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Asymptotically quasiconformal four manifolds

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Abstract. We give a formulation of Yang-Mills gauge theory for open smooth four-dimensional manifolds whose ends are homeomorphic to $S^3 \times [0, \infty)$. We apply this formulation to the study of conformal structures of such manifolds. We introduce the notion of *asymptotically quasiconformal homeomorphic manifolds* and show that there exist manifolds which are mutually homeomorphic but not asymptotically quasiconformal homeomorphic.

Introduction.

On topological manifolds, one can introduce several kinds of additional structures. Of particular interest is the relation among smooth, quasiconformal and topological structures, whose nature drastically change according to the dimension of manifolds. Sullivan showed the equivalence between the topological structure and the quasiconformal structure for manifolds of dimension larger than 5 ([**Su**]). On the other hand Donaldson and Sullivan ([**DS**]) developed Yang-Mills gauge theory for quasiconformal four manifolds. As an application they gave an example of a simply connected closed topological four manifold which does not have a quasiconformal structure, and, as another application, an example of a pair of smooth closed oriented four manifolds which are homeomorphic but not quasiconformal to each other.

As for the existence problem, it is known that the nature of smooth structure on a punctured four manifold is quite different from that on a closed four manifold: while a closed topological manifold does not always have a smooth structure, a punctured four manifold, i.e. the complement of a point in a (topological) closed four manifold, always has a smooth structure ([**Q**]).

It would be quite natural to try to compare smooth and/or topological structures with quasiconformal structure on punctured four manifolds. An open subset of a smooth manifold has a natural quasiconformal structure. In this paper we investigate the quasiconformal structures on open subsets of smooth four manifolds which are homeomorphic to punctured topological four manifolds. We give

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a framework to compare two such quasiconformal structures by using Yang-Mills gauge theory.

When the manifold is not compact, it would be appropriate to introduce some modifications in the definition of equivalence relation between the two quasiconformal structures. We will actually define two distinct, but closely related, notions: "asymptotically quasiconformal homeomorphism" and "asymptotically q.c. equivalence". When both manifolds are punctured "smooth" four manifolds, we defined the notion of "asymptotically q.c. equivalence". When one of the two manifolds is a punctured "smooth" four manifold, we defined the notion of "asymptotically quasiconformal homeomorphism". We give these subtly different notions for punctured four manifolds because of the following reason.

Let us take two closed smooth four manifolds M and M' which are homeomorphic, but are not necessary quasiconformal equivalent to each other. We would like to define the "appropriate notion of equivalence relation" between the quasiconformal structures on $M \setminus \{\text{pt}\}$ and $M' \setminus \{\text{pt}\}$ so that it does not directly imply the quasiconformal equivalence between the closed manifolds M and M', where we equip with the cylindrical metrics on them.

We will define the notion of "asymptotically q.c. equivalence" using a family of quasiconformal homeomorphisms $\{I_l\}_l$ which are given on increasing and exhausting compact subsets $\{K_l\}_l$ of $M \setminus \{\text{pt}\}$ whose ranges are exhausting compact subsets on $M' \setminus \{\text{pt}\}$ as $l \to \infty$. Notice that if the conformal constants of $\{I_l\}$ are uniformly bounded and each I_{l+1} is an extension of I_l, then the family gives a global quasiconformal homeomorphism between M and M'. In our definition of asymptotically q.c. equivalence, we will use a family $\{I_l\}$ for which we do "not" assume that each I_{l+1} is an extension of I_l but assume a weaker asymptotical condition.

In this paper we equip with cylindrical metrics g on $\hat{M} = M \setminus \{\text{pt}\}$ and g' on $\hat{M}' = M' \setminus \{\text{pt}\}$ to develop a gauge theoretical method to analyze the family $\{I_l\}$ of quasiconformal mappings between them. We study and compare the behaviour of Yang-Mills moduli spaces over $(\hat{M}, g)$ and $(\hat{M}', g')$ using the family $\{I_l\}$. As $l \to \infty$, the quasiconformal homeomorphism I_l becomes more complicated. Under such complicated behaviour, it is necessary to manipulate the various moduli spaces of ASD connections with multiple bubbles at the same time in order to compare the moduli spaces over $\hat{M}$ and $\hat{M}'$.

In practice, we reconstruct ASD moduli spaces so that their ambient spaces are families of uniformly bounded $L^{2+\epsilon}$ or $L^{4+2\epsilon}$-functions. Then we will formulate an "asymptotic ASD connection", consisting of a family of approximately ASD connections satisfying some conditions. The moduli space of "asymptotic ASD connections" $(AsM(\hat{M}, g))$ contains the ordinary ASD moduli space as a subset. If the ordinary ASD moduli space is not compact, then the asymptotic ASD connection allows "bubbles". In the definition of "asymptotic ASD connection" we need to specify a parameter ($\boldsymbol{T} = \{\boldsymbol{T}(\boldsymbol{l}, \boldsymbol{j})\}$) which consists of data for some translation scaling. The union of the asymptotic ASD moduli space for all the parameters ($PAsM(\hat{M}, g)$) is the "parametrized asymptotic ASD moduli space" which are fibred over the parameter space. This extended moduli spaces will be used to compare two cylindrical manifolds which are mutually "asymptotically quasiconformal homeomorphic".

The main purpose of this paper is to investigate certain types of degenerations of smooth structures on four manifolds using families of quasiconformal mappings from gauge theoretic view points. The family of mappings would cause appearance of bubbling, and it is natural to introduce the asymptotic ASD moduli spaces. One advantage of our formulation is, as we will show in this paper, that the correspondence between the asymptotic ASD moduli spaces can be explicitly described by the correspondence between the moduli spaces on the closed four manifolds. While it would be logically possible to avoid using these bubbling function spaces to obtain our topological applications, such description would become rather complicated. Moreover the framework to formulate asymptotic ASD moduli spaces could be potentially extended to more wider generalization of quasiconformal mappings on open manifolds, e.g. "non-uniform case" without the condition (5) below, though we do not pursue this direction in this paper.

Here we introduce two main constructions. Firstly we introduce a canonical method to construct asymptotic function spaces from (basically) any kinds of function spaces. Secondly as an underlying function spaces, we study elliptic theory on parametrized Banach spaces over cylindrical four manifolds.

Let (K, g) and (K', g') be compact Riemannian manifolds respectively. Then we say that a homeomorphism $I : (K, g) \cong (K', g')$ is quasiconformal and has the constant bounded by C, if the following numbers are pointwisely estimated:

$$H(\varphi)(x) = \limsup_{r \to 0} \frac{\max\{|\varphi(y) - \varphi(x)| : |x - y| = r\}}{\min\{|\varphi(y) - \varphi(x)| : |x - y| = r\}} \leq C$$

where $\varphi = I, I^{-1}$. $\log H(I)(x)$ are equal to $d([g], [I^*(g')])(x)$, the conformal distance (4.A), which measure pointwise-distances between one metric and its pull-back by the quasiconformal homeomorphism. On the construction of gauge theory by use of quasiconformal mappings, a key role in the analysis is played by the Gehring's theorem ([**G**]); if $I : (K, g) \cong (K', g')$ is a quasiconformal homeomorphism, then $\nabla I \in L^{4+\delta}$ and $\delta > 0$ is essentially determined by $\text{ess sup}\, H(I)$. Notice that the pull-backed Riemannian metric $I^*(g')$ can be approximated by a smooth family g_i in L^2 (Lemma 4.2). If we denote $I_i = \text{id} : (K, g_i) \cong (K, I^*(g'))$, then $\text{ess sup}\, H(I_i)$ are uniformly bounded. Particularly $H(I_i)^2 - 1 \to 0$ hold in all

L^n.

With these facts in mind, let us introduce the notion of asymptotic morphisms below. Let us take two closed, oriented smooth four manifolds M and M', and consider two corresponding cylindrical manifolds $(\hat{M}, g)$ and $(\hat{M}', g')$. Then we introduce asymptotic morphisms between these Riemannian manifolds as follows.

Let us take two exhaustions by compact subsets:

$$K_0(') \subset\subset K_1(') \subset\subset \cdots \subset \hat{M}(\hat{M}').$$

An *asymptotic morphism* from $(\hat{M}, g)$ to $(\hat{M}', g')$ with respect to $\{K_l\}_l$ and $\{K'_l\}_l$ consists of a family of quasiconformal homeomorphisms $I_l : (K_l, g) \cong (K'_l, g')$ and $\delta > 0$ satisfying:

(1) $|H(I_l)^2 - 1|L^N_{\text{loc}} \to 0$ for all $N > 0$,

(2) $d(I_{l'}(x), I_l(x)) \to 0$ as $l, l' \to \infty$ for $x \in K_l \cap K_{l'}$,

(3) $\{|\nabla I_l|L^{4+\delta}_{\text{loc}}\}$ are uniformly bounded, and $|\nabla(I_{l'} - I_l)|L^{4+\delta}_{\text{loc}} \to 0$,

(4) for any $x \in \hat{M}'$, there exists $l(x) \geq 0$ such that image of I_l contains x for all $l \geq l(x)$. Moreover $I_l(x) \to \infty$ hold as $x \to \infty$.

A *uniformly bounded* asymptotic morphism consists of an asymptotic morphism with the additional conditions:

(5) $d(x, I_l(x))$ are uniformly bounded with respect to x.

Here we identify both end $\hat{M}$ and end $\hat{M}'$ with $S^3 \times [0, \infty)$ by use of the smooth local polar coordinates around the removed points on both M and M'. The condition (5) is required in order to control the weighted norms in 4.C.

If the family $\{I_l^{-1}\}_l$ consists of another asymptotic morphism, then we say that $(\hat{M}, g)$ and $(\hat{M}', g')$ are *asymptotically q.c. equivalent*.

THEOREM 0.1.

(1) *Suppose there exists a uniformly bounded asymptotically q.c. equivalence as above, such that for any* $[A_l] \in As\mathfrak{M}(\hat{M}('), g('); \boldsymbol{T})$,

$$\lim_{r \to 0} \limsup_{x \to \infty} \sup_{k,l} \left| I_l(^{-1})^*(F_{A_k}) \right| L^2(B_r(x)) = 0. \quad *$$

Then it induces a homeomorphism:

$$\left\{ I_l^* \right\}_l : PAs\mathfrak{M}(\hat{M}, g) \cong PAs\mathfrak{M}(\hat{M}', g')$$

where g *and* g' *are sufficiently near the cylindrical metrics.*

(2) *There is a natural embedding*:

$$\mathfrak{M}(\hat{M}, g) \hookrightarrow As\mathfrak{M}(\hat{M}, g; \boldsymbol{T}).$$

Notice that (2) always holds independently of the assumption of (1).

Roughly speaking each connected component of the asymptotic moduli spaces may be mapped on different types in the other spaces, since it may happen that the component will split into several islands passing through $\{I_l^*\}$. This is the main reason why we have to introduce such bubbling moduli spaces instead of the standard one.

Notice that we do not assume genericity of Riemannian metrics. The above proposition is verified in Section 7. As we will see in 2.A, the underlying Banach spaces are different from the ordinary ones. It would be very interesting to use non-uniformly bounded Banach spaces (ϵ collapsed one, 3.C). Fredholm theory works for such case, however gauge group action seems problematic.

The above assumption $*$ is automatically satisfied, if one additionally assumes *uniform bound of infinitesimal volumes* by mapping by $\{I_l\}$, in the sense:

$$\lim_{\epsilon \to 0} \sup_l \sup_{x \in I_l} \operatorname{vol} I_l(B_\epsilon(x)) = 0.$$

Corollary 0.1. *Suppose there exists an asymptotically q.c. equivalence between $(\hat{M}, g)$ and $(\hat{M}', g')$ which satisfies uniform bound of infinitesimal volumes as above. Then the induced morphism*:

$$\{I_l^*\}_l : PAs\mathfrak{M}(E, \hat{M}, g) \cong PAs\mathfrak{M}(E', \hat{M}', g')$$

gives a homeomorphism.

These notions all depend on the Riemannian metric g on $\hat{M}$. However these properties hold under C^0 perturbations of g in the sense of the equality $d([g],[g']) = d([\varphi^*(g)], [\varphi^*(g')])$, where $\varphi : \hat{M} \to Gl(T\hat{M})$ is any bounded continuous section. The important case is when $\varphi = d\Psi$ for some diffeomorphism Ψ near id.

Let M be a smooth, closed and oriented four manifold, and take an open subset $R \subset M$ whose closure $\bar{R} = \text{cl}R \subset M$ is homeomorphic to the smooth disk. Thus $\check{M} = M \backslash \bar{R}$ and $\hat{M} = M \backslash$ pt are mutually homeomorphic. For example one can find some open subset $\check{M} \subset M = 3(S^2 \times S^2)$ which also admits a smooth embedding into K3 surface, where it corresponds to the hyperbolic matrix $3H$ in the decomposition of the intersection form $-2E_8 \oplus 3H$. It is known that $\check{M}$ is not diffeomorphic to $\hat{M}$ in this case.

Let us have an exhaustion $K_0 \subset\subset K_1 \subset\subset \cdots \subset \check{M}$ by compact subsets, and choose a family of Riemannian metrics g_l on each K_l, where g_l and g_k may be mutually different. We equip a small perturbation g of any cylindrical metric on $\hat{M}$. We will similarly introduce a notion, asymptotically quasiconformal homeomorphism $\{I_l : (K_l, g_l) \to (\hat{M}, g)\}_l$ (Section 8).

THEOREM 0.2. *There exists a pair (M, R) as above such that there are no family $\{(K_l, g_l)\}_l$ on $\check{M} \equiv M \backslash \bar{R}$ which admit asymptotically quasiconformally homeomorphisms to $(\hat{M}, g)$.*

Here we point out that asymptoticity of $*$ homomorphisms are elegantly introduced in the C^* algebras world in [**CH**].

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1. Function spaces.

1.A. Function spaces.

In this paper we always assume that M is simply connected. Let $(\hat{M}, g)$ be a complete Riemannian four manifold whose end is isometric to $(S^3 \times [0, \infty), g|S^3 \times dt^2)$, where $g|S^3$ is a fixed smooth metric on S^3. Let us take a positive number $\delta > 0$ and choose a smooth function $w : \hat{M} \to [0, \infty)$ with $w(m, t) = t\delta$ on the end. Sometimes we denote $w = w(\delta)$ to specify the constant. Recall that the weighted Sobolev space $L^p_w(\hat{M})$ is a Banach space with the norm $|u|^p = \int_{\hat{M}} \exp(w)|u|^p$. Similarly one defines weighted Sobolev k spaces as $(W^p_k)_w(\hat{M}) = \{u : \nabla^i(u) \in L^p_w(\hat{M}; \Lambda^*),\ i \leq k\}$.

Let us take a small $\epsilon \geq 0$ and a pair of positive constants $0 < \delta' \leq \delta$. We put $w' = w(\delta')$ and $w = w(\delta)$. Then we define the Banach spaces by:

$$B^0(\hat{M}) = \big\{u \in L^{4+2\epsilon}_{2w'}(\hat{M}),\ du \in L^{4+2\epsilon}_{2w} : |u|B^0 = |u|L^{4+2\epsilon}_{2w'} + |du|L^{4+2\epsilon}_{2w}\big\},$$

$$B^1(\hat{M}) = \big\{u \in L^{4+2\epsilon}_{2w}(\hat{M}, \Lambda^1),\ du \in L^{2+\epsilon}_{w} : |u|B^1 = |u|L^{4+2\epsilon}_{2w} + |du|L^{2+\epsilon}_{w}\big\},$$

$$B^2(\hat{M}) = L^{2+\epsilon}_{w}\big(\hat{M}; \Lambda^2_+\big).$$

1.B. AHS complex over $\boldsymbol{Y} \times \boldsymbol{R}$.

In this section we study the analytic aspects of the Atiyah-Hitchin-Singer complex, where $\delta' = \delta$ and $d^+ = (1/2)(1 + *)d$:

$$0 \longrightarrow B^0(\hat{M}) \xrightarrow{d} B^1(\hat{M}) \xrightarrow{d^+} B^2(\hat{M}) \longrightarrow 0.$$

We have a simple lemma:

LEMMA 1.1. *Let us fix $\delta > 0$. If the AHS complex above is Fredholm for $\delta' = \delta$, then the same is true for any $0 < \delta' < \delta$.*

This lemma follows from the next estimates:

$$|du|B^1(\hat{M}) = |du|L^{4+2\epsilon}_{2w}(\hat{M}) \geq C|u|L^{4+2\epsilon}_{2w}(\hat{M}) \geq C|u|L^{4+2\epsilon}_{2w'}(\hat{M}).$$

We will choose a smaller $\delta' < \delta$ in 3.D.

Let (Y, g) be a Riemannian 3-manifold and denote also by g as the product one on $Y \times \boldsymbol{R}$, where we equip with the standard metric on $\boldsymbol{R}$. Let $\Lambda^1(Y \times \boldsymbol{R})$ and $\Lambda^2_+(Y \times \boldsymbol{R})$ be the exterior differentials of 1 forms and self-dual 2 forms with respect to g. Then we have the natural identification:

$$\begin{aligned} &\Lambda^1(Y \times \boldsymbol{R}) = p^*(\Lambda^1(Y)) \oplus p^*(\Lambda^0(Y)), \\ &\Lambda^2_+(Y \times \boldsymbol{R}) = p^*(\Lambda^1(Y)) \end{aligned}$$

where $p : Y \times \boldsymbol{R} \to Y$ is the projection. The isomorphisms are given by:

$$u + vdt \leftrightarrow (u, v), \quad *_Y u + u \wedge dt \leftrightarrow u.$$

Let us put $X = Y \times \boldsymbol{R}$. Using L^2 adjoint operator, we get the elliptic operator $P = d^* \oplus d^+ : \Lambda^1(X) \to \Lambda^0(X) \oplus \Lambda^2_+(X)$. Passing through the identification, this is expressed as:

$$P = d^* \oplus d^+ : p^*(\Lambda^1(Y) \oplus \Lambda^0(Y)) \to p^*(\Lambda^1(Y) \oplus \Lambda^0(Y)).$$

Let us use t as the coordinate on $\boldsymbol{R}$. Then one has the following expression:

$$P = -\frac{d}{dt} + \begin{pmatrix} *_Y d & d \\ d^* & 0 \end{pmatrix} \equiv -\frac{d}{dt} + Q.$$

Q is an elliptic self-adjoint differential operator on $L^2(Y; \Lambda^1(Y) \oplus \Lambda^0(Y))$. Let us define an isometry (later we will restrict on $m = 2, 4$):

$$I : L^{m+\epsilon}(\hat{M}, \Lambda^*) \to L^{m+\epsilon}_w(\hat{M}, \Lambda^*)$$

by $I(u) = \exp(-w/(m+\epsilon))u$. Then a simple calculation shows the equality:

$$P(w) \equiv I^{-1}PI = P + \frac{1}{m+\epsilon}\frac{dw}{dt} : W^{m+\epsilon}_{k+1}(\hat{M};\Lambda^1) \to W^{m+\epsilon}_k\big(\hat{M};\Lambda^0 \oplus \Lambda^2_+\big)$$

$$= -\frac{d}{dt} + \begin{pmatrix} *_M d & d \\ d^* & 0 \end{pmatrix} + \frac{1}{m+\epsilon}\frac{dw}{dt} \equiv -\frac{d}{dt} + Q(-\delta) \text{ on the end}$$

where:

$$Q(\delta) = \begin{pmatrix} *_Y d - \dfrac{\delta}{m+\epsilon} & d \\ d^* & -\dfrac{\delta}{m+\epsilon} \end{pmatrix} : W^{m+\epsilon}_{k+1}(Y,\Lambda^1 \oplus \Lambda^0) \to W^{m+\epsilon}_k(Y,\Lambda^1 \oplus \Lambda^0).$$

There is a similar expression:

$$P^*(w) = \frac{d}{dt} + Q(\delta)$$

for $P^* = d \oplus (d^+)^*$, where $(d^+)^*$ is the formal adjoint operator with respect to the L^2 inner product. We will denote by $(d^+)^*_w$ the formal adjoint of d^+ with respect to L^2_w inner product.

1.C. Computation of cohomology groups.

This section is devoted to verify the following:

Proposition 1.1. *For a small choice of $\delta > 0$ and $0 \leq \epsilon \leq \epsilon_0$, the AHS complex is Fredholm with $H^0 = H^1 = 0$ and $\dim H^2 = b^2_+(M)$.*

Proof. Since the proof is long, we divide it into 3 subsections.

1.C.1. Case of H^0.

First let us consider $i = 0$. We show that $d : B^0(\hat{M}) \to B^1(\hat{M})$ has closed range. Recall $Q(\delta)$ is an elliptic operator on $L^{m+\epsilon}(S^3)$ defined using $g|S^3$. We show that:

$$P^* = d \oplus (d^+)^* : \big(W^{4+\epsilon}_1\big)_{2w}\big(\hat{M};\Lambda^0 \oplus \Lambda^2_+\big) \to L^{4+\epsilon}_{2w}(\hat{M};\Lambda^1)$$

has closed range with finite dimensional kernel.

Sublemma 1.1. *Let us choose a small $\delta > 0$ and any $0 \leq \epsilon \leq \epsilon_0$. We put $p = 4 + 2\epsilon$ or $2 + \epsilon$. Then:*

(1) $Q(\delta) : W_1^p(S^3) \to L^p(S^3)$ *satisfies the estimates* $|Q(\delta)(u)| \geq C|u|$ *for all* u *and for some* $C = C(\delta) > 0$.
(2) $Q(\delta) + d/dt : W_1^p(S^3 \times \boldsymbol{R}) \to L^p(S^3 \times \boldsymbol{R})$ *is invertible* ([**LM**]).

Proof of Sublemma 1.1. Let us take a family $u_i \in W_1^{4+2\epsilon}(S^3)$ with $|u_i| = 1$, and suppose $|Q(\delta)u_i|L^{4+2\epsilon}(S^3) \to 0$ as $i \to \infty$. From the three dimensional Sobolev embedding, the inclusion $W_1^{4+2\epsilon}(S^3) \hookrightarrow L^{4+2\epsilon}(S^3)$ is compact (see [**GT**, p. 167]), and a subsequence of $\{u_i\}_i$ converges to $u_\infty \in L^{4+2\epsilon}(S^3)$. On the other hand, by the elliptic estimate, there is a positive constant $C > 0$ with $|u_i|L^{4+2\epsilon}(S^3) > C$. In particular u_∞ is non-zero. Again by the elliptic estimate, u_∞ lies in $W_1^{4+2\epsilon}(S^3)$ and satisfies $Q(\delta)u_\infty = 0$. By elliptic regularity, u_∞ is smooth, but there are no such solution over S^3.

For $p = 2 + \epsilon$ case, the process is similar. This verifies Sublemma 1.1. □

Remark 1.1. In Proposition 3.2, we will verify a similar result as (2) above using different kinds of Banach spaces.

Let us consider (2) above. Let us put $\Delta = Q(\delta)^2$, $A = -d/dt + Q(\delta)$ and $A^* = d/dt + Q(\delta)$. Then we consider:

$$A^*A = AA^* = -\frac{d^2}{dt^2} + \Delta : W_2^{4+2\epsilon}(S^3 \times \boldsymbol{R}) \to L^{4+2\epsilon}(S^3 \times \boldsymbol{R}).$$

In order to verify (2), it is enough to see that the above operator is invertible, since then the estimates below hold:

$$|A^*(u)|L^{4+2\epsilon} = |A^*A(v)|L^{4+2\epsilon} \geq C|v|W_2^{4+2\epsilon} \geq C|u|W_1^{4+2\epsilon}$$

where $v = A^*(AA^*)^{-1}(u) \in W_2^{4+2\epsilon}$.

Let us return to the proof of Proposition 1.1. Let us choose a compactly supported cut off function $\varphi : \hat{M} \to [0,1]$ with Supp $d\varphi \subset [0,1] \times S^3 \subset \hat{M}$. Then one has the estimates:

$$\begin{aligned}
|u|\big(W_1^{4+\epsilon}\big)_{2w} &\leq |\varphi u|\big(W_1^{4+\epsilon}\big)_{2w} + |(1-\varphi)u|\big(W_1^{4+\epsilon}\big)_{2w} \\
&\leq |\varphi u|\big(W_1^{4+\epsilon}\big)_{2w} + C|P^*[(1-\varphi)u]|L_{2w}^{4+\epsilon} \\
&\leq C\big\{|\varphi u|\big(W_1^{4+\epsilon}\big)_{2w} + |(1-\varphi)P^*(u)|L_{2w}^{4+\epsilon} + |[1-\varphi, P^*]u|L_{2w}^{4+\epsilon}\big\}, \\
|\varphi u|\big(W_1^{4+\epsilon}\big)_{2w} &\leq C\big\{|\varphi P^*(u)|L_{2w}^{4+\epsilon} + |\varphi u|L_{2w}^{4+\epsilon} + |[\varphi, P^*]u|L_{2w}^{4+\epsilon}\big\}.
\end{aligned}$$

Notice that Supp φ, Supp $[1-\varphi, P^*]$ and Supp $[\varphi, P^*]$ are all compact.

Since the inclusion $W_1^{4+2\epsilon}(K) \hookrightarrow L^{4+2\epsilon}(K)$ is compact for all $\epsilon \geq 0$ and for any compact subset K, it follows that P^* has closed range with finite dimensional kernel. In particular $d : B^0 \to B^1$ has closed range, since $|df|B^1 = |df|L^{4+2\epsilon}_{2w}$. Thus one has verified the result $H^0 = 0$ for $\epsilon \geq 0$.

1.C.2. Case of H^1.

Next we show $H^1 = 0$.

Sublemma 1.2 ([**T1**]). *Suppose that f satisfies $df \in L^p_w$. Then there is a constant c uniquely determined by f satisfying the following estimates*:

$$\left[\int \exp\left(\frac{4w}{4-p}\right)|f-c|^{4p/(4-p)}\right]^{(4-p)/4} \leq C(p,\delta)\int \exp(w)|df|^p, \quad p \in (2,4),$$

$$\left(\int \exp(2w)|f-c|^4\right)^{1/2} \leq C\int \exp(w)|df|^2,$$

$$|f-c|C^0(S^3 \times \{t\}) \leq C(t)\exp\left(-\frac{\delta}{4+\epsilon}t\right),$$

$$p = 4+\epsilon > 4, \quad C(t) \to 0.$$

Let us return to the proof of Proposition 1.1. Let us take $u \in H^1$. Thus $d^+(u) = 0$ holds. One can check $d(u) \in L^2(\hat{M};\Lambda^2)$ by Hölder's inequality, since $\exp(-\mu w)$ is integrable on $\hat{M}$ for any positive $\mu > 0$. Then $d(u) = 0$ holds by integrating by parts. Let us consider the complex $\{(W^p_k)_{\mathrm{loc}}(\hat{M};\Lambda^*), d_*\}$ and their cohomology groups H^*. In general in order to get the isomorphism of H^* with the usual de Rham cohomology groups, it is enough to check the existence of:

(1) partition of unity in $(W^p_k)_{\mathrm{loc}}$ spaces and
(2) Poincaré lemma in $(W^p_k)_{\mathrm{loc}}(D;\Lambda^*)$ for small disks D (see [**W**]).

