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Nonlinear instability of linearly unstable standing waves for nonlinear Schrödinger equations

By Vladimir Georgiev and Masahito Ohta

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Abstract. We study the instability of standing waves for nonlinear Schrödinger equations. Under a general assumption on nonlinearity, we prove that linear instability implies orbital instability in any dimension. For that purpose, we establish a Strichartz type estimate for the propagator generated by the linearized operator around standing wave.

1. Introduction.

In this paper we study the instability of standing waves for nonlinear Schrödinger equations

$$i\partial_t u + \Delta u + g(|u|^2)u = 0, \quad (t,x) \in \boldsymbol{R} \times \boldsymbol{R}^N, \tag{1}$$

where u is a complex-valued function of (t,x), and g is a real-valued function. A typical example of nonlinearity is $g(|u|^2)u = |u|^{p-1}u$ with $1 < p < 2^* - 1$, where $2^* = 2N/(N-2)$ if $N \geq 3$ and $2^* = \infty$ if $N = 1, 2$. Precise assumptions on the nonlinearity will be made later. By a standing wave we mean a solution of (1) of the form $u(t,x) = e^{i\omega t}\varphi(x)$, where $\omega \in \boldsymbol{R}$ and $\varphi \in H^1(\boldsymbol{R}^N) \setminus \{0\}$ is a solution of the stationary problem

$$-\Delta\varphi + \omega\varphi - g(|\varphi|^2)\varphi = 0, \quad x \in \boldsymbol{R}^N. \tag{2}$$

For the special case $g(|u|^2)u = |u|^{p-1}u$ with $1 < p < 2^* - 1$, the following results are well-known. For each $\omega > 0$, the stationary problem (2) has a unique positive radial solution in $H^1(\boldsymbol{R}^N)$ (see [**35**], [**3**] for existence, and [**24**] for uniqueness). We call it ground state. When $N \geq 2$, other than the ground state, there exist infinitely many solutions of (2) in $H^1(\boldsymbol{R}^N)$. We call them excited states. For

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the ground state φ of (2) with $\omega > 0$, the standing wave $e^{i\omega t}\varphi$ is orbitally stable if $1 < p < 1 + 4/N$, while it is orbitally unstable if $1 + 4/N \le p < 2^* - 1$ (see [**2**], [**5**], [**37**]). For more general nonlinearity, Shatah and Strauss [**33**] gave a general condition for orbital instability of ground state-standing waves for (1) constructing suitable Lyapunov functionals (see also [**19**] and [**16**], [**26**], [**30**], [**31**]). We remark that these results are mostly limited to ground states and are not applicable to excited states. Here, we recall the definition of orbital stability and instability of standing waves.

Definition 1. We say that the standing wave $e^{i\omega t}\varphi$ is *orbitally stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^1(\boldsymbol{R}^N)$ and $\|u_0 - \varphi\|_{H^1} < \delta$, then the solution $u(t)$ of (1) with $u(0) = u_0$ exists globally and satisfies

$$\inf_{(\theta,y)\in \boldsymbol{R}\times\boldsymbol{R}^N} \left\| u(t) - e^{i\theta}\varphi(\cdot + y)\right\|_{H^1} < \varepsilon$$

for all $t \ge 0$. Otherwise, $e^{i\omega t}\varphi$ is called *orbitally unstable* or *nonlinearly unstable*.

While, $e^{i\omega t}\varphi$ is said to be *linearly unstable* if the linearized operator $A = JH$ around the standing wave has an eigenvalue with positive real part (for the definition of J and H, see (3) and (7) below). The linear instability of standing waves for (1) was studied by Jones [**22**] and Grillakis [**17**], [**18**] (see also [**20**], [**27**], [**29**]). In particular, for the case $g(|u|^2)u = |u|^{p-1}u$ with $1 + 4/N < p < 2^* - 1$, it is proved in [**17**] that for any radially symmetric, real-valued solution φ of (2) with $\omega > 0$, $e^{i\omega t}\varphi$ is linearly unstable. The result in [**17**] guarantees that among radially symmetric solutions, one can find oscillating solutions (i.e. solutions changing the sign) and these solutions shall generate excited states $e^{i\omega t}\varphi$. On the other hand, Mizumachi [**27**], [**29**] considered complex-valued solutions of (2) in $\boldsymbol{R}^2$ of the form $\varphi_m(x) = e^{im\theta}\phi(r)$, where m is a positive integer, and r, θ are the polar coordinates in $\boldsymbol{R}^2$ (see [**21**], [**25**] for existence of φ_m). It is proved that if $p > 3$ then for any m, $e^{i\omega t}\varphi_m$ is linearly unstable ([**27**]), and that if $1 < p < 3$ then for sufficiently large m, $e^{i\omega t}\varphi_m$ is linearly unstable ([**29**]).

However, it is a highly nontrivial problem whether linear instability implies orbital instability for (1), especially in higher dimensional case (see [**11**], [**12**], [**28**], [**34**]). Even in two dimensional case, some technical difficulties arise from the estimates of nonlinear terms (see Lemma 13 of [**7**]). For the case $N \le 3$, a satisfactory answer for this problem was given by Colin, Colin and Ohta [**8**]. The main idea in [**8**] is to employ time derivative in the estimates of nonlinear terms without using space derivatives directly, and to apply the H^2-regularity of H^1-solutions for (1). However, the proof of [**8**] is based on the L^2-estimate on the propagator e^{tA} generated by the linearized operator A, and the restriction $N \le 3$

comes from the embedding $H^2(\boldsymbol{R}^N) \hookrightarrow L^\infty(\boldsymbol{R}^N)$.

