{v}(\varepsilon, x)$ with $\varepsilon \neq 0$ converges around $x = \infty$. By (1.1), it is continued meromorphically to the whole complex plane and must be a rational solution of (1.1). Then the corresponding $\mathrm{dP_{II}}$ with $a_0 = -i\varepsilon^3$, $a_1 = -i\varepsilon^3 \alpha$ also admits the rational solution $y_n = i\varepsilon \hat{v}(\varepsilon, i\varepsilon n)$, and, by [**12**, Theorem 2.2], we have $a_1/a_0 = \alpha \in \boldsymbol{Z}$. In case $\varepsilon = 0$, the convergence of $\hat{v}(0, x) = \hat{v}_0(x)$ implies $\alpha \in \boldsymbol{Z}$ (see [**5**]). Thus the *only if* part of (2) has been verified. To show the *if* part, suppose that $\alpha \in \boldsymbol{Z}$ and that $\varepsilon \neq 0$. Let $v_{\alpha,\varepsilon}(x)$ be the corresponding rational solution of (1.1) [**4**], [**10**]. It is easy to see that $v_{\alpha,\varepsilon}(x) = -\alpha x^{-1} + O(x^{-2})$ around $x = \infty$, and hence the Laurent series expansion of it must coincide with $\hat{v}(\varepsilon, x)$, since the coefficients of $\hat{v}(\varepsilon, x)$ are uniquely determined as shown above. This fact implies the convergence of $\hat{v}(\varepsilon, x)$. The case $\varepsilon = 0$ is treated by the same argument for $\mathrm{P_{II}}$. Thus we obtain Theorem 2.1.

4. Lemmas concerning asymptotic series.

Let $\Sigma(\theta, r)$ denote the sector defined by

$$\Sigma(\theta, r): \quad |\arg x - \pi| < \theta, \quad |x| > r$$

with $r > 0$, $\theta > 0$. Recall that the formal series $\hat{v}(\varepsilon, x)$ is decomposed as in (2.3). By [**14**, Theorem 9.6] or [**13**, Theorem 5.1], there exists a function $\psi_*(\varepsilon, x)$ with the properties:

(1) $\psi_*(\varepsilon, x)$ is holomorphic for $|\varepsilon| < \varepsilon_0$, $x \in \Sigma(2\pi/3, 1)$;
(2) $\psi_*(\varepsilon, x) \sim \hat{v}_*(\varepsilon, x)$ uniformly for $|\varepsilon| < \varepsilon_0$ as $x \to \infty$ through $\Sigma(2\pi/3, 1)$.

Since the solution $v_{\mathrm{II}}(x)$ of $\mathrm{P_{II}}$ admits asymptotic expression (2.2) in a sector of opening angle $\theta = \pi/2 + \delta$, where δ is a small positive number, we have the following:

Lemma 4.1. *The function*

$$\psi(\varepsilon, x) := v_{\mathrm{II}}(x) + \varepsilon^2 \psi_*(\varepsilon, x)$$

is holomorphic and bounded for $|\varepsilon| < \varepsilon_0$, $x \in \Sigma(\pi/2 + \delta, r')$, and admits the asymptotic representation $\psi(\varepsilon, x) \sim \hat{v}(\varepsilon, x)$ uniformly for $|\varepsilon| < \varepsilon_0$ as $x \to \infty$ through $\Sigma(\pi/2 + \delta, r')$, where r' is a sufficiently large positive number.

Recall (3.1) with $v(x) = \hat{v}(\varepsilon, x) = \sum_{j \ge 1} c_j(\varepsilon) x^{-j}$ as a formal series, and define $\tilde{c}_m(\varepsilon) \in \boldsymbol{C}[\varepsilon^2]$ by

$$\begin{aligned}\sum_{m \ge 3} \tilde{c}_m(\varepsilon) x^{-m} &= \hat{v}(\varepsilon, x + i\varepsilon) + \hat{v}(\varepsilon, x - i\varepsilon) - 2\hat{v}(\varepsilon, x) \\ &= \sum_{j \ge 1} c_j(\varepsilon) x^{-j} \big((1 + i\varepsilon x^{-1})^{-j} + (1 - i\varepsilon x^{-1})^{-j} - 2\big). \qquad (4.1)\end{aligned}$$

By Lemma 4.1, for every positive integer N, we have

$$\psi(\varepsilon, x) = \sum_{j=1}^{N} c_j(\varepsilon) x^{-j} + O(x^{-N-1})$$

in $\Sigma(\pi/2 + \delta, r')$, and hence

$$\begin{aligned}&\psi(\varepsilon, x + i\varepsilon) + \psi(\varepsilon, x - i\varepsilon) - 2\psi(\varepsilon, x) \\ &\quad = \sum_{j=1}^{N} c_j(\varepsilon) x^{-j} \big((1 + i\varepsilon x^{-1})^{-j} + (1 - i\varepsilon x^{-1})^{-j} - 2\big) + O(x^{-N-1}) \\ &\quad = \sum_{m=3}^{N} \tilde{c}_m(\varepsilon) x^{-m} + O(x^{-N-1})\end{aligned}$$

uniformly for $|\varepsilon| < \varepsilon_0$ as far as $x \pm i\varepsilon \in \Sigma(\pi/2 + \delta, r')$. This implies

$$\psi(\varepsilon, x + i\varepsilon) + \psi(\varepsilon, x - i\varepsilon) - 2\psi(\varepsilon, x) \sim \sum_{m \ge 3} \tilde{c}_m(\varepsilon) x^{-m} \qquad (4.2)$$

uniformly for $|\varepsilon| < \varepsilon_0$ as $x \to \infty$ through $\Sigma(\pi/2, \tilde{r}')$ if $\tilde{r}'$ is sufficiently large. The following lemma gives another asymptotic expression. Note that this expression cannot be derived directly from (4.2).

LEMMA 4.2. *Let $v_{**}(\varepsilon, x)$ be a function such that*

$$\psi(\varepsilon, x + i\varepsilon) + \psi(\varepsilon, x - i\varepsilon) - 2\psi(\varepsilon, x) = -\varepsilon^2 v_{\mathrm{II}}''(x) + \varepsilon^4 v_{**}(\varepsilon, x).$$

Then it admits an asymptotic representation of the form

$$v_{**}(\varepsilon, x) \sim \sum_{m\geq 3} \tilde{c}^*_m(\varepsilon) x^{-m}$$

with $\tilde{c}^*_m(\varepsilon) \in \boldsymbol{C}[\varepsilon^2]$ *uniformly for* $|\varepsilon| < \varepsilon_0$ *as* $x \to \infty$ *through* $\Sigma(\pi/2, \tilde{r}')$.

PROOF. Since the asymptotic expansions of $v_{\mathrm{II}}(x)$ and $\psi_*(\varepsilon, x)$ are valid in $\Sigma(\pi/2 + \delta, r')$, we have, for each $m \in \boldsymbol{N}$,

$$v^{(m)}_{\mathrm{II}}(x) \sim \hat{v}^{(m)}_0(x) = \sum_{j\geq 1} (-1)^m (j)_m c_j(0) x^{-j-m}, \tag{4.3}$$

$$\frac{\partial^m}{\partial x^m}\psi_*(\varepsilon, x) \sim \frac{\partial^m}{\partial x^m}\hat{v}_*(\varepsilon, x) = \sum_{j\geq 1} (-1)^m (j)_m c^*_j(\varepsilon) x^{-j-m} \tag{4.4}$$

as $x \to \infty$ through $\Sigma(\pi/2, \tilde{r}')$. Let N be a given positive integer. Note that

$$\begin{aligned} &v_{\mathrm{II}}(x + i\varepsilon) + v_{\mathrm{II}}(x - i\varepsilon) - 2v_{\mathrm{II}}(x) \\ &\quad = -\varepsilon^2 v''_{\mathrm{II}}(x) + \frac{2}{4!}\varepsilon^4 v^{(4)}_{\mathrm{II}}(x) + \cdots + \frac{2(-1)^N}{(2N)!}\varepsilon^{2N} v^{(2N)}_{\mathrm{II}}(x) + R_{2N}(\varepsilon, x) \end{aligned} \tag{4.5}$$

with

$$R_{2N}(\varepsilon, x) := \frac{1}{(2N+1)!}\int_0^{i\varepsilon} (i\varepsilon - t)^{2N+1}\big(v^{(2N+1)}_{\mathrm{II}}(x+t) + v^{(2N+1)}_{\mathrm{II}}(x-t)\big)dt.$$

By (4.3), we have $v^{(m)}_{\mathrm{II}}(x) = O(x^{-m-1})$ for every $m \in \boldsymbol{N}$, and $R_{2N}(\varepsilon, x) = O(\varepsilon^{2N+2}x^{-2N-2})$. Hence substitution of (4.3) with $4 \leq m \leq 2N$ into (4.5) yields the asymptotic representation

$$\varepsilon^{-4}\big(v_{\mathrm{II}}(x + i\varepsilon) + v_{\mathrm{II}}(x - i\varepsilon) - 2v_{\mathrm{II}}(x) + \varepsilon^2 v''_{\mathrm{II}}(x)\big) \sim \sum_{m\geq 3} \tilde{c}^0_m(\varepsilon) x^{-m} \tag{4.6}$$

with $\tilde{c}^0_3(\varepsilon) = \tilde{c}^0_4(\varepsilon) = 0$ uniformly for $|\varepsilon| < \varepsilon_0$ as $x \to \infty$ through $\Sigma(\pi/2, \tilde{r}')$. Here the coefficients $\tilde{c}^0_m(\varepsilon) \in \boldsymbol{C}[\varepsilon^2]$ are uniquely determined. Similarly, using (4.4), we derive

$$\varepsilon^{-2}\big(\psi_*(\varepsilon, x + i\varepsilon) + \psi_*(\varepsilon, x - i\varepsilon) - 2\psi_*(\varepsilon, x)\big) \sim \sum_{m\geq 3} \tilde{c}^\sharp_m(\varepsilon) x^{-m} \tag{4.7}$$

with $\tilde{c}^{\sharp}_m(\varepsilon) \in \boldsymbol{C}[\varepsilon^2]$. From (4.6) and (4.7), it follows that

$$\psi(\varepsilon, x + i\varepsilon) + \psi(\varepsilon, x - i\varepsilon) - 2\psi(\varepsilon, x) = -\varepsilon^2 v''_{\mathrm{II}}(x) + \varepsilon^4 v_{**}(\varepsilon, x)$$

with $v_{**}(\varepsilon, x) \sim \sum_{m\ge 3}(\tilde{c}^0_m(\varepsilon) + \tilde{c}^{\sharp}_m(\varepsilon))x^{-m}$, which completes the proof. □

5. Equivalent equation.

For $|\varepsilon| < \varepsilon_0$, equation (1.1) is written in the form

$$v(x + i\varepsilon) + v(x - i\varepsilon) - 2v(x) + \varepsilon^2 xv(x) = \Xi(\varepsilon, x, v(x)), \tag{5.1}$$

where

$$\Xi(\varepsilon, x, v) = -\alpha\varepsilon^2 - \frac{\varepsilon^2 v^2}{1 + \varepsilon^2 v^2}((2 - \varepsilon^2 x)v - \alpha\varepsilon^2).$$

The function $\Xi(\varepsilon, x, v)$ is holomorphic for $|\varepsilon| < \varepsilon_0$, $|v| < \varepsilon_0^{-1}$, $|x| > 1$, and is expanded into a convergent series of the form

$$\Xi(\varepsilon, x, v) = -\alpha\varepsilon^2 - 2\varepsilon^2 v^3 + \varepsilon^4 \sum_{l\ge 2} \Xi_l(\varepsilon, x)v^l.$$

Here $\Xi_2(\varepsilon, x) = \alpha$, $\Xi_3(\varepsilon, x) = x$, and $\Xi_l(\varepsilon, x)$ $(l \ge 4)$ are polynomials in x and ε^2 satisfying $\Xi_{2\nu}(\varepsilon, x) = O(\varepsilon^{2\nu-4})$ and $\Xi_{2\nu+1}(\varepsilon, x) = O(\varepsilon^{2\nu-4}x)$ uniformly for $|\varepsilon| < \varepsilon_0$, $|x| > 1$. Let us substitute $v(x) = w(x) + \psi(\varepsilon, x)$ into (5.1). If $|w(x) + \psi(\varepsilon, x)|$ is sufficiently small, we obtain

$$w(x + i\varepsilon) + w(x - i\varepsilon) - (2 - \varepsilon^2 x)w(x) = g(\varepsilon, x, w(x))$$

with

$$\begin{aligned} g(\varepsilon, x, w) = \Xi(\varepsilon, &x, w + \psi(\varepsilon, x)) \\ &- \psi(\varepsilon, x + i\varepsilon) - \psi(\varepsilon, x - i\varepsilon) + 2\psi(\varepsilon, x) - \varepsilon^2 x\psi(\varepsilon, x) \\ = \varepsilon^4 &\sum_{l\ge 2} \Xi_l(\varepsilon, x)(w + \psi(\varepsilon, x))^l - \alpha\varepsilon^2 - 2\varepsilon^2(w + \psi(\varepsilon, x))^3 \\ &- \psi(\varepsilon, x + i\varepsilon) - \psi(\varepsilon, x - i\varepsilon) + 2\psi(\varepsilon, x) - \varepsilon^2 x\psi(\varepsilon, x). \end{aligned}$$

Then

$$\begin{aligned} g_0(\varepsilon,x) := g(\varepsilon,x,0) = & -\psi(\varepsilon,x+i\varepsilon)-\psi(\varepsilon,x-i\varepsilon)+2\psi(\varepsilon,x) \\ & -\varepsilon^2\bigl(\alpha+x\psi(\varepsilon,x)+2\psi(\varepsilon,x)^3\bigr)+\varepsilon^4\sum_{l\ge 2}\Xi_l(\varepsilon,x)\psi(\varepsilon,x)^l. \end{aligned}$$

Let us take $r''=r''(\varepsilon_0)>\tilde{r}'$ so large that $|\psi(\varepsilon,x)|<\varepsilon_0^{-1}/2$ for $|\varepsilon|<\varepsilon_0$, $x\in\Sigma(\pi/2,r'')$. By Lemma 4.2, $g_0(\varepsilon,x)$ is written as follows:

$$\begin{aligned} g_0(\varepsilon,x) &= \varepsilon^2 v''_{\mathrm{II}}(x)-\varepsilon^4 v_{**}(\varepsilon,x) \\ &\quad -\varepsilon^2\Bigl(\alpha+x\bigl(v_{\mathrm{II}}(x)+\varepsilon^2\psi_*(\varepsilon,x)\bigr)+2\bigl(v_{\mathrm{II}}(x)+\varepsilon^2\psi_*(\varepsilon,x)\bigr)^3\Bigr) \\ &\quad +\varepsilon^4\sum_{l\ge 2}\Xi_l(\varepsilon,x)\bigl(v_{\mathrm{II}}(x)+\varepsilon^2\psi_*(\varepsilon,x)\bigr)^l \\ &= \varepsilon^2\bigl(v''_{\mathrm{II}}(x)-\alpha-xv_{\mathrm{II}}(x)-2v_{\mathrm{II}}(x)^3\bigr)+\varepsilon^4 g_0^*(\varepsilon,x) \\ &= \varepsilon^4 g_0^*(\varepsilon,x), \end{aligned}$$

where $g_0^*(\varepsilon,x)$ admits an asymptotic expression of the form

$$g_0^*(\varepsilon,x)\sim\sum_{j\ge 1}\tilde{c}_j^{**}(\varepsilon)x^{-j},\qquad \tilde{c}_j^{**}(\varepsilon)\in\boldsymbol{C}[\varepsilon^2]$$

uniformly for $|\varepsilon|<\varepsilon_0$ as $x\to\infty$ through $\Sigma(\pi/2,r'')$. Since $\hat{v}(\varepsilon,x)$ satisfies (5.1) as a formal series, by (4.1), (4.2) and Lemma 4.1, we have $g_0(\varepsilon,x)\sim 0$ uniformly for $|\varepsilon|<\varepsilon_0$. Hence we conclude that $\tilde{c}_j^{**}(\varepsilon)=0$ for every $j\ge 1$, namely that $\varepsilon^{-4}g_0(\varepsilon,x)\sim 0$. Furthermore, if $|w|<\varepsilon_0^{-1}/2$,

$$\begin{aligned} g(\varepsilon,x,w)-g_0(\varepsilon,x) &= \varepsilon^4\sum_{l\ge 2}\Xi_l(\varepsilon,x)\bigl((w+\psi(\varepsilon,x))^l-\psi(\varepsilon,x)^l\bigr) \\ &\qquad -2\varepsilon^2\bigl((w+\psi(\varepsilon,x))^3-\psi(\varepsilon,x)^3\bigr) \\ &= \varepsilon^2\sum_{l\ge 1} g_l(\varepsilon,x)w^l, \end{aligned}\tag{5.2}$$

where

$$\varepsilon^2 g_l(\varepsilon,x)=\frac{1}{l!}\,\frac{\partial^l}{\partial w^l}g(\varepsilon,x,w)\Big|_{w=0}.$$

The coefficients $g_l(\varepsilon,x)$ are holomorphic for $|\varepsilon|<\varepsilon_0$, $x\in\Sigma(\pi/2,r'')$ and expressed by asymptotic series in x^{-1}. Using

$$g_1(\varepsilon, x) = -6\psi(\varepsilon, x)^2 + \varepsilon^2 \sum_{l \geq 2} l\, \Xi_l(\varepsilon, x)\psi(\varepsilon, x)^{l-1},$$

we obtain the estimate $g_1(\varepsilon, x) = O(|x^{-2}| + |\varepsilon^2 x^{-1}|) = O(x^{-1})$. For $l \geq 2$ we have $g_2(\varepsilon, x) = O(\varepsilon^2)$ and $g_l(\varepsilon, x) = O(\varepsilon^{l-3}x)$ $(l \geq 3)$ uniformly for $|\varepsilon| < \varepsilon_0$, $x \in \Sigma(\pi/2, r'')$. By the further change of the unknown $w(x) = x^{-1}u(x)$, we get

$$\begin{aligned} &u(x + i\varepsilon) - (2 - \varepsilon^2 x)(1 + i\varepsilon x^{-1})u(x) + \frac{1 + i\varepsilon x^{-1}}{1 - i\varepsilon x^{-1}} u(x - i\varepsilon) \\ &\quad = (x + i\varepsilon)g(\varepsilon, x, x^{-1}u(x)) \\ &\quad = (x + i\varepsilon)g_0(\varepsilon, x) + \varepsilon^2 \sum_{l \geq 1} (1 + i\varepsilon x^{-1})x^{-l+1} g_l(\varepsilon, x)u(x)^l. \end{aligned}$$

Thus we have

Proposition 5.1. *By $v(x) = x^{-1}u(x) + \psi(\varepsilon, x)$, equation* (1.1) *is changed into*

$$u(x + i\varepsilon) - (2 - \varepsilon^2 x)(1 + i\varepsilon x^{-1})u(x) + \frac{1 + i\varepsilon x^{-1}}{1 - i\varepsilon x^{-1}} u(x - i\varepsilon) = G(\varepsilon, x, u(x)). \quad (5.3)$$

The function $G(\varepsilon, x, u)$ is holomorphic for $|\varepsilon| < \varepsilon_0$, $x \in \Sigma(\pi/2, r'')$, $|x^{-1}u| < \varepsilon_0^{-1}/2$, and is expanded into the convergent series

$$G(\varepsilon, x, u) = G_0(\varepsilon, x) + \varepsilon^2 \sum_{l \geq 1} G_l(\varepsilon, x)u^l$$

with the coefficients $G_l(\varepsilon, x)$ satisfying

$$\begin{gathered} \varepsilon^{-4}G_0(\varepsilon, x) \sim 0, \qquad G_1(\varepsilon, x) = O(x^{-1}), \\ G_2(\varepsilon, x) = O(\varepsilon^2 x^{-1}), \qquad G_l(\varepsilon, x) = O(\varepsilon^{l-3}x^{2-l}) \quad (l \geq 3) \end{gathered}$$

uniformly for $|\varepsilon| < \varepsilon_0$ as $x \to \infty$ through $\Sigma(\pi/2, r'')$, where r'' is a sufficiently large positive number.