Thus there is $f \in (W_1^{4+2\epsilon})_{\mathrm{loc}}(\hat{M})$ with $df = u$. (Recall that M is simply connected by the assumption.) Then by Sublemma 1.2, u lies in the closure of $d((W_1^{4+2\epsilon})_{2w}(\hat{M})) \subset B^1(\hat{M})$. Since d has closed range, this implies that there is $f \in L^{4+2\epsilon}_{2w}$ with $df = u$. This verifies $H^1 = 0$.

1.C.3. Case of H^2.

Let us consider the case H^2. We take three steps. Firstly we verify that Cokernel of $d^+((W_1^{2+\epsilon})_w(\hat{M};\Lambda^1)) \subset L^{2+\epsilon}_w(\hat{M};\Lambda^2_+)$ is finite dimensional. Secondly we construct a bounded map $Q : L^{2+\epsilon}_w \to B^1$ with $d^+Q|\operatorname{im} d^+ = \mathrm{id}$. Finally we verify that Cokernel of $d^+(B^1) \subset L^{2+\epsilon}_w(\hat{M};\Lambda^2_+)$ has $b^2_+(M)$ dimension.

Sublemma 1.3. $d^+ : (W_1^{2+\epsilon})_w(\hat{M};\Lambda^1) \to L^{2+\epsilon}_w(\hat{M};\Lambda^2_+)$ *has closed range* *with finite dimensional cokernel, for all* $0 \leq \delta \leq \delta_0$ *and all* $0 \leq \epsilon \leq \epsilon_0$.

PROOF OF SUBLEMMA 1.3. Recall that self-dual 2 forms over $S^3 \times \boldsymbol{R}$ can be regarded as 1 forms over S^3. Let $(d^+)^*$ be the formal adjoint operator (without weight function) on $S^3 \times \boldsymbol{R}$. Then one has the equality:

$$\Delta = d^+ \circ (d^+)^* = -\frac{d^2}{dt^2} + \Delta_1^3 : W_2^{2+\epsilon}(S^3 \times \boldsymbol{R}; p^*\Lambda^1) \to L^{2+\epsilon}(S^3 \times \boldsymbol{R}; p^*\Lambda^1)$$

where Δ_1^3 is the Laplace operator on 1 forms over S^3. This operator has closed range with null kernel since Δ_1^3 is invertible over S^3 (see also Proposition 3.2).

It is also true for $\Delta_1^3 \pm \delta$ for all small δ. In particular a parallel argument as 1.C.1 shows that $\Delta_w \equiv d^+ \circ (d^+)^*_w : (W_2^{2+\epsilon})_w(\hat{M}; \Lambda_+^2) \to L_w^{2+\epsilon}(\hat{M}; \Lambda_+^2)$ also has closed range and finite dimensional kernel. (Here one uses compactness of the inclusion $W_2^{2+\epsilon} \hookrightarrow W_1^{2+\epsilon}$).

Let us see d^+ has finite dimensional cokernel. At $\epsilon = 0$, cokernel $d^+ = \ker(d^+)^*_w$ holds using Hilbert space structure, where $H = \mathrm{Ker}(d^+)^*_w \cap L_w^2$ has $b_+^2(M)$ dimension. Any element $u \in H$ lies in also $L_w^{2+\epsilon}$, since it satisfies exponential decay estimate (see Sublemma 3.1 or the inequality below).

Let us take $u \in \mathrm{Coker}\,\Delta_w \subset L_w^{2+\epsilon}$. We choose a family of cut off functions φ_i with exhausting supports. Then each $u_i = \varphi_i u$ lies also in L_w^2. Thus there are $v_i \in H$ and $\alpha_i \in \mathrm{Ker}\, d_w^* \cap (W_1^2)_w$ with $u_i = d^+(\alpha_i) + v_i \in L_w^2 \cap L_w^{2+\epsilon}$. Now we claim $\alpha_i \in (W_1^{2+\epsilon})_w$. For this it is enough to see $\alpha_i \in L_w^{2+\epsilon}$ by the elliptic estimate. Since Δ_w has closed range, this implies $u \in H$, and so $\dim \mathrm{Coker}\,\Delta_w \leq \dim H = b_+^2(M)$.

Let us verify the claim. The following argument will be called later as a *local to global method*. By normalizing one makes $|\alpha_i|(W_1^2)_w = 1$. Recall end $\hat{M}$ is isometric to $S^3 \times [0, \infty)$. By the Sobolev inequality, there is a constant C with:

$$C^{-1}|\alpha_i|L^{2+\epsilon}(S^3 \times [n, n+1]) \leq |\alpha_i|W_1^2(S^3 \times [n, n+1]) \leq \exp\left(-\frac{n\delta}{2}\right) \leq 1.$$

Then we have inequality:

$$\begin{aligned}
|\alpha_i|^{2+\epsilon} L_w^{2+\epsilon} &= \sum_n \exp(n\delta)|\alpha_i|^{2+\epsilon} L^{2+\epsilon}(S^3 \times [n, n+1]) \\
&\leq C \sum_n \exp(n\delta)|\alpha_i|^{2+\epsilon} W_1^2(S^3 \times [n, n+1]) \\
&\leq C \sum_n \exp(n\delta)|\alpha_i|^2 W_1^2(S^3 \times [n, n+1]) \\
&= C|\alpha_i|^2 \big(W_1^2\big)_w = C|\alpha_i|^{2+\epsilon} \big(W_1^2\big)_w.
\end{aligned}$$

Thus we get an inequality $|\alpha_i|L^{2+\epsilon}_w \leq C|\alpha_i|(W^2_1)_w$. Again by multiplying constants, one may remove the norm condition above. Thus this inequality holds in general. This verifies the claim. This completes the proof of Sublemma 1.3. □

SUBLEMMA 1.4. *Let* $Q' : L^2_w(\hat{M}) \cap \operatorname{im} d^+ \to (W^2_1)_w(\hat{M})$ *be the inverse to* d^+. *Then there is an extension* $Q : L^{2+\epsilon}_w \cap \operatorname{im} d^+ \to (W^{2+\epsilon}_1)_w$.

PROOF OF SUBLEMMA 1.4. Recall that $d^+((W^{2+\epsilon}_1)_w) \subset L^{2+\epsilon}_w$ has finite codimension for all sufficiently small $0 \leq \delta \leq \delta_0$ and small $0 \leq \epsilon$. Let us denote $H(\delta, \epsilon) = \operatorname{Coker} d^+ \subset L^{2+\epsilon}_w(\hat{M}; \Lambda^2_+)$. Any element $u \in H(0) = H(0,0)$ lies in $(W^{2+\epsilon}_k)_w$ for all k, since u satisfies the exponential decay estimate (see Sublemma 3.1) and the Sobolev estimate.

Firstly we claim the inclusions $H(0) \subset H(\delta, \epsilon)$ for $\delta, \epsilon > 0$ (more precisely the projection to the quotient space $L^{2+\epsilon}_w / d^+((W^{2+\epsilon}_1)_w)$ is injective). Suppose not. Then there is $u \in H(0)$ satisfying $u = d^+(\alpha)$, $\alpha \in (W^{2+\epsilon}_1)_w$. Let us take another $0 \leq \delta' < \delta$ and the corresponding weight function w'. We see $\alpha \in (W^2_1)_{w'}$ below. This shows $u = 0$ since $H(\delta', 0)$ is unchanged under small perturbations of $\delta' \geq 0$. Recall Hölder's inequality $\int ab \leq (\int a^p)^{1/p} (\int b^q)^{1/q}$ for $1/p + 1/q = 1$. Then we apply this for

$$ab = \exp(w')|\alpha|^2, \quad b^q = \exp(w)|\alpha|^{2+\epsilon}.$$

Then we solve $a = \exp(w' - q^{-1}w)|\alpha|^{2 - q^{-1}(2+\epsilon)}$. Letting $q = (2+\epsilon)/2$, one has $a = \exp(w' - q^{-1}w)$. With respect to ϵ, one chooses a sufficiently small $\delta' < \delta$ so that $\delta' - q^{-1}\delta < 0$. Then a^p is integrable. This shows $\exp(w')|\alpha|^2$ is also integrable. Thus we get the inclusion $H(0) \subset H(\delta, \epsilon)$ and verified the claim.

Next let us consider $\nabla = d^*_{w'} \oplus d^+ : (W^{2+\epsilon}_1)_{w'} \to L^{2+\epsilon}_{w'}$. We claim that for $\delta' > 0$, there is a constant $C > 0$ with bound:

$$|u|\big(W^{2+\epsilon}_1\big)_{w'} \leq C|\nabla(u)|L^{2+\epsilon}_{w'}.$$

$\Delta_{w'} = d^*_{w'} d : (W^{2+\epsilon}_2)_{w'} \to L^{2+\epsilon}_{w'}$ gives an isomorphism on functions, which can be checked by use of the parallel argument as 1.C.1. In particular $d^*_{w'} : (W^{2+\epsilon}_1)_{w'} \to L^{2+\epsilon}_{w'}$ has closed range. Since d^+ also has closed range, it is enough to check $\operatorname{Ker} \nabla = 0$. Let us take $\alpha \in \operatorname{Ker} \nabla$. On the end $\hat{M}$, $I^{-1} \nabla I : W^{2+\epsilon}_1 \to L^{2+\epsilon}$ has the following form where I is in 1.B:

$$-\frac{d}{dt} + \begin{pmatrix} *_M d + \dfrac{\delta'}{m+\epsilon} & d \\ d^* & -\delta' \dfrac{m+\epsilon-1}{m+\epsilon} \end{pmatrix}$$

where the second term is a self-adjoint elliptic operator over S^3. Moreover for small δ', there are no kernels (see [**K1**, Lemma 6.3]). Thus $I^{-1}(\alpha) \in W_1^{2+\epsilon}$ satisfies the exponential decay (use a similar argument as the proof of Sublemma 3.1). In particular $\alpha \in (W_1^{2+\epsilon})_w$ holds for some $0 < \delta' < \delta$.

Now as before the equality $d(\alpha) = 0$ holds. Let us put $m = 4/(2-\epsilon)$. Then by Sublemma 1.2, there exists $f \in L_{mw}^{m(2+\epsilon)}$ with $df = \alpha$. One may assume $f \in L_{w'}^2$. Since f satisfies $d_{w'}^* \circ df = 0$, this implies $df = \alpha = 0$. This shows the claim.

Now we verify the equality $H(0) = H(\delta, \epsilon)$. Suppose not. Then there is some element $u \in H(\delta, \epsilon)$ but not in $H(0)$. One may assume $u \in L_{w'}^2$ and write $u = d^+(\alpha)$, $\alpha \in (W_1^2)_{w'}$ and $d_{w'}^*(\alpha) = 0$. From the Sobolev embedding $(W_1^2)_{\mathrm{loc}} \hookrightarrow L_{\mathrm{loc}}^{2+\epsilon}$, $u_i = d^+(\varphi_i \alpha)$ and $\mu_i = d_{w'}^*(\varphi_i \alpha)$ converges to u and zero respectively in $L_{w'}^{2+\epsilon}$. From the above claim, one sees $\alpha \in (W_1^{2+\epsilon})_{w'}$. This shows that the image of $H(\delta, \epsilon) \to H(\delta', \epsilon)$ is exactly $H(0)$ (again more precisely the image of the canonical map $L_w^{2+\epsilon}/d^+((W_1^{2+\epsilon})_w) \to L_{w'}^{2+\epsilon}/d^+((W_1^{2+\epsilon})_{w'})$ coincides with $H(0) \subset L_{w'}^{2+\epsilon}/d^+((W_1^{2+\epsilon})_{w'})$).

We claim that for all sufficiently small pairs, the above maps $H(\delta, \epsilon) \to H(\delta', \epsilon)$ are all isomorphisms. For this it is enough to verify that the bounded maps:

$$\bar{d}^+ : \left(W_1^{2+\epsilon}\right)_w \to L_w^{2+\epsilon}/H(0)$$

are surjective for all small δ, where $\bar{d}^+$ is the composition of d^+ with the quotient maps. Passing through the isomorphisms $I(\delta) : W_k^{2+\epsilon} \cong (W_k^{2+\epsilon})_w$, one may assume that all the operators act on $W_k^{2+\epsilon}$. Let us denote $d^+(\delta) = I(\delta)^{-1} d^+ I(\delta) : W_1^{2+\epsilon} \to L^{2+\epsilon}$. Notice that $d^+(\delta)$ is a family of continuous operators with respect to δ. Then surjectivity of the above maps are equivalent to the one of the family of continuous maps:

$$\bar{d}^+(\delta) : W_1^{2+\epsilon} \to L^{2+\epsilon}/I(\delta)(H(0)).$$

By the condition, this is surjective at $\delta = 0$, and by the elliptic regularity, $I(\delta)(H(0)) \to H(0)$ in $L^{2+\epsilon}$ as $\delta \to 0$. Since surjectivity is an open condition, and $\bar{d}^+(\delta)$ are continuous family of bounded maps, we conclude that they are surjective for all small $\delta > 0$. This shows that for sufficiently small $\epsilon \in [0, \epsilon_0]$ and $0 \le \delta \le \delta_0$, $H(\epsilon, \delta) = H(0, 0)$.

It is known that $d^+((W_1^2)_w) \subset L_w^2$ has codimension $b_+^2(M)$ ([**T1**]). Thus the equalities, $\operatorname{codim}\{d^+((W_1^{2+\epsilon})_w) \subset L_w^{2+\epsilon}\} = \dim H(0) = b_+^2(M)$ hold.

Recall that $\Delta_w = d_w^* d : (W_2^{2+\epsilon})_w \to L_w^{2+\epsilon}$ gives an isomorphism. Let us take $\alpha \in \operatorname{Ker} d^+ \subset (W_1^{2+\epsilon})_w(\hat{M}; \Lambda^1)$. Then $d(d_w^* d)^{-1} d_w^*(\alpha) = \alpha$ since $|\nabla| \ge C > 0$. In particular $\operatorname{Ker} d^+$ is equal to $\operatorname{im} d$ and $H^1 = 0$. Then by the open mapping theorem, the following map gives an isomorphism between Banach spaces:

$$d\Delta_w^{-1} d_w^* \oplus d^+ : \left(W_1^{2+\epsilon}\right)_w \to \operatorname{Ker} d^+ \oplus \operatorname{im} d^+ \subset \left(W_1^{2+\epsilon}\right)_w \oplus L_w^{2+\epsilon}.$$

(Continuity and injectivity are clear. For surjectivity, let us take $(df, d^+(v)) \in \operatorname{Ker} d^+ \oplus \operatorname{im} d^+$. Then $df - d\Delta_w^{-1} d_w^*(v) + v$ maps to (df, d^+v)).

Let $H \subset L_w^{2+\epsilon}$ be the cokernel of d^+. Then since H is finite dimensional, the isomorphism $L_w^{2+\epsilon} \cong \operatorname{im} d^+ \oplus H$ holds between Banach spaces. Let us define a map Q by $Q(d^+(v)) = v - d\Delta_w^{-1} d_w^*(v)$ for $v \in (W_1^{2+\epsilon})_w$. Then this gives a bounded map:

$$Q : L_w^{2+\epsilon}\left(\hat{M}; \Lambda_+^2\right) \cap \operatorname{im} d^+ \to \operatorname{Ker} d_w^* \cap \left(W_1^{2+\epsilon}\right)_w (\hat{M}; \Lambda^1)$$

which satisfies $d^+Q \mid \operatorname{im} d^+ = \mathrm{id}$. This completes the proof of Sublemma 1.4. $\square$

Let us return to the proof of Proposition 1.1. Using the Sobolev inequality and the above Q, one has the estimates on $\operatorname{im} d^+$:

$$\begin{aligned}
|u|^{4+2\epsilon} L_w^{2+\epsilon} &\geq C |Q(u)|^{4+2\epsilon} \left(W_1^{2+\epsilon}\right)_w \\
&= \frac{C}{2} |Q(u)|^{4+2\epsilon} \left(W_1^{2+\epsilon}\right)_w + \frac{C}{2} |Q(u)|^{4+2\epsilon} \left(W_1^{2+\epsilon}\right)_w \\
&\geq C' |u|^{4+2\epsilon} L_w^{2+\epsilon} + C' \left(\sum_{i \in \boldsymbol{N}} \exp(i\delta) |Q(u)|^{2+\epsilon} W_1^{2+\epsilon}([i, i+1] \times S^3) \right)^2 \\
&\geq C' |u|^{4+2\epsilon} L_w^{2+\epsilon} + C' \left(\sum_{i \in \boldsymbol{N}} \exp(i\delta) |Q(u)|^{2+\epsilon} L^{4+2\epsilon}([i, i+1] \times S^3) \right)^2 \\
&\geq C' |Q(u)|^{4+2\epsilon} B^1.
\end{aligned}$$

This shows that one has a continuous extension:

$$Q : H^\perp \subset L_w^{2+\epsilon}\left(\hat{M}; \Lambda_+^2\right) \to B^1$$

satisfying d^+Q is the identity. In particular this verifies that $d^+ : B^1 \to L_w^{2+\epsilon}$ has closed range.

Now the estimate $\dim H^2 \leq b_+^2(M)$ holds, since $Q : L_w^{2+\epsilon} \to (W_1^{2+\epsilon})_w$ has an extension to $Q : L_w^{2+\epsilon} \to B^1$ with $Q|H = 0$ and $d^+Q|\operatorname{im} d^+$ is the identity. Suppose there is a vector $\alpha \in L_{2w}^{4+2\epsilon}$ with $d^+(\alpha) \in H$. Then as before α lies in $L_{w'}^2$. One may assume $\inf_f |\alpha - df| L_{w'}^2 = |\alpha| L_{w'}^2$. From this $d_{w'}^* \alpha = 0$ (differentiate

$|\alpha + tdf|^2 L^2_{w'})$. Then from the elliptic estimate, it follows $\alpha \in (W^2_1)_{w'}$. This shows $\alpha = 0$. This completes the proof of Proposition 1.1. □

REMARK 1.2. One gets a relation of the range of Q and an inequality:

$$\operatorname{im} Q \subset B^1(\hat{M}, \epsilon) \cap \big(W_1^{2+\epsilon}\big)_w(\hat{M}) \cap \operatorname{Ker} d^*_w,$$
$$|d^- Qu| L_w^{2+\epsilon}(\hat{M}) \leq C|d^+ Qu| L_w^{2+\epsilon}(\hat{M}).$$

2. Asymptotic functional spaces.

2.A. Asymptotic Banach spaces.

Let $(\hat{M}, g)$ be a cylindrical manifold as before, whose end is isometric to $(S^3 \times [0, \infty), g|S^3 + dt^2)$.

2.A.1. Thickened Banach spaces.

Let $K \subset \hat{M}$ be a compact subset with smooth boundary ∂K, and $\operatorname{int} K$ be its interior. Then let $(W^p_k)_w(K)_0$ be the Banach spaces where $C^\infty_{\mathrm{cp}}(\operatorname{int} K)$ are dense in them, with their norms:

$$|u|^p \big(W_k^p\big)_w(K)_0 = \sum_{i \leq k} \int_K \exp(w) |\nabla^i u|^p.$$

Notice the inclusion $(W^p_k)_w(K)_0 \subset (W^p_k)_w(\hat{M})$. One defines $B^i(K)_0$ using $L^p_w(K)_0$ similarly as in 1.A. These satisfy the inclusions $B^i(K)_0 \subset B^i(\hat{M})$.

Let us fix small constants $\epsilon > 0$, $\delta \geq \delta' > 0$ and the corresponding weight functions w and w'. Then we introduce the *thickened Banach spaces* using the norms:

$$|u| \hat{L}^4_{2w}(\hat{M}) \equiv \sup_{0 \leq \epsilon' \leq \epsilon} |u| L^{4+2\epsilon'}_{2w}(\hat{M}).$$

Similarly we define $\hat{L}^2_w = \sup_{0 \leq \epsilon' \leq \epsilon} |\ \ | L^{2+\epsilon'}_w(\hat{M})$, or $(\hat{W}^2_k)_w$. Then we put:

$$\bar{B}^0(\hat{M}, w, \epsilon) = \bar{B}^0(\hat{M}) = \big\{u \in \hat{L}^4_{2w'}(\hat{M}),\ du \in \hat{L}^4_{2w}(\hat{M})\big\},$$
$$\bar{B}^1(\hat{M}, w, \epsilon) = \big\{u \in \hat{L}^4_{2w}(\hat{M}),\ d(u) \in \hat{L}^2_w(\hat{M})\big\},$$
$$\bar{B}^2(\hat{M}, w, \epsilon) = \hat{L}^2_w\big(\hat{M}; \Lambda^2_+\big).$$

2.A.2. Asymptotic Banach spaces with one bubble.

Below let us introduce another types of Banach spaces which we call the *asymptotic functional spaces*. Our construction is general, and basically it also works by use of any functional spaces. Here we choose $\bar{B}^*$ as the underlying functional spaces for the construction of the asymptotic functional spaces.

Firstly we define the asymptotic Banach spaces $\hat{B}^*(\hat{M};1)$ with one bubble, and then generalize them to $\hat{B}^*(\hat{M};n)$ with n bubbles. Later when we construct moduli spaces, we will use $\hat{B}^*(\hat{M}) \equiv \hat{B}^*(\hat{M};n)$ for some $n > 0$ as the underlying functional spaces.

Let us take an exhaustion by compact subsets:

$$K_0 \subset K_1 \subset \cdots \subset \hat{M}.$$

Here one always assume that the exhaustion is of bounded geometry (boundaries are uniformly smooth). For each $l \geq 0$, let us take a cut off function $\varphi_l : \hat{M} \to [0,1]$ satisfying:

(1) $\varphi_l \mid \hat{M} \backslash S^3 \times [l, \infty) \equiv 1$,
(2) $\varphi_l \mid S^3 \times [l+1, \infty) \equiv 0$,
(3) $|\varphi_l| C^i$ are uniformly bounded with respect to l for any $i \geq 0$.

By sliding indices, if necessarily, one may assume $\partial K_l \subset S^3 \times [3l, \infty) \subset \operatorname{end} \hat{M}$.

Let $w_l : S^3 \times \boldsymbol{R} \to [0, \infty)$ be a weight function defined by $w_l(m,t) = \delta|l - t|$. We will use the same notation for the restriction $w_l : S^3 \times \boldsymbol{R}_+ = \operatorname{end} \hat{M} \to [0, \infty)$. Moreover let:

$$T_l : S^3 \times [-l, \infty)(\subset S^3 \times \boldsymbol{R}) \to S^3 \times [0, \infty) \subset (\operatorname{end} \hat{M})$$

be the linear translation defined by $T_l(m,k) = (m, k+l)$.

Let us consider the set $[u_l]$, $u_l \in \bar{B}^*(K_l)_0$ satisfying the followings:

(1) $\varphi_l u_l$ converges to some $v_1 \in \bar{B}^*(\hat{M}, w, \epsilon)$,
(2) $(1-\varphi_l) u_l \in \bar{B}^*(\hat{M}, w_{2l}, \epsilon)$ and $T_{2l}^*((1-\varphi_l)u_l)$ converges to some $v_2 \in \bar{B}^*(S^3 \times \boldsymbol{R}, w_0, \epsilon)$.

Then one defines linear spaces $\bar{B}^*(\{K_l\}_l)$ as follows:

$$\bar{B}^*(\{K_l\}_l) = \{[u_l] : \text{satisfying the above (1), (2)}\}.$$

Let us equip a norm on $\bar{B}^*(\{K_l\}_l)$ as:

$$
\begin{aligned}
|[u_l]|\bar{B}^*(\{K_l\}_l) &= \sup_l \|u_l\|_l \bar{B}^*(K_l)_0 \\
&\equiv \sup_l \big\{ |\varphi_l u_l| \bar{B}^*(\hat{M}, w, \epsilon) + |(1-\varphi_l)u_l| \bar{B}^*\big(\hat{M}, w_{2l}, \epsilon\big) \big\} \\
&= \sup_l \big\{ |\varphi_l u_l| \bar{B}^*(\hat{M}, w, \epsilon) + \big|T_{2l}^*(1-\varphi_l)u_l\big| \bar{B}^*(S^3 \times \boldsymbol{R}, w_0, \epsilon) \big\}.
\end{aligned}
$$

Sometimes we use the notation:

$$
\begin{aligned}
\|u_l - u_{l'}\| \bar{B}^* \equiv\ & |\varphi_l u_l - \varphi_{l'} u_{l'}| \bar{B}^*(\hat{M}, w, \epsilon) \\
& + \big|T_{2l}^*(1-\varphi_l)u_l - T_{2l'}^*(1-\varphi_{l'})u_{l'}\big| \bar{B}^*(S^3 \times \boldsymbol{R}, w_0, \epsilon).
\end{aligned}
$$

Then the above two conditions (1) and (2) are equivalent to the convergence $\lim_{l,l' \to \infty} \|u_l - u_{l'}\| \bar{B}^* = 0$.

Let us define the vector subspaces $\bar{B}_0^*(\{K_l\}_l) \subset \bar{B}^*(\{K_l\}_l)$:

$$
\bar{B}_0^*(\{K_l\}_l) = \big\{ [u_l] \in \bar{B}^*(\{K_l\}_l) : \|u_l\|_l \bar{B}^*(K_l)_0 \to 0 \text{ as } l \to \infty \big\}.
$$

Lemma 2.1. *With respect to the above metric,*

(1) $\bar{B}^*(\{K_l\}_l)$ *are Banach spaces,* (*but not reflexive*) *and*
(2) $\bar{B}_0^*(\{K_l\}_l)$ *are closed subspaces of* $\bar{B}^*(\{K_l\}_l)$ *respectively.*

Proof.

(1) We check completeness. For $\alpha = 0, 1, 2, \ldots$, suppose a family $[u_l^\alpha] \in \bar{B}^*(\{K_l\}_l)$ consists of a Cauchy sequence. Thus for every $\epsilon > 0$, there is a large α_0 so that for all $\alpha, \beta \geq \alpha_0$ and all $l = 0, 1, \ldots$, uniform bounds $\|u_l^\alpha - u_l^\beta\|_l \bar{B}^* < \epsilon$ hold.

For each l, let $\beta \to \infty$. Then we get some sequence $[u_l^\infty]$. We claim that this gives an element in $\bar{B}^*(\{K_l\}_l)$. Let us check convergence property. In fact for $l, l' \geq l_0$, one gets the estimates:

$$
\big\|u_l^\infty - u_{l'}^\infty\big\| \bar{B}^* \leq \big\|u_l^\infty - u_l^{\alpha_0}\big\| \bar{B}^* + \big\|u_l^{\alpha_0} - u_{l'}^{\alpha_0}\big\| \bar{B}^* + \big\|u_{l'}^{\alpha_0} - u_{l'}^\infty\big\| \bar{B}^* \leq 3\epsilon.
$$

This shows that $[u_l^\infty]$ satisfies the defining condition above.