The main goal of this work is to show that linear instability implies orbital instability for (1) in any dimension $N \geq 1$ (see Theorem 2 below). In particular, for the case $g(|u|^2)u = |u|^{p-1}u$ with $1+4/N < p < 2^*-1$, it follows from the linear instability result of [**17**] and our Theorem 2 that for any radially symmetric, real-valued solution φ of (2) with $\omega > 0$, $e^{i\omega t}\varphi$ is orbitally unstable in any dimension.

Our approach is based on appropriate Strichartz type estimate for the propagator e^{tA} and gives the possibilities for further generalization. We have chosen the model of the nonlinear Schrödinger equation (1) for simplicity, but even in this case one needs to apply spectral mapping result $\sigma(e^A) = e^{\sigma(A)}$ discussed in the work of Gesztesy, Jones, Latushkin and Stanislavova [**13**] which is based on the abstract result known as the Gearhart-Greiner-Herbst-Prüss theorem (see [**1**], [**14**], [**32**]). If one considers complex-valued solutions of (2), then the assertion

$$\text{linear instability} \Longrightarrow \text{orbital instability}$$

depends on the possible generalization of the property $\sigma(e^A) = e^{\sigma(A)}$ for the linearized operator A around complex-valued excited states. Since our goal is to give general argument working for complex-valued excited states as well, we have to make suitable generalization of the result in [**13**] (see Section 4).

Here, we give an outline of the paper more precisely. In what follows, we often identify $z \in \boldsymbol{C}$ with ${}^t(\Re z, \Im z) \in \boldsymbol{R}^2$, and write $z = {}^t(\Re z, \Im z)$. We define $f(z) = -g(|z|^2)z$ for $z \in \boldsymbol{R}^2$. Then, (1) is rewritten as

$$\partial_t u = J(-\Delta u + f(u)), \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad u = \begin{bmatrix} \Re u \\ \Im u \end{bmatrix}. \tag{3}$$

We assume that $f \in C^1(\boldsymbol{R}^2, \boldsymbol{R}^2)$, and denote the derivative of f at $z \in \boldsymbol{R}^2$ by $Df(z)$, which is a 2×2-real symmetric matrix and is given by

$$Df(z) = -\begin{bmatrix} 2g'(|z|^2)(\Re z)^2 + g(|z|^2) & 2g'(|z|^2)\Re z \Im z \\ 2g'(|z|^2)\Re z \Im z & 2g'(|z|^2)(\Im z)^2 + g(|z|^2) \end{bmatrix}. \tag{4}$$

For nonlinearity, we assume the following.

(H1): g is a real-valued continuous function on $[0,\infty)$, and $f(z) = -g(|z|^2)z$ is decomposed as $f = f_1 + f_2$ with $f_j \in C^1(\boldsymbol{R}^2, \boldsymbol{R}^2)$, $f_j(0) = 0$, $Df_j(0) = O$, $j = 1, 2$, and there exist constants C and $1 < p_j < 2^* - 1$ such that

$$\left|Df_j(z_1) - Df_j(z_2)\right| \le C \begin{cases} |z_1 - z_2|^{p_j - 1} & \text{if } 1 < p_j \le 2 \\ (|z_1|^{p_j-2} + |z_2|^{p_j-2})|z_1 - z_2| & \text{if } p_j > 2 \end{cases} \tag{5}$$

for all z_1, $z_2 \in \boldsymbol{R}^2$.

Remark that the typical example $f(z) = -|z|^{p-1}z$ satisfies (H1) for $1 < p < 2^* - 1$ (see Lemma 2.4 of [**15**]). Moreover, the Cauchy problem for (1) is locally well-posed in $H^1(\boldsymbol{R}^N)$ (see [**23**] and [**4**, Chapter 4]).

For a solution of (2), we assume the following.

(H2): $\omega > 0$ is a constant and $\varphi \in H^1(\boldsymbol{R}^N)$ is a complex-valued nontrivial solution of (2).

For the existence of solutions of (2), see, e.g., [**3**], [**21**], [**25**], [**35**]. By the elliptic regularity theory, we see that $\varphi \in H^2(\boldsymbol{R}^N) \cap C^2(\boldsymbol{R}^N)$ and $\varphi(x)$ decays to 0 exponentially as $|x| \to \infty$. Remark that we consider not only real-valued solutions of (2) but also complex-valued solutions, and that by (4), $Df(\varphi)$ is a diagonal matrix if φ is real-valued, but not in general.

By a change of variables $u(t) = e^{i\omega t}(\varphi + v(t))$ in (1) or (3), we have

$$\partial_t v = Av + h(v), \tag{6}$$

where $v = {}^t(\Re v, \Im v)$, $A = JH$, $h(v) = J[f(\varphi + v) - f(\varphi) - Df(\varphi)v]$, and

$$H = H_0 + Df(\varphi), \quad H_0 = \begin{bmatrix} -\Delta + \omega & 0 \\ 0 & -\Delta + \omega \end{bmatrix}. \tag{7}$$

For the linearized operator $A = JH$, we assume the following.