6. Proof of Theorem 2.2.

If the domain $H(r)$ is stable under both operations $x \mapsto x \pm i\varepsilon$, or if every $x \in H(r)$ satisfies $x \pm i\varepsilon \in H(r)$, then $\varepsilon \in \boldsymbol{R}$. For this reason we consider (5.3) for $(\varepsilon, x) \in E(\varepsilon_0) \times H(r)$. In this and the remaining sections, ε denotes a *positive* parameter.

6.1. Associated linear equation.

To prove Theorem 2.2, we construct a solution of (5.3) such that $\varepsilon^{-2}u(x) \sim 0$ in any closed sector contained in $|\arg x - \pi| < \pi/2$. Let us regard (5.3) as a nonlinear perturbation of the linear equation

$$u(x+i\varepsilon) - (2-\varepsilon^2 x)(1+i\varepsilon x^{-1})u(x) + \frac{1+i\varepsilon x^{-1}}{1-i\varepsilon x^{-1}}u(x-i\varepsilon) = 0. \tag{6.1}$$

To find linearly independent solutions of (6.1), we set

$$\boldsymbol{u}(x) = \begin{pmatrix} u(x) \\ u(x) - (1+i\varepsilon x^{-1})(1-i\varepsilon x^{-1})^{-1}u(x-i\varepsilon) \end{pmatrix}$$

and consider a system of difference equations of the form

$$\begin{aligned} \boldsymbol{u}(x+i\varepsilon) &= A(\varepsilon, x)\boldsymbol{u}(x), \\ A(\varepsilon, x) &= \begin{pmatrix} 1-\varepsilon^2 x(1+i\varepsilon x^{-1}) + 2i\varepsilon x^{-1} & 1 \\ -\varepsilon^2 x(1+i\varepsilon x^{-1}) & 1 \end{pmatrix}, \end{aligned} \tag{6.2}$$

where $u(x)$ is the unknown of (6.1). The characteristic equation for $A(\varepsilon, x)$ is

$$\rho^2 - (2-\varepsilon^2 x)(1+i\varepsilon x^{-1})\rho + (1+2i\varepsilon x^{-1}) = 0$$

admitting the roots

$$\begin{aligned} \rho^*_\pm(\varepsilon, x) := &\left(1 + e^{-\pi i}\frac{\varepsilon^2 x}{2}\right)(1+i\varepsilon x^{-1}) \\ &\pm \sqrt{\left(\frac{\varepsilon^4 x^2}{4} + e^{-\pi i}\varepsilon^2 x\right)(1+i\varepsilon x^{-1})^2 + \varepsilon^2 x^{-2}}, \end{aligned} \tag{6.3}$$

where the square root is chosen so that $\mathrm{Re}\sqrt{\cdot} > 0$ as $\mathrm{Re}(e^{-\pi i}\varepsilon^2 x) \to +\infty$. The simplified quadratic equation $\rho^2-(2-\varepsilon^2x)\rho+1=0$ admits the roots $\rho_\pm(\varepsilon,x)$ given by (2.4). A relation between $\rho_\pm(\varepsilon,x)$ and $\rho^*_\pm(\varepsilon,x)$ will be discussed in the final section.

PROPOSITION 6.1. *Let $r^{(3)}$ be a sufficiently large positive number. Then $\rho_\pm(\varepsilon,x)$ have the properties below:*

(1) *$\rho_\pm(\varepsilon,x)$ are continuous in $\varepsilon\in E(\varepsilon_0)$ and holomorphic in $x\in H(r^{(3)})$;*
(2) *$|\rho_-(\varepsilon,x)|<1<|\rho_+(\varepsilon,x)|$ and $\operatorname{Re}\rho_\pm(\varepsilon,x)>0$ for $(\varepsilon,x)\in E(\varepsilon_0)\times H(r^{(3)})$;*
(3) *$\rho_\pm(\varepsilon,x)=(e^{-\pi i}\varepsilon^2x)^{\pm1}(1\mp2(\varepsilon^2x)^{-1}+O((\varepsilon^2x)^{-2}))$ as $\varepsilon^2x\to\infty$ through $E(\varepsilon_0)\times H(r^{(3)})$;*
(4) *$\rho_\pm(\varepsilon,x)=1\pm\varepsilon(e^{-\pi i}x)^{1/2}(1+O(\varepsilon x^{1/2}))$ as $\varepsilon^2x\to0$ through $E(\varepsilon_0)\times H(r^{(3)})$;*
(5) *if the sequence $\{(\varepsilon_n,x_n)\mid n\in\boldsymbol{N}\}\subset E(\varepsilon_0)\times H(r^{(3)})$ satisfies $|\rho_-(\varepsilon_n,x_n)|\to1$ or $|\rho_+(\varepsilon_n,x_n)|\to1$ as $n\to\infty$, then $\varepsilon_n^2x_n\to0$ as $n\to\infty$.*

PROOF. Since $\rho_\pm(\varepsilon,x)$ do not have branch points as far as $\varepsilon^2x\neq0,4$, property (1) immediately follows, and properties (3) and (4) are easily checked. We may set

$$\rho_-(\varepsilon,x)=\rho_0(\varepsilon,x)^{-1}e^{-i\theta(\varepsilon,x)},\quad \rho_+(\varepsilon,x)=\rho_0(\varepsilon,x)e^{i\theta(\varepsilon,x)}.$$

Here $\rho_0(\varepsilon,x)>0$, $\theta(\varepsilon,x)\in\boldsymbol{R}$, and $\rho_0(\varepsilon,x)$ satisfies $\rho_0(\varepsilon,x)\to+\infty$ as $\varepsilon^2x\to\infty$. Note that

$$\operatorname{Re}\big(\rho_-(\varepsilon,x)+\rho_+(\varepsilon,x)\big)=\big(\rho_0(\varepsilon,x)^{-1}+\rho_0(\varepsilon,x)\big)\cos\theta(\varepsilon,x)=\operatorname{Re}(2-\varepsilon^2x)>2$$

for $(\varepsilon,x)\in E(\varepsilon_0)\times H(r^{(3)})$. This implies $|\theta(\varepsilon,x)|<\pi/2$ and $\rho_0(\varepsilon,x)\neq1$, which yield property (2). If $\rho_0(\varepsilon_n,x_n)\to1$, then, by the inequality above, we have $\theta(\varepsilon_n,x_n)\to0$, so that $-\varepsilon_n^2x_n=\rho_-(\varepsilon_n,x_n)+\rho_+(\varepsilon_n,x_n)-2\to0$ as $n\to\infty$. Thus (5) is verified. □

Concerning a fundamental matrix solution for (6.2) we have the following result, which will be proved in the final section.

PROPOSITION 6.2. *System (6.2) possesses a fundamental matrix solution of the form*

$$U(\varepsilon,x)=\begin{pmatrix}1&1\\1-\rho_+^*(\varepsilon,x)&1-\rho_-^*(\varepsilon,x)\end{pmatrix}\big(I+P(\varepsilon,x)\big)\begin{pmatrix}\zeta_-(\varepsilon,x)&0\\0&\zeta_+(\varepsilon,x)\end{pmatrix}$$

with

$$\zeta_\pm(\varepsilon, x) := \exp\biggl(\frac{1}{i\varepsilon}\int_{x_0}^{x}\varphi_\pm(\varepsilon, t)dt\biggr), \quad x_0 \in H(\tilde{r}),$$

which is continuous in $\varepsilon \in E(\varepsilon_0)$ and holomorphic in $x \in H(\tilde{r})$, provided that $\tilde{r}$ is sufficiently large. Here $P(\varepsilon, x) = (P_{ij}(\varepsilon, x))$ $(i, j = 1, 2)$ and $\varphi_\pm(\varepsilon, x)$ are a square matrix and functions, respectively, with the properties:

(1) $P(\varepsilon, x) = O((\operatorname{Re} x)^{-1/2})$ *uniformly for $\varepsilon \in E(\varepsilon_0)$ as $x \to \infty$ through $H(\tilde{r})$;*
(2) *for $i = 1, 2$, $P_{i1}(\varepsilon, x) = O(x^{-1/2})$ (respectively, $P_{i2}(\varepsilon, x) = O(x^{-1/2})$) in $H(\tilde{r}) \cap \{x \mid \operatorname{Im} x > 0\}$ (respectively, $H(\tilde{r}) \cap \{x \mid \operatorname{Im} x < 0\}$);*
(3) $\varphi_\pm(\varepsilon, x) = (1 + O(x^{-1}))\log \rho_\pm(\varepsilon, x)$ *uniformly for $(\varepsilon, x) \in E(\varepsilon_0) \times H(\tilde{r})$;*
(4) $\operatorname{Re}\varphi_-(\varepsilon, x) < 0$ *and* $\operatorname{Re}\varphi_+(\varepsilon, x) > 0$ *in $E(\varepsilon_0) \times H(\tilde{r})$.*

In addition the relation

$$\zeta_-(\varepsilon, x + i\varepsilon)\zeta_+(\varepsilon, x + i\varepsilon) = \zeta_-(\varepsilon, x)\zeta_+(\varepsilon, x)(1 + O(\varepsilon x^{-1})) \tag{6.4}$$

holds in $E(\varepsilon_0) \times H(\tilde{r})$.

From Proposition 6.2, we immediately obtain linearly independent solutions of (6.1).

Corollary 6.3. *Equation* (6.1) *admits linearly independent solutions $u_\pm(\varepsilon, x)$ expressed as*

$$u_\pm(\varepsilon, x) = \bigl(1 + O((\operatorname{Re} x)^{-1/2})\bigr)\zeta_\pm(\varepsilon, x)$$

in $E(\varepsilon_0) \times H(\tilde{r})$. If $\operatorname{Im} x > 0$ (respectively, $\operatorname{Im} x < 0$), the error term $O((\operatorname{Re} x)^{-1/2})$ of $u_-(\varepsilon, x)$ (respectively, $u_+(\varepsilon, x)$) may be replaced by $O(x^{-1/2})$.

Remark 6.1. By Proposition 6.1, (3) and (4), in $E(\varepsilon_0) \times H(\tilde{r})$, the solutions $u_\pm(\varepsilon, x)$ behave as

$$u_\pm(\varepsilon, x) = \exp\biggl(\mp\frac{2}{3}x^{3/2}(1 + o(1))\biggr)$$

as $\varepsilon^2 x \to 0$, $x \to \infty$, and

$$u_\pm(\varepsilon, x) = \exp\biggl(\mp\frac{ix}{\varepsilon}\bigl(\log(\varepsilon^2 x) - 1 - \pi i + o(1)\bigr)\biggr)$$

as $\varepsilon^2 x \to \infty$.

Since $|1-\varepsilon^2 x/4| \geq |1-\mathrm{Re}(\varepsilon^2 x)/4| > 1$, we have

$$\rho_+^*(\varepsilon,x)-\rho_-^*(\varepsilon,x) = 2\sqrt{\left(\frac{\varepsilon^4 x^2}{4}-\varepsilon^2 x\right)(1+i\varepsilon x^{-1})^2+\varepsilon^2 x^{-2}}$$

$$\gg \varepsilon x^{1/2}(1+i\varepsilon x^{-1})\sqrt{\frac{\varepsilon^2 x}{4}-1+O(x^{-3})} \gg \varepsilon x^{1/2}$$

uniformly in $E(\varepsilon_0)\times H(\tilde{r})$, and hence $\det U(\varepsilon,x) \gg \varepsilon x^{1/2}\zeta_-(\varepsilon,x)\zeta_+(\varepsilon,x)$. Then, by (6.4), the Casorati determinant for $u_-(\varepsilon,x)$ and $u_+(\varepsilon,x)$ is

$$\begin{aligned}\Delta(\varepsilon,x) &:= \begin{vmatrix} u_-(\varepsilon,x) & u_+(\varepsilon,x) \\ u_-(\varepsilon,x+i\varepsilon) & u_+(\varepsilon,x+i\varepsilon)\end{vmatrix} \\ &= \frac{1}{\chi(\varepsilon,x)}\begin{vmatrix} u_-(\varepsilon,x+i\varepsilon) & u_+(\varepsilon,x+i\varepsilon) \\ u_-(\varepsilon,x+i\varepsilon)-\chi(\varepsilon,x)u_-(\varepsilon,x) & u_+(\varepsilon,x+i\varepsilon)-\chi(\varepsilon,x)u_+(\varepsilon,x)\end{vmatrix} \\ &= \chi(\varepsilon,x)^{-1}\det U(\varepsilon,x+i\varepsilon) \gg \varepsilon x^{1/2}\zeta_-(\varepsilon,x)\zeta_+(\varepsilon,x), \end{aligned} \tag{6.5}$$

where $\chi(\varepsilon,x) = 1+2i\varepsilon x^{-1}$.

6.2. Summation equation.

Let us consider the summation equation

$$\begin{aligned}\omega(\varepsilon,x) &= \mathscr{S}(\varepsilon,x;\omega(\varepsilon,x)) \\ &:= -\sum_{k=-\infty}^{-1}\frac{u_-(\varepsilon,x)u_+(\varepsilon,x+ki\varepsilon)}{\Delta(\varepsilon,x+ki\varepsilon)}G(\varepsilon,x+ki\varepsilon,\omega(\varepsilon,x+ki\varepsilon)) \\ &\quad -\sum_{k=0}^{\infty}\frac{u_-(\varepsilon,x+ki\varepsilon)u_+(\varepsilon,x)}{\Delta(\varepsilon,x+ki\varepsilon)}G(\varepsilon,x+ki\varepsilon,\omega(\varepsilon,x+ki\varepsilon)),\end{aligned} \tag{6.6}$$

which is derived by Lagrange's method of variation of constants (cf. [**1**, Section 5]). Every solution of (6.6) satisfies (5.3), provided that the right-hand member converges. We would like to construct a solution of (6.6) such that $\varepsilon^{-2}\omega(\varepsilon,x)\sim 0$ in any closed sector contained in $|\arg x-\pi|<\pi/2$. By Proposition 5.1, rechoosing $\tilde{r}$ larger if necessary, we may suppose that $G(\varepsilon,x,u)$ has the following properties:

(a) for $\varepsilon\in E(\varepsilon_0)$, $x\in H(\tilde{r})$, $|x^{-1}u|<\varepsilon_0^{-1}/3$,

$$|G(\varepsilon,x,u)| \ll |G_0(\varepsilon,x)|+|\varepsilon^2 x^{-1}u|, \tag{6.7}$$

(b) for $\varepsilon \in E(\varepsilon_0)$, $x \in H(\tilde{r})$, $|x^{-1}v_1|, |x^{-1}v_2| < \varepsilon_0^{-1}/3$,

$$|G(\varepsilon, x, v_2) - G(\varepsilon, x, v_1)| \ll \varepsilon^2 x^{-1} |v_2 - v_1|. \tag{6.8}$$

By Proposition 5.1, for every positive integer ν, there exists a positive number M_ν depending on ν such that

$$|G_0(\varepsilon, x)| \leq M_\nu \varepsilon^4 |x|^{-\nu} \tag{6.9}$$

uniformly in $E(\varepsilon_0) \times H(\tilde{r})$. Furthermore we need the following fact (cf. [**1**, p. 24]):

LEMMA 6.4. *If $a > 1$, then*

$$\sum_{k=-\infty}^{\infty} \frac{1}{|x + ki\varepsilon|^a} \ll \varepsilon^{-1} (\operatorname{Re} x)^{-a+1}$$

uniformly in $E(\varepsilon_0) \times H(\tilde{r})$. Furthermore,

$$\sum_{k=0}^{\infty} \frac{1}{|x + ki\varepsilon|^a} \ll \varepsilon^{-1} x^{-a+1}$$

uniformly in $E(\varepsilon_0) \times (H(\tilde{r}) \cap \{x \mid \operatorname{Im} x > 0\})$.