(2) Suppose a family $[u_l^\alpha] \in \bar{B}_0^*(\{K_l\}_l)$ satisfies the convergence $\sup_l \|u_l^\alpha - u_l^\beta\|_l \bar{B}^*(K_l)_0 \to 0$ as $\alpha, \beta \to \infty$. Thus for any $\epsilon > 0$, there is α_0 so that for all $\alpha, \beta \geq \alpha_0$, the estimate $\sup_l \|u_l^\alpha - u_l^\beta\|_l \bar{B}^*(K_l)_0 \leq \epsilon$ holds. Letting $\beta \to \infty$, one has:

$$
\sup_l \big\|u_l^\alpha - u_l^\infty\big\|_l \bar{B}^*(K_l)_0 \leq \epsilon.
$$

This shows $[u_l^\infty] \in \bar{B}_0^*(\{K_l\}_l)$. This verifies Lemma 2.1. $\square$

Now we define the *asymptotic Banach spaces with one bubble* by:

$$\hat{B}^*(\{K_l\}_l) \equiv \bar{B}^*(\{K_l\}_l)/\bar{B}_0^*(\{K_l\}_l)$$

where we equip the quotient norms on $\hat{B}^*(\{K_l\}_l)$.

Let us study some basic properties of $\hat{B}^*(\{K_l\}_l)$. Let us introduce another norms on $\hat{B}^*(\{K_l\}_l)$:

$$|[u_l]|_1 = \limsup_{l\to\infty} \|u_l\|_l \bar{B}^*(K_l)_0.$$

We denote by $|\ \ |_0$ the quotient norms before:

$$|[u_l]|_0 = \inf\Big\{\sup_l \|u_l + v_l\|_l \bar{B}^*(K_l)_0 : [v_l] \in \bar{B}_0^*(\{K_l\}_l)\Big\}.$$

LEMMA 2.2. *$|\ \ |_0$ and $|\ \ |_1$ are isometric.*

PROOF. Let us take $[u_l] \in \hat{B}^*(\{K_l\}_l)$ with $|[u_l]|_1 = 1$. Then for any $\epsilon > 0$, there is l_0 with $1-\epsilon \le \sup_{l\ge l_0} \|u_l\|_l \bar{B}^*(K_l)_0 \le 1+\epsilon$. Let us put:

$$v_l = \begin{cases} -u_l & l < l_0 \\ 0 & l \ge l_0. \end{cases}$$

Then $[v_l] \in \bar{B}_0^*(\{K_l\}_l)$, and one has:

$$\sup_l \|u_l + v_l\|_l \bar{B}^*(K_l)_0 = \sup_{l\ge l_0} \|u_l\|_l \bar{B}^*(K_l)_0 \le 1+\epsilon.$$

This shows $|[u_l]|_0 \le 1$. Suppose $|[u_l]|_0 = 1-\delta$, $\delta > 0$. Then there is $[v_l] \in \bar{B}_0^*(\{K_l\}_l)$ with $\sup_l \|u_l + v_l\|_l \bar{B}^*(K_l)_0 \le 1-\delta/2$. Let us take $4\epsilon < \delta$ and a large l with:

$$\|v_l\|_l \bar{B}^*(K_l)_0 \le \frac{\delta}{4}, \quad \|u_l\|_l \bar{B}^*(K_l)_0 \ge 1-\epsilon.$$

Then one has the estimates:

$$\begin{aligned}\|u_l\|_l \bar{B}^*(K_l)_0 &\leq \|u_l + v_l\|_l \bar{B}^*(K_l)_0 + \|v_l\|_l \bar{B}^*(K_l)_0 \\ &\leq 1 - \frac{\delta}{2} + \frac{\delta}{4} = 1 - \frac{\delta}{4} < 1 - \epsilon.\end{aligned}$$

This is a contradiction. This verifies the proof of Lemma 2.2. □

Let us compare $\hat{B}^*(\{K_l\}_l, w, \epsilon)$ and $\bar{B}^*(\hat{M}, w, \epsilon)$. These are different Banach spaces, and roughly speaking the latter are the subspaces of the former without bubble. In fact one has the following:

PROPOSITION 2.1. *There exist natural isometric embeddings*:

$$i : \bar{B}^*(\hat{M}, w, \epsilon) \hookrightarrow \hat{B}^*(\{K_l\}_l, w, \epsilon)$$

with closed image, but i are not surjective.

PROOF. Let us take $u \in \bar{B}^*(\hat{M})$, and let φ_l be the family of cut off functions in 2.A. Define the linear map i by $i(u) = [\varphi_l u] \in \hat{B}^*(\{K_l\}_l, w, \epsilon)$. By Lemma 2.2, i is isometric, and so closedness follows (see Remark 3.1 in 3.D).

Now let us consider a family $u_l \in \bar{B}^*(K_l)_0$ defined as follows; recall that $\hat{M}$ contains the isometric subspace $(S^3 \times [0, \infty), g|S^3 \times dt^2)$. Let us take any $u \in C^\infty_{\mathrm{cp}}(S^3 \times (0,1))$, and $S_t : S^3 \times [t, t+1] \cong S^3 \times [0,1]$ be the isometry by the translation. Then one puts $u_l = S^*_{2l}(u) \in C^\infty_{\mathrm{cp}}(S^3 \times (2l, 2l+1))$. It is clear that $[u_l]$ defines a non-zero element in $\hat{B}^*(\{K_l\}_l, w, \epsilon)$, but is not contained in the image of i. This completes the proof of Proposition 2.1. □

It is easy to get the following:

COROLLARY 2.1. *There are isometries between Banach spaces*:

$$\hat{B}^*(\{K_l\}_l) \cong \bar{B}^*(\hat{M}, w, \epsilon) \oplus \bar{B}^*(S^3 \times \boldsymbol{R}, w_0, \epsilon).$$

2.A.3. Asymptotic Banach spaces with more bubbles.

Above $\hat{B}^*(\{K_l\}_l)$ are extended Banach spaces containing with one bubble. Let us extend them so that they contain n bubbles for $n \geq 1$.

Let us take families of indices $\{a(l,j)\}_{l,j}$ for $l = 0, 1, \dots$ and $j = 0, 1, \dots, n$ satisfying:

(1) $a(l,j) < a(l, j+1)$, and $a(l, j+1) - a(l,j) \to \infty$ as $l \to \infty$,
(2) $\hat{M} \backslash S^3 \times [a(l,n)+1, \infty) \subset K_l$, and $d(a(l,n), \partial K_l) \to \infty$ as $l \to \infty$.

For $j = 1, \dots, n$, let us put:

$$b(l,j)=\frac{1}{2}(a(l,j)+a(l,j-1)).$$

$a(l,j)$ and $b(l,j)$ correspond to the 'bottom' and the 'top' of the bubbles respectively.

Let us put $w(l,j)=w_{b(l,j)}$ where w_l are in 2.A.2. Also we use the linear translations $T(l,j):S^3\times\boldsymbol{R}\to S^3\times\boldsymbol{R}$ by $T(l,j)(m,0)=(m,b(l,j))$. All the above data are determined by the family $\{a(l,j)\}$. Let us put the set of the parameters as:

$$P=\{\boldsymbol{T}=\{T(l,j)\}:(1)\text{ above }\}$$

where we equip a topology by uniform convergence. When we use some $\boldsymbol{T}\in P$, we may change the indices of exhaustion $\{K_l\}_l$ by $\{K_{m(l)}\}_l$, $m(l)\gg l$ if necessarily so that (2) above is also satisfied (see Lemma 2.3 below).

Now we consider the set $[u_l]$, $u_l\in\bar{B}^*(K_l,w,\epsilon)_0$ satisfying the followings:

(1)′ $\varphi_{a(l,0)}u_l$ converges to some $v_1\in\bar{B}^*(\hat{M},w,\epsilon)$,
(2)′ $(\varphi_{a(l,j)}-\varphi_{a(l,j-1)})u_l\in\bar{B}^*(K_l,w(l,j),\epsilon)_0$ and $T(l,j)^*((\varphi_{a(l,j)}-\varphi_{a(l,j-1)})u_l)$ converge to some $v_j\in\bar{B}^*(S^3\times\boldsymbol{R},w_0,\epsilon)$ for $j=1,\dots,n$.

We equip the norms by:

$$\begin{aligned}|[u_l]|\bar{B}^*(\{K_l\}_l;n)&=\sup_l\|u_l\|_l\\&\equiv\sup_l\Big\{|\varphi_{a(l,0)}u_l|\bar{B}^*(K_l,w,\epsilon)_0\\&\qquad+\sum_{j=1}^n|(\varphi_{a(l,j)}-\varphi_{a(l,j-1)})u_l|\bar{B}^*(K_l,w(l,j),\epsilon)_0\Big\}.\end{aligned}$$

The set of $[u_l]$ as above form Banach spaces $\bar{B}^*(\{K_l\}_l;n)$. By the same way as in 2.A.2, one gets closed subspaces $\bar{B}^*_0(\{K_l\}_l;n)$. Later we will fix the indices $\{a(l,j)\}$ and the corresponding $\{T(l,j)\}$, and will not denote them specifically.

Definition 2.1. Let us fix $n\geq 0$. Then we put:

$$\hat{B}^*(\hat{M},w,\epsilon)=\hat{B}^*(\hat{M},w,\epsilon;n)=\bar{B}^*(\{K_l\}_l;n)/\bar{B}^*_0(\{K_l\}_l;n).$$

As before one gets the isometries:

$$\begin{aligned}\hat{B}^*(\hat{M},\epsilon) \cong \bar{B}^*(\hat{M},w,\epsilon) &\oplus \bar{B}^*(S^3 \times \boldsymbol{R}, w, \epsilon) \\ &\oplus \cdots (n \text{ times}) \cdots \oplus \bar{B}^*(S^3 \times \boldsymbol{R}, w, \epsilon).\end{aligned}$$

Notice the filtration of the isometric embeddings:

$$\bar{B}^*(\hat{M},w,\epsilon) = \hat{B}^*(\hat{M},w,\epsilon;0) \subset \hat{B}^*(\hat{M},w,\epsilon;1) \subset \cdots \subset \hat{B}^*(\hat{M},w,\epsilon;n).$$

Let us take a subsequence $k(0) < k(1) < \cdots$ with $l \ll k(l)$, and consider another exhaustion $K'_0 \subset K'_1 \subset \cdots$ where $K'_l = K_{k(l)}$. Then we have two kinds of the natural maps:

$$\begin{aligned}&i : \hat{B}^*(\{K_l\}) \to \hat{B}^*\big(\{K'_l\}\big), \quad [u_l] \in \bar{B}^*(K_l)_0 \to [u_l] \in \bar{B}^*\big(K'_l\big)_0, \\ &j : \hat{B}^*(\{K_l\}) \to \hat{B}^*\big(\{K'_l\}\big), \quad [u_l] \in \bar{B}^*(K_l)_0 \to [u_{k(l)}] \in \bar{B}^*\big(K'_l\big)_0.\end{aligned}$$

Lemma 2.3. *Both i and j give isometries.*

Proof. The following two properties are clear; i is an isometric injection and j is a surjection. Let us see j is isometric. By definition, the estimate $|[u_l]| \geq |j([u_l])|$ holds. Take $[u_l] \in \hat{B}^*(\{K_l\})$ and choose subindices $m(l)$ with $\lim_l \|u_{m(l)}\|_{m(l)} = |[u_l]|$. For any small $\epsilon > 0$, there exists l_0 so that for all $l \geq l_0$, $\|u_{m(l)} - u_{k(l)}\| < \epsilon$. Thus one gets an estimate:

$$\|u_{k(l)}\|_{k(l)} \geq \|u_{m(l)}\|_{m(l)} - \|u_{m(l)} - u_{k(l)}\| \geq \|u_{m(l)}\|_{m(l)} - \epsilon.$$

This verifies that j is also isometric.

Now let us check that i is a surjection. For simplicity of the notations, we only check it for $\hat{B}^*(\{K_l\}_l;1)$. The general case is similar. Let us take $[u_l] \in \hat{B}^*(\{K'_l\}_l;1)$ such that $\varphi_l u_l$ and $T^*_{2l}(1-\varphi_l)u_l$ converge to v_1 and v_2 respectively. It is enough to find some $[w_l] \in \hat{B}^*(\{K_l\}_l;1)$ so that $\varphi_l w_l$ and $T^*_{2l}((1-\varphi_l)w_l)$ both converge to v_1 and v_2 respectively.

Let $S_l : S^3 \times [0,\infty)(\subset \hat{M}) \cong S^3 \times [-l,\infty)(\subset S^3 \times \boldsymbol{R})$ be the family of translations, and we take families of cut off functions p_l on $\hat{M}$ and q_l on $S^3 \times \boldsymbol{R}$ satisfying $\operatorname{Supp} p_l \subset \operatorname{Supp} p_{l+1} \cdots \subset \hat{M}$ and $\operatorname{Supp} q_l \subset \operatorname{Supp} q_{l+1} \cdots \subset S^3 \times \boldsymbol{R}$. Moreover one may choose these so that $\operatorname{Supp} p_l \subset K_l$ and $S_l^{-1}(\operatorname{Supp} q_l) \subset K_l$. Then one gets $w_l = p_l v_1 + S^*_{2l}(q_l v_2) \in \bar{B}^*(K_l)_0$ and clearly $[w_l] \in \hat{B}^*(\{K_l\}_l)$ satisfies the desired condition. This completes the proof of Lemma 2.3. □

3. AHS complex.

3.A. Here we show that the AHS complex between $\hat{B}^*(\{K_l\}_l)$ is Fredholm. In order to verify it, we will take two steps. Firstly we consider the case of the asymptotic Banach spaces defined by use of more simple function spaces. Then we treat the case of $\hat{B}^*(\{K_l\}_l)$.

Recall the Banach spaces $B^*(\hat{M}, w, \epsilon)$ defined in 1.A, and φ_l, w_l and T_l in 2.A.2. Let us fix $\epsilon, \delta > 0$ and $n \geq 0$ where n is the number of the bubbles. Let us put:

$$B^*(\{K_l\}_l, \epsilon) = \left\{[u_l] : u_l \in B^*(K_l, \epsilon)_0, \text{ satisfying } (1)', (2)' \text{ below}\right\}$$

$(1)'$ $\varphi_l u_l$ converges to some $v_1 \in B^*(\hat{M}, w, \epsilon)$,
$(2)'$ $(1-\varphi_l)u_l \in B^*(\hat{M}, w_{2l}, \epsilon)$ and $T_{2l}^*((1-\varphi_l)u_l)$ converges to some $v_2 \in B^*(S^3 \times \boldsymbol{R}, w_0, \epsilon)$.

Let us define the norms $|[u_l]|B^*(\{K_l\}_l, \epsilon) = \sup_l \|u_l\|_l B^*$ by the same way as in 2.A.2, where $\|u_l\|_l B^* = |\varphi_l u_l|B^* + |T_{2l}^*(1-\varphi_l)u_l|B^*$. Then similarly we have their closed subspaces $B_0^*(\{K_l\}_l, \epsilon)$, and put:

$$\tilde{B}^*(\{K_l\}_l, \epsilon) \equiv B^*(\{K_l\}_l, \epsilon)/B_0^*(\{K_l\}_l, \epsilon).$$

3.B. $\tilde{B}^*$ case.

3.B.1. Firstly let us verify Fredholmness of the AHS complex for $\tilde{B}^*(\{K_l\}_l)$, and consider the bounded complex:

$$0 \longrightarrow \tilde{B}^0(\{K_l\}_l) \xrightarrow{d} \tilde{B}^1(\{K_l\}_l) \xrightarrow{d^+} \tilde{B}^2(\{K_l\}_l) \longrightarrow 0.$$

Proposition 3.1. *The above complex has closed range.*

Proof. We only consider the one bubble case $n = 1$. The general case is similar. Our proof is based on Proposition 1.1.

Let us consider the image $d(\tilde{B}^0) \subset \tilde{B}^1$, and suppose the convergence to some $[w_l] \in \tilde{B}^1$:

$$|[d(u_l^\alpha)] - [w_l]|_1 \to 0 \quad \text{as } \alpha \to \infty.$$

By Proposition 1.1, $d : (W_1^{4+2\epsilon})_w \to L_w^{4+2\epsilon}(\Lambda^1)$ has closed range with null kernel over both $\hat{M}$ and $S^3 \times \boldsymbol{R}$. So by use of cut off functions, one gets the estimates for all $l \geq l_0$, where l_0 are determined by $[u_l]$:

$$\|du_l\|_l B^1 \geq C\|u_l\|_l B^0. \quad *$$

By definition for small $\epsilon > 0$, there is α_0 so that $\lim_{l\to\infty} \|d(u_l^\alpha) - w_l\|_l B^1 < \epsilon$ hold for all $\alpha \geq \alpha_0$. Thus for each α, there is $l(\alpha)$ so that the set $\{du^\alpha_{l(\alpha)}\}_\alpha$ consists of Cauchy sequence in $\| \quad \|B^1$ topology, where as in 2.A.2 for different indices, we define

$$\|u_l - u_{l'}\|B^* \equiv |\varphi_l u_l - \varphi_{l'} u_{l'}|B^*(\hat{M}, w, \epsilon) \\ + \left|T^*_{2l}(1-\varphi_l)u_l - T^*_{2l'}(1-\varphi_{l'})u_{l'}\right|B^*(S^3 \times \boldsymbol{R}, w_0, \epsilon).$$

Moreover one may assume the estimates $*$ for all $u^\alpha_{l(\alpha)}$. Thus $[v_\alpha = u^\alpha_{l(\alpha)}]$ defines an element in $B^0(\{K_{l(\alpha)}\}_\alpha)$. By the condition, $[dv_\alpha - w_{l(\alpha)}] \in B^1_0(\{K_{l(\alpha)}\})$. The same argument as the proof of Lemma 2.3 verifies that the canonical maps $j : \tilde{B}^*(\{K_l\}_l) \cong \tilde{B}^*(\{K_{l(\alpha)}\}_\alpha)$ are all isomorphisms. This verifies that $d : \tilde{B}^0(\{K_l\}_l) \to \tilde{B}^1(\{K_l\}_l)$ has closed range.

Next let us consider $d^+(\tilde{B}^1) \subset \tilde{B}^2$. The following lemma completes the proof of Proposition 3.1:

Lemma 3.1. *There exist a decomposition $\tilde{B}^2 \cong \operatorname{im} d^+ \oplus H$ and a bounded map $Q : \tilde{B}^2 \to \tilde{B}^1$ with $d^+Q|\operatorname{im} d^+$ is the identity, where $H = H(0)$ is the finite dimensional space in Proposition* 1.1.

Proof of Lemma 3.1. Recall the decomposition $L^{2+\epsilon}_w(\hat{M}; \Lambda^2_+) = \operatorname{im} d^+ \oplus H$ in Sublemma 1.4, and the bounded map Q with $d^+ \circ Q = \mathrm{id}$ in the proof of Proposition 1.1:

$$Q : \operatorname{im} d^+ \cap L^{2+\epsilon}_w(\hat{M}; \Lambda^2_+) \to B^1(\hat{M}; \epsilon).$$

With respect to the above decomposition, we use the Banach norm $|w|_0 \equiv |w^1|_1 \oplus |w^2|_2$ on $w = w^1 \oplus w^2 \in L^{2+\epsilon}_w(M; \Lambda^2_+) = \operatorname{im} d^+ \oplus H$.

For any $u \in H$ with $|u|B^2 = 1$, $\lim_l |(1-\varphi_l)u|B^2 = 0$ holds since H is finite dimensional. For any element $u = [u_l] \in \tilde{B}^2$, one has the decompositions as $\varphi_l u_l = u^1_l \oplus u^2_l \in B^2(\hat{M}, \epsilon)$ where $u^1_l \in \operatorname{im} d^+ \subset L^{2+\epsilon}_w(\hat{M}; \Lambda^2_+)$ and $u^2_l \in H$. With respect to the family $\{Q(u^1_l)\}_l$, let us take some subindices $k(l) \geq l$ and put $\psi_l = \varphi_{k(l)}$ so that the estimates $|Q(u^1_l) - \psi_l Q(u^1_l)|B^1(\hat{M}, \epsilon) \to 0$ hold. In the Banach space $B^2(\{K_{k(l)}\}_l)$, both the sequences $[v^1_l = d^+(\psi_l Q(u^1_l))]$ and $[\psi_l u^2_l]$ satisfy the condition $(1)'$ above. So these defines elements in $\operatorname{im} d^+$, $\tilde{H} \subset \tilde{B}^2(\{K_{k(l)}\}_l)$ respectively.

Since $d^+ : B^1(S^3 \times \boldsymbol{R}, \epsilon) \to B^2(S^3 \times \boldsymbol{R}, \epsilon)$ is surjective, there is a bounded inverse $Q : B^2(S^3 \times \boldsymbol{R}, \epsilon) \to B^1(S^3 \times \boldsymbol{R}, \epsilon)$ such that $d^+ \circ Q$ is the identity. Then by choosing a subsequence $\{k(l)'\}_l$ as above, one may assume that the family $[w_l \equiv (1-\psi_l')(T_{2k(l)'}^{-1})^* Q(T_{2l}^*(1-\varphi_l)u_l)]$ gives an element in $\tilde{B}^1(\{K_{k(l)'}\}_l)$.

Thus one assigns $[u_l] \to [d^+\psi_l' Q(u_l^1) + d^+ w_l] \oplus [\psi_l u_l^2]$. Passing through the isometries $i : \tilde{B}^*(\{K_l\}_l) \cong \tilde{B}^*(\{K_{k(l)'}\}_l)$, one gets a map $\tilde{B}^2(\{K_l\}_l) \to \operatorname{im} d^+ \oplus H$ so that it is the identity on $\operatorname{im} d^+$. Since the norms are invariant under change of indices, this map gives the isomorphism.

Suppose $[u_l^2] \in B_0^2$. Then one puts $Q([u_l]) = [\psi_l' Q(u_l) + w_l] \in \tilde{B}^1$. This is the desired map. This completes the proof of Lemma 3.1. □

3.B.2. Computation of cohomology groups.

Next let us compute the cohomology groups. Let us take a subsequence $k(0) < k(1) < \cdots$ satisfying $l \ll k(l)$, the corresponding exhaustion $K_0' \subset K_1' \subset \cdots$ with $K_l' = K_{k(l)}$, and the Banach spaces $\tilde{B}^*(\{K_l\})$ and $\tilde{B}^*(\{K_l'\})$. Their cohomology groups are mutually isomorphic.

By Proposition 1.1, the cohomology groups of the AHS complex:

$$0 \longrightarrow B^0(\hat{M}, g, \epsilon, w) \xrightarrow{d} B^1(\hat{M}, g, \epsilon, w) \xrightarrow{d^+} B^2(\hat{M}, g, \epsilon, w) \longrightarrow 0$$

satisfy $H^0 = H^1 = 0$ and $\dim H^2 = b_2^+(M)$. Let us consider the corresponding complex:

$$0 \longrightarrow \tilde{B}^0(\{K_l\}_l) \xrightarrow{d} \tilde{B}^1(\{K_l\}_l) \xrightarrow{d^+} \tilde{B}^2(\{K_l\}_l) \longrightarrow 0.$$

We show that the same results hold in this case. Firstly let us take $u = [u_l] \neq 0 \in H^0$. By definition, there is a positive constant $C > 0$ so that both estimates $\|u_l\|_l B^0 \geq C$ and $\|d(u_l)\|_l B^1 \to 0$ are satisfied for all large $l \gg 0$. This is a contradiction since $d : (W_1^{4+2\epsilon})_{2w} \to L_{2w}^{4+2\epsilon}(\Lambda^1)$ has closed range with null kernel. This shows $H^0 = 0$.

Next let us choose $u = [u_l] \in H^1$. Then $\lim_{l\to\infty} \|d^+(u_l)\|_l B^2 = 0$ holds. Let us check that there are $f_l \in B^0(K_l)_0$ with $\lim_{l\to\infty} \|u_l - df_l\|_l B^1(\hat{M}) = 0$. Suppose $\varphi_l u_l$ and $T_{2l}^*(1-\varphi_l)u_l$ converge to v_1 and v_2 in $B^1(\hat{M}, w, \epsilon)$ and $B^1(S^3 \times \boldsymbol{R}, w, \epsilon)$ respectively. Then both v_i satisfy $d^+(v_i) = 0$. Thus there are $f \in B^0(\hat{M})$ and $g \in B^0(S^3 \times \boldsymbol{R})$ with $df = v_1$ and $dg = v_2$. Using the notations in the proof of Lemma 2.3, one gets $[f_l = p_l f + S_{2l}^*(q_l g)] \in B^0(\{K_l\}_l)$. Clearly $[u_l] = [df_l] \in B^1(\{K_l\}_l)$. This shows $H^1 = 0$.

$\dim H^2(\tilde{B}^*(\{K_l\}_l)) = b_+^2(M)$ holds by Lemma 3.1.

3.C. ϵ collapsed Banach spaces.

Here we define more general asymptotic Banach spaces $\hat{B}^*(\{K_l, \epsilon_l\}_l)$, and in the next section we study the AHS complex over $\hat{B}^*(\{K_l, \epsilon_l\}_l)$.

Let $K_0 \subset K_1 \subset \cdots \subset \hat{M}$ be an exhaustion, and choose a family of small and positive constants:

$$1 \gg \epsilon_0 \geq \epsilon_1 \geq \cdots > 0.$$

Then the Banach spaces $\hat{L}^p_w(K_l) = \hat{L}^p_w(K_l; \epsilon_l)$ $(\hat{L}^p_w(K_l)_0)$ are defined (2.A). The norms on $\hat{L}^4_{2w}$ are given by:

$$|u|\hat{L}^4_{2w}(K_l, \epsilon_l) = \sup_{0 \leq \epsilon \leq \epsilon_l} \big\{ |u| L^{4+2\epsilon}_{2w}(K_l) \big\}.$$

$\hat{L}^2_w$ are also defined similarly. Let us put:

$$\begin{aligned}
\bar{B}^0(K_l, \epsilon_l)_0 &= \big\{ u \in \hat{L}^4_{2w'}(K_l, \epsilon_l)_0, \ du \in \hat{L}^4_{2w}(K_l, \epsilon_l)_0 \big\}, \\
\bar{B}^1(K_l, \epsilon_l)_0 &= \big\{ u \in \hat{L}^4_{2w}(K_l, \epsilon_l)_0, \ d(u) \in \hat{L}^2_w(K_l, \epsilon_l)_0 \big\}, \\
\bar{B}^2(K_l, \epsilon_l)_0 &= \hat{L}^2_w\big(K_l, \epsilon_l; \Lambda^2_+\big)_0.
\end{aligned}$$

We recall the norms $\| \ \|_l$ introduced in 2.A. If one chooses a subfamily $\{\epsilon'_l = \epsilon_{k(l)}\}_l$ with $l \leq k(l)$, then clearly the estimates hold:

$$\|u\|_l \bar{B}^*\big(K_l, \epsilon'_l\big)_0 \leq \|u\|_l \bar{B}^*(K_l, \epsilon_l)_0.$$

In order to define asymptotic Banach spaces, we use the formulation used in 2.A. Let us define:

$$\bar{B}^*(\{K_l, \epsilon_l\}_l) = \big\{ [u_l] : u_l \in \bar{B}^*(K_l, \epsilon_l)_0, \quad (1), (2) \text{ below} \big\}$$

with the norm $|[u_l]| = \sup_l \|u_l\|_l \bar{B}^*(K_l, \epsilon_l)_0$. Let $\{a(i,j)\}$ and $\{T(i,j)\}$ be the families of indices and translations in 2.A. Here $[u_l \in \bar{B}^*(K_l)_0]$ is a sequence satisfying:

(1) $|\varphi_{a(l,0)} u_l - \varphi_{a(l',0)} u_{l'}|\bar{B}^*(K_{l'}, w, \epsilon_{l'})_0 \to 0$ as $l' \geq l \to \infty$,
(2) $u(l,j) \equiv (\varphi_{a(l,j)} - \varphi_{a(l,j-1)}) u_l \in B^*(K_l, w(l,j), \epsilon_l)_0$ and $|T(l,j)^* u(l,j) - T(l',j)^* u(l',j)|\bar{B}^*(S^3 \times \boldsymbol{R}, w_0, \epsilon_{l'}) \to 0$ as $l' \geq l \to \infty$.