(H3): The operator A has an eigenvalue λ_0 such that $\Re\lambda_0 > 0$.

As stated above, sufficient conditions for (H3) are studied by [**17**], [**18**], [**20**], [**22**], [**27**], [**29**]. See also [**6**], [**9**], [**10**], [**38**] for spectral properties of A. We now state the main result of this paper.

Theorem 2. *Assume* (H1)–(H3). *Then, the standing wave* $e^{i\omega t}\varphi$ *of* (1) *is orbitally unstable.*

The rest of the paper is organized as follows. In Section 2, assuming that the propagator e^{tA} satisfies an exponential growth condition (11), we introduce a suitable norm (12) and establish a Strichartz type estimate for e^{tA}. In Section 3, we prove Theorem 2. In the proof, we apply the Strichartz type estimate for e^{tA} proved in Section 2, and we employ time derivative instead of space derivatives in the estimates of nonlinear terms as in [**8**]. Finally, in Section 4, we give some remarks on the spectral mapping theorem for e^A due to Gesztesy, Jones, Latushkin and Stanislavova [**13**].

2. Strichartz estimates.

Let $V_{jk} \in L^\infty(\boldsymbol{R}^N, \boldsymbol{R})$ for $j,k = 1,2$, and we consider linear operators

$$A = A_0 + V, \quad A_0 = JH_0, \quad V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \tag{8}$$

on $L^2(\boldsymbol{R}^N) \times L^2(\boldsymbol{R}^N)$ with domains $D(A_0) = D(A) = H^2(\boldsymbol{R}^N) \times H^2(\boldsymbol{R}^N)$, where J and H_0 are defined in (3) and (7). Let e^{tA_0} and e^{tA} be the strongly continuous groups on $L^2(\boldsymbol{R}^N) \times L^2(\boldsymbol{R}^N)$ generated by A_0 and A respectively, and we define

$$\Gamma_0[f](t) = \int_0^t e^{(t-s)A_0} f(s)\, ds, \quad \Gamma[f](t) = \int_0^t e^{(t-s)A} f(s)\, ds.$$

Moreover, we denote $L^r := L^r(\boldsymbol{R}^N) \times L^r(\boldsymbol{R}^N)$ and $L^q_T Y := L^q((0,T), Y)$ for a Banach space Y. Note that $u(t) = e^{tA}\psi + \Gamma[f](t)$ satisfies

$$\partial_t u = Au + f(t) = A_0 u + Vu + f(t), \quad u(0) = \psi, \tag{9}$$

and $u_0(t) = e^{tA_0}\psi + \Gamma_0[f](t)$ satisfies

$$\partial_t u_0 = A_0 u_0 + f(t) = Au_0 + f(t) - Vu_0, \quad u_0(0) = \psi. \tag{10}$$

We assume that there exist positive constants C and ν such that

$$\|e^{tA}\|_{B(L^2)} \le Ce^{\nu t} \tag{11}$$

for all $t \ge 0$. For $\lambda > 0$, we define functions e_λ^+ and e_λ^- by $e_\lambda^\pm(t) = e^{\pm\lambda t}$ for $t \in \boldsymbol{R}$. Moreover, we define

$$\|f\|_{L_T^{q,\lambda} Y} := e^{\lambda T} \left\| e_\lambda^- f \right\|_{L_T^q Y}. \tag{12}$$

Note that $\|f\|_{L^q_T Y} \le \|f\|_{L_T^{q,\lambda} Y} \le \|f\|_{L_T^{q,\mu} Y}$ for $0 < \lambda < \mu$ and $T > 0$. The Hölder conjugate of q is denoted by q'. For the definition of admissible pairs and the standard Strichartz estimates for $e^{it\Delta}$, see, e.g., [**4**, Section 2.3].

LEMMA 3. *Assume $V \in L^\infty(\boldsymbol{R}^N)$ and (11). Let $0<\nu<\mu$ and let (q,r) be any admissible pair. Then, there exists a constant C independent of ψ, f and T such that $u(t)=e^{tA}\psi+\Gamma[f](t)$ satisfies*

$$\|u(t)\|_{L^2} \le C\big(e^{\nu t}\|\psi\|_{L^2} + e^{\mu t}\|e_\mu^- f\|_{L_T^{q'}L^{r'}}\big)$$

for all $t \in [0,T]$.

PROOF. Let $u_0(t)=e^{tA_0}\psi+\Gamma_0[f](t)$. Then, by (9) and (10), we have

$$\partial_t(u-u_0)=A(u-u_0)+Vu_0, \quad (u-u_0)(0)=0,$$

so $u-u_0=\Gamma[Vu_0]$. By the assumption (11), we have

$$\begin{aligned}\|u(t)-u_0(t)\|_{L^2} &\le \int_0^t \big\|e^{(t-s)A}Vu_0(s)\big\|_{L^2}\,ds \\ &\le C\|V\|_{L^\infty}\int_0^t e^{\nu(t-s)}\|u_0(s)\|_{L^2}\,ds\end{aligned}$$

for all $t\in[0,T]$. Here, by the standard Strichartz estimate for $e^{it\Delta}$, we have