PROOF. In $H(\tilde{r})$,

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} \frac{1}{|x + ki\varepsilon|^a} &= \sum_{k=-\infty}^{\infty} \frac{\varepsilon^{-a}}{\left| \dfrac{x}{i\varepsilon} + k \right|^a} = \sum_{k=-\infty}^{\infty} \frac{\varepsilon^{-a}}{\left(\left(k + \dfrac{\operatorname{Im} x}{\varepsilon} \right)^2 + \left(\dfrac{\operatorname{Re} x}{\varepsilon} \right)^2 \right)^{a/2}} \\
&\leq \frac{2\varepsilon^{-a}}{\left| \dfrac{\operatorname{Re} x}{\varepsilon} \right|^a} + 2 \sum_{m \geq 1} \frac{\varepsilon^{-a}}{\left(m^2 + \left| \dfrac{\operatorname{Re} x}{\varepsilon} \right|^2 \right)^{a/2}} \\
&\ll \frac{2}{|\operatorname{Re} x|^a} + \sum_{m \geq 1} \frac{\varepsilon^{-a}}{\left(m + \left| \dfrac{\operatorname{Re} x}{\varepsilon} \right| \right)^a} \ll \frac{2}{|\operatorname{Re} x|^a} + \varepsilon^{-a} \int_{|\operatorname{Re} x/\varepsilon|}^{\infty} \frac{ds}{s^a} \\
&\ll \frac{2}{|\operatorname{Re} x|^a} + \frac{\varepsilon^{-a}}{\left| \dfrac{\operatorname{Re} x}{\varepsilon} \right|^{a-1}} \ll \varepsilon^{-1} (\operatorname{Re} x)^{-a+1}.
\end{aligned}$$

If $\operatorname{Im} x > 0$, then

$$\sum_{k=0}^{\infty} \frac{1}{|x+ki\varepsilon|^a} = \sum_{k=0}^{\infty} \frac{\varepsilon^{-a}}{\left(\left(k+\dfrac{\operatorname{Im} x}{\varepsilon}\right)^2 + \left|\dfrac{\operatorname{Re} x}{\varepsilon}\right|^2\right)^{a/2}}$$

$$\leq \sum_{k=0}^{\infty} \frac{\varepsilon^{-a}}{\left(k+\dfrac{\operatorname{Im} x}{\varepsilon} + \left|\dfrac{\operatorname{Re} x}{\varepsilon}\right|\right)^a} \ll \frac{1}{|x|^a} + \varepsilon^{-a}\int_{(|\operatorname{Re} x|+\operatorname{Im} x)/\varepsilon}^{\infty} \frac{ds}{s^a}$$

$$\ll \varepsilon^{-1}(|\operatorname{Re} x| + \operatorname{Im} x)^{-a+1} \ll \varepsilon^{-1} x^{-a+1},$$

which implies the desired estimate. □

For each positive integer N and for a positive number r, denote by $\mathscr{F}_N(r)$ the family of functions $u(\varepsilon, x)$ with the properties:

(1) $u(\varepsilon, x)$ is continuous in $\varepsilon \in E(\varepsilon_0)$ and holomorphic in $x \in H(r)$;
(2) $|u(\varepsilon, x)| \leq \varepsilon^2 |\operatorname{Re} x|^{-N}$ in $E(\varepsilon_0) \times H(r)$.

Then concerning the operator $\mathscr{S}(\varepsilon, x;\ \cdot\ )$ we have the following:

Lemma 6.5.

(1) *For every positive integer $N \geq 2$, there exists a positive number r_N such that $\mathscr{S}(\varepsilon, x; u(\varepsilon, x)) \in \mathscr{F}_N(r_N)$ if $u(\varepsilon, x) \in \mathscr{F}_N(r_N)$.*

(2) *There exists a positive number $r_2' \geq r_2$ such that*

$$\big|\mathscr{S}(\varepsilon, x; v_2(\varepsilon, x)) - \mathscr{S}(\varepsilon, x; v_1(\varepsilon, x))\big| \leq \frac{1}{2} \sup_{E(\varepsilon_0)\times H(r_2')} |v_2(\varepsilon, x) - v_1(\varepsilon, x)|$$

if $v_1(\varepsilon, x),\ v_2(\varepsilon, x) \in \mathscr{F}_2(r_2')$.

Proof. By Proposition 6.2, (4) and Corollary 6.3, we have, uniformly for $k \leq -1$,

$$\frac{u_-(\varepsilon, x)u_+(\varepsilon, x+ki\varepsilon)}{\zeta_-(\varepsilon, x+ki\varepsilon)\zeta_+(\varepsilon, x+ki\varepsilon)} \ll \exp\left(\frac{1}{i\varepsilon}\int_{x+ki\varepsilon}^{x} \varphi_-(\varepsilon, t)dt\right)$$

$$\ll \exp\left(\int_k^0 \operatorname{Re} \varphi_-(\varepsilon, x+i\varepsilon s)ds\right) \ll 1$$

in $E(\varepsilon_0) \times H(\tilde{r})$. Similarly, uniformly for $k \geq 0$,

$$\frac{u_-(\varepsilon, x+ki\varepsilon)u_+(\varepsilon, x)}{\zeta_-(\varepsilon, x+ki\varepsilon)\zeta_+(\varepsilon, x+ki\varepsilon)} \ll 1.$$

Suppose that $N \geq 2$, and that $r > \tilde{r}$. Using Lemma 6.4 together with (6.5), (6.7) and (6.9) with $\nu = N+1$, we have, for $u(\varepsilon, x) \in \mathscr{F}_N(r)$,

$$|\mathscr{S}(\varepsilon, x; u(\varepsilon, x))| \ll \sum_{k=-\infty}^{\infty} \varepsilon^3 |\operatorname{Re} x + ki\varepsilon|^{-N-3/2} \ll \varepsilon^2 |\operatorname{Re} x|^{-N-1/2},$$

which implies $\mathscr{S}(\varepsilon, x; u(\varepsilon, x)) \in \mathscr{F}_N(r)$, provided that $r = r_N > \tilde{r}$ is sufficiently large. Thus the first assertion is proved. Using (6.8), we have, for $v_1(\varepsilon, x)$, $v_2(\varepsilon, x) \in \mathscr{F}_2(r)$ with $r \geq r_2$,

$$\begin{aligned}
&\big|\mathscr{S}(\varepsilon, x; v_2(\varepsilon, x)) - \mathscr{S}(\varepsilon, x; v_1(\varepsilon, x))\big| \\
&\quad \ll \sum_{k=-\infty}^{\infty} \varepsilon |x + ki\varepsilon|^{-3/2} \sup_{k \in \boldsymbol{Z}} \big|v_2(\varepsilon, x+ki\varepsilon) - v_1(\varepsilon, x+ki\varepsilon)\big| \\
&\quad \ll |\operatorname{Re} x|^{-1/2} \sup_{E(\varepsilon_0)\times H(r)} |v_2(\varepsilon, x) - v_1(\varepsilon, x)|.
\end{aligned}$$

Choosing $r = r_2' \geq r_2$ sufficiently large, we obtain the second assertion. □

6.3. Construction of an asymptotic solution.

By Lemma 6.5, (1), we may define a sequence $\{\omega_n(\varepsilon, x) \mid n \in \boldsymbol{N}\}$ by

$$\omega_0(\varepsilon, x) \equiv 0, \qquad \omega_{n+1}(\varepsilon, x) = \mathscr{S}(\varepsilon, x; \omega_n(\varepsilon, x))$$

such that, for every integer $N \geq 2$,

$$\{\omega_n(\varepsilon, x) \mid n \in \boldsymbol{N}\} \subset \mathscr{F}_N(r_N). \tag{6.10}$$

From Lemma 6.5, (2), it follows that

$$\sup_{E(\varepsilon_0)\times H(r_2')} \big|\omega_{n+1}(\varepsilon, x) - \omega_n(\varepsilon, x)\big| \leq 2^{-n} \sup_{E(\varepsilon_0)\times H(r_2')} |\omega_1(\varepsilon, x)| \leq 2^{-n}\varepsilon^2 (r_2')^{-2},$$

which implies that $\omega_n(\varepsilon, x)$ converges to some function $\omega_\infty(\varepsilon, x) \in \mathscr{F}_2(r_2')$ uniformly for $(\varepsilon, x) \in E(\varepsilon_0) \times H(r_2')$. Furthermore,

$$\big|\mathscr{S}(\varepsilon, x; \omega_n(\varepsilon, x)) - \mathscr{S}(\varepsilon, x; \omega_\infty(\varepsilon, x))\big| \le \frac{1}{2} \sup_{E(\varepsilon_0)\times H(r_2')} |\omega_n(\varepsilon, x) - \omega_\infty(\varepsilon, x)| \to 0$$

as $n \to \infty$, and hence $\omega_\infty(\varepsilon, x)$ is a solution of (6.6).

Let N be a given positive integer such that $N \ge 2$. By (6.10), the solution $\omega_\infty(\varepsilon, x)$ belongs to $\mathscr{F}_N(r_N)$ as well. Then, for any small positive number δ, we have $\varepsilon^{-2}|\omega_\infty(\varepsilon, x)| \le |\operatorname{Re} x|^{-N} \le (\sin\delta)^{-N}|x|^{-N}$ if $|\arg x - \pi| \le \pi/2 - \delta$, $|x| \ge r_N/\sin\delta$. Since $\varepsilon^{-2}\omega_\infty(\varepsilon, x)$ is bounded in $H(r_2')$, this fact implies $\varepsilon^{-2}\omega_\infty(\varepsilon, x) = O(x^{-N})$ in the sector $|\arg x - \pi| \le \pi/2 - \delta$, $|x| \ge r_2'/\sin\delta$. Therefore $\varepsilon^{-2}\omega_\infty(\varepsilon, x) \sim 0$ uniformly for $\varepsilon \in E(\varepsilon_0)$ as $x \to \infty$ through any closed sector contained in $|\arg x - \pi| < \pi/2$.

Taking $r_0 = r_2' \ge r_2$, we obtain the solution $u(\varepsilon, x) = \omega_\infty(\varepsilon, x)$ of (5.3) in $E(\varepsilon_0) \times H(r_0)$, from which the desired solution $v(\varepsilon, x) = \psi(\varepsilon, x) + x^{-1}\omega_\infty(\varepsilon, x)$ of (1.1) immediately follows. The fact $\omega_\infty(\varepsilon, x) \in \mathscr{F}_N(r_N)$ mentioned above also yields the expression of $v_*(\varepsilon, x)$ in $H(r_0^{(N)})$ with $r_0^{(2)} = r_0$ as in Remark 2.1.

6.4. Uniqueness.

Suppose that $v(\varepsilon, x)$ and $\tilde{v}(\varepsilon, x)$ are as in Theorem 2.2. Then $\tilde{u}(\varepsilon, x) := x(\tilde{v}(\varepsilon, x) - \psi(\varepsilon, x))$ is a solution of (5.3) such that

$$\tilde{u}(\varepsilon, x) = x\big(v(\varepsilon, x) + O(x^{-1}) - \psi(\varepsilon, x)\big) = O(1) \tag{6.11}$$

in $E(\varepsilon_0) \times H(r_0)$. By the same argument as in the proof of Lemma 6.5, (1), we have $\mathscr{S}(\varepsilon, x; \tilde{u}(\varepsilon, x)) = O((\operatorname{Re} x)^{-1/2})$ in $E(\varepsilon_0) \times H(r_0)$. The function $\tilde{u}(\varepsilon, x)$ is also a solution of the inhomogeneous difference equation

$$u(x + i\varepsilon) - (2 - \varepsilon^2 x)(1 + i\varepsilon x^{-1})u(x) + \frac{1 + i\varepsilon x^{-1}}{1 - i\varepsilon x^{-1}} u(x - i\varepsilon) = G(\varepsilon, x, \tilde{u}(\varepsilon, x)),$$

and hence it is written in the form

$$\tilde{u}(\varepsilon, x) = \varpi_-(\varepsilon, x)u_-(\varepsilon, x) + \varpi_+(\varepsilon, x)u_+(\varepsilon, x) + \mathscr{S}(\varepsilon, x; \tilde{u}(\varepsilon, x))$$

in $E(\varepsilon_0) \times H(r_0)$, where $\varpi_\pm(\varepsilon, x)$ are suitably chosen periodic functions with the period $i\varepsilon$. Since $|u_-(\varepsilon, x)|$ and $|u_+(\varepsilon, x)|$ diverge as $\operatorname{Im} x \to -\infty$ and as $\operatorname{Im} x \to +\infty$, respectively, $\varpi_\pm(\varepsilon, x)$ must vanish identically. Hence $\tilde{u}(\varepsilon, x)$ is a solution of (6.6) with estimate (6.11). By the same argument as in the proof of Lemma 6.5, we have

$$|\omega_\infty(\varepsilon,x)-\tilde u(\varepsilon,x)| = \big|\mathscr{S}(\varepsilon,x;\omega_\infty(\varepsilon,x)) - \mathscr{S}(\varepsilon,x;\tilde u(\varepsilon,x))\big|$$
$$\le |\operatorname{Re} x|^{-1/2} \sup_{E(\varepsilon_0)\times H(r_2'')} |\omega_\infty(\varepsilon,x)-\tilde u(\varepsilon,x)|,$$

provided that r_2'' is sufficiently large. Therefore $\omega_\infty(\varepsilon,x)\equiv\tilde u(\varepsilon,x)$, which implies $v(\varepsilon,x)\equiv\tilde v(\varepsilon,x)$. This completes the proof of Theorem 2.2.

7. Proof of Theorem 2.4.

7.1. Another equivalent equation.

Let us substitute $v(x)=v(\varepsilon,x)+x^{-1}y(x)$ into (5.1) or (1.1). Since $v(\varepsilon,x)$ is a solution of (5.1), equation (5.1) is written in the form

$$y(x+i\varepsilon)-(2-\varepsilon^2x)(1+i\varepsilon x^{-1})y(x)+\frac{1+i\varepsilon x^{-1}}{1-i\varepsilon x^{-1}}y(x-i\varepsilon)=\widetilde{G}(\varepsilon,x,y(x))$$

with

$$\begin{aligned}\widetilde{G}(\varepsilon,x,y) &:= (x+i\varepsilon)\big(\Xi(\varepsilon,x,v(\varepsilon,x)+x^{-1}y)-\Xi(\varepsilon,x,v(\varepsilon,x))\big)\\
&= (x+i\varepsilon)\bigg(\varepsilon^4\sum_{l\ge2}\Xi_l(\varepsilon,x)\big((x^{-1}y+v(\varepsilon,x))^l-v(\varepsilon,x)^l\big)\\
&\qquad -2\varepsilon^2\big((x^{-1}y+v(\varepsilon,x))^3-v(\varepsilon,x)^3\big)\bigg)\\
&= \varepsilon^2\sum_{l\ge1}\widetilde{G}_l(\varepsilon,x)y^l, \qquad (7.1)\end{aligned}$$

where

$$\varepsilon^2\widetilde{G}_l(\varepsilon,x)=\frac{1}{l!}\frac{\partial^l}{\partial y^l}\widetilde{G}(\varepsilon,x,y)\bigg|_{y=0}.$$

Observing that $v(\varepsilon,x)=\psi(\varepsilon,x)+O(\varepsilon^2x^{-2})$ uniformly for $\varepsilon\in E(\varepsilon_0)$, $x\in H(r_0)$ (cf. Remark 2.1), and comparing with (5.2), we have $\widetilde{G}_l(\varepsilon,x)-G_l(\varepsilon,x)=O(\varepsilon^2x^{-l})$. Thus, instead of Proposition 5.1, we have

Proposition 7.1. *By $v(x)=x^{-1}y(x)+v(\varepsilon,x)$, equation* (1.1) *is changed into*

$$y(x+i\varepsilon)-(2-\varepsilon^2 x)(1+i\varepsilon x^{-1})y(x)+\frac{1+i\varepsilon x^{-1}}{1-i\varepsilon x^{-1}}y(x-i\varepsilon)=\widetilde{G}(\varepsilon,x,y(x)), \quad (7.2)$$

where $\widetilde{G}(\varepsilon,x,y)$ *is continuous in* $\varepsilon\in E(\varepsilon_0)$ *and holomorphic in* (x,y) *satisfying* $x\in H(r_0)$, $|x^{-1}y|<\varepsilon_0^{-1}/2$, *and is expanded into the convergent series*

$$\widetilde{G}(\varepsilon,x,y)=\varepsilon^2\sum_{l\geq 1}\widetilde{G}_l(\varepsilon,x)y^l$$

with coefficients satisfying

$$\widetilde{G}_1(\varepsilon,x)=O(x^{-1}),\quad \widetilde{G}_2(\varepsilon,x)=O(\varepsilon^2x^{-1}),\quad \widetilde{G}_l(\varepsilon,x)=O(\varepsilon^{l-3}x^{2-l})\quad (l\geq 3)$$

uniformly for $\varepsilon\in E(\varepsilon_0)$ *as* $x\to\infty$ *through* $H(r_0)$.