Recall that one has the closed subspaces $\bar{B}^*_0(\{K_l, \epsilon_l\}_l) \subset \bar{B}^*(\{K_l, \epsilon_l\}_l)$, and the quotient Banach spaces:

$$\hat{B}^*(\{K_l, \epsilon_l\}_l) = \bar{B}^*(\{K_l, \epsilon_l\}_l) / \bar{B}^*_0(\{K_l, \epsilon_l\}_l)$$

in 2.A. If we take another exhaustion $\{K_l'\}_l$ with $K_l \subset K_l'$, then we have a natural embedding $\hat{B}^*(\{K_l, \epsilon_l\}_l) \subset \hat{B}^*(\{K_l', \epsilon_l\}_l)$. Then we take a direct limit and denote as:

$$\hat{B}^*(\hat{M}, g) = \lim \hat{B}^*(\{K_l, \epsilon_l\}_l).$$

When all ϵ_l are the same and equal to $\epsilon > 0$, then $\hat{B}^*(\hat{M}, g)$ are isomorphic to $\hat{B}^*(\{K_l, \epsilon\}_l)$ by Lemma 2.3.

3.D. AHS complex for $\bar{B}^*(\{K_l, \epsilon_l\}_l)$.

Recall that we have used a family of non-increasing constants $\{\epsilon_l\}_l$ in the definition of $\hat{B}^*(\hat{M}, g)$. Here we study analytic properties of the AHS complex $\{\hat{B}^*(\hat{M}, g), d\}$ including the case when the family of constants approaches to zero, $\epsilon_0 \gg \epsilon_1 \gg \cdots \epsilon_l \gg \cdots \to 0$. In particular we show that it is Fredholm and the cohomology groups are the expected ones.

Let us take pairs $(p, p') \in \{(2, 2+\epsilon), (4, 4+2\epsilon)\}$, and Q be a positive second order self-adjoint elliptic operator on L^2 over S^3. Let us introduce the Banach spaces:

$$\dot{L}^p = \mathrm{Max}\{L^p, L^{p'}\}.$$

Namely we have $|u|\dot{L}^p = \max\{|u|L^p, |u|L^{p'}\}$.

For $\dot{L}^4$ spaces, we restrict only on the case; $Q = \Delta_0 + \delta > 0 : \dot{W}_2^4(S^3) \to \dot{L}^4(S^3)$ where Δ_0 is the Laplace operator over functions.

Proposition 3.2.

$$\Delta = -\frac{d^2}{dt^2} + Q : \dot{W}_2^p(S^3 \times \boldsymbol{R}) \to \dot{L}^p(S^3 \times \boldsymbol{R})$$

gives an isomorphism with a uniform bound:

$$|\Delta(u)|\dot{L}^p(S^3 \times \boldsymbol{R}) \geq C|u|\dot{W}_2^p(S^3 \times \boldsymbol{R})$$

where C is independent of choice of small $\epsilon \geq 0$.

Sublemma 3.1.

$$\Delta : W_2^p(S^3 \times \boldsymbol{R}) \to L^p(S^3 \times \boldsymbol{R})$$

is an isomorphism.

Proof of Sublemma 3.1. For $p = 2$, the result is well known, using the exponential decay estimate for vectors with low spectrum.

Let us consider the case $p = 4$, and take $u \in W_2^4(S^3 \times \boldsymbol{R})$. Notice that u is a continuous section by the Sobolev embedding. So we denote the restriction by $u_t \in W_2^4(S^3)$ a.e. on $S^3 \times \{t\}$. Suppose $|\Delta(u)|L^4$ is small. We put $f(t) = \int_{S^3} |u_t|^4 \operatorname{vol}$. Then we have the estimates:

$$\begin{aligned} f(t)' &= 4 \int \langle u', u \rangle |u|^2, \\ f'' &= \int \langle u'', u \rangle |u|^2 + \int |u'|^2 |u|^2 + 2 \int |\langle u', u \rangle|^2 \\ &\geq \int \langle Qu, u \rangle |u|^2 - \mu f \end{aligned}$$

where $\delta > \mu \geq 0$. Here we use the condition $Q = \Delta_0 + \delta$. We have the following:

$$\begin{aligned} \langle \Delta_0 u_t, u_t \rangle |u_t|^2 \operatorname{vol} &= - * d * du_t u_t^3 \operatorname{vol} = -(d * du_t) u_t^3, \\ \int \langle \Delta_0 u_t, u_t \rangle u_t^2 &= \int 3 u_t^2 du_t \wedge * du_t = 3 \int |du_t|^2 u_t^2 \geq 0. \end{aligned}$$

Thus we have an estimate $f'' \geq (\delta - \mu) f$. The rest follows from the standard method, and one can verify the exponential decay for f and Fredholmness for Δ. This completes the proof of Sublemma 3.1. □

Let us return to the proof of Proposition 3.2. Firstly we show that there exists a constant C independent of ϵ with a bound:

$$|\Delta(u)| \dot{L}^p(S^3 \times \boldsymbol{R}) \geq C |u| \dot{L}^p(S^3 \times \boldsymbol{R}).$$

Let us take u with $|u|\dot{L}^p(S^3 \times \boldsymbol{R}) = 1$. When $|u|\dot{L}^p(S^3 \times \boldsymbol{R}) = |u|L^p(S^3 \times \boldsymbol{R})$, then the result follows from the above sublemma.

Suppose $|u|\dot{L}^2(S^3 \times \boldsymbol{R}) = |u|L^{2+\epsilon}(S^3 \times \boldsymbol{R})$. We decompose:

$$|u|^{2+\epsilon} L^{2+\epsilon}(S^3 \times \boldsymbol{R}) = \sum_n |u|^{2+\epsilon} L^{2+\epsilon}(S^3 \times [n, n+1]).$$

Then by the Sobolev inequality, there is a constant C independent of ϵ and n with a bound:

$$|u|L^{2+\epsilon}(S^3 \times [n, n+1]) \leq C|u|W_1^2(S^3 \times [n, n+1]).$$

Thus one gets an estimate:

$$\sum_n |u|^2 L^{2+\epsilon}(S^3 \times [n, n+1]) \leq C \sum_n |u|^2 W_1^2(S^3 \times [n, n+1]) = C|u|^2 W_1^2(S^3 \times \boldsymbol{R}).$$

Here we claim the inequality:

$$\sum_n |u|^{2+\epsilon} L^{2+\epsilon}(S^3 \times [n, n+1]) \leq \left[\sum_n |u|^2 L^{2+\epsilon}(S^3 \times [n, n+1]) \right]^{(2+\epsilon)/2}.$$

From this, we get the estimate for $p = 2$:

$$|u|L^{2+\epsilon}(S^3 \times \boldsymbol{R}) \leq C|u|W_1^2(S^3 \times \boldsymbol{R}).$$

In fact let us put $b_n = |u|^2 L^{2+\epsilon}(S^3 \times [n, n+1]) \leq 1$ and $(2+\epsilon)/2 = 1 + \delta$. Then we show the inequality $\sum_n b_n^{1+\delta} \leq (\sum_n b_n)^{1+\delta}$. By normalizing one may assume $\sum_n b_n = 1$. Since $b_n \leq 1$, clearly we have $\sum_n b_n^{1+\delta} \leq \sum_n b_n$. Thus we have the estimate:

$$\sum_n b_n^{1+\delta} \leq \sum_n b_n = \left(\sum_n b_n \right)^{1+\delta}.$$

Now we have the desired inequalities:

$$C|\Delta(u)|L^2 \geq |u|W_2^2 \geq |u|W_1^2 \geq C'|u|L^{2+\epsilon}.$$

This implies the estimate $|\Delta(u)|\dot{L}^2 \geq C|u|\dot{L}^2$.

For $p' = 4$, the method is parallel. Here we note the Sobolev embedding $W_1^4 \hookrightarrow L^{4+2\epsilon}$.

Now we want to obtain the inequality $|\Delta(u)|\dot{L}^p \geq C|u|\dot{W}_2^p$. By the elliptic estimate, we have:

$$|u|W_2^{4+2\epsilon} \leq C\big\{|u|^2 W_1^{4+2\epsilon} + |\Delta(u)|^2 L^{4+2\epsilon}\big\}^{1/2}.$$

Suppose $|u_i|\dot{W}_2^p = 1$ and $|\Delta(u_i)|\dot{L}^p \to 0$. If $|u_i|\dot{W}_2^p = |u_i|W_2^p$, then the assumption $|\Delta(u_i)|L^p \to 0$ contradicts to Sublemma 3.1. So let $|u_i|\dot{W}_2^p = |u_i|W_2^{4+2\epsilon}$. Combining with $|\Delta(u_i)|L^{4+2\epsilon} \to 0$ and the above elliptic estimate, we have

$1-\delta \leq |u_i|W_1^{4+2\epsilon} \leq 1$. Using the Sobolev embeddings:

$$W_2^2 \hookrightarrow W_1^{2+\epsilon}, \quad W_2^4 \hookrightarrow W_1^{4+2\epsilon}$$

and a similar argument as above, we get the estimates $1-\delta \leq |u_i|W_1^{4+2\epsilon} \leq C|u_i|W_2^4$. Combining with the estimate $|u_i|W_2^4 \leq C|\Delta(u_i)|L^4$, this is a contradiction. Thus we get the desired estimate:

$$|\Delta(u_i)|\dot{L}^p \geq C|u_i|\dot{W}_2^p.$$

This completes the proof of Proposition 3.2.

Corollary 3.1. *There exists a bounded* $Q : \hat{L}_w^2(S^3 \times \boldsymbol{R}) \to (\hat{W}_1^2)_w \cap \operatorname{Ker} d_w^*$ *with* $d^+Q = \mathrm{id}$.

Proof. We proceed as in the proof of Proposition 1.1. $\Delta : (\hat{W}_2^2)_w \to \hat{L}_w^2$ satisfies the estimates $|\Delta(f)| \geq C|f|$ for all functions f. Thus $(\hat{W}_1^2)_w$ decomposes as:

$$\left(\hat{W}_1^2\right)_w \cong \operatorname{im} d \oplus \operatorname{Ker} d_w^*$$

by $u \to d\Delta^{-1}d_w^*u \oplus u - d\Delta^{-1}d_w^*u$.

Let us put $\nabla = d^+ \oplus d_w^* : (\hat{W}_1^2)_w \to \hat{L}_w^2$. Then there is a constant $C > 0$ so that the estimates $|\nabla(u)| \geq C|u|$ hold for all u. In particular the bounded map $d^+ : \operatorname{Ker} d_w^* \to \hat{L}_w^2$ satisfies the estimates $|d^+(u)| \geq C|u|$. Now we define the bounded map Q by $Qd^+(v) = v - d\Delta^{-1}d_w^*v$.

This completes the proof of Corollary 3.1. □

From the estimate at the end of Section 1, one gets the following map:

$$Q : \bar{B}^2(S^3 \times \boldsymbol{R}) \to \bar{B}^1(S^3 \times \boldsymbol{R}) \cap \operatorname{Ker} d_w^*$$

with $d^+Q = \mathrm{id}$.

Remark 3.1. Recall that since Δ over $\hat{M}$ can be written as $d_w^* \circ d$, it follows from the above that $d : (\hat{W}_2^4)_{2w} \to (\hat{W}_1^4)_{2w}$ has closed range.

Let us consider $d : (\hat{W}_1^4)_{2w} \to \hat{L}_{2w}^4$. This has also closed range as follows. From Sublemma 3.1, $d : (W_1^2)_{2w} \to L_{2w}^2$ has closed range. Then we have an estimate:

$$|u|^4 L^4_{2w} = \int \exp(2w) u^4 \le 4C \int \exp(2w) u^2 |du|^2 \le 4C|u|^2 L^4_{2w} |du|^2 L^4_{2w}.$$

Thus we get a bound $|u|L^4_{2w} \le C|du|L^4_{2w}$. Then following the proof of Proposition 3.2, we get the desired estimate $|u|\hat{L}^4_{2w} \le C|du|\hat{L}^4_{2w}$.

Lemma 3.2. *Let us take δ and ϵ_0. Then for $0 < \delta' = \delta'(\delta) < \delta$, $0 \le \epsilon_l \le \epsilon_0$, the AHS complex:*

$$0 \longrightarrow \hat{B}^0(\hat{M}) \xrightarrow{d} \hat{B}^1(\hat{M}) \xrightarrow{d^+} \hat{B}^2(\hat{M}) \longrightarrow 0$$

has closed range, where $\hat{B}^*(\hat{M}) = \hat{B}^*(\{K_l, \epsilon_l\}_l, w('))$.

Proof. Recall that for $0 \le \epsilon \le \epsilon_0$, one gets an isomorphism:

$$d : B^0(\hat{M}) \cong \operatorname{im} d \subset B^1(\hat{M}).$$

Let $| \ |^\epsilon_i$, $i = 0, 1$, be the right and left hand sides norms respectively. Then one has equivalences $C^{-1}_\epsilon | \ |^\epsilon_0 \le | \ |^\epsilon_1 \le C_\epsilon | \ |^\epsilon_0$.

Sublemma 3.2. *For $d : B^0(\hat{M};\epsilon) \to B^1(\hat{M};\epsilon)$, there is a uniform bound of the family of constants $C^{-1} \le C_\epsilon \le C$.*

Proof of Sublemma 3.2. We show uniform bounds $|f|B^0(\hat{M};\epsilon) \le C|df|B^1(\hat{M};\epsilon)$ for all $0 \le \epsilon \le \epsilon_0$, which is the same as the uniform estimates $|f|L^{4+2\epsilon}_{2w'}(\hat{M}) \le C|df|L^{4+2\epsilon}_{2w}(\hat{M})$. Let us choose $\delta' < \delta$ which will be determined later, and take the weight function w' with weight δ'. Then there is a constant C independent of ϵ with a bound:

$$| \ |L^4_{2w'}(\hat{M}) \le C| \ |L^{4+2\epsilon}_{2w}(\hat{M}).$$

This follows from the local Hölder estimates $L^4_{\mathrm{loc}} \hookrightarrow L^{4+2\epsilon}_{\mathrm{loc}}$ and the local to global method in the proof of Sublemma 1.3 (see below, or proof of Proposition 4.1). We show another estimate $|u|L^4_{2w'} \le C|du|L^4_{2w}$. Using Hilbert space structure, one easily gets the estimates $|u|(W^2_1)_w \le C|du|L^2_w$ (see [**K1**, 6.A]). Then let us verify the following estimates:

$$|u|L^4_{2w'} \le C|u|(W^2_1)_{w'} \le C|du|L^2_{w'} \le |du|L^4_{2w}.$$

Notice the local Sobolev embedding $(W^2_1)_{\mathrm{loc}} \hookrightarrow L^4_{\mathrm{loc}}$. Then the first and the last inequalities follow from the next estimates respectively:

$$\sum_n a_n^4 \exp(2n\delta') \leq \Big(\sum_n a_n^2 \exp(n\delta')\Big)^2,$$

$$\begin{aligned}\sum_n a_n^2 \exp(n\delta') &= \sum_n \big[a_n^2 \exp(n\delta)\big][\exp(n(\delta' - \delta))] \\ &\leq C\Big(\sum_n a_n^4 \exp(2n\delta)\Big)^{1/2}.\end{aligned}$$

Finally we show a uniform bound $|f|L^{4+2\epsilon}_{2w} \leq C|f|(W^4_1)_{2w}$. Notice the local Sobolev embedding $|f|L^{4+2\epsilon}_{\mathrm{loc}} \leq C|f|(W^4_1)_{\mathrm{loc}}$. Let us follow the local to global method. By normalizing one may assume $|f|(W^4_1)_{2w} = C^{-1} < 1$. In particular for each n, it follows $a_n \equiv |f|L^{4+2\epsilon}(S^3 \times [n, n+1]) \leq Cb_n \equiv |f|W^4_1(S^3 \times [n, n+1]) \leq 1$. Let us put $1 - \mu = 4/(4 + 2\epsilon)$. Then one gets the estimate:

$$\begin{aligned}|f|L^{4+2\epsilon}_{2w} &= \Big(\sum_n a_n^{4+2\epsilon} \exp(2n\delta)\Big)^{(4+2\epsilon)^{-1}} \\ &\leq \Big(\sum_n a_n^4 \exp(2n\delta)\Big)^{(4+2\epsilon)^{-1}} \\ &\leq C\Big(\sum_n b_n^4 \exp(2n\delta)\Big)^{(4+2\epsilon)^{-1}} = C|f|^{1-\mu}\big(W^4_1\big)_{2w} \\ &= C|f|\big(W^4_1\big)_{2w}|f|^{-\mu}\big(W^4_1\big)_{2w} \leq C'|f|\big(W^4_1\big)_{2w}.\end{aligned}$$

This verifies the claim.

Now suppose a family f_i and $\epsilon_i \to 0$, satisfies $|f_i|L^{4+2\epsilon_i}_{2w'} = 1$ and $|df_i|L^{4+2\epsilon_i}_{2w} \to 0$. Then one gets the estimate:

$$\begin{aligned}|f_i|L^{4+2\epsilon}_{2w'} &\leq C|f_i|\big(W^4_1\big)_{2w'} \\ &\leq C|f_i|L^4_{2w'} + C\delta_i \\ &\leq C|df_i|L^4_{2w''} + C\delta_i \leq C|df_i|L^{4+2\epsilon_i}_{2w} + C\delta_i \to 0.\end{aligned}$$

This is a contradiction. This completes the proof of Sublemma 3.2. □

Let us return to the proof of Lemma 3.2. Suppose there exists a family $\{[u^\alpha_l]\}_\alpha \subset \hat{B}^0(\{K_l, \epsilon_l\}_l)$ with $|[u^\alpha_l]|\hat{B}^0 = 1$ and $\lim_\alpha |[du^\alpha_l]|\hat{B}^1 \to 0$. Let us take a decreasing family δ_α and assume $\limsup_l \|du^\alpha_l\|_l\bar{B}^1 < \delta_\alpha$, while $\limsup_l \|u^\alpha_l\|_l\bar{B}^1 = 1$. In particular for some arbitrarily large $l = l(\alpha)$, one has the estimates:

$$\|du_l^\alpha\|_l \bar{B}^1 \leq \delta_\alpha, \quad \|u_l^\alpha\|_l \bar{B}^0 \geq 1-\delta_\alpha.$$

But then one must have the inequality by Sublemma 3.2:

$$1-\delta_\alpha \leq \|u_l^\alpha\|_l \bar{B}^0(K_l,\epsilon_l)_0 \leq C\|du_l^\alpha\|_l \bar{B}^1(K_l,\epsilon_l)_0 \leq C\delta_\alpha.$$

This is a contradiction. Thus $d : \hat{B}^0(\{K_l,\epsilon_l\}_l) \to \hat{B}^1(\{K_l,\epsilon_l\}_l)$ has closed range.

We have another method to verify that d has closed range, which also works for d^+.

Let us take any subindices $\{k(l)\}_l$ and consider the restriction map:

$$j : \hat{B}^*(\{K_l,\epsilon_l\}_l) \to \hat{B}^*\big(\{K_{k(l)},\epsilon_{k(l)}\}_l\big),$$
$$j_0 : \bar{B}^*(\{K_l,\epsilon_l\}_l) \to \bar{B}^*\big(\{K_{k(l)},\epsilon_{k(l)}\}_l\big).$$

We claim that j gives an isometric isomorphism between Banach spaces. Since j_0 is surjective, j is also the same. Let us take $[u_l] \in \hat{B}^*(\{K_l,\epsilon_l\}_l)$. We choose subindices $\{m(l)\}$ and $\{k(l)'\} \subset \{k(l)\}$ so that $\lim_l \|u_{m(l)}\|_{m(l)} = |[u_l]|\hat{B}^*(\{K_l,\epsilon_l\}_l)$ and $\lim_l \|u_{k(l)'}\|_{k(l)'} = |j([u_l])|\hat{B}^*(\{K_{k(l)},\epsilon_{k(l)}\}_l)$. By the definition, for any small $\mu > 0$, there is a large l_0 such that for all $l' \geq l \geq l_0$, $\|u_{l'} - u_l\|\bar{B}^*(K_{l'},\epsilon_{l'}) \leq \mu$. Thus one gets $\|u_{k(l)'} - u_{m(l')}\|\bar{B}^*(K_{m(l')},\epsilon_{m(l')}) \leq \mu$. Then we have estimates:

$$\begin{aligned}
&-\|u_{k(l)'} - u_{m(l')}\|\bar{B}^*\big(K_{m(l')},\epsilon_{m(l')}\big) + \|u_{m(l')}\|\bar{B}^*\big(K_{m(l')},\epsilon_{m(l')}\big)\\
&\leq \|u_{k(l)'}\|\bar{B}^*\big(K_{m(l')},\epsilon_{m(l')}\big)\\
&\leq \|u_{k(l)'} - u_{m(l')}\|\bar{B}^*\big(K_{m(l')},\epsilon_{m(l')}\big) + \|u_{m(l')}\|\bar{B}^*\big(K_{m(l')},\epsilon_{m(l')}\big).
\end{aligned}$$

Thus we get the estimate:

$$\begin{aligned}
|[u_l]|\hat{B}^*(\{K_l,\epsilon_l\}_l) &= \lim_l \|u_{m(l')}\|\bar{B}^*\big(K_{m(l')},\epsilon_{m(l')}\big)\\
&\leq \lim_l \|u_{k(l)'}\|\bar{B}^*\big(K_{k(l)'},\epsilon_{k(l)'}\big) = |j([u_l])|\hat{B}^*\big(\{K_{k(l)},\epsilon_{k(l)}\}_l\big).
\end{aligned}$$

Since j is distance decreasing, this verifies the claim.

Let us take a compactly supported cut off function $\varphi_0 : \hat{M} \to [0,1]$ with $\varphi_0|K_0 \equiv 1$. One may assume $(\hat{M}\backslash K_0, g)$ is isometric to $(S^3 \times [0,\infty), g|S^3 + dt^2)$. Let us consider $d : \hat{B}^0(\{K_l,\epsilon_l\}_l) \to \hat{B}^1(\{K_l,\epsilon_l\}_l)$. By Sublemma 3.2, one has a uniform bound:

$$C\|d(1-\varphi_0)u_l\|\bar{B}^1(K_l;\epsilon_l)_0 \geq \|(1-\varphi_0)u_l\|\bar{B}^0(K_l;\epsilon_l)_0.$$

Let us take a family $[u_l^\alpha] \in \bar{B}^0$ with $|[u_l^\alpha]|\bar{B}^0 = 1$, $|[du_l^\alpha]|\bar{B}^1(\{K_l,\epsilon_l\}_l) \to 0$ as $\alpha \to \infty$. We show that:

$$d : \bar{B}^0(\{K_l,\epsilon_l\}_l) \to \bar{B}^1(\{K_l,\epsilon_l\}_l) \qquad *$$

has closed range asymptotically in the following sense; $\{[\varphi_0 u_{k(l,\alpha)}^{a(\alpha)}]\}_\alpha$ converges in $L^{4+2\epsilon_0}(K_0)_0$ for some subindices $k(l,\alpha)$ and $a(\alpha)$. Here the estimates $\cdots k(l,m) \geq \cdots \geq k(l,1) \geq l$ hold for all $l \in \boldsymbol{N}$. Notice that $\{[du_{k(l,\alpha)}^{a(\alpha)}]\}_\alpha$ also converges to 0 in $\bar{B}^1$.

We have estimates:

$$\begin{aligned}
\big|[u_l^\alpha]\big|\bar{B}^0 &\leq \big|[\varphi_0 u_l^\alpha]\big|\bar{B}^0 + \big|[(1-\varphi_0)u_l^\alpha]\big|\bar{B}^0 \\
&\leq \big|[\varphi_0 u_l^\alpha]\big|\bar{B}^0 + C\big|[d(1-\varphi_0)u_l^\alpha]\big|\bar{B}^1 \\
&\leq \big|[\varphi_0 u_l^\alpha]\big|\bar{B}^0 + C\big|[du_l^\alpha]\big|\bar{B}^1 + C\big|[d\varphi_0 \wedge u_l^\alpha]\big|\bar{B}^1 \\
&\leq \big|[\varphi_0 u_l^\alpha]\big|\bar{B}^0 + C\big|[du_l^\alpha]\big|\bar{B}^1 + C\big|[u_l^\alpha]\big|\bar{B}^1(\{K_l \cap \text{ Supp } d\varphi_0\}_l) \\
&\leq C\big|[\varphi_0' u_l^\alpha]\big|\bar{B}^0 + C\big|[du_l^\alpha]\big|\bar{B}^1
\end{aligned}$$

where $\operatorname{Supp}\varphi_0 \subset \operatorname{Supp}\varphi_0'$. Let us put $\epsilon = \epsilon_0$. Then $\varphi_0 u_l^\alpha \in L^{4+2\epsilon}(K_0)_0$ for all α, l by the compact embedding $W_1^4(K_0) \hookrightarrow \hat{L}^4(K_0)$. Thus it is enough to obtain a convergent sequence $\{[\varphi_0 u_{k(l,\alpha)}^{a(\alpha)}]\}_\alpha \subset \bar{B}^0(K_0)_0 \subset L^{4+2\epsilon}(K_0)_0$.

Let us put $v_l^\alpha = \varphi_0 u_l^\alpha$ and choose any decreasing sequence $\delta_i(>0) \to 0$. Since the sets $\{v_l^\alpha\}_l$ consists of Cauchy sequences in $W_1^4(K_0)_0$, they converge to $v^\alpha \in W_1^4(K_0)_0$ for each α. Firstly choose a subsequence $a(i)$ with $|v^{a(i)} - v^{a(j)}|L^{4+2\epsilon}(K_0)_0 < \max\{\delta_i,\delta_j\}$ for all i,j. Then we put $w_l^i = v_l^{a(i)}$ and $w^i = v^{a(i)}$.

Let us choose subindices $k(l,1)$ so that both the estimates $|w_{k(l,1)}^1 - w^1|W_1^4(K_0) < \delta_1$ and $|du_{k(l,1)}^{a(1)}|L_{\text{loc}}^{4+\epsilon}(K_0) < \delta_1$ hold for all l. Next choose subindices $\{k(l,2)\}_l \subset \{k(l,1)\}_l$ so that $|w_{k(l,2)}^2 - w^2|W_1^4(K_0) < \delta_2$ and $|du_{k(l,2)}^{a(2)}|L^{4+2\epsilon}(K_0) < \delta_2$ hold for all l. We inductively choose $k(l,i)$ as above. Then the family $\{[z_l^i = w_{k(l,i)}^i]\}_l$ satisfy the estimates:

$$\begin{aligned}
\big|z_l^i - z_l^j\big|L^{4+2\epsilon}(K_0)_0 \leq \big|z_l^i - w^i\big|L^{4+2\epsilon}(K_0)_0 + |w^i - w^j|L^{4+2\epsilon}(K_0)_0 \\
+ \big|w^j - z_l^j\big|L^{4+2\epsilon}(K_0)_0 \leq \delta_i + \delta_i + \delta_j.
\end{aligned}$$

This shows that $\{[z_l^i = \varphi_0 u_{k(l,i)}^{a(i)}]\}_l$ is a convergent sequence. Notice the inclusion $\{k(l,i)\}_l \subset \{k(l,i-1)\}_l$.