$$\|u_0(t)\|_{L^2} \le C\big(\|\psi\|_{L^2}+\|f\|_{L_t^{q'}L^{r'}}\big) \le C\big(\|\psi\|_{L^2}+e^{\mu t}\big\|e_\mu^- f\big\|_{L_T^{q'}L^{r'}}\big)$$

for all $t\in[0,T]$. Thus,

$$\begin{aligned}\|u(t)\|_{L^2} &\le \|u_0(t)\|_{L^2}+\|u(t)-u_0(t)\|_{L^2} \\ &\le \|u_0(t)\|_{L^2}+C\int_0^t e^{\nu(t-s)}\|\psi\|_{L^2}\,ds + Ce^{\nu t}\int_0^t e^{(\mu-\nu)s}\big\|e_\mu^- f\big\|_{L_T^{q'}L^{r'}}\,ds \\ &\le C\big(e^{\nu t}\|\psi\|_{L^2}+e^{\mu t}\big\|e_\mu^- f\big\|_{L_T^{q'}L^{r'}}\big)\end{aligned}$$

for all $t\in[0,T]$. This completes the proof. □

PROPOSITION 4. *Assume $V\in L^\infty(\boldsymbol{R}^N)$ and (11). Let $0<\lambda<\nu<\mu$, and let (q_1,r_1) and (q_2,r_2) be any admissible pairs. Then, there exists a constant C independent of ψ, f and T such that $u(t)=e^{tA}\psi+\Gamma[f](t)$ satisfies*

$$\|u\|_{L_T^{q_1,\lambda}L^{r_1}} \le C\Big(e^{\nu T}\|\psi\|_{L^2}+\|f\|_{L_T^{q_2',\mu}L^{r_2'}}\Big).$$

Proof. We put $v(t)=e^{-\lambda t}u(t)$. Then, by (9), we have

$$\partial_t v = A_0 v + (V-\lambda)v + e^{-\lambda t} f(t), \quad v(0)=\psi.$$

By the standard Strichartz estimate for $e^{it\Delta}$, we have

$$\left\|e_\lambda^- u\right\|_{L_T^{q_1}L^{r_1}} = \|v\|_{L_T^{q_1}L^{r_1}} \le C\Big(\|\psi\|_{L^2} + \|(V-\lambda)v\|_{L_T^1 L^2} + \left\|e_\lambda^- f\right\|_{L_T^{q_2'}L^{r_2'}}\Big).$$

Here, by Lemma 3, we have

$$\begin{aligned}\|(V-\lambda)v\|_{L_T^1L^2} &\le (\|V\|_{L^\infty}+\lambda)\|v\|_{L_T^1L^2} \le C\int_0^T e^{-\lambda t}\|u(t)\|_{L^2}\,dt \\ &\le C\int_0^T \Big\{e^{(\nu-\lambda)t}\|\psi\|_{L^2} + e^{(\mu-\lambda)t}\left\|e_\mu^- f\right\|_{L_T^{q_2'}L^{r_2'}}\Big\}\,dt \\ &\le C\Big\{e^{(\nu-\lambda)T}\|\psi\|_{L^2} + e^{(\mu-\lambda)T}\left\|e_\mu^- f\right\|_{L_T^{q_2'}L^{r_2'}}\Big\}.\end{aligned}$$

Moreover, since $\|e_\lambda^- f\|_{L_T^{q_2'}L^{r_2'}} \le e^{(\mu-\lambda)T}\|e_\mu^- f\|_{L_T^{q_2'}L^{r_2'}}$, we obtain the desired estimate. □

3. Proof of Theorem 2.

In this section we assume (H1)–(H3), and prove Theorem 2. For $j=1,2$, we put

$$h_j(v) = J[f_j(\varphi+v) - f_j(\varphi) - Df_j(\varphi)v], \quad r_j = p_j+1,$$

and let (q_j, r_j) be the corresponding admissible pair. Note that $h(v)=h_1(v)+h_2(v)$ in (6).

Lemma 5. *There exist $\lambda^*\in \boldsymbol{C}$ and $\chi\in H^2(\boldsymbol{R}^N,\boldsymbol{C})^2$ such that $\Re\lambda^*>0$, $A\chi=\lambda^*\chi$ and $\|\chi\|_{L^2}=1$. Moreover, e^{tA} satisfies* (11) *for some ν with $\Re\lambda^*<\nu<(1+\alpha)\Re\lambda^*$, where*

$$\alpha := \min\{1, r_1-2, r_2-2\}. \tag{13}$$

Proof. Since $Df(\varphi)$ decays exponentially at infinity, Weyl's essential spectrum theorem implies that $\sigma_{\mathrm{ess}}(A)\subset\{z\in\boldsymbol{C}:\Re z=0\}$. Moreover, the number of eigenvalues of $A=JH$ in $\{z\in\boldsymbol{C}:\Re z>0\}$ is finite (see, e.g., Theorem

5.8 of [**20**]). Therefore, by (H3), there exists an eigenvalue λ^* of A such that $\Re\lambda^* = \max\{\Re z : z \in \sigma(A)\} > 0$. Further, by the spectral mapping theorem due to Gesztesy, Jones, Latushkin and Stanislavova [**13**], we have $\sigma(e^A) = e^{\sigma(A)}$. Here we need some modification of [**13**] when φ is not real-valued. We shall discuss it in Section 4. Then, the spectral radius of e^A is $e^{\Re\lambda^*}$. Finally, by Lemma 3 of [**34**], we see that e^{tA} satisfies (11) for some ν with $\Re\lambda^* < \nu < (1+\alpha)\Re\lambda^*$. □

LEMMA 6. *There exists a constant C such that*

$$\|h_j(v)\|_{L^2} + \|h_j(v)\|_{L^{r_j'}} \le C\big(\|v\|_{H^2} + \|v\|_{H^2}^{r_j-2}\big)\|v\|_{H^2}$$

for all $v \in H^2(\boldsymbol{R}^N)$.