To prove Theorem 2.4 consider the summation equation

$$y(\varepsilon,x)=\sigma u_-(\varepsilon,x)+\mathscr{S}_-(\varepsilon,x;y(\varepsilon,x)) \quad (7.3)$$

with

$$\mathscr{S}_-(\varepsilon,x;y(\varepsilon,x)):=\sum_{k=0}^{\infty}\frac{u_-(\varepsilon,x)u_+(\varepsilon,x+ki\varepsilon)-u_+(\varepsilon,x)u_-(\varepsilon,x+ki\varepsilon)}{\Delta(\varepsilon,x+ki\varepsilon)}$$
$$\times\widetilde{G}(\varepsilon,x+ki\varepsilon,y(\varepsilon,x+ki\varepsilon)).$$

Note that $u_-(\varepsilon,x)$ depends on the base point x_0 in the expression of $\zeta_-(\varepsilon,x)$, which will be specified soon. Every solution of (7.3) satisfies (7.2). Putting $y(\varepsilon,x)=u_-(\varepsilon,x)\omega(\varepsilon,x)$ in (7.3), we have

$$\omega(\varepsilon,x)=\mathscr{S}_-^*(\varepsilon,x;\omega(\varepsilon,x)) \quad (7.4)$$

with

$$\mathscr{S}_-^*(\varepsilon,x;\omega(\varepsilon,x)):=\sigma+\sum_{k=0}^{\infty}\left(\frac{u_+(\varepsilon,x+ki\varepsilon)}{\Delta(\varepsilon,x+ki\varepsilon)}-\frac{u_+(\varepsilon,x)u_-(\varepsilon,x+ki\varepsilon)}{u_-(\varepsilon,x)\Delta(\varepsilon,x+ki\varepsilon)}\right)$$
$$\times u_-(\varepsilon,x+ki\varepsilon)\widetilde{G}^*(\varepsilon,x+ki\varepsilon,\omega(\varepsilon,x+ki\varepsilon)),$$

where

$$\widetilde{G}^*(\varepsilon, x, y) = \varepsilon^2 \sum_{l \ge 1} \widetilde{G}_l(\varepsilon, x) u_-(\varepsilon, x)^{l-1} y^l.$$

7.2. Construction of solutions.

Let δ_0 be a sufficiently small positive number. Then, by Propositions 6.1 and 6.2,

$$\varphi_-(\varepsilon, x) = i\varepsilon x^{1/2}\bigl(1 + O(|\varepsilon x^{1/2}| + |x^{-1}|)\bigr) \tag{7.5}$$

uniformly in $E(\varepsilon_0) \times H(r_0)$ as far as $|\varepsilon^2 x| < \delta_0$. Let δ ($< \pi/2$) be a given positive number as in Theorem 2.4. In $E(\varepsilon_0) \times H(r_0)$, consider the function

$$Q(\varepsilon, x) := \frac{1}{i\varepsilon}\int_{x_0}^{x} \varphi_-(\varepsilon, t)dt = \frac{2}{3}x^{3/2}\bigl(1 + O(|\varepsilon x^{1/2}| + |x^{-1}|)\bigr) - \frac{2}{3}x_0^{3/2}$$

with $x_0 := r_0 e^{i(\pi-\delta)}/\cos\delta$, where a constant related to $O(\,\cdot\,)$ is independent of x_0. By $Z = (2/3)x^{3/2}$ the line $\arg x = \pi - \delta$ is mapped to the line $\arg Z = 3\pi/2 - 3\delta/2$. Hence, choosing $\tilde{r}_0 = \tilde{r}_0(x_0) > r_0$ sufficiently large, and rechoosing δ_0 smaller if necessary, we may suppose that, as far as $|\varepsilon^2 x| < \delta_0$, by the conformal mapping $Z = Q(\varepsilon, x)$ the part of this line contained in $H(\tilde{r}_0)$ is mapped to a curve along which $\operatorname{Re} Z$ monotonically decreases.

Let R be a given number such that $R \ge 1$. To treat (7.4) we choose the path of integration in the expressions of $\zeta_\pm(\varepsilon, x)$ as follows: for $x \in S_-(\delta, r, R)$ with $r \ge \tilde{r}_0$, set $\Gamma(x_0, x) := [x_0, (x)_\delta] \cup [(x)_\delta, x]$. Here $(x)_\delta := \operatorname{Re} x - i \operatorname{Re} x \cdot \tan\delta$ is the point at which the line $\arg x = \pi - \delta$ intersects the vertical line passing through x, $[x_0, (x)_\delta]$ is a segment joining x_0 to $(x)_\delta$, and $[(x)_\delta, x]$ is one joining $(x)_\delta$ to x along the vertical line. Then we set

$$\zeta_\pm(\varepsilon, x) = \exp\biggl(\frac{1}{i\varepsilon}\int_{\Gamma(x_0, x)} \varphi_\pm(\varepsilon, t)dt\biggr). \tag{7.6}$$

Let $s_-(r) := re^{i(\pi-\delta)}/\cos\delta$ and $s_-(r+R) := (r+R)e^{i(\pi-\delta)}/\cos\delta$ be the vertices of $S_-(\delta, r, R)$ on the line $\arg x = \pi - \delta$, and let $[s_-(r), s_-(r+R)]$ denote the edge of $S_-(\delta, r, R)$ joining these vertices. For each $r \ge \tilde{r}_0$ we may choose $\tilde{\varepsilon}_{r,R} = \tilde{\varepsilon}_{r,R}(\delta)$ so small that, for every $(\varepsilon, x) \in E(\tilde{\varepsilon}_{r,R}) \times [s_-(r), s_-(r+R)]$, the inequality $|\varepsilon^2 x| \le \delta_0/2$ is valid. Then we have

Lemma 7.2. *There exists a positive number M_0 independent of R such that, for every $r \ge \tilde{r}_0$, the inequality $|\zeta_-(\varepsilon, x)| \le M_0$ holds uniformly for $(\varepsilon, x) \in E(\tilde{\varepsilon}_{r,R}) \times S_-(\delta, r, R)$.*

Proof. Set $\tilde{x}_0 := \tilde{r}_0 e^{i(\pi-\delta)}/\cos\delta$. If $(\varepsilon, x) \in E(\tilde{\varepsilon}_{r,R}) \times [s_-(r), s_-(r+R)]$, then

$$|\zeta_-(\varepsilon,x)| = |\zeta_-(\varepsilon,\tilde{x}_0)|\bigl|\exp\bigl(Q(\varepsilon,x) - Q(\varepsilon,\tilde{x}_0)\bigr)\bigr| \le |\zeta_-(\varepsilon,\tilde{x}_0)| \le M_0$$

uniformly in $E(\varepsilon_0)$, where M_0 is some positive number, because, as shown above, the image of $[x_0,(x)_\delta] \cap H(\tilde{r}_0)$ under the mapping $Z = Q(\varepsilon,x)$ is a curve along which $\operatorname{Re} Z$ is monotone decreasing. Then, by Proposition 6.2, (4), for $(\varepsilon,x) \in E(\tilde{\varepsilon}_{r,R}) \times S_-(\delta,r,R)$

$$\begin{aligned}|\zeta_-(\varepsilon,x)| &= |\zeta_-(\varepsilon,(x)_\delta)|\left|\exp\left(\frac{1}{i\varepsilon}\int_{(x)_\delta}^{x}\varphi_-(\varepsilon,t)dt\right)\right| \\ &\le M_0\exp\left(\frac{1}{\varepsilon}\int_0^{\operatorname{Im} x - \operatorname{Im}(x)_\delta}\operatorname{Re}\varphi_-(\varepsilon,(x)_\delta + is)ds\right) \le M_0,\end{aligned}$$

which completes the proof. □

Since

$$\widetilde{G}^*(\varepsilon,x,y) = \varepsilon^2\sum_{l\ge1}\widetilde{G}_l(\varepsilon,x)u_-(\varepsilon,x)^{-1}\bigl(u_-(\varepsilon,x)y\bigr)^l,$$

by Proposition 7.1 and Lemma 7.2, it satisfies, for every $r \ge \tilde{r}_0$,

$$\bigl|\widetilde{G}^*(\varepsilon,x,y)\bigr| \ll \varepsilon^2 x^{-1} y \tag{7.7}$$

and

$$\bigl|\widetilde{G}^*(\varepsilon,x,y_2) - \widetilde{G}^*(\varepsilon,x,y_1)\bigr| \ll \varepsilon^2 x^{-1}|y_2 - y_1|$$

uniformly for $(\varepsilon,x) \in E(\tilde{\varepsilon}_{r,R}) \times S_-(\delta,r,R)$, $|y|, |y_1|, |y_2| < ry_0$. Here y_0, which is sufficiently small, and each constant related to the order estimates above are independent of r and R. By Proposition 6.2, (4) together with (6.5), for every $k \ge 0$,

$$\frac{u_-(\varepsilon,x+ki\varepsilon)u_+(\varepsilon,x+ki\varepsilon)}{\Delta(\varepsilon,x+ki\varepsilon)} \ll \varepsilon^{-1}(x+ki\varepsilon)^{-1/2}, \tag{7.8}$$

$$\begin{aligned}
&\frac{u_-(\varepsilon, x+ki\varepsilon)^2 u_+(\varepsilon,x)}{u_-(\varepsilon,x)\Delta(\varepsilon, x+ki\varepsilon)} \\
&\quad \ll \frac{u_-(\varepsilon, x+ki\varepsilon)u_+(\varepsilon,x)}{u_-(\varepsilon,x)u_+(\varepsilon, x+ki\varepsilon)} \cdot \varepsilon^{-1}(x+ki\varepsilon)^{-1/2} \\
&\quad \ll \varepsilon^{-1}(x+ki\varepsilon)^{-1/2} \exp\biggl(\frac{1}{i\varepsilon}\int_x^{x+ki\varepsilon} \bigl(\varphi_-(\varepsilon,t)-\varphi_+(\varepsilon,t)\bigr)dt\biggr) \\
&\quad \ll \varepsilon^{-1}(x+ki\varepsilon)^{-1/2} \exp\biggl(\int_0^k \mathrm{Re}\bigl(\varphi_-(\varepsilon, x+i\varepsilon s)-\varphi_+(\varepsilon, x+i\varepsilon s)\bigr)ds\biggr) \\
&\quad \ll \varepsilon^{-1}(x+ki\varepsilon)^{-1/2}. \qquad (7.9)
\end{aligned}$$

For each $\sigma \neq 0$, let $\mathscr{F}_\sigma(r, \tilde{\varepsilon}_{r,R})$ be the family of functions $y(\sigma,\varepsilon,x)$ continuous in $\varepsilon \in E(\tilde{\varepsilon}_{r,R})$, holomorphic in $x \in S_-(\delta, r, R)$ and satisfying $|y(\sigma,\varepsilon,x)| \leq 2|\sigma|$ uniformly for $(\varepsilon, x) \in E(\tilde{\varepsilon}_{r,R}) \times S_-(\delta,r,R)$. Then, using the estimates above, by the same argument as in the proof of Lemma 6.5, we easily check the following property concerning $\mathscr{S}_-^*(\varepsilon, x; \cdot\,)$:

Lemma 7.3. *There exists a positive number r_σ such that $\mathscr{S}_-^*(\varepsilon, x; y(\sigma,\varepsilon,x)) \in \mathscr{F}_\sigma(r_\sigma, \varepsilon_{\sigma,R})$ if $y(\sigma,\varepsilon,x) \in \mathscr{F}_\sigma(r_\sigma, \varepsilon_{\sigma,R})$ and that*

$$\begin{aligned}
&\bigl|\mathscr{S}_-^*(\varepsilon, x; y_2(\sigma,\varepsilon,x)) - \mathscr{S}_-^*(\varepsilon, x; y_1(\sigma,\varepsilon,x))\bigr| \\
&\quad \leq \frac{1}{2} \sup_{E(\varepsilon_{\sigma,R})\times S_-(\delta, r_\sigma, R)} \bigl|y_2(\sigma,\varepsilon,x) - y_1(\sigma,\varepsilon,x)\bigr| \qquad (7.10)
\end{aligned}$$

if $y_1(\sigma,\varepsilon,x), y_2(\sigma,\varepsilon,x) \in \mathscr{F}_\sigma(r_\sigma, \varepsilon_{\sigma,R})$, where $\varepsilon_{\sigma,R} = \varepsilon_{\sigma,R}(\delta) := \tilde{\varepsilon}_{r_\sigma,R} = \tilde{\varepsilon}_{r_\sigma,R}(\delta)$.

It is easy to see that r_σ and $\varepsilon_{\sigma,R}$ may be chosen so that $r_\sigma \leq r_{\sigma'}$, $\varepsilon_{\sigma,R} \geq \varepsilon_{\sigma',R}$ if $|\sigma'| \geq |\sigma|$.

Define a sequence $\{\omega_n(\sigma,\varepsilon,x) \mid n \in \boldsymbol{N}\}$ by

$$\omega_0(\sigma,\varepsilon,x) \equiv 0, \quad \omega_{n+1}(\sigma,\varepsilon,x) = \mathscr{S}_-^*(\varepsilon, x; \omega_n(\sigma,\varepsilon,x)).$$

Then, by Lemma 7.3, $\omega_n(\sigma,\varepsilon,x) \in \mathscr{F}_\sigma(r_\sigma, \varepsilon_{\sigma,R})$ converges to some $\omega_\infty(\sigma,\varepsilon,x) \in \mathscr{F}_\sigma(r_\sigma, \varepsilon_{\sigma,R})$ uniformly for $(\varepsilon,x) \in E(\varepsilon_{\sigma,R}) \times S_-(\delta, r_\sigma, R)$, which is a solution of (7.4). Substituting $\omega_\infty(\sigma,\varepsilon,x)$ into the right-hand member of (7.4) and using Lemma 6.4, (7.7), (7.8) and (7.9), we obtain $\omega_\infty(\sigma,\varepsilon,x) = \sigma(1+O(x^{-1/2}))$, since $S_-(\delta, r_\sigma, R)$ is contained in the upper half-plane. By Corollary 6.3 equation (7.2) admits the solution

$$y(x)=u_-(\varepsilon,x)\omega_\infty(\sigma,\varepsilon,x)=\sigma\zeta_-(\varepsilon,x)(1+O(x^{-1/2})).$$

Note that a constant related to $O(x^{-1/2})$ is independent of R. This yields the expression of $V_-(\sigma,\varepsilon,x)=x^{-1}u_-(\varepsilon,x)\omega_\infty(\sigma,\varepsilon,x)$ in Theorem 2.4. Let R' be a given number such that $R'>R$. Then, for $\varepsilon<\varepsilon_{\sigma,R'}$, equation (7.4) admits a solution $\omega_\infty^{R'}(\sigma,\varepsilon,x)=\sigma(1+O(x^{-1/2}))$ in $S_-(\delta,r_\sigma,R')$. Using (7.10), by the same argument as in Section 6.4, we can show the uniqueness property, that is, $\omega_\infty^{R'}(\sigma,\varepsilon,x)\equiv\omega_\infty(\sigma,\varepsilon,x)$ in $S_-(\delta,r_\sigma,R)$. Consequently, $V_-(\sigma,\varepsilon,x)$ may be continued analytically to $S_-(\delta,r_\sigma,R')$ if ε is sufficiently small. The function $V_+(\sigma,\varepsilon,x)$ in $S_+(\delta,r_\sigma,R)$ is constructed by the same argument. Thus (1) of Theorem 2.4 is verified.

For another pair (δ',R') such that $S_-(\delta,r_\sigma(\delta),R)\cap S_-(\delta',r_\sigma(\delta'),R')\neq\emptyset$, equation (7.4) admits a solution $\omega_\infty^{\delta',R'}(\sigma,\varepsilon,x)=\sigma(1+O(x^{-1/2}))$. Since $\zeta_-(\varepsilon,x)$ is given by (7.6), $u_-(\varepsilon,x)$ depends on the base point $x_0=r_0e^{i(\pi-\delta)}/\cos\delta$, and we denote it by $u_{-,\delta}(\varepsilon,x)$. For $\varepsilon<\min\{\varepsilon_{\sigma,R'}(\delta'),\varepsilon_{\sigma,R}(\delta)\}$, there exists a constant multiplier $C_{\delta\delta'}$ such that $u_{-,\delta}(\varepsilon,x)=C_{\delta\delta'}u_{-,\delta'}(\varepsilon,x)$. Then (7.3) with $(\sigma,u_{-,\delta}(\varepsilon,x))$ coincides with the equation with $(\sigma C_{\delta\delta'},u_{-,\delta'}(\varepsilon,x))$, and, by (7.10) restricted to $S_-(\delta,r_\sigma(\delta),R)\cap S_-(\delta',r_\sigma(\delta'),R')$, we have $\omega_\infty^{\delta',R'}(\sigma C_{\delta\delta'},\varepsilon,x)\equiv\omega_\infty(\sigma,\varepsilon,x)$. This fact implies that $V_-(\sigma,\varepsilon,x)$ has the property as in Remark 2.3.

7.3. Relation to truncated solutions of $\mathbf{P}_{II}$.

The following proposition describes the relation between (6.1) and the Airy equation.

Proposition 7.4. *There exists a sequence $\{\varepsilon_n\mid n\in\boldsymbol{N}\}$ such that $u_\pm(\varepsilon_n,x)$ converge uniformly on every compact set contained in $H(\tilde r_0)$ as $\varepsilon_n\to 0$, and the limit functions $u_{\pm,0}(x)$ are linearly independent solutions of the differential equation $(x^{-1}u)''-u=0$ admitting the expressions*

$$x^{-1}u_{\pm,0}(x)=C_\pm x^{-1/4}\exp\bigg(\mp\frac{2}{3}x^{3/2}\bigg)(1+O(x^{-3/2})),$$

respectively, as $x\to\infty$ through $H(\tilde r_0)$, where $C_\pm\neq 0$ are constants of integration.