Now let us consider $d : \hat{B}^0 \to \hat{B}^1$. Suppose a family $\{[u_l^\alpha]\}_\alpha \subset \hat{B}^0$ satisfies $|[u_l^\alpha]|\hat{B}^0 = 1$ and $|[du_l^\alpha]|\hat{B}^1 \to 0$. Thus there are arbitrarily large $l = l(\alpha)$ satisfying $\|du_l^\alpha\|_l \bar{B}^1(K_l) \leq \delta_\alpha$ and $\|u_l^\alpha\|_l \bar{B}^0(K_l) \geq 1 - \delta_\alpha$. We put $f_l = \varphi_0 u_{k(l,l)}^{a(l)} = z_l^l$ and restrict $\{[z_l^\alpha = \varphi_0 u_{k(l,l)}^\alpha]_l\}_\alpha \subset \hat{B}^0(\{K_{k(l,l)}\}_l) \subset \hat{B}^0(\{K_l\}_l)$. Then we claim that the family converges to $[f_l] \in \hat{B}^0(\{K_{k(l,l)}\}_l)$. Since we consider $\hat{B}^*$ norm, one may assume $\alpha, \beta \leq l, l'$. By choice, we have conditions:

$$\left|z_l^\alpha - z_{l'}^\alpha\right|\bar{B}^0 < 2\delta_\alpha, \quad \left|z_l^\alpha - z_l^\beta\right|\bar{B}^0 < 3\max(\delta_\alpha, \delta_\beta).$$

Thus we get the claim. This verifies closedness of d.

Let us consider d^+. Let us take $[u_l^\alpha] \in \hat{B}^1$ satisfying $|[u_l]|\hat{B}^1 = 1$ and $|[du_l^\alpha] - [w_l]|\hat{B}^2 \to 0$ as $\alpha \to \infty$. Notice that the Laplace operator $\Delta : W_2^{2+\epsilon}(S^3; \Lambda^1) \to L^{2+\epsilon}(S^3; \Lambda^1)$ admits a uniform bound $|\Delta| \geq C$ where C is independent of ϵ. In particular $d^+ : (\hat{W}_1^2)_w(S^3 \times \boldsymbol{R}) \cap \operatorname{Ker} d_w^* \to \hat{L}_w^2(S^3 \times \boldsymbol{R})$ admits a uniform bound $|d^+| \geq C$.

Let us fix a smooth Riemannian metric h on M such that there is an isometry $(\hat{M} \cap \operatorname{Supp} \varphi_0, g) \cong (\bar{M} \subset M, h)$ where $\hat{M} = \bar{M} \cup_{S^3} S^3 \times [0, \infty)$. Then with respect to h, one constructs another Banach spaces $\hat{B}^*(M, h, \{\epsilon_l\}_l)$.

Sublemma 3.3. *With respect to the function spaces $\hat{B}^*(M, h, \{\epsilon_l\}_l)$, the cohomology groups are isomorphic to the de Rham's one.*

Proof of Sublemma 3.3. In fact one can check the Poincaré lemma. □

Let us return to the proof of Lemma 3.2. Thus there is $Q_1 : \operatorname{im} d^+ \subset \hat{B}^2(M, h) \to \operatorname{Ker} d^* \subset \hat{B}^1(M, h)$ with $d^+ \circ Q_1 = \mathrm{id}$. Let $(S^3 \times \boldsymbol{R}, g = g|S^3 + dt^2)$ be the cylinder. Then there is also $Q_2 : \hat{B}^2(S^3 \times \boldsymbol{R}) \to \hat{B}^1(S^3 \times \boldsymbol{R})$ with $d^+ \circ Q_2 = \mathrm{id}$.

Let $I : H(0) \subset \hat{B}^2(\hat{M})$ be as before. Then using a cut off function ψ, one may represent any element $u \in H^2$ by a compactly supported form, $u' = u - d^+(\psi Q_2(\psi u)) \in H(0)'$. These forms consists of $H^2(\hat{B}^*(M, h), d_*)$. Let us put $\psi = \varphi_0^{1/2}$, $\psi' = (1 - \varphi_0)^{1/2}$ and:

$$Q' = \psi Q_1 \psi + \psi' Q_2 \psi' : \operatorname{im} d^+ \subset \hat{B}^2(\hat{M}) \to \hat{B}^1(\hat{M}).$$

We follow the argument in [**DS**, p. 212].

In Corollary 3.1, we have constructed Q_2 so that each u_l lies in $\operatorname{Ker} d_w^*$ for any $u = [u_l] \in$ image Q_2. In particular $u_l \in (W_1^2)_{\mathrm{loc}}$. Thus from the compact embedding $(W_1^2)_{\mathrm{loc}} \hookrightarrow L_{\mathrm{loc}}^{2+\epsilon}$, it follows that $d^+ \circ Q' - \mathrm{id} : \operatorname{im} d^+ \subset \hat{B}^2(\hat{M}) \to$

$\operatorname{im} d^+ \subset \hat{B}^2(\hat{M})$ is compact. So $d^+ \circ Q'$ is Fredholm of index 0.

Let $w_1, \dots, w_l$ and $d^+(\alpha_1), \dots, d^+(\alpha_l)$ be the linear independent kernel and cokernel sets for d^+Q' respectively. Then we put:

$$Q = \left(Q' + \sum_{i=1}^{l} (w_i, \quad)\alpha_i\right) T^{-1} : \operatorname{im} d^+ \subset \hat{B}^2(\hat{M}) \to \hat{B}^1(\hat{M})$$

where $T = d^+ \circ Q' + \sum_{i=1}^{l} (w_i, \quad) d^+(\alpha_i)$.

Q satisfies $d^+ \circ Q = \mathrm{id}$. This implies the existence of a bounded map:

$$Q : \operatorname{im} d^+ \subset \hat{B}^2(\hat{M}) \to \hat{B}^1(\hat{M})$$

with $d^+ \circ Q = \mathrm{id}$. Thus d^+ has closed range. This completes the proof of Lemma 3.2. □

Proposition 3.3. *The AHS complex between $\hat{B}^*(\hat{M})$ has its cohomology groups as*:

$$H^0\big(\hat{B}^*(\{K_l, \epsilon_l\}_l)\big) = 0, \quad H^1\big(\hat{B}^*(\{K_l, \epsilon_l\}_l)\big) = 0,$$
$$H^2\big(\hat{B}^*(\{K_l, \epsilon_l\}_l)\big) \cong H = H(0).$$

In particular $\dim H^2 = b^2_+(M)$.

Proof. $H^0 = 0$ follows from combination of the proof of Proposition 1.1 and 3.B.

Next we claim that there exists $Q = Q(\hat{M}) : \hat{L}^2(\hat{M}, \epsilon') \cap \operatorname{im} d^+ \to \bar{B}^1(\hat{M}, \epsilon')$ with $d^+ \circ Q = \mathrm{id}$ and $|Q| \leq C$ where C is independent of $0 \leq \epsilon' \leq \epsilon$. Assuming this for the moment, we proceed as follows. For H^2, it also follows from 3.B and existence of Q above.

Let us consider H^1. For simplicity we consider only the case $n = 1$. Then we take $[\alpha_l] \in H^1$, and so $\lim_l \|d^+(\alpha_l)\|_l = 0$ holds.

Since $\varphi_l \alpha_l - Q(\hat{M})(d^+(\varphi_l \alpha_l)) \in \operatorname{Ker} d^+ \cap \bar{B}^1(\hat{M}, \epsilon_l)$, there are $f_l \in \bar{B}^0(\hat{M}, \epsilon_l)$ with $\varphi_l \alpha - df_l = Q(\hat{M})(d^+(\varphi_l \alpha_l))$ hold, where $|f_l|\bar{B}^0(\hat{M}, \epsilon_l)$ are uniformly bounded by Sublemma 3.2. Moreover by the assumption, $|Q(\hat{M})(d^+(\varphi_l \alpha_l))|\bar{B}^1(\hat{M}, \epsilon_l) \to 0$ hold. Similarly there are $g_l \in \bar{B}^0(S^3 \times \boldsymbol{R}, \epsilon_l)$ with $T^*_{2l}((1 - \varphi_l)\alpha_l) - dg_l = Q(S^3 \times \boldsymbol{R})(T^*_{2l}(d^+(1 - \varphi_l)\alpha_l))$.

Let us choose subindices $k(l) \gg l$ (with respect to $\{\alpha_l\}_l$). Then $I : \hat{B}^*(\hat{M}) \to \hat{B}^*(\hat{M})$ by $[\alpha_l] \in \hat{B}^*(\{K_l, \epsilon_l\}_l) \to [\varphi_l \alpha_l + (T^{-1}_{2(k(l)-l)})^*((1 - \varphi_l)\alpha_l)] \in \hat{B}^*(\{K_{k(l)}, \epsilon_l\}_l)$ gives an isomorphism. Then we put $\beta_l = \varphi_{k(l)} f_l + (1 -$

$\varphi_{k(l)})((T_{2k(l)}^{-1})^* g_l)$. Then we have $I([\alpha_l]) - [d\beta_l] \in \bar{B}_0^1(\{K_{k(l)}, \epsilon_l\}_l)$. This verifies $[\alpha_l] = 0 \in H^1$.

Now we verify the claim. Recall $\nabla_i = (d^*)_w \oplus d^+ : (\hat{W}_1^2)_w(\hat{M}, \epsilon_i) \to \hat{L}_w^2(\hat{M}, \epsilon_i)$ satisfy the estimates $|\nabla_i(u)| \geq C_i|u|$ for all u. It is enough to see uniform bound $C_i \geq C$. Suppose contrary. Let us take $u_i \in (\hat{W}_1^2)_w(\hat{M}, \epsilon_i)$ with $|\nabla_i(u_i)|\hat{L}_w^2(\hat{M}, \epsilon_i) \to 0$. Notice uniform bound:

$$|\nabla_i((1-\varphi_0)(u_i))|\hat{L}_w^2(\hat{M}, \epsilon_i) \geq C|(1-\varphi_0)(u_i)|\big(\hat{W}_1^2\big)_w(\hat{M}, \epsilon_i). \quad *$$

Then as in the proof of Proposition 1.1, we have a familiar estimate:

$$|u_i|\big(\hat{W}_1^2\big)_w(\hat{M}, \epsilon_i) \leq C\big\{|\varphi_0 u_i|L^{2+\epsilon}(K_0)_0 + |\nabla_i(u_i)|\hat{L}_w^2(\hat{M}, \epsilon_i)\big\}.$$

Then using the compact embedding $(W_1^{2+\epsilon'})_{\text{loc}} \hookrightarrow L_{\text{loc}}^{2+\epsilon}$, $\epsilon' \leq \epsilon$, we find some $u_\infty \in \operatorname{Ker} \nabla \cap (W_1^2)_w$. This must be zero. This shows $|u_i|L^{2+\epsilon}(\operatorname{Supp}\varphi_0) \to 0$. Combining with the above $*$, one finds $|u_i|\hat{L}_w^2(\hat{M}, \epsilon_i) \to 0$. Since $\nabla_i(u_i) \to 0$, it follows from the elliptic estimate, $|u_i|(\hat{W}_1^2)_w(\hat{M}, \epsilon_i) \to 0$. This completes the proof of Proposition 3.3. $\square$

4. Asymptotic conformality.

4.A. Quasiconformal mappings.

Let g_1 and g_2 be two Riemannian metrics on the same manifold $\hat{M}$. Then the conformal distance is defined as:

$$d([g_1],[g_2])(x) = \sup_{\zeta,\psi} \log\left\{\frac{|\zeta|_2}{|\psi|_2} : \ |\zeta|_1 = |\psi|_1 = 1,\ \zeta, \psi \in T_x\hat{M}\right\}.$$

Let $D, D' \subset \boldsymbol{R}^4$ be compact domains with Riemannian metrics g and g' respectively. Then a quasiconformal homeomorphism $f : (D,g) \cong (D',g')$, satisfies the following properties:

(1) f is almost everywhere differentiable with $\nabla f \in L_{\text{loc}}^4(D)$. The following estimate holds:

$$\int_D |\nabla f|^4 \leq 16K^3 \operatorname{vol}(D'), \quad K = \operatorname{ess\,sup} H(f).$$

(2) (Gehring's theorem [**G**]) The derivative ∇f is in fact in $L_{\text{loc}}^{4+\delta}$, where $\delta > 0$ is determined only by the quasiconformal constant, and ∇f is in the distributional sense: $\int_D h(\partial f/\partial x_i) = -\int_D f(\partial h/\partial x_i)$ for any $h \in C_c^\infty(\operatorname{int} D)$.

(3) $d([f^*(g')],[g])$ is a bounded measurable function and for the quasiconformal constants $H(f)$ in the introduction, the following pointwise equalities hold:

$$H(f)(x) = \exp(d([f^*(g')],[g])_x).$$

(4) Let $\Lambda^2(g) = \Lambda^+(g) \oplus \Lambda^-(g)$ be the decomposition with respect to g. The projection to the anti-self-dual part $pr_-(g) : \Lambda^2 \to \Lambda^2_-(g)$ depends only on the conformal class of g.

$\Lambda^-(f^*(g'))$ is determined by a bounded measurable bundle map $\mu = \mu(f^*(g'),g) : \Lambda^-(g) \to \Lambda^+(g)$ so that $\Lambda^-(f^*(g'))$ can be expressed by the graph $\{(x,\mu(x)) : x \in \Lambda^-(g)\} = \Lambda^-(f^*(g'))$. The following pointwise estimate holds ([**DS**, p. 187]):

$$\frac{1+|\mu|}{1-|\mu|}(x) \leq H(f)(x)^2 \leq \left(\frac{1+|\mu|}{1-|\mu|}(x)\right)^2.$$

(5) Let us put $\tau = \tau(g,f^*(g')) = pr_-(g) - pr_-(f^*(g'))$. One can choose some canonical basis $\{\alpha^i_\pm\}^3_{i=1}$ on $\Lambda^2_\pm(g)$ so that for $\alpha_\pm \in \{\alpha^i_\pm\}_i \subset \Lambda^\pm(g)$, the equalities hold (see [**DS**, p. 186]):

$$pr_-(\alpha_-) = p(\alpha_-)^{-1}\{\alpha_- + \mu(\alpha_-)\}$$

where $p(a) = 1 + |\mu(a)|^2/|a|^2$, and $-|x|^2 + |\mu(x)|^2 - \mu(x)\alpha^+ = 0$ for $pr_-(\alpha_+) = (x,\mu(x))$ with respect to g-norm. So one gets the pointwise norm estimates:

$$|\tau(g,f^*(g'))| = |pr_-(g) - pr_-(f^*(g'))| \leq H(f)^2 - 1.$$

Lemma 4.1 ([**G**], [**DS**]). *Let $f : (D,g) \to (D',g')$ be a quasiconformal homeomorphism and let $\delta > 0$ be in* (2) *above.*

(1) *For $p = 1,2$, let $\alpha \in L^{(4/p)+\epsilon'}(\Lambda^p(D'))$. Then $f^*(\alpha) \in L^{(4/p)+\epsilon}(\Lambda^p(D))$, where $\epsilon = \epsilon(p,\delta,\epsilon') > \delta\epsilon'$. Moreover there is a constant $C = C(\epsilon',D,f,g,g')$ with $C \to [4(\text{ess sup}\, H(f))^{3/2}]^{p/2}$ as $\epsilon' \to 0$ satisfying*:

$$|f^*(\alpha)|L^{(p/4)+\epsilon}(D,g) \leq C|\alpha|L^{(p/4)+\epsilon'}(D',g').$$

(2) *Let $\alpha \in W^4_1(D',g')$. Then*

$$\int_D |f^*(\alpha)|^{4+\epsilon} \operatorname{vol}_g \leq C|\alpha|^{4+\epsilon} W^4_1(D',g')|\nabla f^{-1}|^4 L^{4+\delta}(D',g').$$

PROOF. For (1), $p = 1, 2$, one uses conformal invariance of $L^{4/p}$ norms on p forms and $\nabla f \in L^{4+\delta}$. For (2) one can verify the estimate as follows; recall the Sobolev embedding, $L^N \hookrightarrow W_1^4$ for all N (see [**GT**, p. 167]). Then by the Gehring's theorem (2) above, one gets the estimate for a function $\alpha \in W_1^4$:

$$\begin{aligned}
\int |f^*(\alpha)|^{4+\epsilon} \operatorname{vol}_g &\leq \int |f^*(\alpha)|^{4+\epsilon} f^* |\nabla f^{-1}|^4 |\nabla f|^4 \operatorname{vol}_g \\
&\leq C \int |f^*(\alpha)|^{4+\epsilon} f^* |\nabla f^{-1}|^4 f^*(\operatorname{vol}_h) \\
&\leq C|\alpha|^{4+\epsilon} L^N(D') |\nabla f^{-1}|^4 L^{4+\delta}(D') \\
&\leq C|\alpha|^{4+\epsilon} W_1^4(D', h) |\nabla f^{-1}|^4 L^{4+\delta}(D', h), \\
&\qquad\qquad \left[\left(\frac{N}{4+\epsilon} \right)^{-1} + \left(\frac{4+\delta}{4} \right)^{-1} = 1 \right].
\end{aligned}$$

Thus one gets (2) above as desired. This completes the proof of Lemma 4.1. □

REMARK 4.1. Let $f : D \cong D'$ be a quasiconformal homeomorphism and $\Phi : D' \cong D''$ be a diffeomorphism. Let C and C' be the quasiconformal constants of f and $\Phi \circ f$ respectively. If ∇f lies in $L^{4+\delta}_{\mathrm{loc}}$, then $\nabla(\Phi \circ f) \in L^{4+\delta}_{\mathrm{loc}}$, even though C' may be larger than C.

LEMMA 4.2 (see [**GT**, p. 147]). *Let $D, D' \subset \boldsymbol{R}^4$ be compact subsets and $f : D \cong D' \subset \boldsymbol{R}^4$ be a quasiconformal homeomorphism between them. Then for any smooth Riemannian metric g on D', $f^*(g)$ can be approximated by a family of smooth Riemannian metrics g_i in the sense*:

(1) *$g_i \to f^*(g)$ in $L^2(D)$,*
(2) *quasiconformal constants $K(g_i, f^*(g))$ are uniformly bounded, and*
(3) *$H(g_i, f^*(g))^2 - 1, \mu(f^*(g), g_i)$, $|pr_-(f^*(g)) - pr_-(g_i)| \to 0$ in all L^N.*

PROOF. We only consider (3). Let us consider $H(g_i, f^*(g))^2 - 1$. Its pointwise norms are uniformly bounded a.e. Let us take local orthonormal basis $e_1, \ldots, e_4$ with respect to g_i. Then one can write $f^*(g) = \sum g^i_{a,b} e^*_a \otimes e^*_b$, where $g^i_{a,b} - \delta_{a,b} \to 0$ hold in L^2. Let us choose local sections ζ and ξ with pointwise norms $|\zeta|_{g_i} = |\xi|_{g_i} = 1$. For any small $\delta \gg \epsilon > 0$, if we choose a sufficiently large i, then the set $D(\delta) \subset D$ satisfying $|g^i_{a,b} - \delta_{a,b}| > \delta$ on $D(\delta)$ has its measure less than ϵ. Moreover we have the estimates $(1-\delta)|\zeta|^2_{g_i} \leq |\zeta|^2 \leq (1+\delta)|\zeta|^2_{g_i}$ on $D \backslash D(\delta)$. Combining with a uniform bound $\operatorname{ess\,sup} |\zeta|/|\xi| \leq K$, we get the estimate:

$$\left|\frac{|\zeta|^2}{|\xi|^2}-1\right|^2 L^2(D) \leq \left|\frac{|\zeta|^2}{|\xi|^2}-1\right|^2 L^2(D\backslash D(\delta)) + (K^2-1)^2\epsilon$$
$$\leq \left(\frac{2\delta}{1-\delta}\right)\mathrm{vol}(D) + (K^2-1)^2\epsilon.$$

This shows $H(g_i, f^*(g))^2 - 1 \to 0$ in L^2. For μ and τ, one can verify similar estimates by use of the inequalities (5) above.

Next let us verify L^n convergence. By use of the interpolation argument, L^n convergence follows from the L^2 convergence and the uniform boundedness of ess sup $K(g_i, f^*(g))$ in (2).

Recall the intermediate inequality (see [**GT**, p. 146]):

$$|u|L^q \leq \epsilon|u|L^r + \epsilon^{-(1-\lambda)/\lambda}|u|L^p$$

where $\epsilon > 0$, $\lambda < 1$, $1/q = \lambda/p + (1-\lambda)/r$, $p \leq q \leq r$.

We show that if $|u_i|L^\infty \leq C$ and $|u_i|L^2 \to 0$, then $|u_i|L^n \to 0$ for all n. For any small $\epsilon \gg \delta > 0$, let us choose large i so that $|u_i|L^2 < \delta$ hold. We apply the above estimate for $1 < p \leq 2$, $q = n$, $\lambda = 1/n$. With respect to u_i, we choose a sufficiently large r with $|u_i|L^r \leq |u_i|L^\infty \mathrm{vol}(D) \leq C'$. Then these data (q, r, λ) determines $1 < p \leq 2$. Now we have an estimate:

$$|u_i|L^n \leq \epsilon|u_i|L^r + \epsilon^{-(1-\lambda)/\lambda}|u|_i L^p$$
$$\leq C'\epsilon + C''\epsilon^{-(1-\lambda)/\lambda}|u_i|L^2 \leq C'\epsilon + C''\epsilon^{-(1-\lambda)/\lambda}\delta.$$

This implies (3). This completes the proof of Lemma 4.2. □

Remark 4.2. We do not know whether the above approximation in (3) could be done in L^∞. However one may expect that collapsing of smooth structure can cause a phenomena that $\mu(f^*(g), g_i)|L^N$ approaches to zero non-uniformly with respect to i and N.

4.B. Regularization.

Let $F_i : (D, g) \cong (D', g')$ be a family of quasiconformal mappings with $|H(F_i)^2 - 1|L^N_{\mathrm{loc}} \to 0$ for all large $N \gg 0$. Recall $\tau_i = \tau(g, F_i^*(g')) = pr_-(g) - pr_-(F_i^*(g'))$.

Let α be a smooth 1 form on D' with $d^+(\alpha) = 0$ and $d(\alpha) = w \neq 0$ for all points ([**DS**, p. 196]). Let us take a constant $\delta > 0$.

Lemma 4.3. *Suppose $|F_i^*(w)|L^{2+\delta}_{\mathrm{loc}}$ are uniformly bounded. If F_i and ∇F_i converge in C^0_{loc} and L^4_{loc} respectively, then $\nabla F_\infty \equiv \lim \nabla F_i$ in $L^{4+\delta'}_{\mathrm{loc}}$ for some*

$0 < \delta' < \delta$.

Proof. By the condition, there are measurable sets $D_i \subset D$ satisfying $m(D_i) \to 0$ and ess $\sup|\tau_i|D\backslash D_i < c$. Then there is C with a pointwise bound $|\nabla F_i|^{4+\delta'} \leq C|F_i^*(w)|^{2+\delta}$ on $D\backslash D_i$. Thus $\nabla F_i|D\backslash D_i \in L^{4+\delta'}_{\mathrm{loc}}$. Then we show that the limit $\nabla F_i \equiv \nabla F_\infty$ lie in $L^{4+\delta'}_{\mathrm{loc}}$ for $\delta' = \delta'(c)$. Let us choose small $\epsilon > 0$. Then one takes a subsequence $k(l)$ with $m(D(\epsilon)) < \epsilon$, $D(\epsilon) = \bigcup_l D_{k(l)}$ and $\nabla F_{k(l)}|D\backslash D(\epsilon) \in L^{4+\delta'}_{\mathrm{loc}}$. Recall the inequality $|u|L^q \leq \epsilon|u|L^r + \epsilon^{-(1-\lambda)/\lambda}|u|L^p$ where $q^{-1} = \lambda/p + (1-\lambda)/r$, $p \leq q \leq r$. Since $|\nabla F_{k(l)}|L^{4+\delta'}_{\mathrm{loc}}(D\backslash D(\epsilon))$ are uniformly bounded, one may apply this inequality to $u = \nabla F_{k(l)} - \nabla F_{k(l')}$. Then one sees that for all $\epsilon > 0$ and some $0 < \delta'' < \delta'$, $|\nabla F_\infty|L^{4+\delta''}_{\mathrm{loc}}(D\backslash D(\epsilon))$ are uniformly bounded. We change the notation δ'' by δ'. Then by letting $\epsilon \to 0$, one knows $\nabla F_\infty \in L^{4+\delta'}_{\mathrm{loc}}$. By changing indices, one may assume $|\nabla F_\infty - \nabla F_i|L^{4+\delta'}_{\mathrm{loc}}(D\backslash D_i) \to 0$. This completes the proof of Lemma 4.3. □

Now we consider the converse direction.

Lemma 4.4. *Suppose all ∇F_i above converge in $L^{4+\delta}_{\mathrm{loc}}$. Then there are constants C, $0 \leq \epsilon < \epsilon'$ such that for any $u \in \bar{B}^*(D', \epsilon')_{\mathrm{loc}}$, there is a bound*:

$$\left|F_\infty^*(u)\right|\bar{B}^*(D, \epsilon)_{\mathrm{loc}} \leq C|u|\bar{B}^*(D', \epsilon')_{\mathrm{loc}}.$$

Proof. As above there are measurable sets $D_i \subset D$ satisfying $m(D_i) \to 0$ and ess $\sup|\tau_i|D\backslash D_i < c$. Then there is C with a pointwise bound $|\nabla F_i|^4 \leq C\det(\nabla F_i)$ on $D\backslash D_i$. Then as in [**DS**, p. 198], one gets the estimate:

$$\left|F_i^*(u)\right|\bar{B}^*(D\backslash D_i, \epsilon) \leq C|u|\bar{B}^*(D'\backslash D'_i, \epsilon').$$

Now choose subindices $k(l)$ with $\lim_l m(\bigcup_{l' \geq l} D_{k(l')}) = 0$. We put $E_l = \bigcup_{l' \geq l} D_{k(l')} \subset D$. Then for all triple $l'' \geq l' \geq l$, there are uniform estimates:

$$\begin{aligned}&\left|(F_{k(l'')} - F_{k(l')})^*(u)\right|\bar{B}^*(D\backslash E_l, \epsilon)\\&\quad\leq C\big(|\nabla(F_{k(l'')} - F_{k(l')})|L^{4+\delta}(D)\big)|u|\bar{B}^*\big(D'\backslash E'_l, \epsilon'\big).\end{aligned}$$

This shows $|F_\infty^*(u)|\bar{B}^*(D\backslash E_l, \epsilon) \leq C|u|\bar{B}^*(D'\backslash E'_l, \epsilon')$. Since $m(E_l) \to 0$, this gives the result of Lemma 4.4. □

4.C. Induced morphisms.