PROOF. Since

$$h_j(v) = J\int_0^1 \big\{Df_j(\varphi+\theta v) - Df_j(\varphi)\big\}v\,d\theta,$$

it follows from (5) that

$$\|h_j(v)\|_{L^2} + \|h_j(v)\|_{L^{r_j'}} \le C\begin{cases} \|v\|_{H^2}^{r_j-1} & \text{if } 2 < r_j \le 3, \\ \big(\|\varphi\|_{H^2}^{r_j-3} + \|v\|_{H^2}^{r_j-3}\big)\|v\|_{H^2}^2 & \text{if } r_j > 3, \end{cases}$$

which implies the desired estimate. □

In what follows, let λ and μ be numbers satisfying

$$0 < \lambda < \Re\lambda^* < \nu < \mu < (1+\alpha)\lambda, \tag{14}$$

and we define

$$\|v\|_{X_T} = \|v\|_{L_T^{\infty,\lambda}H^2} + \|\partial_t v\|_{L_T^{q_1,\lambda}L^{r_1}} + \|\partial_t v\|_{L_T^{q_2,\lambda}L^{r_2}}.$$

LEMMA 7. *Let $v(t)$ be an H^2-solution of (6) in $[0,\infty)$. Then, there exists a constant C independent of v and T such that*

$$\begin{aligned}\|v\|_{X_T} \le\ & C\Big(\|v\|_{L_T^{\infty,\lambda}L^2} + \|\partial_t v\|_{L_T^{\infty,\lambda}L^2} + \|\partial_t v\|_{L_T^{q_1,\lambda}L^{r_1}} + \|\partial_t v\|_{L_T^{q_2,\lambda}L^{r_2}}\Big) \\ & + C\big(\|v\|_{X_T}^2 + \|v\|_{X_T}^{r_1-1} + \|v\|_{X_T}^{r_2-1}\big).\end{aligned}$$

PROOF. By Lemma 6, we have

$$\begin{aligned}\|v(t)\|_{H^2} &\le C\big(\|v(t)\|_{L^2}+\|Av(t)\|_{L^2}\big)\\ &\le C\big(\|v(t)\|_{L^2}+\|\partial_t v(t)\|_{L^2}+\|h(v(t))\|_{L^2}\big)\\ &\le C\big(\|v(t)\|_{L^2}+\|\partial_t v(t)\|_{L^2}+\|v(t)\|_{H^2}^2+\|v(t)\|_{H^2}^{r_1-1}+\|v(t)\|_{H^2}^{r_2-1}\big)\end{aligned}$$

for all $t\in[0,T]$. Thus,

$$\begin{aligned}\|v\|_{L_T^{\infty,\lambda}H^2} \le C&\Big(\|v\|_{L_T^{\infty,\lambda}L^2}+\|\partial_t v\|_{L_T^{\infty,\lambda}L^2}\Big)\\ &+C\Big(\|v\|_{L_T^{\infty,\lambda}H^2}^2+\|v\|_{L_T^{\infty,\lambda}H^2}^{r_1-1}+\|v\|_{L_T^{\infty,\lambda}H^2}^{r_2-1}\Big),\end{aligned}$$

which implies the desired estimate. □

LEMMA 8. *There exists a constant independent of v and T such that*

$$\|h_j(v)\|_{L_T^{q_j',\mu}L^{r_j'}} \le C\big(\|v\|_{X_T}^2+\|v\|_{X_T}^{r_j-1}\big).$$

PROOF. By Lemma 6, we have

$$e^{-\mu t}\|h_j(v(t))\|_{L^{r_j'}} \le Ce^{(2\lambda-\mu)t}\big\|e_\lambda^- v\big\|_{L_T^\infty H^2}^2+Ce^{((r_j-1)\lambda-\mu)t}\big\|e_\lambda^- v\big\|_{L_T^\infty H^2}^{r_j-1}$$

for all $t\in[0,T]$. Moreover, by (13) and (14), we have

$$\begin{aligned}e^{\mu T}\big\|e_\mu^- h_j(v)\big\|_{L_T^{q_j'}L^{r_j'}} &\le Ce^{2\lambda T}\big\|e_\lambda^- v\big\|_{L_T^\infty H^2}^2+Ce^{(r_j-1)\lambda T}\big\|e_\lambda^- v\big\|_{L_T^\infty H^2}^{r_j-1}\\ &\le C\big(\|v\|_{X_T}^2+\|v\|_{X_T}^{r_j-1}\big),\end{aligned}$$

which implies the desired estimate. □

LEMMA 9. *There exists a constant C independent of v and T such that*

$$\|\partial_t h_j(v)\|_{L_T^{q_j',\mu}L^{r_j'}} \le C\big(\|v\|_{X_T}^2+\|v\|_{X_T}^{r_j-1}\big).$$