Proof. Let $\{X_\nu\mid\nu\in\boldsymbol{N}\}$ be a sequence of compact sets satisfying $X_1\subset X_2\subset\cdots\subset X_\nu\subset\cdots$ and $\bigcup_{\nu=1}^\infty X_\nu=H(\tilde r_0)$. Note that $u_\pm(\varepsilon,x)$ are holomorphic in $x\in\widetilde X_1$ and bounded in $E(\varepsilon_0)\times\widetilde X_1$ (cf. Remark 6.1), where $\widetilde X_1$ is some bounded domain containing X_1. By the Vitali-Montel theorem there exists a sequence $\{\varepsilon_m^{(1)}\mid m\in\boldsymbol{N}\}\subset E(\varepsilon_0)$ such that $u_\pm(\varepsilon_m^{(1)},x)$ converge uniformly on X_1 to functions $u_{\pm,0}^{X_1}(x)$, respectively, as $\varepsilon_m^{(1)}\to 0$. We may choose a subsequence $\{\varepsilon_m^{(2)}\mid m\in\boldsymbol{N}\}\subset\{\varepsilon_m^{(1)}\mid m\in\boldsymbol{N}\}$ such that $u_\pm(\varepsilon_m^{(2)},x)$ converge uniformly on X_2

to $u^{X_2}_{\pm,0}(x)$ which are the analytic continuations of $u^{X_1}_{\pm,0}(x)$ to X_2, respectively. Repetition of this procedure yields $\{\varepsilon^{(\nu)}_m \mid m \in \boldsymbol{N}\}$ $(\nu \geq 1)$ related to X_ν. Then we obtain the sequence $\{\varepsilon_n := \varepsilon^{(n)}_n \mid n \in \boldsymbol{N}\}$ such that $u_\pm(\varepsilon_n, x)$ together with their derivatives converge to holomorphic functions $u_{\pm,0}(x)$ in $H(\tilde{r}_0)$ uniformly on every X_ν. Write (6.1) in the form

$$\begin{aligned} -xw(x) &= \varepsilon^{-2}\big(w(x+i\varepsilon) + w(x-i\varepsilon) - 2w(x)\big) \\ &= -w''(x) + \frac{\varepsilon^{-2}}{3!}\int_0^{i\varepsilon} (i\varepsilon - t)^3\big(w^{(3)}(x+t) + w^{(3)}(x-t)\big)dt \end{aligned}$$

with $w(x) = x^{-1}u(x)$, and substitute $\varepsilon = \varepsilon_n$ and $w(x) = x^{-1}u_\pm(\varepsilon_n, x)$. For $x \in X_1$ we take the limit $\varepsilon_n \to 0$ to conclude that $x^{-1}u_{\pm,0}(x)$ are solutions of the Airy equation $w'' - xw = 0$. By (7.5), (7.6) and the corresponding expression of $\varphi_+(\varepsilon, x)$, or by Remark 2.4, if δ_0 is sufficiently small, then

$$u_\pm(\varepsilon, x) = \exp\left(\mp \frac{2}{3}x^{3/2}\big(1 + O(|\varepsilon x^{1/2}| + |x^{-1}|)\big)\right) \tag{7.11}$$

uniformly for $(\varepsilon, x) \in E(\varepsilon_0) \times H(\tilde{r}_0)$ satisfying $|\varepsilon^2 x| < \delta_0$. For each X_ν, let n_ν be an integer such that, for every $n \geq n_\nu$, the inequality $|\varepsilon_n^2 x| < \delta_0$ holds on X_ν. Put $\varepsilon = \varepsilon_n$ for $n \geq n_\nu$ in (7.11) and take the limit $\varepsilon_n \to 0$. Then we have

$$x^{-1}u_{\pm,0}(x)\exp\left(\pm \frac{2}{3}x^{3/2}\right) \ll \exp(O(x^{1/2})) \tag{7.12}$$

uniformly for $x \in X_\nu$, where every constant related to the order estimates is independent of X_ν. Hence $x^{-1}u_{\pm,0}(x)$ satisfy (7.12) uniformly in $H(\tilde{r}_0)$, which implies that these are Airy functions expressible as in the proposition. $\square$

Let $\{\varepsilon_n \mid n \in \boldsymbol{N}\}$ be the sequence given by Proposition 7.4 and let $\{R_\nu \mid \nu \in \boldsymbol{N}\}$ be a sequence satisfying $1 < R_1 < R_2 < \cdots < R_\nu < \cdots$ and $R_\nu \to \infty$. Recall the sequence of compact sets $\{X_\nu \mid \nu \in \boldsymbol{N}\}$ considered in the proof of Proposition 7.4. To the function $xV_-(\sigma, \varepsilon, x) = \sigma u_-(\varepsilon, x)(1 + O(x^{-1/2}))$ and the sequence $\{X_\nu \cap S_-(\delta, r_\sigma, R_\nu) \mid \nu \in \boldsymbol{N}\}$ in place of $u_-(\varepsilon, x)$ and $\{X_\nu \mid \nu \in \boldsymbol{N}\}$, we apply the same reasoning as in the proof of Proposition 7.4 to obtain a subsequence $\{\varepsilon_{n(k)} \mid k \in \boldsymbol{N}\} \subset \{\varepsilon_n \mid n \in \boldsymbol{N}\}$ such that, for each ν, $V_-(\sigma, \varepsilon_{n(k)}, x)$ with $\varepsilon_{n(k)} < \varepsilon_{\sigma,R_\nu}$ converges uniformly on every compact set contained in $S_-(\delta, r_\sigma, R_\nu)$ as $\varepsilon_{n(k)} \to 0$. Here a constant related to $O(x^{-1/2})$ is independent of R_ν. Let us renumber the sequence thus obtained as $\{\varepsilon_n \mid n \in \boldsymbol{N}\}$. Then, for every $R > 1$, $V_-(\sigma, \varepsilon_n, x)$ with

$\varepsilon_n < \varepsilon_{\sigma,R}$ converges on every compact set contained in $S_-(\delta, r_\sigma, R)$, and the limit function admits the analytic continuation $V_{-,0}(\sigma, x)$ to $S_-(\delta, r_\sigma, +\infty)$ satisfying $xV_{-,0}(\sigma, x) = \sigma u_{-,0}(x)(1+O(x^{-1/2}))$ in $S_-(\delta, r_\sigma, +\infty)$. The desired expression of

$$\lim_{\varepsilon_n \to 0} v_-(\sigma, \varepsilon_n, x) = \lim_{\varepsilon_n \to 0} \big(v(\varepsilon_n, x) + V_-(\sigma, \varepsilon_n, x)\big) = v_{\mathrm{II}}(x) + V_{-,0}(\sigma, x)$$

in $S_-(\delta, r_\sigma, +\infty)$ is obtained by using Proposition 7.4.

It remains to show that the limit function satisfies $\mathrm{P_{II}}$. In view of (7.3) we set

$$\begin{aligned}\Psi(\varepsilon, x, \xi) &:= \frac{u_-(\varepsilon, x)u_+(\varepsilon, \xi) - u_+(\varepsilon, x)u_-(\varepsilon, \xi)}{\Delta(\varepsilon, \xi)} \widetilde{G}(\varepsilon, \xi, \xi V_-(\sigma, \varepsilon, \xi)) \\ &= u_-(\varepsilon, x)\left(\frac{u_+(\varepsilon, \xi)}{\Delta(\varepsilon, \xi)} - \frac{u_+(\varepsilon, x)u_-(\varepsilon, \xi)}{u_-(\varepsilon, x)\Delta(\varepsilon, \xi)}\right) \\ &\quad \times u_-(\varepsilon, \xi)\widetilde{G}^*\left(\varepsilon, \xi, \frac{\xi V_-(\sigma, \varepsilon, \xi)}{u_-(\varepsilon, \xi)}\right).\end{aligned}$$

By the same argument as in deriving (7.8) and (7.9) we can verify the estimates

$$\begin{aligned}\frac{u_-(\varepsilon, x+it)u_+(\varepsilon, x+it)}{\Delta(\varepsilon, x+it)} &\ll \varepsilon^{-1}(x+it)^{-1/2}, \\ \frac{u_-(\varepsilon, x+it)^2 u_+(\varepsilon, x)}{u_-(\varepsilon, x)\Delta(\varepsilon, x+it)} &\ll \varepsilon^{-1}(x+it)^{-1/2}\end{aligned}$$

uniformly for $t \geq 0$ in $E(\varepsilon_{\sigma,R}) \times S_-(\delta, r_\sigma, R)$. Combining these with (7.7), we have

$$\Psi(\varepsilon, x, x+it) \ll u_-(\varepsilon, x) \cdot \varepsilon |x+it|^{-3/2} \tag{7.13}$$

uniformly for $t \geq 0$ in $E(\varepsilon_{\sigma,R}) \times S_-(\delta, r_\sigma, R)$. Let X be a compact set contained in $S_-(\delta, r_\sigma, R)$, and consider the set $[X] := \{x + it \mid x \in X, t \geq 0\} \subset S_-(\delta, r_\sigma, R)$. Note that

$$\Psi_\xi(\varepsilon, x, \xi) = \frac{1}{2\pi i}\int_{|\tilde{\xi}-\xi|=\delta_X/2} \frac{\Psi(\varepsilon, x, \tilde{\xi})}{(\tilde{\xi}-\xi)^2} d\tilde{\xi} \ll \max_{|\tilde{\xi}-\xi|=\delta_X/2} |\Psi(\varepsilon, x, \tilde{\xi})|$$

uniformly for $\xi \in [X]$, where $\delta_X := \min\{1, \mathrm{dist}([X], \partial S_-(\delta, r_\sigma, R))\}$. Using (7.13) together with this fact we have

$$\Psi_\xi(\varepsilon, x, x+it) \ll u_-(\varepsilon, x) \cdot \varepsilon |x+it|^{-3/2}$$

uniformly for $t \geq 0$ in $E(\varepsilon_{\sigma,R}) \times [X]$. Then

$$\begin{aligned}
&\mathscr{S}_-(\varepsilon, x; xV_-(\sigma,\varepsilon,x)) - \int_0^\infty \Psi(\varepsilon, x, x+i\varepsilon s)ds \\
&\quad = \sum_{k=0}^\infty \Psi(\varepsilon, x, x+ki\varepsilon) - \sum_{k=0}^\infty \frac{1}{\varepsilon}\int_{k\varepsilon}^{(k+1)\varepsilon} \Psi(\varepsilon, x, x+it)dt \\
&\quad = \sum_{k=0}^\infty \frac{1}{\varepsilon}\int_{k\varepsilon}^{(k+1)\varepsilon} (\Psi(\varepsilon, x, x+ki\varepsilon) - \Psi(\varepsilon, x, x+it))dt \\
&\quad = \sum_{k=0}^\infty \frac{1}{\varepsilon}\int_{k\varepsilon}^{(k+1)\varepsilon} \left(\int_t^{k\varepsilon} \Psi_\xi(\varepsilon, x, x+i\tilde{t})d\tilde{t}\right)dt \\
&\quad \ll u_-(\varepsilon,x)\sum_{k=0}^\infty \int_{k\varepsilon}^{(k+1)\varepsilon} \varepsilon|x+it|^{-3/2}dt \\
&\quad \ll \varepsilon u_-(\varepsilon,x)\int_0^\infty |x+it|^{-3/2}dt \ll \varepsilon x^{-1/2}u_-(\varepsilon,x)
\end{aligned}$$

uniformly in $E(\varepsilon_{\sigma,R}) \times [X]$. Hence, by (7.3), the function $V_-(\sigma,\varepsilon,x)$ satisfies the relation

$$xV_-(\sigma,\varepsilon,x) = \sigma u_-(\varepsilon,x)(1+O(\varepsilon x^{-1/2})) + \int_0^\infty \Psi(\varepsilon,x,x+i\varepsilon s)ds \tag{7.14}$$

with

$$\begin{aligned}
&\int_0^\infty \Psi(\varepsilon,x,x+i\varepsilon s)ds \\
&\quad = \int_x^\infty \frac{u_-(\varepsilon,x)u_+(\varepsilon,\xi) - u_+(\varepsilon,x)u_-(\varepsilon,\xi)}{i\varepsilon\,\Delta(\varepsilon,\xi)}\widetilde{G}(\varepsilon,\xi,\xi V_-(\sigma,\varepsilon,\xi))d\xi,
\end{aligned}$$

where the path of integration of the last integral is the vertical line $\xi = x+it$ $(t \geq 0)$ starting from x. Put $\varepsilon = \varepsilon_n$ in (7.14). Since

$$u_\pm(\varepsilon_n, x+i\varepsilon_n) - u_\pm(\varepsilon_n,x) = i\varepsilon_n u'_\pm(\varepsilon_n,x) + \int_0^{i\varepsilon_n}(i\varepsilon_n - t)u''_\pm(\varepsilon_n, x+t)dt,$$

using Proposition 7.4 again, we have $(i\varepsilon_n)^{-1}(u_\pm(\varepsilon_n,x+i\varepsilon_n) - u_\pm(\varepsilon_n,x)) \to u'_{\pm,0}(x)$ as $\varepsilon_n \to 0$ uniformly on every compact set contained in $S_-(\delta, r_\sigma, R)$. Observe that

$(i\varepsilon_n)^{-1}\Delta(\varepsilon_n,\xi) \to u_{-,0}(\xi)u'_{+,0}(\xi) - u_{+,0}(\xi)u'_{-,0}(\xi)$ as $\varepsilon_n \to 0$, and that $\Psi(\varepsilon, x, x+it)$ satisfies (7.13) uniformly for $t \geq 0$ in $E(\varepsilon_{\sigma,R}) \times S_-(\delta, r_\sigma, R)$. We take the limit $\varepsilon_n \to 0$ in (7.14) to obtain

$$xV_{-,0}(\sigma,x) = \sigma u_{-,0}(x) + 2\int_x^\infty \frac{u_{-,0}(x)u_{+,0}(\xi) - u_{+,0}(x)u_{-,0}(\xi)}{u_{-,0}(\xi)u'_{+,0}(\xi) - u_{+,0}(\xi)u'_{-,0}(\xi)} \times \xi\big(\big(\xi^{-1}\cdot\xi V_{-,0}(\sigma,\xi) + v_{\mathrm{II}}(\xi)\big)^3 - v_{\mathrm{II}}(\xi)^3\big)d\xi$$

for $x \in [X]$, since, by (7.1), $\lim_{\varepsilon_n \to 0} \varepsilon_n^{-2}\widetilde{G}(\varepsilon_n, \xi, y) = -2\xi\big((\xi^{-1}y + v_{\mathrm{II}}(\xi))^3 - v_{\mathrm{II}}(\xi)^3\big)$. This implies that $y = xV_{-,0}(\sigma,x)$ is a solution of the equation

$$x(x^{-1}y)'' - xy = 2x\big(\big(x^{-1}y + v_{\mathrm{II}}(x)\big)^3 - v_{\mathrm{II}}(x)^3\big).$$

Consequently the limit function $v_{\mathrm{II}}(x) + V_{-,0}(\sigma,x) = \lim_{\varepsilon_n \to 0} v_-(\sigma,\varepsilon_n,x)$ satisfies $\mathrm{P_{II}}$. This completes the proof of Theorem 2.4.

8. Proof of Proposition 6.2.

In this section, we often use the symbol $\overline{f}(\varepsilon,x) := f(\varepsilon, x+i\varepsilon)$, in particular $\overline{x} := x + i\varepsilon$.

8.1. Preliminaries.

Recall the characteristic roots $\rho^*_\pm(\varepsilon,x)$ given by (6.3). There exists a relation between $\rho^*_\pm(\varepsilon,x)$ and $\rho_\pm(\varepsilon,x)$ described as follows:

Lemma 8.1. *If $r_*^{(3)}$ is sufficiently large, then*

$$\rho^*_\pm(\varepsilon,x) = \rho_\pm(\varepsilon,x)\big(1 + O(\varepsilon x^{-1})\big)$$

uniformly in $E(\varepsilon_0) \times H(r_*^{(3)})$.

Proof. For each $\varepsilon \in E(\varepsilon_0)$, the functions $\rho_\pm(\varepsilon,x)$ and $\rho^*_\pm(\varepsilon,x)$ are holomorphic in $x \in H(r_*^{(3)})$ if $r_*^{(3)}$ is sufficiently large. If $|\varepsilon^2 x| < \delta_0$,

$$\sqrt{\left(\frac{\varepsilon^4x^2}{4} - \varepsilon^2 x\right)(1+i\varepsilon x^{-1})^2 + \varepsilon^2 x^{-2}} = (1+i\varepsilon x^{-1})(1+O(x^{-3}))\sqrt{\frac{\varepsilon^4x^2}{4} - \varepsilon^2 x},$$

where δ_0 is sufficiently small. Then

$$\frac{\rho_+^*(\varepsilon,x)}{\rho_+(\varepsilon,x)}(1+i\varepsilon x^{-1})^{-1}-1 \ll \frac{x^{-3}}{\rho_+(\varepsilon,x)}\sqrt{\frac{\varepsilon^4x^2}{4}-\varepsilon^2x} \ll \varepsilon x^{-1}$$

for $|\varepsilon^2 x| < \delta_0$. If $|\varepsilon^2 x| \ge \delta_0$,

$$\sqrt{\left(\frac{\varepsilon^4x^2}{4}-\varepsilon^2x\right)(1+i\varepsilon x^{-1})^2+\varepsilon^2x^{-2}} = (1+i\varepsilon x^{-1})(1+O(\varepsilon^2x^{-2}))\sqrt{\frac{\varepsilon^4x^2}{4}-\varepsilon^2x},$$

since $|1-\varepsilon^2x/4| > 1$. Using this, we also obtain the same estimate. Consequently

$$\rho_+^*(\varepsilon,x) = \rho_+(\varepsilon,x)\big(1+O(\varepsilon x^{-1})\big)$$

uniformly in $E(\varepsilon_0)\times H(r_*^{(3)})$. The relation for $\rho_-^*(\varepsilon,x)$ and $\rho_-(\varepsilon,x)$ immediately follows from $\rho_-^*(\varepsilon,x)\rho_+^*(\varepsilon,x) = 1+2i\varepsilon x^{-1}$ and $\rho_-(\varepsilon,x)\rho_+(\varepsilon,x) = 1$. □

The coefficient $A(\varepsilon,x)$ of (6.2) is diagonalised as follows:

$$T(\varepsilon,x)^{-1}A(\varepsilon,x)T(\varepsilon,x) = \begin{pmatrix}\rho_-^*(\varepsilon,x) & 0\\ 0 & \rho_+^*(\varepsilon,x)\end{pmatrix},$$

where

$$T(\varepsilon,x) := \begin{pmatrix}1 & 1\\ 1-\rho_+^*(\varepsilon,x) & 1-\rho_-^*(\varepsilon,x)\end{pmatrix}.$$

Then, by $\boldsymbol{u}(x) = T(\varepsilon,x)\boldsymbol{v}(x)$, system (6.2) is changed into

$$\begin{aligned}&\boldsymbol{v}(x+i\varepsilon) = B(\varepsilon,x)\boldsymbol{v}(x), \\ &\quad B(\varepsilon,x) = T(\varepsilon,x+i\varepsilon)^{-1}T(\varepsilon,x)\begin{pmatrix}\rho_-^*(\varepsilon,x) & 0\\ 0 & \rho_+^*(\varepsilon,x)\end{pmatrix}.\end{aligned} \tag{8.1}$$

By a straight-forward computation, we have

$$T(\varepsilon,x+i\varepsilon)^{-1}T(\varepsilon,x) = I + h_1(\varepsilon,x)(I-L) + h_2(\varepsilon,x)(K-J).$$

Here

$$h_1(\varepsilon, x) := \frac{\mu(\varepsilon, x) - \overline{\mu}(\varepsilon, x)}{2\overline{\mu}(\varepsilon, x)}, \quad h_2(\varepsilon, x) := \frac{\lambda(\varepsilon, x) - \overline{\lambda}(\varepsilon, x)}{2\overline{\mu}(\varepsilon, x)}$$

with

$$\begin{aligned} \lambda(\varepsilon, x) &:= \frac{\rho_+^*(\varepsilon, x) + \rho_-^*(\varepsilon, x)}{2} = \left(1 - \frac{\varepsilon^2 x}{2}\right)(1 + i\varepsilon x^{-1}), \\ \mu(\varepsilon, x) &:= \frac{\rho_+^*(\varepsilon, x) - \rho_-^*(\varepsilon, x)}{2} = \sqrt{\left(\frac{\varepsilon^4 x^2}{4} - \varepsilon^2 x\right)(1 + i\varepsilon x^{-1})^2 + \varepsilon^2 x^{-2}}, \end{aligned} \tag{8.2}$$

and J, K and L are matrices given by

$$J := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These matrices satisfy

$$\begin{gathered} J^2 = -K^2 = L^2 = I, \\ JK = -KJ = -L, \quad KL = -LK = -J, \quad LJ = -JL = K. \end{gathered}$$

To evaluate $h_1(\varepsilon, x)$ and $h_2(\varepsilon, x)$ we note the following fact.