Let us take two closed smooth four manifolds M and M'. Suppose $(\hat{M}, g)$ and $(\hat{M}', g')$ admit a uniformly bounded asymptotic morphism, consisting of families of quasiconformal homeomorphisms $\{F_l : (K_l', g') \cong (K_l, g)\}_l$. Recall $\tau_l = \tau(F_l^*(g'), g) = pr_-(g) - pr_-(F_l^*(g')) \to 0$ in L^n for all large n.

Let $\hat{B}^*(\hat{M}(')) = \hat{B}^*(\{K_l(')\}_l, \epsilon('))$ and $(\hat{B}^*(\hat{M}(')), d_i)$ be the AHS complex:

$$0 \longrightarrow \hat{B}^0(\hat{M}(')) \xrightarrow{d} \hat{B}^1(\hat{M}(')) \xrightarrow{d^+} \hat{B}^2(\hat{M}(')) \longrightarrow 0.$$

PROPOSITION 4.1. *Choosing sufficiently small constants $\epsilon' < \epsilon$ and $\delta' < \delta$, there are induced morphisms*:

$$\bar{F}^* : \big(\hat{B}^i(\hat{M}, w, \epsilon), d_i\big) \to \big(\hat{B}^i(\hat{M}', w', \epsilon'), d_i\big).$$

SUBLEMMA 4.1. *Let $F : (K, g) \cong (K, h)$ be a quasiconformal homeomorphism between two Riemannian spaces with $|\tau(F^*(h), g)|L^n < \alpha$ for some large n. Then there is $c = c(\alpha, F) > 0$ and $\epsilon' < \epsilon$ with*:

$$|[F^*, d^+]u|L^{2+\epsilon'}(K, g) \leq c|du|L^{2+\epsilon}(K, h), \quad c \to 0 \text{ as } \alpha \to 0.$$

PROOF OF SUBLEMMA 4.1. Let $d|\Lambda^2 = d^+ \oplus d^-$ be the decomposition with respect to g, and express $d^+(F^*(h)) = d^+ + \tau d$. Then one has the equality:

$$\begin{aligned}
[F^*, d^+] &= F^* \circ d^+(h) - d^+(g) \circ F^* = F^* \circ (1 + *(h))d - d^+(g) \circ F^* \\
&= (1 + F^* * (h)(F^{-1})^*)d \circ F^* - d^+(g) \circ F^* \\
&= (d^+(F^*(h)) - d^+(g)) \circ F^* = \tau d \circ F^*.
\end{aligned}$$

Then the following estimates completes the proof of Sublemma 4.1:

$$\begin{aligned}
&|[F^*, d^+]u|L^{2+\epsilon'}(K_l, g) = |\tau d F^* u|L^{2+\epsilon'} \\
&\quad \leq |\tau|L^n |dF^*u|L^{2+\epsilon''} \leq C(|\nabla F|L^{4+\delta})|\tau|L^n|du|L^{2+\epsilon}(K_l, h). \qquad \square
\end{aligned}$$

Let us continue the proof of Proposition 4.1. We claim the following; let f and g be functions over $\hat{M}$ with a uniform bound $|f|L^{2+\epsilon'}_{\text{loc}} \leq C|g|L^{2+\epsilon}_{\text{loc}}$, $\epsilon' < \epsilon$. Then for some choice of $\delta' < \delta$, one gets the global bound:

$$|f|L^{2+\epsilon'}_{w'} \leq C|g|L^{2+\epsilon}_{w}.$$

In fact we have estimates:

$$\begin{aligned}\int |f|^{2+\epsilon'} \exp(w') &\leq \sum_n \exp(\delta' n)|g_i|^{2+\epsilon'} L^{2+\epsilon}_{\mathrm{loc}} \\ &= \sum_n \left(\exp\left(\frac{\delta'' n}{2+\epsilon'}\right)|g_i| L^{2+\epsilon}_{\mathrm{loc}}\right)^{2+\epsilon'} (\exp(\delta' - \delta'')n) \\ &\leq \left(\sum_n \exp(\delta n)|g_i|^{2+\epsilon} L^{2+\epsilon}_{\mathrm{loc}}\right)^{p^{-1}} \left(\sum_n \exp(qn(\delta' - \delta''))\right)^{q^{-1}} \\ &\leq C|g|^{2+\epsilon'} L^{2+\epsilon}_w\end{aligned}$$

where $p = (2+\epsilon)/(2+\epsilon')$, $p^{-1} + q^{-1} = 1$ and $\delta'' = p^{-1}\delta > \delta'$. This verifies the claim.

Let us take $[u_l] \in \hat{B}^*(\hat{M})$. Then by the estimate in Lemma 4.1, Lemma 4.4 and the above claim, one gets:

$$\|F_l^*(u_l)\| \bar{B}^*(K_l', g', w', \epsilon') \leq C\|u_l\| \bar{B}^*(K_l, g, w, \epsilon).$$

(More precisely one should remove some small measurable sets, or use F_∞^*. However as far as considering asymptotic sequences, these do not matter and we will denote as above.)

Let us check $[F_l^*(u_l)]$ defines also an element in $\hat{B}^*(\hat{M}')$. Recall $|\nabla(F_l - F_{l'})|L^{4+\delta} \to 0$. Then we have the inequalities:

$$\begin{aligned}&\|F_l^*(u_l) - F_{l'}^*(u_{l'})\| \bar{B}^*(K_l', g', \epsilon') \\ &\quad = \|F_{l'}^*[u_l - u_{l'}]\| + \|(F_l - F_{l'})^*(u_l)\| \\ &\quad \leq C(F_l, \epsilon)\|u_l - u_{l'}\| \bar{B}^*(K_l, g, \epsilon) + \delta(l, l')\|u_l\| \to 0.\end{aligned}$$

Thus $F^* : \hat{B}^*(\hat{M}, \epsilon) \to \hat{B}^*(\hat{M}', \epsilon')$ gives a bounded map.

Next let us consider the differentials. Notice that $d : \hat{B}^0(\hat{M}) \to \hat{B}^1(\hat{M})$ commutes with F^*.

Next let us consider d^+ and the family $F_l^*(d^+(u_l)) \in \hat{L}^2_{w'}(K_l')_0$. By Lemma 4.4 and the above claim, the family is uniformly bounded. Then we show $\|d^+ F_{l'}^* u_{l'} - d^+ F_l^* u_l\| \bar{B}^2(K_{l'}, g_{l'}, w', \epsilon') \to 0$. In fact one has the estimates:

$$\begin{aligned}&\|d^+ F_{l'}^* u_{l'} - d^+ F_l^* u_l\| \bar{B}^2(K_{l'}, g_{l'}, w', \epsilon') \\ &\leq \|[F_{l'}^*, d^+] u_{l'}\| + \|[F_l^*, d^+] u_l\| + \|F_{l'}^* d^+(u_{l'}) - F_l^* d^+(u_l)\| \\ &\leq \delta(F_l, F_{l'}, \epsilon)\{\|du_{l'}\| \hat{L}^2_w + \|du_l\| \hat{L}^2_w\} + C\|du_l - du_{l'}\| + C\delta(l, l')\|du_l\|.\end{aligned}$$

By Sublemma 4.1 and $|\tau_l|L^n \to 0$, one may assume that the last term converges to zero. Thus F^* gives a morphism from $\{\hat{B}^*(\hat{M},\epsilon), d^*\}$ to $\{\hat{B}^*(\hat{M}',\epsilon'), d^*\}$. This completes the proof of Proposition 4.1. $\square$

Corollary 4.1. *Let $\hat{M}$ and $\hat{M}'$ be asymptotically q.c. equivalent. Let us take $\epsilon > \epsilon' > \epsilon'' > 0$. Then we have the induced maps*:

$$F^* : \hat{B}^*(\hat{M}, w, \epsilon) \to \hat{B}^*(\hat{M}', w', \epsilon'), \quad (F^{-1})^* : \hat{B}^*(\hat{M}', w', \epsilon') \to \hat{B}^*(\hat{M}, w'', \epsilon'')$$

such that the composition $(F^{-1})^* \circ F^*$ gives the identity.

5. Connections and bundles.

5.A. Let $(\hat{M}, g)$ be a cylindrical manifold as before. In all the later sections, we will use the underlying Banach spaces $\hat{B}^*(\hat{M},\epsilon)$ defined in Section 2.A (3.C), where all the constants ϵ_l are the same $\epsilon > 0$.

Let $E^0 \to \hat{M}$ be a G bundle ($G = SU(2)$ or $SO(3)$) and fix a trivialization of E^0 on the end. One chooses another G bundles $E^1, \ldots, E^n$ over $S^3 \times \boldsymbol{R}$, where these also fix trivializations on both ends. E^i are determined by c_2 or p_1. Let us take end connected sums as:

$$E = E^0 \sharp E^1 \sharp \cdots \sharp E^n.$$

Passing through some identifications of the ends, E gives a G bundle over $\hat{M}$ which bubbles along the cylinder.

Recall the transformations $T(l,j)$, the families of indices $a(l,j)$ and the corresponding cut off functions $\varphi_{a(l,j)}$ for $j = 0, 1, \ldots, n$ in 2.A.3. Let us choose a family of compactly supported smooth connections $\{A^0, \ldots, A^n\}$ over E^i respectively. Then by gathering these, one gets a family of the reference connections A_l over E by:

$$A_l^0 = \varphi_{a(l,0)} A^0 + \sum_{j=1}^{n} (T(l,j)^{-1})^* (\varphi_{a(l,j)} - \varphi_{a(l,j-1)}) A^j.$$

Let $I \subset \boldsymbol{R}_+$ be an interval. Then for $g \in G$, we denote the constant gauge transformation by $T(g) \in \mathrm{Aut}(E|S^3 \times I)$, where we regard $S^3 \times I \subset S^3 \times \boldsymbol{R}_+ = \mathrm{end}\, \hat{M}$.

Now let us define the affine Banach space and the Banach Lie group by:

$$
\begin{aligned}
As\mathfrak{A} &= \big\{ \big[A_l^0\big] + a : a = [a_l] \in \hat{B}^1 \big\}, \\
As\mathfrak{G} &= \big\{ u = [u_l] : \big[\nabla_{A_l^0} u_l\big] \in \hat{B}^1,\ u_l \in \bar{B}^0_{\mathrm{loc}}(\mathrm{Aut}(E)), * \big\}
\end{aligned}
$$

where $*$ requires the condition that there is a constant $c = c(u) > 0$ so that $[u_l - T(g)_l] \in \hat{B}^0$ hold, where the locally constant gauge transformations $T(g)_l = \sum_{j=0}^n M(l,j) \times g_j$ are given for some $g = \{g_0(u), \dots, g_n(u)\} \subset G$ and for a family of subsets in end $\hat{M}$:

$$
\begin{aligned}
&M(l,0) = S^3 \times [c, a(l,0) - c], \\
&M(l,j) = S^3 \times [a(l,j) + c, a(l,j+1) - c] \quad (n \geq j \geq 1,\ a(l,n+1) = \infty).
\end{aligned}
$$

We denote $As\mathfrak{A}_n$ and $As\mathfrak{G}_n$ respectively, when we stress the number of the bubblings. Naturally one has stratifications:

$$
As\mathfrak{A}_0 \subset As\mathfrak{A}_1 \subset \cdots, \quad As\mathfrak{G}_0 \subset As\mathfrak{G}_1 \subset \cdots
$$

Notice:

$$
\begin{aligned}
As\mathfrak{A}_0 &= \big\{ A^0 + a : a \in \bar{B}^1(\hat{M}) \big\}, \\
As\mathfrak{G}_0 &= \big\{ u \in As\mathfrak{G} : \nabla_{A_0} u \in \bar{B}^0(\hat{M}) \big\}.
\end{aligned}
$$

It is straightforward to check the following:

LEMMA 5.1.

(1) *$As\mathfrak{G}$ is a Banach Lie group. Its Lie algebra* $As\mathfrak{g} = \mathrm{Lie}\, As\mathfrak{G} = \{[\sigma_l] \in \hat{B}^0_{\mathrm{loc}} : \nabla_{A_l^0}\sigma_l \in \hat{B}^1, *\}$ *is equipped with norm* $\|[\nabla_{A_l^0}\sigma_l]\|\hat{B}^1 + \sum_{j=0}^n |h_j(\sigma)|$, $h_j(\sigma) \in \mathfrak{g}$.
(2) *$As\mathfrak{G}$ acts on $As\mathfrak{A}$ by*:

$$
[u_l]\big([A_l^0] + [a_l]\big) = \big[u_l^*(A_l^0 + a_l)\big].
$$

PROOF.
(1) By the Sobolev embedding $C^0 \hookrightarrow W_1^{4+2\epsilon}$, any element $g \in \hat{B}^0(\hat{M};\epsilon)$ consists of a family of continuous sections $g_l : K_l \subset \hat{M} \to \mathrm{Aut}(E)$. Then since the estimates $1 - 4/(4+2\epsilon) \geq 0$ hold, we get the multiplication map:

$$
\hat{B}^0(\hat{M};\epsilon) \times \hat{B}^0(\hat{M};\epsilon) \to \hat{B}^0(\hat{M};\epsilon).
$$

Then passing through the exponential map, one gets a Banach manifold structure on $As\mathfrak{G}$.

(2) follows from the Sobolev multiplication properties:

$$\hat{B}^0 \times \hat{L}^p_w \to \hat{L}^p_w, \quad \hat{L}^{4+2\epsilon}_{2w} \times \hat{L}^{4+2\epsilon}_{2w} \to \hat{L}^{2+\epsilon}_w$$

where $p = 4+2\epsilon$ or $2+\epsilon$. Then we check the condition:

$$\begin{aligned}
&\big\|u_l^*(A_l^0 + a_l) - u_{l'}^*(A_{l'}^0 + a_{l'})\big\|\bar{B}^1 \\
&\quad \leq C\big[(1 + \|u_l\|\bar{B}^0 + \|u_{l'}\|\bar{B}^0)^2 \|u_l - u_{l'}\|\bar{B}^0\big] \\
&\qquad + \|(u_l - u_{l'})^* a_l\|\bar{B}^1 + \big\|u_{l'}^*(a_l - a_{l'})\big\|\bar{B}^1 \\
&\quad \leq C\big(1 + \|u_l\|\bar{B}^0 + \|u_{l'}\|\bar{B}^0\big)^2 \\
&\qquad \times \big[\|u_l - u_{l'}\|\bar{B}^0 + \|u_l - u_{l'}\|\bar{B}^0 \|a_l\|\bar{B}^1 + \|a_l - a_{l'}\|\bar{B}^1\big] \to 0.
\end{aligned}$$

This completes the proof of Lemma 5.1. $\square$

Let us consider a smooth map and its kernel:

$$F^+ : As\mathfrak{A} \to \hat{B}^2\big(\hat{M}; \Lambda^2_+ \otimes \mathrm{ad}\mathfrak{g}\big), \quad \big[A_l^0 + a_l\big] \to \big[F^+_{A_l^0 + a_l}\big].$$

$$As\hat{\mathfrak{M}} = \big\{[A_l] \in As\mathfrak{A} \mid [F^+_{A_l}] \in \bar{B}^2_0(\{K_l\}_l, \epsilon)\big\}.$$

Notice that any elements in $\bar{B}^2_0(\{K_l\}_l, \epsilon)$ are 0 in $\hat{B}^2(\{K_l\}_l, \epsilon)$.

We call an element $[A_l] \in As\hat{\mathfrak{M}}(\hat{M}, \epsilon)$ an *asymptotic ASD connection*. By Lemma 5.1, $As\mathfrak{G}$ act on $As\hat{\mathfrak{M}}$. Let us denote the quotient spaces by:

$$As\mathfrak{M} \equiv As\hat{\mathfrak{M}}/As\mathfrak{G}.$$

Then one also has the stratifications:

$$\mathfrak{M}(\hat{M}) = As\mathfrak{M}_0 \subset As\mathfrak{M}_1 \subset \cdots \subset As\mathfrak{M}_n = As\mathfrak{M}.$$

Recall that in order to construct the underlying Banach spaces, one has fixed the translations $\boldsymbol{T} = \{T(l,j)\}$ satisfying some properties in 2.A. If we vary the parameter spaces, then we get the parametrized asymptotic ASD moduli space:

$$PAs\mathfrak{M}_k(\hat{M}, g, \epsilon) = \bigcup_{\boldsymbol{T}} As\mathfrak{M}_k(\hat{M}, g, \epsilon; \boldsymbol{T}).$$

When we consider a single asymptotic moduli space, then we will not specify $\boldsymbol{T}$.

5.B. Regularity.

The following lemma is well known.

Lemma 5.2. *Let us take any* $[A] \in \mathfrak{M}(\hat{M}, g)$. *Then there are positive* $\delta_k > 0$ *such that one may represent it by a smooth ASD connection* A *with*:

$$\|A \mid \operatorname{end} \hat{M}\| C^k(S^3, t) \leq C_k \exp(-\delta_k t)$$

identifying $\operatorname{end} \hat{M} \cong S^3 \times [0, \infty)$.

Let us take $[A_l] \in As\hat{\mathfrak{M}}$.

Lemma 5.3.

(1) *An asymptotic ASD connection* $[A_l]$ *converges as* $l \to \infty$ *to* $(A(0), \ldots, A(n))$, *where* $A(0)$ *is a* $\bar{B}^1(\hat{M}, \epsilon)$ *ASD connection, and* $A(i)$ *are* $\bar{B}^1(S^3 \times \boldsymbol{R}, \epsilon)$ *ASD connections for* $1 \leq i \leq n$.
(2) *If two asymptotic ASD connections* $[A_l]$ *and* $[A_l']$ *are* $As\mathfrak{G}(\hat{M}, \epsilon)$ *gauge equivalent with the corresponding* $\{A(j)\}_j$ *and* $\{A(j)'\}_j$ *in* (1), *then each* $A(j)$ *and* $A(j)'$ *are* $\mathfrak{G}(\hat{M}, \epsilon)$ *or* $\mathfrak{G}(S^3 \times \boldsymbol{R}, \epsilon)$ *gauge equivalent.*
(3) *There exists* $[g_l] \in As\mathfrak{G}$ *so that* $[g_l][A_l] \in As\mathfrak{A}$ *can be represented by another* $[A_l']$, *where each* A_l' *are given by cutting and connecting a family of smooth ASD connections* $(A(0), \ldots, A(n))$.

Proof. We only consider one bubble case $n = 1$. The general case is similar.

By definition, $\|F^+_{A_l}\|\bar{B}(\{K_l\}_l) \to 0$ hold as $l \to \infty$, and so A_l converges to ASD connections $A(0)$ and $A(1)$ over $\hat{M}$ and $S^3 \times \boldsymbol{R}$ respectively in $\bar{B}^1$ topology. This verifies (1).

If $\|g_l^* A_l - A_l'\|\bar{B}^1(\{K_l\}_l) \to 0$ hold for some $[g_l] \in As\mathfrak{G}$, then g_l converges to a pair $(g(0), g(1))$ of $\bar{B}^0$ gauge groups over $\hat{M}$ and $S^3 \times \boldsymbol{R}$ respectively so that these satisfy the equalities $g(i)^* A(i) = A(i)'$. This verifies (2).

Let $A(i)$ be as above. By the above lemma, there exist gauge transformations $g(0)$ and $g(1)$ over $\hat{M}$ and $S^3 \times \boldsymbol{R}$ such that $g(i)^*(A(i))$ are smooth ASD connections. We check $g(i) \in \mathfrak{G} \subset As\mathfrak{G}$. On the end one can put $A(i) = d + a_i$ and $g(i)^*(A(i)) = d + a_i'$ where $a_i, a_i' \in \bar{B}^1$. Then one has the estimate:

$$\begin{aligned} |a_i - a_i'|\hat{L}^4_w &= |g(i)^*(A(i)) - A(i)|\hat{L}^4_w = |g(i)^{-1} dg(i) + g(i)^{-1} a_i g(i) - a_i|\hat{L}^4_w \\ &\geq |dg(i)|\hat{L}^4_w - 2|a_i|\hat{L}^4_w. \end{aligned}$$

This shows $dg(i) \in \hat{L}^4_w$ and so $g(i) \in \bar{B}^0$.

Let S_l be the translations in Lemma 2.3. We put $A'_l = d+\varphi_l a'_0+(1-\varphi_l)S^*_{2l}a'_1$. Since $g(i) \in \bar{B}^0$, there are $g'_i \in G$ so that $|g(0)-g'_0|C^0(S^3\times\{T\}), |g(1)-g'_1|C^0(S^3\times\{-T\}) \to 0$ as $T \to \infty$. After multiplying some constant, one may assume $g' = g'_0 = g'_1 \in G$. By use of small perturbations of $g'+\varphi_l(g(0)-g')+(1-\varphi_l)S^*_{2l}(g(1)-g')$, one gets an element $[g_l] \in As\mathfrak{G}(\hat{M})$ satisfying $[g_l][A_l] = [A'_l]$. This verifies (3). This completes the proof of Lemma 5.3. $\square$

Corollary 5.1. *For any pair $\epsilon > \epsilon' > 0$, the natural inclusions give homeomorphisms*:

$$I : As\mathfrak{M}_k(\hat{M}, g, \epsilon) \cong As\mathfrak{M}_k(\hat{M}, g, \epsilon'), \quad 0 \le k \le n.$$

Let us construct a canonical map from the set of ASD connections $\hat{\mathfrak{M}}$ to $As\hat{\mathfrak{M}}$. Let φ_l be the family of cut off functions over $\hat{M}$ in 2.A. Then for $A = A^0 + a$, one can assign $[A_l] \in As\mathfrak{A}(\hat{M})$ with $A_l = A^0 + \varphi_l a$. It follows from Lemma 5.2 that $[A_l]$ defines an element in $As\hat{\mathfrak{M}}$. Thus one gets a map $\hat{I} : \hat{\mathfrak{M}} \to As\hat{\mathfrak{M}}$.

Corollary 5.2. *There exists a surjection* (*essentially an isomorphism*) :

$$As\mathfrak{M}(\hat{M}; E) \to \bigcup_{\{E_l\}} \mathfrak{M}(\hat{M}; E_0) \times_{j=1}^n \mathfrak{M}(S^3 \times \boldsymbol{R}; E_j)$$

where (1) $w_2(E) = w_2(E_0)$ *and* (2) $p_1(E) = \sum_{j=0}^n p_1(E_j)$.

In particular there is a natural injection:

$$I : \mathfrak{M}(\hat{M}, g') \hookrightarrow As\mathfrak{M}(\hat{M})$$

which is induced by $\hat{I}$ above.

5.C. Local structure of the moduli spaces.

Let us consider the action:

$$As\mathfrak{G} \times As\mathfrak{A} \to As\mathfrak{A}.$$

The derivative at (id, A) is given by:

$$(u, a) \to a + d_A(u) \in \hat{B}^1.$$

Lemma 5.4. *For any $A \in As\hat{\mathfrak{M}}(\hat{M})$, $\{\hat{B}^*, d^*_A\}$ is a Fredholm complex with* *the index* $-2p_1(E) - 3(1 + b^2_+(M))$*. We denote their cohomology groups by* $H^*_A = \operatorname{Ker} d^{*+1}_A / \operatorname{im} d^*_A$.

PROOF. Let us take $A = [A_l] \in As\hat{\mathfrak{M}}(\hat{M})$. Firstly the composition $d^+_A \circ d^0_A = 0$ hold from the equality $d^+_A d_A([u_l]) = [F^+_{A_l}, u_l]$ and the Sobolev multiplication $\hat{B}^2 \times \hat{B}^0 \to \hat{B}^2$.

Let us see $\operatorname{im} d^0_A$ is closed. By 5.B, one may choose $[A'_l]$, $As\mathfrak{G}$ gauge equivalent to $[A_l]$ such that $A_l = \varphi_l A_0 + (1-\varphi_l) S^*_{2l} A_1$, where A_j are smooth ASD connections over $\hat{M}$ and $S^3 \times \boldsymbol{R}$ respectively. Then for each l, $d_{A'_l} : \bar{B}^0(\hat{M}; \operatorname{ad}\mathfrak{g}) \to \bar{B}^1(\hat{M}; \Lambda^1 \otimes \operatorname{ad}\mathfrak{g})$ has closed range with finite dimensional kernel (c.f. [**K1**, Lemma 1.5]). Then as the proofs of Proposition 1.1 and Lemma 3.2, one sees that $d^0_A : \hat{B}^0 \to \hat{B}^1$ has closed range. Similarly one can check $d^+_A : \hat{B}^1 \to \hat{B}^2$ has closed range. The index computation is done from Corollary 2.1, Corollary 5.2 and the sum formula. This completes the proof of Lemma 5.4. □

Let us take $A \in As\mathfrak{M}$. There exists $Q_A : \operatorname{im} d^+_A \cap \hat{B}^2 \to \hat{B}^1$ as in the last part of the proof of Lemma 3.2. Let us briefly recall its construction below. Firstly we construct $Q_{A^0} : \operatorname{im} d^+_{A(0)} \cap \bar{B}^2(\hat{M}) \to \bar{B}^1(\hat{M})$ with $d^+_{A(0)} \circ Q_{A(0)} = \operatorname{id}$ for an ASD connection $A(0)$ over $\hat{M}$. Similarly we have $Q_{A(1)}$ over $S^3 \times \boldsymbol{R}$. Suppose $A \in As\mathfrak{M}$ corresponds to a family of ASD connections $(A(0), \ldots, A(n))$. Then we cut and connect $Q_{A(j)}$ as:

$$Q'_A = \sum \psi_j Q_{A(j)} \psi'_j : \operatorname{im} d^+_A \cap \hat{B}^2(\{K_l\}_l; \epsilon) \to \hat{B}^1(\{K_l\}_l; \epsilon).$$

Again as in the Lemma 3.2, one can modify Q'_A to Q_A so that Q_A satisfies $d^+_A \circ Q_A = \operatorname{id}$. This gives the desired map $Q_A : \operatorname{im} d^+_A \cap \hat{B}^2 \to \hat{B}^1$.

Thus $\hat{B}^1$ splits as:

$$\hat{B}^1 = \operatorname{im} Q_A \oplus \ker d^+_A$$

by $u = Q_A d^+_A(u) + (u - Q_A d^+_A(u))$. From this, one has a natural isomorphism:

$$\ker d^+_A / \operatorname{im} d_A \cong \hat{B}^1 / (\operatorname{im} Q_A + \operatorname{im} d_A).$$

In particular $\operatorname{im} Q_A + \operatorname{im} d_A \subset \hat{B}^1$ has finite codimension. By adding a finite dimensional subspace $V_A \subset \hat{B}^2$, one has a tangent space of $As\mathfrak{B} = As\mathfrak{A}/As\mathfrak{G}$ at $[A]$:

$$T_{[A]} As\mathfrak{B} \cong \operatorname{im} Q_A \oplus V_A.$$

Let us put:

$$CG(A) = \{A + a \mid a \in \operatorname{im} Q_A + V_A\},$$
$$I : CG(A) \times As\mathfrak{G} \to As\mathfrak{A} \quad (A', g) \to g^*(A').$$

Then dI is given by:

$$dI = \mathrm{id} \oplus d_A : (\operatorname{im} Q_A \oplus V_A) \times As\mathfrak{g} = \operatorname{Lie} As\mathfrak{G} \to \hat{B}^1.$$

Thus if A is irreducible ($\operatorname{Ker} d_A = 0$), then one gets the following:

Proposition 5.1. *There are neighbourhoods $U \subset As\mathfrak{A}$ of A and $V \subset As\mathfrak{G}$ of* id *such that*:

$$I : U \cap CG(A) \times V \to As\mathfrak{A},$$
$$I : U \cap CG(A) \cap As\hat{\mathfrak{M}} \times V \to As\hat{\mathfrak{M}}$$

are homeomorphisms respectively.