PROOF. Since $\partial_t h_j(v(t))=J\{Df_j(\varphi+v(t))-Df_j(\varphi)\}\partial_t v(t)$, it follows from (5) that

$$\|\partial_t h_j(v(t))\|_{L^{r_j'}} \le C\big(\|v(t)\|_{H^2} + \|v(t)\|_{H^2}^{r_j-2}\big)\|\partial_t v(t)\|_{L^{r_j}}.$$

Thus we have

$$\begin{aligned} e^{-\mu t}\|\partial_t h_j(v(t))\|_{L^{r_j'}} \le{}& Ce^{(2\lambda-\mu)t}\big\|e_\lambda^- v\big\|_{L_T^\infty H^2}\cdot e^{-\lambda t}\|\partial_t v(t)\|_{L^{r_j}} \\ &+ Ce^{((r_j-1)\lambda-\mu)t}\big\|e_\lambda^- v\big\|_{L_T^\infty H^2}^{r_j-2}\cdot e^{-\lambda t}\|\partial_t v(t)\|_{L^{r_j}} \end{aligned}$$

for all $t \in [0,T]$. Moreover, by (13), (14) and the Hölder inequality,

$$\begin{aligned} &e^{\mu T}\big\|e_\mu^-\partial_t h_j(v)\big\|_{L_T^{q_j'}L^{r_j'}} \\ &\le Ce^{2\lambda T}\big\|e_\lambda^- v\big\|_{L_T^\infty H^2}\big\|e_\lambda^-\partial_t v\big\|_{L_T^{q_j}L^{r_j}} + Ce^{(r_j-1)\lambda T}\big\|e_\lambda^- v\big\|_{L_T^\infty H^2}^{r_j-2}\big\|e_\lambda^-\partial_t v\big\|_{L_T^{q_j}L^{r_j}} \\ &\le C\big(\|v\|_{X_T}^2 + \|v\|_{X_T}^{r_j-1}\big). \end{aligned}$$

This completes the proof. □

Proof of Theorem 2. We use the argument in [**20**, Section 6] (see also [**8**], [**34**]). Suppose that the standing wave $e^{i\omega t}\varphi$ of (1) is orbitally stable. For small $\delta > 0$, let $u_\delta(t)$ be the solution of (1) with $u_\delta(0) = \varphi + \delta\Re\chi$, where $\chi \in H^2(\boldsymbol{R}^N, \boldsymbol{C})^2$ is the eigenfunction of A corresponding to the eigenvalue λ^* given in Lemma 5. Note that $A\overline{\chi} = \overline{\lambda^*}\overline{\chi}$. Since either $\Re\chi \notin \ker A$ or $\Im\chi \notin \ker A$, we may assume that $\Re\chi \notin \ker A$. Since we assume that $e^{i\omega t}\varphi$ is orbitally stable in $H^1(\boldsymbol{R}^N)$, the H^1-solution $u_\delta(t)$ of (1) exists globally for sufficiently small $\delta > 0$. Moreover, since φ, $\chi \in H^2(\boldsymbol{R}^N)$, by the H^2-regularity for (1), we see that $u_\delta \in C([0,\infty), H^2(\boldsymbol{R}^N)) \cap C^1([0,\infty), L^2(\boldsymbol{R}^N))$ and $\partial_t u_\delta \in L_T^{q_1}L^{r_1} \cap L_T^{q_2}L^{r_2}$ for all $T > 0$ (see [**23**], [**36**] and also [**4**, Section 5.2]). By the change of variables

$$u_\delta(t) = e^{i\omega t}(\varphi + v_\delta(t)), \tag{15}$$

we see that v_δ has the same regularity as that of u_δ, and satisfies

$$\partial_t v_\delta(t) = Av_\delta(t) + h(v_\delta(t)), \quad v_\delta(0) = \delta\Re\chi,$$

$$v_\delta(t) = \delta\Re\big(e^{\lambda^* t}\chi\big) + \Gamma[h(v_\delta)](t), \tag{16}$$

$$\partial_t v_\delta(t) = \delta\Re\big(\lambda^* e^{\lambda^* t}\chi\big) + e^{tA}h(\delta\Re\chi) + \Gamma[\partial_t h(v_\delta)](t) \tag{17}$$

for all $t \ge 0$. Let $\varepsilon_0 > 0$ be a small positive number to be determined later, let