Lemma 8.2. *Let η be a complex parameter. Then*

$$\mu(\varepsilon, (1+\eta)x) = \mu(\varepsilon, x)(1 + O(\eta)) \tag{8.3}$$

uniformly for $|\eta| < \eta_0$, $\varepsilon \in E(\varepsilon_0)$, $x \in H(r_*'')$, where η_0 is sufficiently small, and r_*'' is sufficiently large. Furthermore,

$$\mu(\varepsilon, (1+\eta)x) - \mu(\varepsilon, x) = -\frac{\eta \varepsilon^2 x \lambda(\varepsilon, x)}{2\mu(\varepsilon, x)}(1 + O(|\eta| + |x^{-1}|)). \tag{8.4}$$

Proof. Note that

$$\mu(\varepsilon, (1+\eta)x) - \mu(\varepsilon, x) = \frac{\mu(\varepsilon, (1+\eta)x)^2 - \mu(\varepsilon, x)^2}{\mu(\varepsilon, (1+\eta)x) + \mu(\varepsilon, x)} \tag{8.5}$$

and that $\mu(\varepsilon, x)^2 = -\varepsilon^2 x \tilde{\mu}(\varepsilon, x)$ with $\tilde{\mu}(\varepsilon, x) := (1 - \varepsilon^2 x/4)(1 + i\varepsilon x^{-1})^2 - x^{-3}$. Then the numerator is

$$
\begin{aligned}
&-(1+\eta)\varepsilon^2 x\tilde{\mu}(\varepsilon,(1+\eta)x)+\varepsilon^2 x\tilde{\mu}(\varepsilon,x) \\
&\quad = -(1+\eta)\varepsilon^2 x\big(\tilde{\mu}(\varepsilon,(1+\eta)x)-\tilde{\mu}(\varepsilon,x)\big)-\eta\varepsilon^2 x\tilde{\mu}(\varepsilon,x) \\
&\quad = -(1+\eta)\varepsilon^2 x\bigg(-\frac{\eta\varepsilon^2 x}{4}+O(\eta x^{-1})\bigg) \\
&\qquad -\eta\varepsilon^2 x\bigg(1-\frac{\varepsilon^2 x}{4}+O(|\varepsilon^3|+|\varepsilon x^{-1}|+|x^{-3}|)\bigg) \\
&\quad = -\eta\varepsilon^2 x\bigg(1-\frac{\varepsilon^2 x}{2}+O(|x^{-1}|+|\varepsilon^3 x\cdot x^{-1}|+|\eta\varepsilon^2 x|)\bigg) \\
&\quad = -\eta\varepsilon^2 x\,\lambda(\varepsilon,x)\big(1+O(|x^{-1}|+|\eta|)\big) \qquad (8.6)
\end{aligned}
$$

uniformly for $|\eta|<\eta_0$, $\varepsilon\in E(\varepsilon_0)$, $x\in H(r''_*)$, if η_0 is sufficiently small and r''_* is sufficiently large. Suppose that $|\varepsilon^2 x|<\delta_0$, where δ_0 is sufficiently small. Substituting this into (8.5) and using $\mu(\varepsilon,(1+\eta)x)+\mu(\varepsilon,x)\gg\varepsilon\sqrt{x}$ and $\varepsilon\sqrt{x}\ll\mu(\varepsilon,x)$, we have

$$
\mu(\varepsilon,(1+\eta)x)-\mu(\varepsilon,x)\ll\eta\varepsilon\sqrt{x}\ll\eta\mu(\varepsilon,x).
$$

For $|\varepsilon^2 x|\ge\delta_0$, using $\mu(\varepsilon,(1+\eta)x)+\mu(\varepsilon,x)\gg\varepsilon^2 x$ and $\mu(\varepsilon,x)\gg\varepsilon^2 x$, we also obtain (8.3). Estimate (8.6) together with (8.3) yields (8.4), which completes the proof. □

Then we have

Lemma 8.3. *The functions $h_1(\varepsilon,x)$ and $h_2(\varepsilon,x)$ are expressed as*

$$
\begin{aligned}
h_1(\varepsilon,x)&=\frac{i\varepsilon^3}{4}\cdot\frac{\lambda(\varepsilon,x)}{\mu(\varepsilon,x)^2}(1+O(x^{-1}))=O(\varepsilon x^{-1}),\\
h_2(\varepsilon,x)&=\bigg(\frac{i\varepsilon^3}{4}+O(|\varepsilon^2 x^{-2}|+|\varepsilon^4 x^{-1}|)\bigg)\frac{1}{\mu(\varepsilon,x)}=O(\varepsilon x^{-1})
\end{aligned}
$$

uniformly in $E(\varepsilon_0)\times H(r''_*)$.

Proof. By Lemma 8.2 with $\eta=i\varepsilon x^{-1}$, we have

$$
\overline{\mu}(\varepsilon,x)-\mu(\varepsilon,x)=-\frac{i\varepsilon^3\lambda(\varepsilon,x)}{2\mu(\varepsilon,x)}(1+O(x^{-1}))
$$

and $\overline{\mu}(\varepsilon, x) = \mu(\varepsilon, x)(1 + O(\varepsilon x^{-1}))$, which yield the estimate for $h_1(\varepsilon, x)$. Using

$$\overline{\lambda}(\varepsilon, x) - \lambda(\varepsilon, x) = -\frac{i\varepsilon^3}{2} + O(\varepsilon^2 x^{-2}),$$

we obtain the expression of $h_2(\varepsilon, x)$. □

8.2. Reduction of system (8.1).

Let us write the coefficient $B(\varepsilon, x)$ of (8.1) in the form

$$B(\varepsilon, x) = b_0(\varepsilon, x)I + b_1(\varepsilon, x)J + b_2(\varepsilon, x)K + b_3(\varepsilon, x)L. \tag{8.7}$$

Here

$$\begin{aligned}
b_0(\varepsilon, x) &:= \lambda(\varepsilon, x) + h_1(\varepsilon, x)\lambda(\varepsilon, x) - h_2(\varepsilon, x)\mu(\varepsilon, x),\\
b_1(\varepsilon, x) &:= \mu(\varepsilon, x) + h_1(\varepsilon, x)\mu(\varepsilon, x) - h_2(\varepsilon, x)\lambda(\varepsilon, x),\\
b_2(\varepsilon, x) &:= -h_1(\varepsilon, x)\mu(\varepsilon, x) + h_2(\varepsilon, x)\lambda(\varepsilon, x),\\
b_3(\varepsilon, x) &:= -h_1(\varepsilon, x)\lambda(\varepsilon, x) + h_2(\varepsilon, x)\mu(\varepsilon, x).
\end{aligned}$$

These quantities are estimated as follows:

Proposition 8.4. *We have*

$$b_0(\varepsilon, x) = \lambda(\varepsilon, x)(1 + O(\varepsilon x^{-1})), \quad b_1(\varepsilon, x) = \mu(\varepsilon, x)(1 + O(|\varepsilon x^{-1}| + |x^{-3}|)),$$
$$b_2(\varepsilon, x) \ll \varepsilon x^{-1}, \quad b_3(\varepsilon, x) \ll \varepsilon x^{-1}$$

uniformly in $E(\varepsilon_0) \times H(r''_*)$. Moreover, the diagonal entries of $B(\varepsilon, x)$ are

$$b_0(\varepsilon, x) \pm b_1(\varepsilon, x) = \rho_{\pm}(\varepsilon, x)\big(1 + O(\varepsilon x^{-1})\big).$$

Proof. From the estimate $h_1(\varepsilon, x) - h_2(\varepsilon, x)\mu(\varepsilon, x)/\lambda(\varepsilon, x) \ll \varepsilon x^{-1}$, the expression of $b_0(\varepsilon, x)$ immediately follows. The function $b_1(\varepsilon, x)$ is written in the form

$$b_1(\varepsilon, x) = \mu(\varepsilon, x)\left(1 + h_1(\varepsilon, x) - h_2(\varepsilon, x)\frac{\lambda(\varepsilon, x)}{\mu(\varepsilon, x)}\right).$$

Using Lemma 8.3 and considering the cases $|\varepsilon^2 x| \geq \delta_0$ and $|\varepsilon^2 x| < \delta_0$, we have

$$h_1(\varepsilon, x) - h_2(\varepsilon, x)\frac{\lambda(\varepsilon, x)}{\mu(\varepsilon, x)} = \frac{\lambda(\varepsilon, x)}{\mu(\varepsilon, x)^2} \cdot O(|\varepsilon^2 x^{-2}| + |\varepsilon^3 x^{-1}|) \ll |\varepsilon x^{-1}| + |x^{-3}|$$

uniformly in $E(\varepsilon_0) \times H(r''_*)$, which implies the expression of $b_1(\varepsilon, x)$. Furthermore,

$$b_2(\varepsilon, x) \ll \frac{\lambda(\varepsilon, x)}{\mu(\varepsilon, x)} \cdot \varepsilon^2(|x^{-2}| + |\varepsilon x^{-1}|) \ll \varepsilon x^{-1}.$$

Note that

$$b_3(\varepsilon, x) = -\frac{i\varepsilon^3}{4} \cdot \frac{\lambda(\varepsilon, x)^2}{\mu(\varepsilon, x)^2}(1 + O(x^{-1})) + \frac{i\varepsilon^3}{4} + O\big(|\varepsilon^2 x^{-2}| + |\varepsilon^4 x^{-1}|\big).$$

If $|\varepsilon^2 x| < \delta_0$, we immediately obtain $b_3(\varepsilon, x) \ll \varepsilon x^{-1}$, since $\varepsilon^3 = \varepsilon x^{-1} \cdot \varepsilon^2 x$. For $|\varepsilon^2 x| \geq \delta_0$, using the estimate

$$\frac{\lambda(\varepsilon, x)^2}{\mu(\varepsilon, x)^2} = \frac{(1 - \varepsilon^2 x/2)^2}{(1 - \varepsilon^2 x/2)^2 - 1 + \varepsilon^2 x^{-2}(1 + i\varepsilon x^{-1})^{-2}} = 1 + O((\varepsilon^2 x)^{-1}),$$

we deduce $b_3(\varepsilon, x) \ll \varepsilon x^{-1}$. The expressions of the diagonal entries of $B(\varepsilon, x)$ immediately follow from Lemmas 8.1 and 8.3. □

To reduce the power exponent of the estimate for $b_3(\varepsilon, x)$, we apply a transformation of the form

$$\boldsymbol{v}(x) = (I + p(\varepsilon, x)K)\boldsymbol{y}(x),$$

which changes (8.1) into

$$\begin{aligned} &\boldsymbol{y}(x + i\varepsilon) = D(\varepsilon, x)\boldsymbol{y}(x), \\ &D(\varepsilon, x) = \frac{1}{1 + \overline{p}(\varepsilon, x)^2}\big(d_0(\varepsilon, x)I + d_1(\varepsilon, x)J + d_2(\varepsilon, x)K + d_3(\varepsilon, x)L\big) \end{aligned} \tag{8.8}$$

with

$$\begin{aligned} d_0(\varepsilon, x) &:= \big(1 + p(\varepsilon, x)\overline{p}(\varepsilon, x)\big)b_0(\varepsilon, x) + \big(\overline{p}(\varepsilon, x) - p(\varepsilon, x)\big)b_2(\varepsilon, x), \\ d_1(\varepsilon, x) &:= \big(1 - p(\varepsilon, x)\overline{p}(\varepsilon, x)\big)b_1(\varepsilon, x) + \big(\overline{p}(\varepsilon, x) + p(\varepsilon, x)\big)b_3(\varepsilon, x), \\ d_2(\varepsilon, x) &:= \big(1 + p(\varepsilon, x)\overline{p}(\varepsilon, x)\big)b_2(\varepsilon, x) - \big(\overline{p}(\varepsilon, x) - p(\varepsilon, x)\big)b_0(\varepsilon, x), \\ d_3(\varepsilon, x) &:= \big(1 - p(\varepsilon, x)\overline{p}(\varepsilon, x)\big)b_3(\varepsilon, x) - \big(\overline{p}(\varepsilon, x) + p(\varepsilon, x)\big)b_1(\varepsilon, x). \end{aligned}$$

The diagonal entries of $D(\varepsilon, x)$ are

$$d_\pm(\varepsilon, x) := \frac{d_0(\varepsilon, x) \pm d_1(\varepsilon, x)}{1 + \overline{p}(\varepsilon, x)^2}.$$

Let us set $p(\varepsilon, x) := (1/2)b_3(\varepsilon, x)/b_1(\varepsilon, x)$. Then we have

PROPOSITION 8.5. *If r_1^* is sufficiently large, then*

$$p(\varepsilon, x) \ll |\varepsilon x^{-1}| + |x^{-3/2}|, \quad \overline{p}(\varepsilon, x) \ll |\varepsilon x^{-1}| + |x^{-3/2}|,$$

and

$$\begin{aligned} d_0(\varepsilon, x) &= \lambda(\varepsilon, x)\big(1 + O(|\varepsilon x^{-1}| + |x^{-3}|)\big), \\ d_1(\varepsilon, x) &= \mu(\varepsilon, x)\big(1 + O(|\varepsilon x^{-1}| + |x^{-3}|)\big), \\ d_2(\varepsilon, x) &\ll \varepsilon x^{-1}, \quad d_3(\varepsilon, x) \ll \varepsilon x^{-2} \end{aligned}$$

uniformly in $E(\varepsilon_0) \times H(r_1^*)$. Furthermore,

$$d_\pm(\varepsilon, x) = \big(b_0(\varepsilon, x) \pm b_1(\varepsilon, x)\big)\big(1 + O(\varepsilon x^{-2})\big) = \rho_\pm(\varepsilon, x)\big(1 + O(\varepsilon x^{-1})\big)$$

uniformly in $E(\varepsilon_0) \times H(r_1^*)$.

PROOF. Note that $\varepsilon x^{1/2} \ll \mu(\varepsilon, x) \ll \varepsilon x^{1/2}$ as $\varepsilon^2 x \to 0$. By the definition of $p(\varepsilon, x)$ together with Proposition 8.4, we have $p(\varepsilon, x) \ll \varepsilon x^{-1}$ if $|\varepsilon^2 x| \geq \delta_0$, and $p(\varepsilon, x) \ll x^{-3/2}$ if $|\varepsilon^2 x| < \delta_0$, since $p(\varepsilon, x) \ll \varepsilon x^{-1}/\mu(\varepsilon, x)$. Thus we have

$$p(\varepsilon, x) \ll \frac{\varepsilon x^{-1}}{\mu(\varepsilon, x)} \ll |\varepsilon x^{-1}| + |x^{-3/2}|$$

in $E(\varepsilon_0) \times H(r''_*)$. By Lemma 8.2, $\big|\mu(\varepsilon, x + \kappa_0|x|e^{i\theta})\big| \geq |\mu(\varepsilon, x)|/2$ for $0 \leq \theta \leq 2\pi$ if $\kappa_0 > 0$ is sufficiently small, and hence we have $p(\varepsilon, t) \ll \varepsilon x^{-1}/\mu(\varepsilon, x)$ on the circle $|t - x| = \kappa_0|x|$, from which it follows that

$$\frac{\partial}{\partial x} p(\varepsilon, x) = \frac{1}{2\pi i}\int_{|t-x|=\kappa_0|x|} \frac{p(\varepsilon, t)}{(t-x)^2} dt \ll \frac{\varepsilon x^{-2}}{\mu(\varepsilon, x)}.$$

Hence,

$$\overline{p}(\varepsilon,x)-p(\varepsilon,x)=\int_x^{x+i\varepsilon}\frac{\partial}{\partial t}p(\varepsilon,t)dt \ll \frac{\varepsilon^2x^{-2}}{\mu(\varepsilon,x)}$$

and

$$\overline{p}(\varepsilon,x)\ll\frac{\varepsilon x^{-1}}{\mu(\varepsilon,x)}\ll|\varepsilon x^{-1}|+|x^{-3/2}|$$

in $E(\varepsilon_0)\times H(r_1^*)$ with $r_1^*=r_*''/(1-\kappa_0)$. Then

$$\begin{aligned}d_3(\varepsilon,x)&=-p(\varepsilon,x)\overline{p}(\varepsilon,x)b_3(\varepsilon,x)-\big(\overline{p}(\varepsilon,x)-p(\varepsilon,x)\big)b_1(\varepsilon,x)\\&\ll\big|(|\varepsilon x^{-1}|+|x^{-3/2}|)^2\varepsilon x^{-1}\big|+\left|\frac{\varepsilon^2x^{-2}}{\mu(\varepsilon,x)}b_1(\varepsilon,x)\right|\ll\varepsilon x^{-2}.\end{aligned}$$

Using

$$\big(\overline{p}(\varepsilon,x)-p(\varepsilon,x)\big)b_0(\varepsilon,x)\ll\frac{\lambda(\varepsilon,x)}{\mu(\varepsilon,x)}\varepsilon^2x^{-2},$$

we obtain the estimate for $d_2(\varepsilon,x)$. Let us write $d_0(\varepsilon,x)\pm d_1(\varepsilon,x)$ in the form

$$d_0(\varepsilon,x)\pm d_1(\varepsilon,x)=\beta_1^{\pm}(\varepsilon,x)+\beta_2^{\pm}(\varepsilon,x)$$

with

$$\begin{aligned}\beta_1^{\pm}(\varepsilon,x)&:=b_0(\varepsilon,x)\pm b_1(\varepsilon,x)+p(\varepsilon,x)\overline{p}(\varepsilon,x)\big(b_0(\varepsilon,x)\mp b_1(\varepsilon,x)\big),\\\beta_2^{\pm}(\varepsilon,x)&:=\big(\overline{p}(\varepsilon,x)-p(\varepsilon,x)\big)b_2(\varepsilon,x)\pm\big(\overline{p}(\varepsilon,x)+p(\varepsilon,x)\big)b_3(\varepsilon,x).\end{aligned}$$