In particular if d_A^+ is surjective, then at $[A]$, $As\mathfrak{G}$ acts freely on $As\hat{\mathfrak{M}}$, and $As\hat{\mathfrak{M}}/As\mathfrak{G}$ are C^∞ manifolds of dimension $-2p_1(E) - 3(1 + b_+^2(M))$.

6. Transversality.

6.A. Let $(\hat{M}, g)$ be a smooth cylindrical manifold, and $E \to \hat{M}$ be an $SO(3)$ bundle with a fixed trivialization on the end. Thus the bundle determines $w_2(E) \in H^2(M : \boldsymbol{Z}_2)$ and $p_1(E) \in H^4(M : \boldsymbol{Z})$. Here we quickly review the construction of $(W_l^2)_w$ ASD moduli spaces over $(E, \hat{M}, g)$.

One has already used a weight function $w : \hat{M} \to [0, \infty)$. Let us choose a compactly supported smooth connection A_0 over E with $A_0 | \mathrm{end} = d$, and put the affine Hilbert spaces and the Hilbert Lie groups ($l \geq 3$):

$$\mathfrak{A}(\hat{M}) = \big\{A_0 + a : a \in \big(W_l^2\big)_w\big(\hat{M}; \operatorname{ad}\mathfrak{g} \otimes \Lambda^1\big)\big\},$$
$$\mathfrak{G}(\hat{M}) = \big\{g \in C^0_{\mathrm{loc}}(\operatorname{Aut} E) : \nabla_{A_0} g \in \big(W_l^2\big)_w(M : \operatorname{ad}\mathfrak{g} \otimes \Lambda^1)\big\}.$$

Let us consider the set of ASD connections $\hat{\mathfrak{M}} = \{A \in \mathfrak{A} : F_A^+ = 0\}$. $\mathfrak{G}$ acts on both $\mathfrak{A}$ and $\hat{\mathfrak{M}}$, and one denotes their quotient spaces by $\mathfrak{B} = \mathfrak{A}/\mathfrak{G}$ and $\mathfrak{M} = \hat{\mathfrak{M}}/\mathfrak{G}$ respectively.

LEMMA 6.1. *Let $A \in \hat{\mathfrak{M}}$. Then the AHS complex*:

$$0 \longrightarrow \left(W^2_{l+1}\right)_w(\hat{M}, g; \mathrm{ad}\mathfrak{g}) \xrightarrow{d_A} \left(W^2_l\right)_w\left(\hat{M}, g; \mathrm{ad}\mathfrak{g} \otimes \Lambda^1\right)$$
$$\xrightarrow{d^+_A} \left(W^2_{l-1}\right)_w\left(\hat{M}, g; \mathrm{ad}\mathfrak{g} \otimes \Lambda^2_+\right) \longrightarrow 0$$

is Fredholm with the index $-2p_1(E) - 3b^2_+(M)$.

This follows from the standard excision method.

6.B. Transversality for $(W^2_k)_w$ spaces.
The aim in 6.B is to verify the following:

PROPOSITION 6.1. *Suppose $b^2_+(M) \geq 1$. Then for some generic choice of g', $\mathfrak{M}(g')$ is a finite dimensional smooth manifold of dimension $-2p_1(E) - 3(1 + b^2_+(M))$ (note M is simply connected).*

In order to construct smooth moduli spaces, one uses K. Uhlenbeck's generic metric theorem. The proof is parallel to [**FU**], [**T1**] as far as one uses Hilbert spaces. Later one will see that this perturbation is also able to apply to L^p_w spaces. Let $(\hat{M}, g, w)$ be a cylindrical manifold with a weight function, and choose a proper map $h : \hat{M} \to [0, \infty)$ with $h(x) \geq (w/2)(x)$. Then one introduces the following Banach manifold:

$$\mathfrak{C} = \left\{\phi \in C^l(Gl(T\hat{M})) : \limsup_i \left(\sum_{j=0}^{l} e^h |\nabla^j(\phi^* g - g)| S^3 \times [i, i+1]\right) = 0\right\}.$$

Let us take $\phi \in \mathfrak{C}$ and put $g' = \phi^* g$. Then one has the AHS complex with respect to g':

$$0 \longrightarrow \left(W^2_{l+1}\right)_w(\hat{M}, g') \xrightarrow{d} \left(W^2_l\right)_w(\hat{M}, g') \xrightarrow{d^+} \left(W^2_{l-1}\right)_w(\hat{M}, g') \longrightarrow 0.$$

This is a Fredholm complex with the same index as the unperturbed one.

LEMMA 6.2. *Suppose M is indefinite. Then by a small perturbation of the Riemannian metric, there are no orbit of reducible connections in $\mathfrak{M}_k(\hat{M})$.*

In order to verify this, we follow [**FU**].

Let us consider the $\mathfrak{G}(\hat{M})$ equivariant map:

$$P_+ : \mathfrak{A}(\hat{M}) \times \mathfrak{C} \to \left(W^2_{l-1}\right)_w(\hat{M}, g)$$

by $P_+(A,\phi) = P_+(g)(\phi^{-1})^* F_A$, where $P_+(g)$ is the projection to the self-dual part with respect to g. Let us put $\bar{\mathfrak{M}}(\hat{M}) = P_+^{-1}(0) \subset \mathfrak{A}(\hat{M}) \times \mathfrak{C}$.

Proposition 6.2 ([**FU**]). *$\bar{\mathfrak{M}}(\hat{M}) \cap (\mathfrak{A}^*(\hat{M}) \times \mathfrak{C})/\mathfrak{G}(\hat{M})$ is a smooth Banach manifold, where $\mathfrak{A}^*$ consists of irreducible connections.*

We sketch its proof. Firstly we see that dP_+ is surjective at any (A,φ) with $P_+(A,\varphi) = 0$. Then it follows that $\bar{\mathfrak{M}}(\hat{M})$ is a Banach manifold on which $\mathfrak{G}(\hat{M})$ acts. Then by making slice for the action, one gets the result.

Now dP_+ splits as:

$$dP_+ = d_1 P_+ \oplus d_2 P_+ : \left(W^2_l\right)_w(\hat{M}, g) \oplus \mathfrak{c} \to \left(W^2_{l-1}\right)_w(\hat{M}, g)$$

where $d_1 P_+(\alpha) = P_+(g)(\varphi^{-1})^* d_A(\alpha)|(u,\varphi)$, $d_2 P_+(r) = P_+(g)((\varphi^{-1})^*(r^* F))$. We show that the differential of P_+ is surjective.

Let us consider the AHS complex:

$$0 \longrightarrow \left(W^2_{l+1}\right)_w(\hat{M}, g) \xrightarrow{d_A} \left(W^2_l\right)_w(\hat{M}, g) \xrightarrow{d_A^+} \left(W^2_{l-1}\right)_w(\hat{M}, g) \longrightarrow 0.$$

Since this is Fredholm, one sees dP_+ has finite codimension. Let us take a representative $u \in \operatorname{Coker} dP_+$ with $(d_A^+)^*_w(u) = 0$. Then one has $d_A(e^w u) = 0$. Then one has the equations, $d_A(F_A) = d_A^*(F_A) = d_A(v) = d_A^*(v) = 0$, where we use $g' = \varphi^*(g)$ metric and $v = \varphi^*(e^w u)$ (here one uses Hilbert space structure of the function space, $(W^2_{l-1})_w(\hat{M}, g'; \Lambda^2_+)$). Then the same argument as [**FU**, p. 56], shows that on open dense subset of $\hat{M}$, F_A can be expressed as $\alpha \otimes a \in \Lambda^2_+ \otimes Ad(E)$, with $|a| = 1$ (pointwise norm) and $d_A(a) = 0$. By the irreducibility, it follows $a = 0$, which contradicts to non-triviality of $p_1(E)$. Thus $dP_+|(A,\varphi)$ is surjective. This completes the proof.

Corollary 6.1. *For a Baire set of $\phi \in \mathfrak{C}$, there are no reducible connection in $\mathfrak{A}(\hat{M})$ with respect to $\phi^*(g)$.*

Proof. Suppose A_0 is reducible, and denote the corresponding $U(1)$ connection by the same A_0. Let P be a $G = U(1)$ bundle. Then as before one puts:

$$\mathfrak{A}(P) = \left\{A_0 + a \mid a \in \left(W^2_l\right)_w(\hat{M}, g)\right\},$$

$$\mathfrak{G}(\hat{M})_0 = \big\{h \in C^0_{\mathrm{loc}}(Y; \mathrm{Aut}(E)) \mid \nabla_{A_0} h \in \big(W^2_l\big)_w(\hat{M}, g'),$$
$$|h - \mathrm{id}\,|C^0(S^3 \times [i, i+1]) \to 0\big\}.$$

Then $\mathfrak{G}(\hat{M})_0$ acts on $\mathfrak{A}(\hat{M})$. Consider $P_+ : \mathfrak{A}(\hat{M}) \times \mathfrak{C} \to (W^2_{l-1})_w(\hat{M}, g)$. dP_+ is surjective at $P_+(A, \varphi) = 0$, since $F_A \neq 0$.

Now one puts $\bar{\mathfrak{M}}(\hat{M}) = P_+^{-1}(0)/\mathfrak{G}(\hat{M})_0$. Then one has a smooth Banach manifold $\bar{\mathfrak{M}}(\hat{M})' = \bar{\mathfrak{M}}(\hat{M})/\mathfrak{G}(\hat{M})_0$. Let us consider a Fredholm map between Banach manifolds $\bar{\pi} : \bar{\mathfrak{M}}(\hat{M})' \to \mathfrak{C}$. Its Fredholm index is the same as the following one:

$$0 \longrightarrow \big(W^2_{l+1}\big)_w(\hat{M}, g) \xrightarrow{d} \big(W^2_l\big)_w(\hat{M}, g) \xrightarrow{d^+} \big(W^2_{l-1}\big)_w(\hat{M}, g) \longrightarrow 0$$

where $H^0(\mathrm{AHS}) = H^1(\mathrm{AHS}) = 0$ and $H^2(\mathrm{AHS}) = b^2_+(M)$. This is also the case when one perturbes the Riemannian metrics slightly. In particular, if $b^2_+ > 0$, then the index is negative. Then the Sard Smale theorem shows that for a Baire set of $\mathfrak{C}$, there are no ASD connections over non-trivial line bundles. This shows that there are no orbits of reducible ASD connections in $\mathfrak{M}(\hat{M})$ for a Baire set of $\mathfrak{C}$. This completes the proof. □

Let us put the projection as $\pi : \bar{\mathfrak{M}}(\hat{M})/\mathfrak{G}(\hat{M}) \to \mathfrak{C}$. Then the direct application of [**FU**] to this case shows:

Corollary 6.2 ([**FU**]). *Suppose $b^2_+ > 0$. Then a Baire set of $\phi \in \mathfrak{C}$ exists such that $\hat{\mathfrak{M}}(\phi) \equiv \pi^{-1}(\phi)$ are smooth finite dimensional manifolds.*

6.C. L^p_w spaces.

Let g' be a small perturbation of g as above. For any ASD connection A with respect to g', $\mathrm{Ker}\{d_A : \mathfrak{g} = \mathrm{Lie}\,\mathfrak{G} \to (W^2_l)_w\} = 0$ and $\mathrm{Coker}\, d^+_A : (W^2_l)_w \to (W^2_{l-1})_w = 0$ hold. Let us recall that we have defined Banach spaces $B^*(\hat{M})$, $\bar{B}^*(\hat{M})$ in 1.A, 2.A respectively (any element lies in $L^{4+2\epsilon}_{2w}(\hat{L})$ or $L^{2+\epsilon}_w(\hat{L})$). Let A be a $\bar{B}^1$ ASD connection, $A \in \mathfrak{A}(\bar{B}^*) = \{A_0 + a | a \in \bar{B}^1(\hat{M}; \mathrm{ad}\mathfrak{g} \otimes \Lambda^1)\}$. One may assume A is smooth.

Lemma 6.3. *For a Baire set of $\phi \in \mathfrak{C}$ and $g' = \phi^*(g)$, the AHS complex*:

$$0 \longrightarrow \bar{B}^0(\hat{M}, g'; \mathrm{ad}\mathfrak{g}) \xrightarrow{d_A} \bar{B}^1(\hat{M}, g'; \mathrm{ad}\mathfrak{g}) \xrightarrow{d^+_A} \bar{B}^2(\hat{M}, g'; \mathrm{ad}\mathfrak{g}) \longrightarrow 0$$

satisfies $\mathrm{Ker}\, d_A = 0$ *and* $\mathrm{Coker}\, d^+_A = 0$.

Proof. When A is a $B^1(\hat{M}, 0)$ ASD connection, one can use the Hilbert space structure of $B^2(\epsilon = 0)$. Then one can follow the parallel argument as above, and gets the same conclusion; for a generic g', there are no $\operatorname{Ker} d_A : B^0(\epsilon = 0) \to B^1(\epsilon = 0)$, and no $\operatorname{Coker} d_A^+ : B^1(\epsilon = 0) \to B^2(\epsilon = 0)$.

Let us use g' above and take $u \in \operatorname{Ker} d_A : \bar{B}^0(\hat{M}, \epsilon) \to \bar{B}^1(\hat{M}, \epsilon)$. Since u satisfies the elliptic equation $(d_A)^*_w \circ d_A(u) = 0$, u has the exponential decay (see the proof of Sublemma 3.1). Thus there is a positive constant $\delta_0 > 0$ so that for the corresponding weight function w_0, u lies in $(W_k^{4+2\epsilon})_{2w_0}$ for all k. Then $u \in (W_l^4)_{2w}$ hold for $0 < \delta < \delta_0$ by the method in the proof of Proposition 4.1. This is a contradiction.

Next let us consider $\operatorname{Coker} d_A^+ : \bar{B}^1(\hat{M}, \epsilon) \to \bar{B}^2(\hat{M}, \epsilon)$. In 5.C, we have obtained $Q_A : \operatorname{im} d_A^+ \cap \bar{B}^2(\hat{M}, \epsilon) \to \bar{B}^1(\hat{M}, \epsilon)$ from $Q_A : \operatorname{im} d_A^+ \cap L^2_w(\hat{M}) \to (W_1^2)_w(\hat{M})$. Since $\operatorname{im} d_A^+ = L^2_w(\hat{M})$ hold for the latter case, one has $Q_A : \bar{B}^2(\hat{M}, \epsilon) \to \bar{B}^1(\hat{M}, \epsilon)$ with $d_A^+ \circ Q_A = \mathrm{id}$ as desired. This completes the proof of Lemma 6.3. □

Remark 6.1. We have perturbation of $\hat{B}^*(\{K_l\}_l, \epsilon)$, the asymptotic Banach spaces as follows. Let g_0 and g_1 be cylindrical metric on $\hat{M}$ and $S^3 \times \boldsymbol{R}$ respectively, and we take g_i', small perturbations as above. Let $g_i'(l)$ be another metrics with:

$$g_0'(l) = \begin{cases} g_0' & \text{on } \hat{M} \backslash S^3 \times [l-1, \infty), \\ g_0 & \text{on } S^3 \times [l, \infty), \end{cases}$$

$$g_1'(l) = \begin{cases} g_1' & \text{on } S^3 \times [-l+1, \infty), \\ g_1 & \text{on } S^3 \times (-\infty, -l]. \end{cases}$$

Then by regarding $\hat{M} = \hat{M} \backslash S^3 \times [l, \infty) \cup S^3 \times [-l, \infty)$, one gets a family of smooth Riemannian metrics $\{g(l)\}_l$ on $\hat{M}$. This perturbation of Riemannian metrics works when one considers only one bubble at infinity, but for the asymptotic Banach spaces with n bubbles, the constructions of families of Riemannian metrics are parallel by use of the family of indices $\{a(i,j)\}$ and translations $T(i,j)$ $i = 0, 1, \ldots,$ $j = 0, \ldots, n$ (2.A). We omit their detailed description.

So far we have used fixed one Riemannian metric g when defining asymptotic Banach spaces. Here we use the family of Riemannian metrics $\{g(l)\}_l$, and formulate the Banach spaces $\bar{B}^*(\{K_l, g(l)\}_l, \epsilon)$ using norms $\|u_l\|_l$ by the following:

$$\begin{aligned} &|[u_l]| \bar{B}^*\big(\{K_l, g(l)\}_l; n\big) \\ &\quad = \sup_l \|u_l\|_l \end{aligned}$$

$$\equiv \sup_l \Big\{ \big|\varphi_{a(l,0)} u_l \big| \bar{B}^*(K_l, w, \epsilon, g(l)) + \sum_{j=1}^n \big| (\varphi_{a(l,j)} - \varphi_{a(l,j-1)}) u_l \big| \bar{B}^*(K_l, w(l,j), \epsilon, g(l)) \Big\}.$$

By the same way as before, one obtains the perturbed asymptotic Banach spaces $\hat{B}^*(\{K_l, g(l)\}, \epsilon)$. They are isomorphic to the previous $\hat{B}^*(\{K_l\}_l, \epsilon)$.

6.D. Perturbation of asymptotic equivalence.

Let $K_0 \subset K_1 \subset \cdots \subset \hat{M}$ be an exhaustion, and take an asymptotic morphism $\{I_l\}_l$ from $(\hat{M}, g)$ to $(\hat{M}', g')$. The morphism is defined using Riemannian metrics, and is not stable under the above perturbation. In order to overcome this, we will use more restrictive perturbation. Let us define:

$$\mathfrak{C} = \Big\{ \Phi \in \mathrm{Diff}_0^l(\hat{M}) : \inf_{\Phi_t} \sup_t \big[|e^h [\mathrm{id} - \Phi_t]| C^0 + \sum_{j=1}^l |e^h \nabla^j (\mathrm{id} - \Phi_t)| C^0 \big] < \infty \Big\}$$

where we take all the family of diffeomorphisms Φ_t with $\phi_0 = \mathrm{id}$ and $\Phi_1 = \Phi$. Thus we have a norm on $T\mathfrak{C}$ defined by $\|\Phi\| = \inf_{\Phi_t} \sup_t [|e^h \Phi_t| C^0 + \sum_{j=1}^l |e^h \nabla^j \Phi_t| C^0]$.

Since $\hat{M}$ is of bounded geometry, there is a small constant $\epsilon > 0$ such that $\mathfrak{C}(\epsilon) = \{\Phi \in \mathfrak{C} : \|\mathrm{id} - \Phi\| \leq \epsilon\}$ is a Banach manifold. Then by a parallel argument, one gets the same conclusion as Corollary 6.2 for $\mathfrak{C}(\epsilon)$ by taking $d\Phi \in C^l(Gl(T\hat{M}))$.

Let us choose any $\Phi \in \mathfrak{C}(\hat{M}, \epsilon)$ and $\Phi' \in \mathfrak{C}(\hat{M}', \epsilon)$. Let us put:

$$h = (d\Phi)^*(g), \quad h' = (d\Phi')^*(g'), \quad J_l = (\Phi')^{-1} \circ I_l \circ \Phi.$$

Then to give another asymptotic morphism, it is immediate to check the five defining conditions in the introduction for the following triples:

$$\{J_l\}_l : (\hat{M}, h) \to (\hat{M}', h').$$

When we perturb asymptotic Banach spaces as above, we choose $\Phi^0 \in \mathfrak{C}(\hat{M}, \epsilon)$, $\Phi^1 \in \mathfrak{C}(S^3 \times \boldsymbol{R}, \epsilon)$ and choose paths of diffeomorphisms Φ_t^0 and Φ_t^1. Then by a similar method as 6.C, one obtains a family of diffeomorphisms $\{\Phi(l)\}_l$. By choosing $h(l) = d\Phi(l)^*(g)$ and $J_l = (\Phi(l)')^{-1} I_l \Phi(l)$ as above, one gets asymptotic Banach spaces $\{\hat{B}^*(\{K_l, \Phi(l)\}_l, \epsilon)\}_*$. The same consideration as 6.C, performs transversality for the AHS complex $\{\hat{B}^*(\{K_l, \Phi(l)\}_l, \epsilon), d_A^*\}_*$.

7. Isomorphisms between moduli spaces.

Let us recall the uniformly bounded asymptotically quasiconformal equivalence introduced in the introduction. Let us take such an equivalence $\{I_l\}_l$ from $(\hat{M}, g)$ to $(\hat{M}', g')$. Let us choose small constants $\epsilon > \epsilon' > \epsilon''$. Here we construct continuous maps by use of Proposition 4.1:

$$\{I_l^*\}_l : \big(As\mathfrak{M}(\hat{M}, g, \epsilon), \mathfrak{M}(\hat{M}, g, \epsilon)\big) \to \big(As\mathfrak{M}(\hat{M}', g', \epsilon'), \mathfrak{M}(\hat{M}', g', \epsilon')\big)$$
$$\{(I_l^{-1})^*\}_l : \big(As\mathfrak{M}(\hat{M}', g', \epsilon'), \mathfrak{M}(\hat{M}', g', \epsilon')\big) \to \big(As\mathfrak{M}(\hat{M}, g, \epsilon''), \mathfrak{M}(\hat{M}, g, \epsilon'')\big).$$

The composition of the above maps gives the identity. By symmetry, one finds that $As\mathfrak{M}(\hat{M}, g)$ and $As\mathfrak{M}(\hat{M}', g')$ are homeomorphic.

Proof of Theorem 0.1. Suppose $\{I_l : K_l' \cong K_l\}_l$ satisfies the condition that for any $[A_l] \in As\mathfrak{M}(\hat{M}('), g('))$,

$$\lim_{r \to 0} \limsup_{x \to \infty} \sup_{k,l} \big|I_l(^{-1})^*(F_{A_k})\big| L^2(B_r(x)) = 0.$$

We verify that for each family of translations $\boldsymbol{T}$, there is another $\boldsymbol{T}'$ such that $\{I_l\}_l$ induces a homeomorphism:

$$\{I_l^*\}_l : As\mathfrak{M}(\hat{M}, g; \epsilon; \boldsymbol{T}) \cong As\mathfrak{M}\big(\hat{M}', g'; \epsilon'; \boldsymbol{T}'\big), \quad \epsilon' < \epsilon$$

for all metrics sufficiently near the cylindrical ones. By the construction, this assignment varies continuously with respect to the pair. Thus this will give the desired homeomorphism between parametrized asymptotic ASD moduli spaces.

Let us take any $[A_l] \in As\mathfrak{M}(\hat{M}, g, \epsilon)$. Passing through the surjection $As\mathfrak{M} \to \bigcup \mathfrak{M}(\hat{M}) \times_n \mathfrak{M}(S^3 \times \boldsymbol{R})$, $[A_l]$ splits into the family of the ASD connections $A(0) \times_{j=1}^n A(j)$.

Recall that in 2.A.3, we have translated the underlying spaces from $S^3 \times [0, \infty) \subset \hat{M}$ into $S^3 \times \boldsymbol{R}$ in defining the function spaces. When such situation comes here, we identify the maps I_l on $\hat{M}$ with $T(l,j)^{-1} \circ I_l \circ T(l,j)$.

For every $l_0 \ll l$, there exist gauge transformations $g_{l_0}^j \in \bar{B}^0_{\mathrm{loc}}(\mathrm{Aut}(E|K_{l_0}'))$ so that the estimates hold:

$$\big|(g_{l_0}^j)^*\big(I_l^*(A_j \mid K_{l_0})\big)\big|\bar{B}^1\big(K_{l_0}', \epsilon'\big)_{\mathrm{loc}} \le C\big|I_l^*(F_{A_j|K_{l_0}})\big|\hat{L}^2\big(K_{l_0}', \epsilon'\big)_{\mathrm{loc}}.$$

In fact there exist gauge transformations $h_{l_0}^j \in \bar{B}^0_{\mathrm{loc}}(\mathrm{Aut}(E|K_{l_0}))$ so that the estimates $|(h_{l_0}^j)^*(A_j|K_{l_0})|\bar{B}^1(K_{l_0}, \epsilon)_{\mathrm{loc}} \le C|F_{A_j|K_{l_0}}|\hat{L}^2(K_{l_0}, \epsilon)_{\mathrm{loc}}$. Then one can choose $g^j_{l_0} = I^*_l(h^j_{l_0}) = h^j_{l_0} \circ I_l$ by Proposition 4.1.

Recall that there is a family of measurable bundle maps $\tau_l \in L^n(K_l)$ with $|\tau_l|L^n \to 0$. Let us put $A_j(l_0, l) \equiv (g^j_{l_0})^*(I^*_l(A_j|K_{l_0})) = d + a_j(l_0, l)$. Then Sublemma 4.1 verifies the following:

Sublemma 7.1 ([**DS**, p. 244]). *A subsequence of $\{A_j(l_0, l)\}_l$ converges to a g' ASD connection in $\bar{B}^1(K'_{l_0}, \epsilon')_{\rm loc}$ as $l \to \infty$ for every l_0.*

In fact by the condition $|\nabla(I_l - I_{l'})|L^{4+\delta}_{\rm loc} \to 0$, one does not need to take subsequences.

Let us continue the proof of Theorem 0.1.

Let us consider $j = 0$. Then it follows from Sublemma 7.1 and Lemma 5.2 that we have the following data:

(1) pair of subindices $\{k_l\}_l, \{m_l\}_l$, $l \ll k_l \ll m_l$,
(2) another translations $\{T'(l, 0)\}_l$

such that the family $\{A_0(l_0, l)\}_{l \geq l_0}$ gives an element in:

$$[A_0(l, m_l)] \in As\mathfrak{M}\big(\big\{K'_{k_l}; T'(l, 0)\big\}_l; \epsilon'\big).$$

One can check that this element does not depend on choice of $\{k_l\}_l$ and $\{m_l\}_l$. Moreover T' can be determined by $[A_l]$.

For $j \geq 1$, the situation is similar. Let $j = 1$, and regard I_l as the maps between $S^3 \times \boldsymbol{R}$ as above. Again there are data as above with $[A_1(l, m(l, 1))] \in As\mathfrak{M}(\{K'_{k(l,1)}; T'(l, 1)\}_l; \epsilon')$. One may assume that both $\{k(l, 1)\}_l$ and $\{m(l, 1)\}_l$ are subindices of $\{k_l\}_l$ and $\{m_l\}_l$ respectively. By the same way for $j \geq 2$, we get also $\{k(l, j)\}$, $\{m(l, j)\}_l$ and $\{T'(l, j)\}$. Let us put $T' = \bigcup_{j=0}^n \{T'(l, j)\}_l$. By gathering and taking subindices, one gets a total element:

$$I(A) = \left[\sum_{j=0}^n A_j(l, m(l, n))\right] \in As\mathfrak{M}\big(\big\{K'_{k(l,n)}; T'\big\}_l; \epsilon'\big).$$

Thus we have chosen $\{k_l\}_l$ and T', and assigned an element $I(A)$. If another $A' \in As\mathfrak{M}(\hat{M}, g, \epsilon)$ is sufficiently near A, then one may choose the same $\{k_l\}_l$ as A, and the corresponding $T'(A')$ are sufficiently near $T' = T'(A)$. By taking countable dense subset of $As\mathfrak{M}(\hat{M}, g, \epsilon)$, one can choose the same $\{k_l\}_l$ for all $A \in As\mathfrak{M}(\hat{M}, g, \epsilon)$ by the diagonal method. Moreover the corresponding T' vary continuously with respect to A.