$k = 1$ if $\Im\lambda^* = 0$, and $k = \exp(2\pi\Re\lambda^*/|\Im\lambda^*|)$ if $\Im\lambda^* \neq 0$, and define T_δ by

$$\log\frac{\varepsilon_0}{k\delta} \leq \Re\lambda^* T_\delta \leq \log\frac{\varepsilon_0}{\delta}, \quad \Im\lambda^* T_\delta \in 2\pi\boldsymbol{Z}. \tag{18}$$

for small $\delta > 0$. First, we prove that there exist constants C_1 and ε_0 independent of δ such that

$$\|v_\delta\|_{X_{T_\delta}} \leq C_1\varepsilon_0 \tag{19}$$

for small δ. For $T \in (0, T_\delta]$, by (16), Proposition 4 and Lemma 8,

$$\begin{aligned}\|v_\delta\|_{L_T^{\infty,\lambda}L^2} &\leq \big\|\delta e^+_{\lambda^*}\chi\big\|_{L_T^{\infty,\lambda}L^2} + C\Big(\|h_1(v)\|_{L_T^{q_1',\mu}L^{r_1'}} + \|h_2(v)\|_{L_T^{q_2',\mu}L^{r_2'}}\Big)\\ &\leq \delta e^{\Re\lambda^* T}\|\chi\|_{L^2} + C\big(\|v_\delta\|^2_{X_T} + \|v_\delta\|^{r_1-1}_{X_T} + \|v_\delta\|^{r_2-1}_{X_T}\big).\end{aligned}$$

Moreover, by (17), Proposition 4 and Lemma 9,

$$\begin{aligned}&\|\partial_t v_\delta\|_{L_T^{\infty,\lambda}L^2} + \|\partial_t v_\delta\|_{L_T^{q_1,\lambda}L^{r_1}} + \|\partial_t v_\delta\|_{L_T^{q_2,\lambda}L^{r_2}}\\ &\quad\leq C\big(\delta e^{\Re\lambda^* T}\|\chi\|_{H^2} + e^{\nu T}\|h(\delta\Re\chi)\|_{L^2} + \|v_\delta\|^2_{X_T} + \|v_\delta\|^{r_1-1}_{X_T} + \|v_\delta\|^{r_2-1}_{X_T}\big).\end{aligned}$$

Here, by Lemma 6 and by (13) and (14),

$$\begin{aligned}e^{\nu T}\|h(\delta\Re\chi)\|_{L^2} &\leq Ce^{\nu T}\big(\delta^2\|\chi\|^2_{H^2} + \delta^{r_1-1}\|\chi\|^{r_1-1}_{H^2} + \delta^{r_2-1}\|\chi\|^{r_2-1}_{H^2}\big)\\ &\leq C\big(\delta e^{\Re\lambda^* T}\big)^{1+\alpha}.\end{aligned}$$

Therefore, by Lemma 7 and (18),

$$\|v_\delta\|_{X_T} \leq C\big(\varepsilon_0 + \varepsilon_0^{1+\alpha} + \|v_\delta\|^2_{X_T} + \|v_\delta\|^{r_1-1}_{X_T} + \|v_\delta\|^{r_2-1}_{X_T}\big) \tag{20}$$

for all $T \in (0, T_\delta]$. Since $\limsup_{T\to+0}\|v_\delta\|_{X_T} \leq C\delta$ and $\|v_\delta\|_{X_T}$ is continuous in T, by (20) we see that there exist constants C_1 and ε_0 independent of δ such that (19) holds for small δ. Next, by (16), (19), Proposition 4 and Lemma 8,

$$\begin{aligned}\big\|v_\delta(T_\delta) - \delta\Re\big(e^{\lambda^* T_\delta}\chi\big)\big\|_{L^2} &\leq C\Big(\|h_1(v)\|_{L_{T_\delta}^{q_1',\mu}L^{r_1'}} + \|h_2(v)\|_{L_{T_\delta}^{q_2',\mu}L^{r_2'}}\Big)\\ &\leq C\big(\|v_\delta\|^2_{X_{T_\delta}} + \|v_\delta\|^{r_1-1}_{X_{T_\delta}} + \|v_\delta\|^{r_2-1}_{X_{T_\delta}}\big) \leq C\varepsilon_0^{1+\alpha}.\end{aligned} \tag{21}$$

Let $(\Re\chi)^\perp$ be the projection of $\Re\chi$ onto the orthogonal complement of $\mathrm{Span}\{i\varphi, \nabla\varphi\}$ in $L^2(\boldsymbol{R}^N, \boldsymbol{R})^2$. Note that we identify $i\varphi = (0, \varphi)$ and $\varphi = (\varphi, 0)$. Since $\mathrm{Span}\{i\varphi, \nabla\varphi\} \subset \ker A$ and $\Re\chi \notin \ker A$, we see that $(\Re\chi)^\perp \neq 0$. By (18) and (21), we have

$$\begin{aligned}
&\left|(v_\delta(T_\delta), (\Re\chi)^\perp)_{L^2} - \delta e^{\Re\lambda^* T_\delta} \|(\Re\chi)^\perp\|_{L^2}^2\right| \\
&\quad = \left|\big(v_\delta(T_\delta) - \delta\Re\big(e^{\lambda^* T_\delta}\chi\big), (\Re\chi)^\perp\big)_{L^2}\right| \leq C\varepsilon_0^{1+\alpha} \|(\Re\chi)^\perp\|_{L^2},
\end{aligned}$$

and we can take a small ε_0 such that

$$(v_\delta(T_\delta), (\Re\chi)^\perp)_{L^2} \geq \frac{\varepsilon_0}{2k} \|(\Re\chi)^\perp\|_{L^2}^2. \tag{22}$$

Finally, we put

$$\Theta_\delta = \inf_{(\theta, y) \in \boldsymbol{R} \times \boldsymbol{R}^N} \left\|u_\delta(T_\delta) - e^{i\theta}\varphi(\cdot + y)\right\|_{L^2}.$$