Suppose that $|\varepsilon^2x|$ is small. Observe that $b_0(\varepsilon,x)\pm b_1(\varepsilon,x)=1+O(\varepsilon x^{1/2})$, and that

$$\begin{aligned}&\beta_1^{\pm}(\varepsilon,x)-\Big(b_0(\varepsilon,x)\pm b_1(\varepsilon,x)+\overline{p}(\varepsilon,x)^2\big(b_0(\varepsilon,x)\mp b_1(\varepsilon,x)\big)\Big)\\&\quad=\overline{p}(\varepsilon,x)\big(p(\varepsilon,x)-\overline{p}(\varepsilon,x)\big)\big(b_0(\varepsilon,x)\mp b_1(\varepsilon,x)\big)\\&\quad\ll\overline{p}(\varepsilon,x)\big(p(\varepsilon,x)-\overline{p}(\varepsilon,x)\big)\\&\quad\ll x^{-1}\cdot\frac{\varepsilon^2x^{-2}}{\mu(\varepsilon,x)}\ll\varepsilon x^{-3}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\beta_1^{\pm}(\varepsilon,x)}{1+\overline{p}(\varepsilon,x)^2} &= \frac{b_0(\varepsilon,x)\pm b_1(\varepsilon,x)+\overline{p}(\varepsilon,x)^2(b_0(\varepsilon,x)\mp b_1(\varepsilon,x))}{1+\overline{p}(\varepsilon,x)^2}+O(\varepsilon x^{-3}) \\ &= b_0(\varepsilon,x)\pm b_1(\varepsilon,x)\mp\frac{2\overline{p}(\varepsilon,x)^2 b_1(\varepsilon,x)}{1+\overline{p}(\varepsilon,x)^2}+O(\varepsilon x^{-3}) \\ &= b_0(\varepsilon,x)\pm b_1(\varepsilon,x)+O\biggl(\frac{\mu(\varepsilon,x)\varepsilon^2x^{-2}}{\mu(\varepsilon,x)^2}\biggr)+O(\varepsilon x^{-3}) \\ &= \bigl(b_0(\varepsilon,x)\pm b_1(\varepsilon,x)\bigr)(1+O(\varepsilon x^{-2}))\end{aligned}$$

as $\varepsilon^2 x\to 0$. Combining this with

$$\begin{aligned}\beta_2^{\pm}(\varepsilon,x) &\ll \bigl(|\overline{p}(\varepsilon,x)|+|p(\varepsilon,x)|\bigr)\bigl(|b_2(\varepsilon,x)|+|b_3(\varepsilon,x)|\bigr) \\ &\ll \frac{\varepsilon^2x^{-2}}{\mu(\varepsilon,x)}\ll \varepsilon x^{-2}\bigl(b_0(\varepsilon,x)\pm b_1(\varepsilon,x)\bigr),\end{aligned}$$

we obtain the expressions of $d_{\pm}(\varepsilon,x)=(\beta_1^{\pm}(\varepsilon,x)+\beta_2^{\pm}(\varepsilon,x))/(1+\overline{p}(\varepsilon,x)^2)$ as $\varepsilon^2x\to 0$. On the other hand, if $|\varepsilon^2 x|>\delta_0$, where $\delta_0>0$ is a sufficiently small fixed number, we have

$$\begin{aligned}&\bigl(\beta_1^{\pm}(\varepsilon,x)-(b_0(\varepsilon,x)\pm b_1(\varepsilon,x))\bigr)(b_0(\varepsilon,x)\pm b_1(\varepsilon,x))^{-1} \\ &\quad\ll \rho_{\mp}(\varepsilon,x)\rho_{\pm}(\varepsilon,x)^{-1}\overline{p}(\varepsilon,x)p(\varepsilon,x)\ll \rho_{+}(\varepsilon,x)^2\frac{\varepsilon^2x^{-2}}{\mu(\varepsilon,x)^2}\ll \varepsilon^2x^{-2}\end{aligned}$$

and

$$\beta_2^{\pm}(\varepsilon,x)(b_0(\varepsilon,x)\pm b_1(\varepsilon,x))^{-1}\ll \rho_{+}(\varepsilon,x)\frac{\varepsilon x^{-1}}{\mu(\varepsilon,x)}\cdot\varepsilon x^{-1}\ll\varepsilon^2x^{-2}.$$

From these combined with Proposition 8.4 the expressions of $d_{\pm}(\varepsilon,x)$ in the domain $|\varepsilon^2x|>\delta_0$ immediately follow, since $1+\overline{p}(\varepsilon,x)^2=1+O(|\varepsilon^2x^{-2}|+|x^{-3}|)=1+O(\varepsilon x^{-2})$ if $|\varepsilon^2x|>\delta_0$. Thus we obtain the desired expressions of the diagonal entries uniformly valid in $E(\varepsilon_0)\times H(r_1^*)$. This completes the proof. $\square$

By a further transformation of the form

$$\boldsymbol{y}(x)=(I+q(\varepsilon,x)L)\tilde{\boldsymbol{y}}(x),$$

system (8.8) is taken into

$$\begin{aligned}
\tilde{\boldsymbol{y}}(x+i\varepsilon) &= \widetilde{D}(\varepsilon,x)\tilde{\boldsymbol{y}}(x), \qquad (8.9)\\
\widetilde{D}(\varepsilon,x) &= \frac{1}{(1+\overline{p}(\varepsilon,x)^2)(1-\overline{q}(\varepsilon,x)^2)}\\
&\quad \times \big(\tilde{d}_0(\varepsilon,x)I + \tilde{d}_1(\varepsilon,x)J + \tilde{d}_2(\varepsilon,x)K + \tilde{d}_3(\varepsilon,x)L\big)
\end{aligned}$$

with

$$\begin{aligned}
\tilde{d}_0(\varepsilon,x) &:= \big(1-q(\varepsilon,x)\overline{q}(\varepsilon,x)\big)d_0(\varepsilon,x) - \big(\overline{q}(\varepsilon,x)-q(\varepsilon,x)\big)d_3(\varepsilon,x),\\
\tilde{d}_1(\varepsilon,x) &:= \big(1+q(\varepsilon,x)\overline{q}(\varepsilon,x)\big)d_1(\varepsilon,x) - \big(\overline{q}(\varepsilon,x)+q(\varepsilon,x)\big)d_2(\varepsilon,x),\\
\tilde{d}_2(\varepsilon,x) &:= \big(1+q(\varepsilon,x)\overline{q}(\varepsilon,x)\big)d_2(\varepsilon,x) - \big(\overline{q}(\varepsilon,x)+q(\varepsilon,x)\big)d_1(\varepsilon,x),\\
\tilde{d}_3(\varepsilon,x) &:= \big(1-q(\varepsilon,x)\overline{q}(\varepsilon,x)\big)d_3(\varepsilon,x) - \big(\overline{q}(\varepsilon,x)-q(\varepsilon,x)\big)d_0(\varepsilon,x).
\end{aligned}$$

We put $q(\varepsilon,x) := (1/2)d_2(\varepsilon,x)/d_1(\varepsilon,x)$. By an argument analogous to that of the proof of Proposition 8.5, we have

$$q(\varepsilon,x) \ll \frac{\varepsilon x^{-1}}{\mu(\varepsilon,x)} \ll |\varepsilon x^{-1}| + |x^{-3/2}|, \quad \overline{q}(\varepsilon,x) \ll \frac{\varepsilon x^{-1}}{\mu(\varepsilon,x)} \ll |\varepsilon x^{-1}| + |x^{-3/2}|$$

$$\overline{q}(\varepsilon,x) - q(\varepsilon,x) \ll \frac{\varepsilon^2 x^{-2}}{\mu(\varepsilon,x)}$$

uniformly in $E(\varepsilon_0)\times H(r_2^*)$, where r_2^* is a sufficiently large positive number. Then,

$$\begin{aligned}
\tilde{d}_0(\varepsilon,x) &= \lambda(\varepsilon,x)\big(1+O(|\varepsilon x^{-1}|+|x^{-3}|)\big),\\
\tilde{d}_1(\varepsilon,x) &= \mu(\varepsilon,x)\big(1+O(|\varepsilon x^{-1}|+|x^{-3}|)\big),\\
\tilde{d}_2(\varepsilon,x) &\ll \varepsilon x^{-2}, \quad \tilde{d}_3(\varepsilon,x) \ll \varepsilon x^{-2}.
\end{aligned}$$

Note that the diagonal entries of $\widetilde{D}(\varepsilon,x)$ are given by

$$\tilde{d}_\pm(\varepsilon,x) := \frac{\tilde{d}_0(\varepsilon,x) \pm \tilde{d}_1(\varepsilon,x)}{(1+\overline{p}(\varepsilon,x)^2)(1-\overline{q}(\varepsilon,x)^2)}$$

with

$$\begin{aligned}&\tilde{d}_0(\varepsilon,x)\pm\tilde{d}_1(\varepsilon,x)\\&\quad=d_0(\varepsilon,x)\pm d_1(\varepsilon,x)-q(\varepsilon,x)\overline{q}(\varepsilon,x)\bigl(d_0(\varepsilon,x)\mp d_1(\varepsilon,x)\bigr)\\&\qquad-\bigl(\overline{q}(\varepsilon,x)-q(\varepsilon,x)\bigr)d_3(\varepsilon,x)\mp\bigl(\overline{q}(\varepsilon,x)+q(\varepsilon,x)\bigr)d_2(\varepsilon,x).\end{aligned}$$

By the same argument as in the case of $d_\pm(\varepsilon,x)$, we derive

$$\frac{\tilde{d}_0(\varepsilon,x)\pm\tilde{d}_1(\varepsilon,x)}{1-\overline{q}(\varepsilon,x)^2}=\bigl(d_0(\varepsilon,x)\pm d_1(\varepsilon,x)\bigr)(1+O(\varepsilon x^{-2}))$$

uniformly in $E(\varepsilon_0)\times H(r_2^*)$. Hence

$$\begin{aligned}\tilde{d}_\pm(\varepsilon,x)&=\frac{d_0(\varepsilon,x)\pm d_1(\varepsilon,x)}{1+\overline{p}(\varepsilon,x)^2}\bigl(1+O(\varepsilon x^{-2})\bigr)=d_\pm(\varepsilon,x)\bigl(1+O(\varepsilon x^{-2})\bigr)\\&=\bigl(b_0(\varepsilon,x)\pm b_1(\varepsilon,x)\bigr)\bigl(1+O(\varepsilon x^{-2})\bigr)=\rho_\pm(\varepsilon,x)\bigl(1+O(\varepsilon x^{-1})\bigr).\end{aligned}$$

Similarly we can find a couple of transformations

$$\tilde{\boldsymbol{y}}(x)=(I+p_*(\varepsilon,x)K)\tilde{\tilde{\boldsymbol{y}}}(x),\quad\tilde{\tilde{\boldsymbol{y}}}(x)=(I+q_*(\varepsilon,x)L)\boldsymbol{w}(x)$$

with

$$p_*(\varepsilon,x)\ll|\varepsilon x^{-2}|+|x^{-5/2}|,\quad q_*(\varepsilon,x)\ll|\varepsilon x^{-2}|+|x^{-5/2}|$$

such that, by the composite of them, the magnitude of off-diagonal entries of $\widetilde{D}(\varepsilon,x)$ is reduced to $O(\varepsilon x^{-3})$. Thus we obtain

Proposition 8.6. *Let r_* be a sufficiently large positive number. In $E(\varepsilon_0)\times H(r_*)$, by the transformation*

$$\boldsymbol{u}(x)=T(\varepsilon,x)\bigl(I+\widetilde{P}(\varepsilon,x)\bigr)\boldsymbol{w}(x)$$

system (6.2) *is reduced to a system of the form*

$$\boldsymbol{w}(x+i\varepsilon)=F(\varepsilon,x)\boldsymbol{w}(x),\quad F(\varepsilon,x)=\begin{pmatrix}f_-(\varepsilon,x)&f_{12}(\varepsilon,x)\\f_{21}(\varepsilon,x)&f_+(\varepsilon,x)\end{pmatrix}.\tag{8.10}$$

Here

(1) $\widetilde{P}(\varepsilon, x)$ *is a square matrix such that* $\widetilde{P}(\varepsilon, x) \ll x^{-1}$ *in* $E(\varepsilon_0) \times H(r_*)$;
(2) $f_{12}(\varepsilon, x) \ll \varepsilon x^{-3}$, $f_{21}(\varepsilon, x) \ll \varepsilon x^{-3}$ *and*

$$f_{\pm}(\varepsilon, x) = \rho_{\pm}(\varepsilon, x)\big(1 + O(\varepsilon x^{-1})\big)$$

in $E(\varepsilon_0) \times H(r_*)$.

8.3. Exponential part of a fundamental matrix solution.

To construct a fundamental matrix solution of (8.10) we need some lemmas. The following is a variant of the Euler-Maclaurin formula, which enables us to calculate an approximate sum of a given function.

LEMMA 8.7. *For a given function* $\phi(x)$ *and for a given number* $\epsilon \neq 0$, *the integral*

$$\Phi(x) := \frac{1}{\epsilon} \int_{x_0}^{x} \left(\phi(t) - \frac{\epsilon}{2}\phi'(t) + \frac{\epsilon^2}{12}\phi''(t) \right) dt$$

satisfies

$$\Phi(x + \epsilon) - \Phi(x) = \phi(x) + \frac{1}{24\epsilon} \int_{x}^{x+\epsilon} (x + \epsilon - t)^2 (x - t)^2 \phi^{(4)}(t) dt.$$

PROOF. For $l = 0, 1, 2$, put

$$\Phi_l(x) := \frac{1}{\epsilon} \int_{x_0}^{x} \phi^{(l)}(t) dt.$$

Then

$$\begin{aligned}
&\Phi_0(x + \epsilon) - \Phi_0(x) \\
&\quad = \phi(x) + \frac{\epsilon}{2}\phi'(x) + \frac{\epsilon^2}{6}\phi''(x) + \frac{\epsilon^3}{24}\phi^{(3)}(x) + \frac{1}{24\epsilon} \int_{x}^{x+\epsilon} (x + \epsilon - t)^4 \phi^{(4)}(t) dt, \\
&-\frac{\epsilon}{2}\big(\Phi_1(x + \epsilon) - \Phi_1(x)\big) \\
&\quad = -\frac{\epsilon}{2}\phi'(x) - \frac{\epsilon^2}{4}\phi''(x) - \frac{\epsilon^3}{12}\phi^{(3)}(x) - \frac{1}{12} \int_{x}^{x+\epsilon} (x + \epsilon - t)^3 \phi^{(4)}(t) dt, \\
&\frac{\epsilon^2}{12}\big(\Phi_2(x + \epsilon) - \Phi_2(x)\big) \\
&\quad = \frac{\epsilon^2}{12}\phi''(x) + \frac{\epsilon^3}{24}\phi^{(3)}(x) + \frac{\epsilon}{24} \int_{x}^{x+\epsilon} (x + \epsilon - t)^2 \phi^{(4)}(t) dt.
\end{aligned}$$

Summing these equalities, we obtain the desired formula. □

LEMMA 8.8. *Let η be a complex parameter. Then*

$$\rho_{\pm}(\varepsilon, x+\eta x) = \rho_{\pm}(\varepsilon, x)\,(1+O(\eta))$$

uniformly for $(\varepsilon, x) \in E(\varepsilon_0) \times H(r_*)$ as $\eta \to 0$.