Now we have assigned $[A'_l] \in As\mathfrak{M}(\hat{M}', g', \epsilon')$ to $[A_l] \in As\mathfrak{M}(\hat{M}, g, \epsilon)$. We show that this is well defined. We have chosen $g_l^j \in C^0_{\mathrm{loc}}(\mathrm{Aut}(E|K_l'))$ to construct $A_j(l)$. Suppose another family $\{h_l^j\}_l$ plays the same role as $\{g_l^j\}_l$. Let us put $g_l' = (h_l^j)^{-1} g_l^j$ (for the moment we omit to denote j), $A_j(l) = (g_l^j)^*(I_l^*(A_j)) = d + a_j(l)$ and $A_j'(l) = (h_l^j)^*(I_l^*(A_j)) = d + a_j'(l)$. Notice the equality $(g_l')^*(A'(l)) = (g_l')^*(h_l^*(I_l^*(A))) = A(l)$. First we have an estimate:

$$\begin{aligned}\big\|d\big(g_l'\big)\big\|\hat{L}^4_{2w} &\leq C\big\|\big(g_l'\big)^*(A'(l))\big\|\hat{L}^4_{2w} + C\big\|a_j'(l)\big\|\hat{L}^4_{2w} \\ &= C\big\{\|a_j(l)\|\hat{L}^4_{2w} + \big\|a_j'(l)\big\|\hat{L}^4_{2w}\big\}.\end{aligned}$$

Thus $\|g_l'\|\bar{B}^0$ are uniformly bounded. By taking a subsequence, one may assume $\{g_l'\}_l$ converges in C^0. Then we have the estimates:

$$\begin{aligned}&\big\|d\big(g_l' - g_{l'}'\big)\big\|\hat{L}^4_{2w} \\ &\quad\leq C\big\{\big\|\big(g_l' - g_{l'}'\big)^*\big[(h_l^*(I_l^*(A)))\big]\big\|\hat{L}^4_{2w} + \big\|\big(g_l' - g_{l'}'\big)^*\big(a_j'(l)\big)\big\|\hat{L}^4_{2w}\big\} \\ &\quad\leq C\big\{\big\|\big(g_l'\big)^*(A'(l)) - \big(g_{l'}'\big)^*(A'(l'))\big\|\hat{L}^4_{2w} \\ &\qquad\quad + \big\|\big(g_{l'}'\big)^*[A'(l) - A'(l')]\big\|\hat{L}^4_{2w} + \big|g_l' - g_{l'}'\big|C^0\big\|a_j'(l)\big\|\hat{L}^4_{2w}\big\} \\ &\quad\leq C\big\{\|A(l) - A(l')\|\hat{L}^4_{2w} + \|A'(l) - A'(l')\|\hat{L}^4_{2w} + \big|g_l' - g_{l'}'\big|C^0\big\|a_j'(l)\big\|\hat{L}^4_{2w}\big\}.\end{aligned}$$

The above estimate gives an element $[g_l'] \in As\mathfrak{G}(\hat{M}', g', \epsilon')$. Thus we have verified that the above assignment is independent of choice of local gauge transformations g_l^j.

Next suppose $[A_l], [B_l] \in As\mathfrak{M}(\hat{M}, g, \epsilon)$ are $As\mathfrak{G}(\hat{M}, g, \epsilon)$ equivalent by $[g_l]$. Choose $[h_l]$ and $[h_l']$ for $[A_l]$ and $[B_l]$ so that one gets $[A_l'], [B_l'] \in As\mathfrak{M}(\hat{M}', g', \epsilon')$ as above. Let us put $u_l = g_l \circ I_l$. Then clearly $I_l^*(B_l) = I_l^*(g_l^*(A_l)) = u_l^*(I_l^*(A_l))$. Then one sees $[(h_l^{-1})(u_l h_l')] \in As\mathfrak{G}$ by comparing $[h_l^*(I_l^*(A_l))]$ with $[(h_l')^* u_l^*(I_l^*(A_l))]$ by the above argument. Thus $[A_l']$ and $[B_l']$ are $As\mathfrak{G}(\hat{M}', g', \epsilon')$ equivalent.

Now we have a well defined map $I^* : As\mathfrak{M}(\hat{M}, g, \epsilon) \to As\mathfrak{M}(\hat{M}', g', \epsilon')$. One also gets another map $(I^{-1})^* : As\mathfrak{M}(\hat{M}', g', \epsilon') \to As\mathfrak{M}(\hat{M}, g, \epsilon'')$ by the same way. Let us take $[A_l] \in As\mathfrak{M}(\hat{M}, g, \epsilon)$. The composition is given as $[g_l^*(A_l)] \in As\mathfrak{M}(\hat{M}, g, \epsilon'')$. The same argument as above verifies $[g_l] \in As\mathfrak{G}(\hat{M}, g, \epsilon'')$. This verifies that the composition $(I^{-1})^* \circ I^*$ gives the identity, and by symmetry $I^* \circ (I^{-1})^* = \mathrm{id}$. Thus combining with Corollary 5.1, this completes the proof of Theorem 0.1. $\square$

8. Large deformation of ASD moduli spaces.

8.A. Asymptotically quasiconformal homeomorphism.

Let M be a closed C^∞ oriented four manifold, and for some $R \subset M$ as in the introduction, one gets $\check{M} = M\backslash \bar{R} \subset M$ and $\hat{M} = M\backslash\mathrm{pt}$. Thus $\check{M}$ and $\hat{M}$ are smooth submanifolds of M.

By perturbing a cylindrical metric on $\hat{M}$, here we will use any generic metric as in Section 6.

An *asymptotically quasiconformal homeomorphism* (ACH) between $\check{M}$ and $\hat{M}$ consists of the following data:

(1) exhaustion by compact subsets $K_0 \subset\subset K_1 \subset\subset \cdots \subset \check{M}$,
(2) a family of Riemannian metrics (K_l, g_l) of uniformly bounded geometry, $(|(g_l - g_{l'})|C^\infty(K_l)$ are uniform),
(3) quasiconformal embeddings $I_l : (K_l, g_l) \hookrightarrow (\hat{M}, g)$

so that these data satisfy the following:

(4) for any $x \in \hat{M}$, there exists $l(x) \geq 0$ such that the image of I_l contains x for all $l \geq l(x)$, and $I_l(m) \to \infty$ as $m \to \infty$,
(5) for all $m \in \check{M}$, $d(I_l(m), I_{l'}(m)) \to 0$ as $l, l' \to \infty$,
(6) $|H(I_l^{\pm 1})^2 - 1|L^N_{\mathrm{loc}} \to 0$ for all large $N \gg 0$,
(7) $|\nabla(I_{l'}^{\pm 1} - I_l^{\pm 1})|L^{4+\delta}_{\mathrm{loc}} \to 0$ for $l' \geq l$ and some $\delta > 0$.

8.B. Large deformation of ASD moduli spaces.

Let us take an asymptotically quasiconformally homeomorphism $\{I_l : (K_l, g_l) \to (\hat{M}, g)\}_l$ from $\check{M}$ to $(\hat{M}, g)$.

Let us take a generic family of Riemannian metrics h_l on M with $|g_l - h_l|C^\infty(K_l) \to 0$. Notice that by changing g_l by h_l, the data $I_l : (K_l, h_l) \hookrightarrow (\hat{M}, g)$ satisfies the defining conditions for ACH.

Let us choose an $SO(3)$ bundle $E \to M$, and construct a family of ASD moduli spaces $\{\mathfrak{M}(M, h_l)\}_l$. Suppose they are all non-empty, and take any elements $[A_l] \in \mathfrak{M}(M, h_l)$. Here we will use Banach spaces $B(\hat{M}, \epsilon)$ in 1.A. Recall that for an $SO(3)$ bundle $E' \to \hat{M}$ with a fixed trivialization on the end, one has the affine Banach space $\mathfrak{A} = \{A_0 + a | a \in B^1(\hat{M}, \epsilon)\}$, where A_0 is a compactly supported smooth connection. Using $\mathfrak{A}$, one gets the ASD moduli space $\mathfrak{M}(\hat{M}, \epsilon)$.

Proposition 8.1. *A subsequence $\{A'_l = (I_l^{-1})^*(A_l)\}_l$ converges to an element $A_\infty \in \mathfrak{M}(\hat{M}, \epsilon)$, after gauge transformation.*

Proof. Here one may assume that all A_l are smooth. After gauge transformation and taking subsequence, one may assume (1) $\{A_l\}$ converges on every compact subset and (2) uniform estimates $|A_l|(W_k^p)_{\text{loc}} \leq C|F_{A_l}|L^2_{\text{loc}}$ hold by taking local trivializations.

Let us consider a family $\{(I_l^{-1})^*(A_l)\}_l$ over $\hat{M}$. By Lemma 4.1(1), one gets the estimate:

$$\begin{aligned}
&\big|\big(I_l^{-1}\big)^*(A_l) - \big(I_{l'}^{-1}\big)^*(A_{l'})\big|L^{4+2\epsilon'}_{\text{loc}} \\
&\quad\leq \big|\big(I_l^{-1}\big)^*(A_l) - \big(I_l^{-1}\big)^*(A_{l'})\big|L^{4+2\epsilon'}_{\text{loc}} + \big|\big(I_l^{-1} - I_{l'}^{-1}\big)^*(A_{l'})\big|L^{4+2\epsilon'}_{\text{loc}} \\
&\quad\leq C\big(\big|\nabla I_l^{-1}\big|L^{4+\delta}_{\text{loc}}\big)|A_l - A_{l'}|L^{4+2\epsilon}_{\text{loc}} + C\big(\big|\nabla\big(I_l^{-1} - I_{l'}^{-1}\big)\big|L^{4+\delta}_{\text{loc}}\big)|A_{l'}|L^{4+2\epsilon}_{\text{loc}} \to 0. \\
&\big|\big(I_l^{-1}\big)^*(A_l)\big|L^{4+2\epsilon'}_{\text{loc}} \leq C\big(\big|\nabla I_l^{-1}\big|L^{4+\delta}_{\text{loc}}\big)|A_l|L^{4+2\epsilon}_{\text{loc}} \leq C\big(\big|\nabla I_l^{-1}\big|L^{4+\delta}_{\text{loc}}\big)|F_{A_l}|L^2_{\text{loc}}.
\end{aligned}$$

Thus the family $\{(I_l^{-1})^*(A_l)\}_l$ converges to A_∞ in $L^{4+2\epsilon'}_{\text{loc}}$ topology. By considering similarly $\{F_{A_l'}\}_l$ also converges in $L^{2+\epsilon'}_{\text{loc}}$.

Let $(\ \ )^+$ be the projection to the self-dual part with respect to g. Then we have the equality $F^+_{A_l'} + \tau_l(F_{A_l'}) = 0$, where $|\tau_l|L^n \to 0$ for large n. Combining with the estimate:

$$\big|\tau_l F_{A_l'}\big|L^2 \leq C|\tau_l|L^n|F_{A_l}|L^{2+\epsilon} \qquad (n^{-1} + (2+\epsilon')^{-1} = 1)$$

it follows that A_∞ satisfies the g-ASD equation $F^+_{A_\infty} = 0$. After gauge transformation by h, one may assume $h^*(A_\infty)$ satisfies the exponential decay estimates in W_k^2 norm. Since $F_{h^*(A_\infty)} = h^*(F_{A_\infty})$, the L^2 norms are unchanged under gauge transformation. In particular F_{A_∞} satisfies exponential decay estimate in L^2 norm.

Let us consider the following estimates:

$$\begin{aligned}
\big|\big(I_l^{-1}\big)^*(A_l)\big|L^{4+2\epsilon'}_{\text{loc}} &\leq C|F_{A_l}|L^2_{\text{loc}} \\
&= C\big|I_l^*\big(F_{(I_l^{-1})^*(A_l)}\big)\big|L^2_{\text{loc}} \leq C'\big|F_{(I_l^{-1})^*(A_l)}\big|L^2_{\text{loc}}, \\
\big|F_{(I_l^{-1})^*(A_l)}\big|L^{2+\epsilon'}_{\text{loc}} &= \big|\big(I_l^{-1}\big)^*(F_{A_l})\big|L^{2+\epsilon'}_{\text{loc}} \leq C|F_{A_l}|L^{2+\epsilon}_{\text{loc}} \\
&\leq C'|F_{A_l}|L^2_{\text{loc}} = C'\big|I_l^*\big(F_{(I_l^{-1})^*(A_l)}\big)\big|L^2_{\text{loc}} \leq C''\big|F_{(I_l^{-1})^*(A_l)}\big|L^2_{\text{loc}}.
\end{aligned}$$

From these estimate, one gets an element $A_\infty \in \mathfrak{M}(\hat{M}, g, \epsilon')$. This completes the proof of Proposition 8.1. □

8.C. Donaldson's invariant.

Let M be an oriented simply connected smooth closed four manifold. Let h be any element in $H_2(M; \boldsymbol{Z})$. One may represent h by a smoothly embedded oriented closed surface. In fact by Hurewicz surjection, h has a representative by a smooth immersion $f : S^2 \to M$. One may assume that the self intersection number of f is zero. For each Whitney pair of self intersection points (p_0, p_1), one has an embedded boundary of a Whitney disk $l = l_0 \cup l_1$, where l_i are arcs with two points $l_0 \cap l_1 = \{p_0, p_1\}$ as the Whitney pair. Remove small neighbourhoods $B_\epsilon(p_i)$, and add a small cylinder $S^1 \times [0,1]$ along $\partial B_\epsilon(p_i)$ which contains l_0 in the cylinder. The result is an embedded oriented closed surface which represents the same homology class as h (notice that $H_2(M; \boldsymbol{Z})$ is torsion free).

Let $\Sigma_1, \dots, \Sigma_m \subset M$ be embedded surfaces representing homology classes $h_i = [\Sigma_i] \in H_2(M; \boldsymbol{Z})$. Then one can construct $Q(E; M, h_1, \dots, h_m)$, the Donaldson's invariant We follow [**FS2**]. Let E be an $SO(3)$ bundle over M, and take any embedded oriented closed surface $\Sigma \subset M$. Let $\mathfrak{B}_k^*(\Sigma; E \mid \Sigma) = \mathfrak{A}(E \mid \Sigma)_k^* / \mathfrak{G}_{k+1}(E \mid \Sigma)$ be the gauge equivalence classes of irreducible Sobolev k connections. Similarly one has $\tilde{\mathfrak{B}}_k^*(\Sigma; E \mid \Sigma) = \mathfrak{A}(E \mid \Sigma)_k^* / \mathfrak{G}_{k+1}(E \mid \Sigma)_0$, where $\mathfrak{G}_{k+1}(E \mid \Sigma)_0 \subset \mathfrak{G}_{k+1}(E \mid \Sigma)$ is a subgroup with $g(x_0)$ is the identity for a fixed point $x_0 \in \Sigma$. Thus one has a fibration:

$$SO(3) \hookrightarrow \tilde{\mathfrak{B}}_k^*(\Sigma; E \mid \Sigma) \to \mathfrak{B}_k^*(\Sigma; E \mid \Sigma)$$

which gives an $SO(3)$ principal bundle β over $\mathfrak{B}_k^*(\Sigma; E \mid \Sigma)$. Now one has the following:

Sublemma 8.1 ([**FS2**]).

(1) *If $\langle w_2(E), \Sigma \rangle \neq 0$, then β lifts to a $U(2)$ bundle $\tilde{\beta}$.*
(2) *For Σ, there is another embedded surface $\Sigma' \subset M$ such that restriction of any irreducible ASD connection over E to $E \mid \Sigma$ is also irreducible.*

For our application, it is enough to assume the condition (1) in the above sublemma. Thus let us choose a lift $\tilde{\beta}$ over $\mathfrak{B}_k^*(\Sigma; E \mid \Sigma)$. One has $c_1(\tilde{\beta}) = w_2(\beta) \bmod 2$. Let us choose a complex line bundle Λ over $\mathfrak{B}_k^*(\Sigma; E \mid \Sigma)$ with $c_1(\Lambda) = c_1(\tilde{\beta})$.

By the above Sublemma (2), one has the restriction map:

$$r : \mathfrak{M}_k^*(E) \to \mathfrak{B}_k^*(\Sigma; E \mid \Sigma).$$

One chooses a generic section s of Λ. Then by the transversality argument ([**DK**, p. 192]), one has a smooth manifold $\mathfrak{M}_k^*(E) \cap r^{-1}(\operatorname{Ker} s)$ of codimension 2. One may generalize this construction. Suppose $\dim \mathfrak{M}_*(E) = 2m$, and choose l embedded surfaces $\Sigma_1, \dots, \Sigma_m \subset M$. As above one has determinant bundles over $\mathfrak{B}_k^*(\Sigma_i; E \mid \Sigma_i)$. Then for generic sections, $s_1, \dots, s_m$, one has a smooth submanifold of dimension 0 as:

$$\mathfrak{M}_*(E) \cap r_1^{-1}(\mathrm{Ker}\, s_1) \cap \cdots \cap r_m^{-1}(\mathrm{Ker}\, s_m).$$

By choosing these embeddings with general positions, one may assume that this 0 dimensional manifold consists of finite oriented points. Thus one may count the number of points so that one obtains an integer. This number is, by definition, the Donaldson's invariants $Q(M, E; [\Sigma_1], \ldots, [\Sigma_m])$ (modulo constant).

8.D. Proof of Theorem 0.2.

Let X be a simply connected smooth four manifold with even type intersection form:

$$(H_2(X : \boldsymbol{Z}), \sigma) \cong \big(H_2(a| - E_8| : \boldsymbol{Z}) \oplus H_2(b(S^2 \times S^2) : \boldsymbol{Z}), -aE_8 \oplus bH\big)$$

where σ is the intersection form of X and H is the one of $S^2 \times S^2$. We take an $SO(3)$ bundle $E \to X$. Let $w_2(E) = \hat{\alpha} = \hat{\alpha}_1 \oplus \hat{\alpha}_2$ and take $[\Sigma_i] = h_i^1 \oplus h_i^2 \in H_2(X : \boldsymbol{Z})$. Then we get the following:

THEOREM 8.1. *Suppose all* $h_i^1 = 0$, $1 \leq i \leq m$ *and* $\hat{\alpha}_i \neq 0$, $i = 1, 2$. *Then for* $M = b(S^2 \times S^2)$, *there exists a pair* (M, R) *with the following property; suppose* $\check{M}$ *admits an asymptotically quasiconformal equivalence to* $(\hat{M}, g)$. *Then the Donaldson's invariant vanishes* $Q(E, X; [\Sigma_1], \ldots, [\Sigma_m]) = 0$.

PROOF. Let us fix a marking of $H^2(X; \boldsymbol{Z})$ as above, and take a smooth embedding $\check{M} = b(S^2 \times S^2) \backslash \bar{R} \hookrightarrow X$ by a construction using Casson handles. Where $\bar{R}$ is a topologically closed disk, and this embedding is compatible with the forms. By taking any integral lift $\alpha \in H^2(X; \boldsymbol{Z})$, it gives a lift of the $SO(3)$ bundle on another $U(2)$ bundle $\bar{E}$. Then for any $h \in H_2(X : \boldsymbol{Z})$ and its mod 2 reduction $[h]_2$, one has the equality $\langle w_2(E), [h]_2 \rangle = \langle c_1(\bar{E}), h \rangle \bmod 2$.

Suppose $\check{M}$ is an open manifold on which one can equip an asymptotically quasiconformal homeomorphism to $(\hat{M}, g)$. Choose any complete Riemannian metrics of bounded geometry h on the interiors of $N = X \backslash \check{M}$.

Let $\Sigma_1, \ldots, \Sigma_{m'} \subset \hat{M}$ be embedded surfaces, and take another $SO(3)$ bundle $E' \to \hat{M}$. Then one has a smooth moduli spaces $\mathfrak{M}(\hat{M}, E')$ of B^1 ASD connections over E' of dimension $2m$ and $\mathfrak{M}(E', \hat{M}; [\Sigma_1], \ldots, [\Sigma_{m'}])$ defined similarly as before. This space has a formal dimension $2(m - m')$.

Now let $\Sigma_1, \ldots, \Sigma_m \subset \check{M} \subset X$ be as in the theorem. One may assume the general position of these; for any $\Sigma_i, \Sigma_j, \Sigma_k$, one has empty intersection $\Sigma_i \cap \Sigma_j \cap \Sigma_k = \phi$. Suppose the Donaldson's invariant does not vanish $Q(E, X; [\Sigma_1], \ldots, [\Sigma_m]) \neq 0$. This shows that for a generic choice of Riemannian metric h

on X, and also generic choice of these surfaces, the 0 dimensional moduli space $\mathfrak{M}(E, X; [\Sigma_1], \dots, [\Sigma_m])$ is non-empty, where its algebraic number is equal to the above invariant Q.

Let us choose exhausting families of compact subsets as:

$$K_0 \subset\subset K_1 \subset\subset \cdots \subset \check{M}, \quad K'_0 \subset\subset K'_1 \subset\subset \cdots \subset N = X \backslash \check{M}.$$

Choose a family of smooth Riemannian metrics g'_l on X with $(I_l^{-1})^* g'_l \sim g$, $g'_l|K'_l = h|K'_l$ as in the definition of ACH, and choose a family of small constants $\delta_l > 0$, $l = 0, 1, \dots$, with $\delta_l \to 0$. In order to construct a smooth $\mathfrak{M}(E, X; [\Sigma_1], \dots, [\Sigma_m])$, one needs to perturb g'_l slightly. Let us choose a family of generic Riemannian metrics g_l on X with $|g_l - g'_l|W_k^2(K_l \cup K'_l) \leq \delta_l$. Then one has a family of non-empty smooth manifolds:

$$\big\{\mathfrak{M}(E, (X, g_l); [\Sigma_1], \dots, [\Sigma_m])\big\}_{l \geq 0}.$$

Let us choose any sequence $[A_l] \in \mathfrak{M}(E, (X, g_l); [\Sigma_1], \dots, [\Sigma_m])$, $l = 0, 1, \dots$. Then there is a subsequence which we also denote by $[A_l]$ so that $(I_l^{-1})^*(A_l)$ converges to $[A_\infty] \in \mathfrak{M}(\hat{M}, \epsilon')$, an ASD connection by Proposition 8.1. On the other hand, $A_l \mid N$ converges to an $h-$ASD connection A'_∞ over (N, h) with $|F_{A'_\infty}|L^2 < \infty$.

Now let $\{p_1, \dots, p_s\} \subset \check{M}$ be the bubbling points. Then at most $2s$ surfaces can contain these points. Thus by reordering surfaces, one has:

$$A_\infty \in \mathfrak{M}\big(E', \hat{M}, \epsilon'; [\Sigma_1], \dots, [\Sigma_{m-2s}]\big).$$

Let us see that A_∞ defines an $SO(3)$ bundle E' on $\hat{M}$ with strictly smaller $|p_1(E')|$ than $|p_1(E)|$. This follows from positivity $|F_{A'_\infty}|L^2(N, h) > 0$, which can be seen as below. One has chosen E so that each $\hat{\alpha}_j \neq 0$ where $w_2(E) = \hat{\alpha} = \sum_{j=1}^2 \hat{\alpha}_j \in \oplus_a \boldsymbol{Z}_2^8 \oplus_{2b} \boldsymbol{Z}_2 \cong H^2(X; \boldsymbol{Z}_2)$. Let $\alpha_j \in H^2(X; \boldsymbol{Z})$ be the corresponding lift. Let us take homology classes $h = h_1 + h_2 \in H_2(X; \boldsymbol{Z})$ such that the mod 2 reduction of $\langle \alpha_j, h_j \rangle_2 \in \boldsymbol{Z}_2$ are both non-zero $(= 1)$. Notice $w_2(E') = \hat{\alpha}_1 \in H^2(\hat{M}; \boldsymbol{Z}_2)$. Let $\Sigma_j \subset X$ be embedded surfaces which represent h_j. $E \mid \Sigma_1$ is isomorphic to $E' \mid \Sigma_1$. In particular one has $w_2(E') \neq 0$. If the family $\{F_{A_i}\}_i$ concentrates at some points $p \in N$, then one has $|p_1(E')| \leq |p_1(E)| - 4$, which follows from the following two properties: (1) p_1 is the minus of the L^2 curvature norms of ASD connections, and (2) the L^2 curvature norms of ASD connections over bundles with non zero w_2 is non zero over simply connected manifolds.

Now suppose it does not concentrate on N. Let $\Sigma_2 \subset N$ as above. Then one has the equality $\langle w_2(E), \Sigma_2 \rangle = \langle c_1(\bar{E}), \Sigma_2 \rangle = \langle c_1(A_i), \Sigma_2 \rangle \neq 0 \bmod 2$. Since there are no concentrating points, one has also $\langle c_1(A'_\infty), \Sigma_2\rangle \neq 0$. Then clearly $\limsup_i |F_{A_i}|L^2(N, g_i \,|\, N) \geq 4\pi$. Thus in any situation, one has $|p_1(E')| \leq |p_1(E)| - 2$.

Now one has $\mathfrak{M}(E, \hat{M}; [\Sigma_1], \ldots, [\Sigma_{m-2s}])$, non-empty moduli space of ASD connections. One may assume that this space contains no reducible orbits. Recall $\dim \mathfrak{M} \leq \dim \operatorname{Ker} d_A^+ / \operatorname{im} d_A - 4 - 2m$, while $\dim \operatorname{Ker} d_A^+ / \operatorname{im} d_A - 2m = 0$. Thus the formal dimension of this space is negative. This is a contradiction. This completes the proof of the Theorem 8.1. □

Proof of Theorem 0.2. First we will find some X, an $SO(3)$ bundle $E \to X$ and some embedded surface $\Sigma \subset X$ with non-zero Donaldson's invariant $Q(X, E, [\Sigma], \ldots, [\Sigma]) \neq 0$. Donaldson verified that one finds such pair (X, E) if X is a simply connected algebraic surface, $|p_1(E)|$ is sufficiently large and $w_2(E)$ is mod 2 reduction of $(1,1)$ form.

Next one can choose a particular marking $(H_2(X : \boldsymbol{Z}), \sigma) \cong (H_2(a| - E_8| : \boldsymbol{Z}) \oplus H_2(b(S^2 \times S^2) : \boldsymbol{Z}), -aE_8 \oplus bH)$ so that for the corresponding decomposition $[\Sigma] = h_1 + h_2$ and $w_2(E) = \alpha_1 + \alpha_2$, h_1 vanishes and $\alpha_i \neq 0$, if the Picard number of X is sufficiently large. This follows from the existence of appropriate automorphisms of even type lattices. For example projective Kummer surface satisfies this condition (see [**K1**, Section 3]).

The combination of these two steps completes the proof of Theorem 0.2. □

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