Then, by (15), $\Theta_\delta = \inf_{(\theta,y)\in\boldsymbol{R}\times\boldsymbol{R}^N} \|v_\delta(T_\delta) + \varphi - e^{i\theta}\varphi(\cdot + y)\|_{L^2}$, and there exists $(\theta_\delta, y_\delta) \in \boldsymbol{R} \times \boldsymbol{R}^N$ such that $\Theta_\delta = \|v_\delta(T_\delta) + \varphi - e^{i\theta_\delta}\varphi(\cdot + y_\delta)\|_{L^2}$. Moreover, since $\Theta_\delta \leq \|v_\delta(T_\delta)\|_{L^2} \leq C_1\varepsilon_0$, we have $\|\varphi - e^{i\theta_\delta}\varphi(\cdot + y_\delta)\|_{L^2} \leq 2C_1\varepsilon_0$. Thus, $|(\theta_\delta, y_\delta)| = O(\varepsilon_0)$ and

$$e^{i\theta_\delta}\varphi(\cdot + y_\delta) - \varphi = i\theta_\delta\varphi + y_\delta \cdot \nabla\varphi + o(\varepsilon_0),$$

which together with (22) implies that

$$\begin{aligned}
&\big(v_\delta(T_\delta) + \varphi - e^{i\theta_\delta}\varphi(\cdot + y_\delta), (\Re\chi)^\perp\big)_{L^2} \\
&\quad = \big(v_\delta(T_\delta), (\Re\chi)^\perp\big)_{L^2} - \big(i\theta_\delta\varphi + y_\delta \cdot \nabla\varphi, (\Re\chi)^\perp\big)_{L^2} - o(\varepsilon_0) \\
&\quad \geq \frac{\varepsilon_0}{4k} \|(\Re\chi)^\perp\|_{L^2}^2
\end{aligned}$$

for some small ε_0. Therefore,

$$\inf_{(\theta, y) \in \boldsymbol{R} \times \boldsymbol{R}^N} \left\|u_\delta(T_\delta) - e^{i\theta}\varphi(\cdot + y)\right\|_{H^1} \geq \Theta_\delta \geq \frac{\varepsilon_0}{4k} \|(\Re\chi)^\perp\|_{L^2}$$

for all δ small. This contradiction proves that $e^{i\omega t}\varphi$ is orbitally unstable. $\square$

4. Remark on spectral mapping theorem.

In this section, we assume that $V_{jk} \in C(\boldsymbol{R}^N, \boldsymbol{R})$ and there exist positive constants ε and C such that

$$|V_{jk}(x)| \le Ce^{-2\varepsilon|x|} \tag{23}$$

for all $x \in \boldsymbol{R}^N$ and $j, k = 1, 2$. We consider the linear operator $A = A_0 + V$ defined by (8). Then we have the following.

PROPOSITION 10. *For each $N \ge 1$ one has $\sigma(e^A) = e^{\sigma(A)}$.*

In [**13**], Proposition 10 is proved for the case $V_{11} = V_{22} = 0$. We modify the proof of Theorem 1 of [**13**] to prove Proposition 10 for general case. As we have stated in Section 1, this generalization is needed to treat the case where a solution φ of (2) is not real-valued.

PROOF OF PROPOSITION 10. For $\xi = a + i\tau$ with a, $\tau \in \boldsymbol{R}\setminus\{0\}$, we denote

$$L(\xi) = \begin{bmatrix} \xi & -D \\ D & \xi \end{bmatrix}, \quad D = -\Delta + \omega.$$

Then, we have $-\xi^2 \notin \sigma(D^2)$ and

$$L(\xi)^{-1} = \begin{bmatrix} \xi[\xi^2 + D^2]^{-1} & D[\xi^2 + D^2]^{-1} \\ -D[\xi^2 + D^2]^{-1} & \xi[\xi^2 + D^2]^{-1} \end{bmatrix}.$$

We also have $\xi - A = L(\xi) - V = L(\xi)[I - L(\xi)^{-1}V]$. Here we decompose $V = WB$ by

$$W = e^{\varepsilon|x|}V, \quad B = e^{-\varepsilon|x|}I.$$

By (23), all entries of W and B are exponentially decaying continuous functions. Moreover, each entry of $BL(\xi)^{-1}W$ has a form

$$P_1(x)\xi[\xi^2 + D^2]^{-1}Q_1(x) + P_2(x)D[\xi^2 + D^2]^{-1}Q_2(x),$$

where P_1, P_2, Q_1 and Q_2 are real-valued continuous functions decaying exponentially. Therefore, by Lemma 6 of [**13**], we see that $\|BL(\xi)^{-1}W\| \to 0$ as $|\tau| \to \infty$. Then the rest of the proof of Proposition 10 is the same as in the proof of Theorem 1 of [**13**]. □

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Vladimir GEORGIEV
Dipartimento di Matematica
Università degli Studi di Pisa
Largo Bruno Pontecorvo 5
I-56127 Pisa, Italy
E-mail: georgiev@dm.unipi.it

Masahito OHTA
Department of Mathematics
Saitama University
Saitama 338-8570, Japan
E-mail: mohta@mail.saitama-u.ac.jp