PROOF. Since $\rho_{\pm}(\varepsilon, x) = \tilde{\rho}_{\pm}(-\varepsilon^2 x)$ with $\tilde{\rho}_{\pm}(z) := 1 + z/2 \pm \sqrt{z + z^2/4}$, it is sufficient to show that

$$\tilde{\rho}_{\pm}(z+\eta z) = \tilde{\rho}_{\pm}(z)\,(1+O(\eta)) \tag{8.11}$$

uniformly for $\operatorname{Re} z > 0$ as $\eta \to 0$. Note that $\tilde{\rho}_{\pm}(z)$ satisfy

$$\tilde{\rho}_{-}(z) \gg z^{-1} \quad \text{as } z \to \infty, \qquad \tilde{\rho}_{+}(z) \gg z \quad \text{as } z \to \infty,$$
$$\tilde{\rho}_{\pm}(z) \gg 1 \quad \text{as } z \to 0,$$

and

$$\tilde{\rho}'_{-}(z) \ll z^{-2} \quad \text{as } z \to \infty, \qquad \tilde{\rho}'_{+}(z) \ll 1 \quad \text{as } z \to \infty,$$
$$\tilde{\rho}'_{\pm}(z) \ll z^{-1/2} \quad \text{as } z \to 0.$$

Then we have

$$\begin{aligned}
&\tilde{\rho}_{-}(z+\eta z) - \tilde{\rho}_{-}(z) \ll \eta z \cdot z^{-2} \ll \eta \tilde{\rho}_{-}(z) && \text{as } z \to \infty,\\
&\tilde{\rho}_{+}(z+\eta z) - \tilde{\rho}_{+}(z) \ll \eta z \cdot 1 \ll \eta \tilde{\rho}_{+}(z) && \text{as } z \to \infty,\\
&\tilde{\rho}_{\pm}(z+\eta z) - \tilde{\rho}_{\pm}(z) \ll \eta z \cdot z^{-1/2} \ll \eta \tilde{\rho}_{\pm}(z) && \text{as } z \to 0.
\end{aligned}$$

In a compact set contained in $\{z \mid \operatorname{Re} z \geq 0, z \neq 0\}$, we have the uniform estimates $\tilde{\rho}_{\pm}(z) \gg 1$ and $\tilde{\rho}'_{\pm}(z) \ll 1$, from which (8.11) immediately follows. This completes the proof of the lemma. □

By Lemma 8.7 with $\phi(x) = \log f_{\pm}(\varepsilon, x)$ and $\epsilon = i\varepsilon$, we obtain

$$\tau_{\pm}(\varepsilon, x) := \frac{1}{i\varepsilon} \int_{x_0}^{x} \varphi_{\pm}(\varepsilon, t)\,dt$$

with

$$\varphi_\pm(\varepsilon,x) := \log f_\pm(\varepsilon,x) - \frac{i\varepsilon}{2}\bigl(\log f_\pm(\varepsilon,x)\bigr)_x + \frac{(i\varepsilon)^2}{12}\bigl(\log f_\pm(\varepsilon,x)\bigr)_{xx}, \tag{8.12}$$

which satisfy

$$\begin{aligned}&\tau_\pm(\varepsilon,x+i\varepsilon) - \tau_\pm(\varepsilon,x)\\ &\quad = \log f_\pm(\varepsilon,x) + \frac{1}{24i\varepsilon}\int_x^{x+i\varepsilon}(x-t+i\varepsilon)^2(x-t)^2\bigl(\log f_\pm(\varepsilon,t)\bigr)_{(4t)}dt\end{aligned} \tag{8.13}$$

($_x = \partial/\partial x$, $_{(4t)} = \partial^4/\partial t^4$). Here $x_0 \in H(r_*)$ is some fixed point. By Proposition 8.6 and Lemma 8.8, for a sufficiently small fixed positive number κ_0, we have, on the circle $|t-x| = \kappa_0|x|$,

$$\log f_\pm(\varepsilon,t) - \log f_\pm(\varepsilon,x) \ll \bigl|\log\rho_\pm(\varepsilon,t) - \log\rho_\pm(\varepsilon,x)\bigr| + |\varepsilon x^{-1}| \ll 1,$$

and hence

$$\begin{aligned}\bigl(\log f_\pm(\varepsilon,x)\bigr)_x &= \frac{1}{2\pi i}\int_{|t-x|=\kappa_0|x|}\frac{\log f_\pm(\varepsilon,t)}{(t-x)^2}dt\\ &\ll \bigl(\bigl|\log f_\pm(\varepsilon,x)\bigr|+1\bigr)x^{-1} \ll \bigl(\bigl|\log\rho_\pm(\varepsilon,x)\bigr|+1\bigr)x^{-1}.\end{aligned}$$

Thus we obtain

$$\varepsilon\bigl(\log f_\pm(\varepsilon,x)\bigr)_x \ll \varepsilon x^{-1}\bigl(\bigl|\log\rho_\pm(\varepsilon,x)\bigr|+1\bigr)$$

uniformly in $E(\varepsilon_0)\times H(r_*)$. Similarly,

$$\begin{aligned}&\varepsilon^2\bigl(\log f_\pm(\varepsilon,x)\bigr)_{xx} \ll \varepsilon^2 x^{-2}\bigl(\bigl|\log\rho_\pm(\varepsilon,x)\bigr|+1\bigr),\\ &\varepsilon^4\bigl(\log f_\pm(\varepsilon,t)\bigr)_{(4t)} \ll \varepsilon^4 t^{-4}\bigl(\bigl|\log\rho_\pm(\varepsilon,t)\bigr|+1\bigr) \ll \varepsilon^4 t^{-4}\bigl(1+\bigl|\log(\varepsilon^2 t)\bigr|\bigr).\end{aligned}$$

Observing the fact

$$\log\rho_\pm(\varepsilon,x) \gg 1 \quad \text{if } |\varepsilon^2 x| \ge \delta_0, \qquad \log\rho_\pm(\varepsilon,x) \gg \varepsilon x^{1/2} \quad \text{if } |\varepsilon^2 x| \le \delta_0$$

(cf. Proposition 6.1, (4), (5)), we have

$$\log f_\pm(\varepsilon,x) = \log\rho_\pm(\varepsilon,x) + O(\varepsilon x^{-1}) = (1+O(x^{-1}))\log\rho_\pm(\varepsilon,x),$$

and $\varepsilon x^{-1} \ll x^{-1}\log\rho_{\pm}(\varepsilon,x)$. By these inequalities, functions (8.12) and (8.13) are expressed as

$$\varphi_{\pm}(\varepsilon,x) = \big(1+O(x^{-1})\big)\log\rho_{\pm}(\varepsilon,x), \tag{8.14}$$

$$\tau_{\pm}(\varepsilon,x+i\varepsilon) - \tau_{\pm}(\varepsilon,x) = \log f_{\pm}(\varepsilon,x) + O(\varepsilon^3 x^{-4}\log x) \tag{8.15}$$

in $E(\varepsilon_0)\times H(r_*)$. Then

$$\zeta_{\pm}(\varepsilon,x) := \exp\big(\tau_{\pm}(\varepsilon,x)\big) = \exp\bigg(\frac{1}{i\varepsilon}\int_{x_0}^{x}\varphi_{\pm}(\varepsilon,t)dt\bigg)$$

satisfy

$$\begin{aligned}\zeta_{\pm}(\varepsilon,x+i\varepsilon) &= \zeta_{\pm}(\varepsilon,x)\exp\big(\log f_{\pm}(\varepsilon,x) + O(\varepsilon^3x^{-4}\log x)\big)\\ &= \zeta_{\pm}(\varepsilon,x)f_{\pm}(\varepsilon,x)\big(1+O(\varepsilon^3x^{-4}\log x)\big)\\ &= \zeta_{\pm}(\varepsilon,x)\big(f_{\pm}(\varepsilon,x)+O(\varepsilon x^{-3}\log x)\big)\end{aligned} \tag{8.16}$$

in $E(\varepsilon_0)\times H(r_*)$. Hence the diagonal matrix $\operatorname{diag}\big(\zeta_{-}(\varepsilon,x),\, \zeta_{+}(\varepsilon,x)\big)$ is a fundamental matrix solution for a system

$$\boldsymbol{w}(x+i\varepsilon) = F_0(\varepsilon,x)\boldsymbol{w}(\varepsilon,x) \tag{8.17}$$

with

$$F_0(\varepsilon,x) = \begin{pmatrix} f_{-}^{0}(\varepsilon,x) & 0\\ 0 & f_{+}^{0}(\varepsilon,x)\end{pmatrix},$$

$$f_{\pm}^{0}(\varepsilon,x) = f_{\pm}(\varepsilon,x)\big(1+O(\varepsilon^3x^{-4}\log x)\big) = f_{\pm}(\varepsilon,x)+O(\varepsilon x^{-3}\log x).$$

The following fact plays an important role in constructing our solution.

Lemma 8.9. $\operatorname{Re}\varphi_{-}(\varepsilon,x) < 0$ *and* $\operatorname{Re}\varphi_{+}(\varepsilon,x) > 0$ *in* $E(\varepsilon_0)\times H(r_*)$.

Proof. Since $\log\rho_{\pm}(\varepsilon,x) = \pm\varepsilon(e^{-\pi i}x)^{1/2}(1+o(1))$ as $\varepsilon^2x\to 0$, we have

$$\big|\arg(\log\rho_{+}(\varepsilon,x))\big| = \big|\arg(e^{-\pi i}x)^{1/2} + \arg(1+o(1))\big| \le \frac{\pi}{4}+o(1),$$

$$\big|\arg(\log\rho_{-}(\varepsilon,x)) - \pi\big| \le \frac{\pi}{4}+o(1),$$

and hence $\big|\mathrm{Re}(\log\rho_\pm(\varepsilon,x))\big| \ge (1/2)\big|\mathrm{Im}(\log\rho_\pm(\varepsilon,x))\big|$ for $|\varepsilon^2 x| < \delta_0$ in $E(\varepsilon_0)\times H(r_*)$, if δ_0 is sufficiently small. Then, by (8.14),

$$\begin{aligned}\pm\mathrm{Re}\,\varphi_\pm(\varepsilon,x) &= \pm\mathrm{Re}(\log\rho_\pm(\varepsilon,x)) + O\big(x^{-1}\,\mathrm{Im}(\log\rho_\pm(\varepsilon,x))\big)\\ &= \pm\mathrm{Re}(\log\rho_\pm(\varepsilon,x))\cdot(1+O(x^{-1}))>0.\end{aligned}$$

By Proposition 6.1, (2) and (5), we have $|\rho_-(\varepsilon,x)| \le 1-\delta_0'$, $|\rho_+(\varepsilon,x)| \ge 1+\delta_0'$ and $|\arg\rho_\pm(\varepsilon,x)| < \pi/2$ for $|\varepsilon^2 x| \ge \delta_0$ in $E(\varepsilon_0)\times H(r_*)$, where δ_0' is some positive number. Then, by (8.14), we also obtain $\pm\mathrm{Re}\,\varphi_\pm(\varepsilon,x)>0$ for $|\varepsilon^2 x|\ge\delta_0$. □

8.4. Construction of a fundamental matrix solution.

Comparing (8.10) with (8.17), we write (8.10) in the form

$$\boldsymbol{w}(x+i\varepsilon) = F(\varepsilon,x)\boldsymbol{w}(x), \quad F(\varepsilon,x) = F_0(\varepsilon,x)+\Omega(\varepsilon,x), \tag{8.18}$$

where $\Omega(\varepsilon,x)$ is a square matrix such that $\Omega(\varepsilon,x) = O(\varepsilon x^{-5/2})$ in $E(\varepsilon_0)\times H(r_*)$. Put $\boldsymbol{w}(x) = \zeta_-(\varepsilon,x)\boldsymbol{z}(x)$ in (8.18). Then it becomes

$$\boldsymbol{z}(x+i\varepsilon) = (F_0^*(\varepsilon,x)+\Omega^*(\varepsilon,x))\boldsymbol{z}(x) \tag{8.19}$$

with

$$\begin{aligned}F_0^*(\varepsilon,x) &= \begin{pmatrix}1 & 0\\ 0 & \dfrac{f_+^0(\varepsilon,x)}{f_-^0(\varepsilon,x)}\end{pmatrix},\\ \Omega^*(\varepsilon,x) &= O(\varepsilon x^{-3/2}),\end{aligned}\tag{8.20}$$

because $\zeta_-(\varepsilon,x)\zeta_-(\varepsilon,x+i\varepsilon)^{-1} = f_-^0(\varepsilon,x)^{-1} \ll \rho_-(\varepsilon,x)^{-1} \ll |\varepsilon^2 x|+1$. The system $\boldsymbol{z}(x+i\varepsilon) = F_0^*(\varepsilon,x)\boldsymbol{z}(x)$ admits the fundamental matrix solution

$$\Lambda(\varepsilon,x) := \mathrm{diag}\big(1,\ \zeta_+(\varepsilon,x)/\zeta_-(\varepsilon,x)\big).$$

To treat (8.19) we note that a linear system $\boldsymbol{z}(x+i\varepsilon) = F(x)\boldsymbol{z}(x)+\boldsymbol{f}(x)$ admits the solution

$$\boldsymbol{z}(x) = Y(x)\boldsymbol{c} + Y(x)\sum_{k=-\infty}^{0} Y(x+ki\varepsilon)^{-1}\boldsymbol{f}(x+(k-1)i\varepsilon)$$

or

$$\boldsymbol{z}(x) = Y(x)\boldsymbol{c} - Y(x)\sum_{k=0}^{\infty} Y(x+(k+1)i\varepsilon)^{-1}\boldsymbol{f}(x+ki\varepsilon),$$

provided that the summation on the right-hand side converges, where $Y(x)$ denotes a fundamental matrix solution for the homogeneous system $\boldsymbol{z}(x+i\varepsilon) = F(x)\boldsymbol{z}(x)$ and $\boldsymbol{c}$ is a constant column vector. Consider the summation equation

$$\begin{aligned}\boldsymbol{z}(x) &= \mathscr{L}(\varepsilon, x; \boldsymbol{z}(x)) \\ &:= \begin{pmatrix}1\\0\end{pmatrix} - \Lambda(\varepsilon,x)\sum_{k=0}^{\infty}\Lambda(\varepsilon, x+(k+1)i\varepsilon)^{-1}\Omega^*(\varepsilon, x+ki\varepsilon)\boldsymbol{z}(x+ki\varepsilon). \end{aligned}\tag{8.21}$$

Every solution of this equation satisfies (8.19). For $r > r_*$, let $\mathscr{F}_0(r)$ denote the family of vector functions $\boldsymbol{z}(\varepsilon,x) = {}^t(z_1(\varepsilon,x), z_2(\varepsilon,x))$ continuous in $\varepsilon \in E(\varepsilon_0)$, holomorphic in $x \in H(r)$ and satisfying $|z_1(\varepsilon,x) - 1| \le 1$, $|z_2(\varepsilon,x)| \le 1$ in $E(\varepsilon_0) \times H(r)$. By Lemma 8.9 the (2.2)-entry of $\Lambda(\varepsilon,x)\Lambda(\varepsilon,x+(k+1)i\varepsilon)^{-1}$ satisfies

$$\begin{aligned}&\left|\frac{\zeta_-(\varepsilon, x+(k+1)i\varepsilon)\zeta_+(\varepsilon,x)}{\zeta_-(\varepsilon,x)\zeta_+(\varepsilon,x+(k+1)i\varepsilon)}\right| \\ &\quad = \left|\exp\biggl(\frac{1}{i\varepsilon}\int_x^{x+(k+1)i\varepsilon}\bigl(\varphi_-(\varepsilon,t) - \varphi_+(\varepsilon,t)\bigr)dt\biggr)\right| \\ &\quad \le \exp\biggl(\int_0^{k+1}\mathrm{Re}\bigl(\varphi_-(\varepsilon, x+i\varepsilon s) - \varphi_+(\varepsilon, x+i\varepsilon s)\bigr)ds\biggr) \le 1 \end{aligned}\tag{8.22}$$

in $E(\varepsilon_0) \times H(r)$, if r is sufficiently large. Furthermore, by (8.20) and Lemma 6.4,

$$\sum_{k=0}^{\infty}\bigl\|\Omega^*(\varepsilon, x+ki\varepsilon)\bigr\| \ll \varepsilon\sum_{k=0}^{\infty}|x+ki\varepsilon|^{-3/2} \ll (\mathrm{Re}\,x)^{-1/2}, \tag{8.23}$$

where $\|\cdot\|$ denotes the standard norm of matrices. By these inequalities, there exists a sufficiently large positive number $\tilde{r}$ such that the linear operator $\mathscr{L}(\varepsilon, x; \cdot)$ has the following properties:

(1) $\mathscr{L}(\varepsilon, x; \boldsymbol{z}(\varepsilon,x)) \in \mathscr{F}_0(\tilde{r})$ if $\boldsymbol{z}(\varepsilon,x) \in \mathscr{F}_0(\tilde{r})$;
(2) for any $\boldsymbol{z}(\varepsilon,x)$, $\tilde{\boldsymbol{z}}(\varepsilon,x) \in \mathscr{F}_0(\tilde{r})$,

$$\bigl\|\mathscr{L}(\varepsilon,x;\tilde{\boldsymbol{z}}(\varepsilon,x)) - \mathscr{L}(\varepsilon,x;\boldsymbol{z}(\varepsilon,x))\bigr\| \le \frac{1}{2}\sup_{E(\varepsilon_0)\times H(\tilde{r})}\bigl\|\tilde{\boldsymbol{z}}(\varepsilon,x) - \boldsymbol{z}(\varepsilon,x)\bigr\|.$$

Then the sequence defined by

$$\boldsymbol{z}_0(\varepsilon,x) = {}^t(1,0), \qquad \boldsymbol{z}_{n+1}(\varepsilon,x) = \mathscr{L}(\varepsilon,x;\boldsymbol{z}_n(\varepsilon,x))$$

converges to a solution $\boldsymbol{z}_\infty(\varepsilon,x)$ of (8.21) or (8.19) such that $\boldsymbol{z}_\infty(\varepsilon,x) \in \mathscr{F}_0(\tilde{r})$. Moreover, by (8.21), (8.22) and (8.23) it is easy to see that

$$\boldsymbol{z}_\infty(\varepsilon,x) = {}^t(1,0) + O((\operatorname{Re} x)^{-1/2})$$

in $E(\varepsilon_0) \times H(\tilde{r})$. Therefore (8.18) admits a solution expressed as

$$\boldsymbol{w}_-(\varepsilon,x) = \zeta_-(\varepsilon,x)\begin{pmatrix} 1 + O((\operatorname{Re} x)^{-1/2}) \\ O((\operatorname{Re} x)^{-1/2}) \end{pmatrix}.$$

Similarly we obtain another solution of the form

$$\boldsymbol{w}_+(\varepsilon,x) = \zeta_+(\varepsilon,x)\begin{pmatrix} O((\operatorname{Re} x)^{-1/2}) \\ 1 + O((\operatorname{Re} x)^{-1/2}) \end{pmatrix}.$$

Then

$$W(\varepsilon,x) = \big(\boldsymbol{w}_-(\varepsilon,x) \;\; \boldsymbol{w}_+(\varepsilon,x)\big) = \big(I + O((\operatorname{Re} x)^{-1/2})\big)\begin{pmatrix} \zeta_-(\varepsilon,x) & 0 \\ 0 & \zeta_+(\varepsilon,x) \end{pmatrix}$$

is a fundamental matrix solution for (8.18). This combined with the transformation of Proposition 8.6 yields a fundamental matrix solution for (6.2) written in the form

$$\begin{aligned} U(\varepsilon,x) &= T(\varepsilon,x)\big(I + \widetilde{P}(\varepsilon,x)\big)W(\varepsilon,x) \\ &= T(\varepsilon,x)(I + P(\varepsilon,x))\begin{pmatrix} \zeta_-(\varepsilon,x) & 0 \\ 0 & \zeta_+(\varepsilon,x) \end{pmatrix} \end{aligned}$$

as in Proposition 6.2. Furthermore, if $\operatorname{Im} x > 0$, then, by Lemma 6.4, we have $\sum_{k=0}^{\infty} \|\Omega^*(\varepsilon, x + ki\varepsilon)\| \ll x^{-1/2}$ instead of (8.23). Using this, we may deduce

$$\boldsymbol{w}_-(\varepsilon,x) = \zeta_-(\varepsilon,x)\begin{pmatrix} 1 + O(x^{-1/2}) \\ O(x^{-1/2}) \end{pmatrix}$$

for $\operatorname{Im} x > 0$, which implies (2) of Proposition 6.2. The property (4) of Proposition 6.2 is given by Lemma 8.9, and relation (6.4) follows from (8.16) combined with Proposition 8.6, (2). Thus we obtain Proposition 6.2.

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