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Loewner matrices of matrix convex and monotone functions

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Abstract. The matrix convexity and the matrix monotony of a real C^1 function f on $(0,\infty)$ are characterized in terms of the conditional negative or positive definiteness of the Loewner matrices associated with f, $tf(t)$, and $t^2f(t)$. Similar characterizations are also obtained for matrix monotone functions on a finite interval (a,b).

Introduction.

In matrix/operator analysis quite important are the notions of matrix/operator monotone and convex functions initiated in 1930's by Löwner [**12**] and Kraus [**11**]. For a real C^1 function on an interval (a,b) it was proved in [**12**] that f is matrix monotone of order n (i.e., $A \le B$ implies $f(A) \le f(B)$ for $n \times n$ Hermitian matrices A, B with eigenvalues in (a,b)) if and only if the matrix

$$L_f(t_1,\dots,t_n) := \left[\frac{f(t_i)-f(t_j)}{t_i-t_j}\right]_{i,j=1}^n$$

of divided differences of f is positive semidefinite for any choice of $t_1,\dots,t_n$ from (a,b). The above matrix $L_f(t_1,\dots,t_n)$ is called the Pick matrix or else the Loewner (= Löwner) matrix associated with f. The characterization of matrix convex functions of similar kind was obtained in [**11**] in terms of divided differences of the second order. Almost a half century later in 1982 a modern treatment of operator (but not matrix) convex functions was developed by Hansen and Pedersen [**7**]. The most readable exposition on the subject is found in [**2**].

Recently in [**3**] Bhatia and the second-named author of this paper presented new characterizations for operator convexity of nonnegative functions on $[0,\infty)$ in terms of the conditional negative or positive definiteness (whose definitions are in Section 1) of the Loewner matrices. More precisely, the main results in [**3**] are

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stated as follows: A nonnegative C^2 function f on $[0,\infty)$ with $f(0)=f'(0)=0$ is operator convex if and only if $L_f(t_1,\dots,t_n)$ is conditionally negative definite for all $t_1,\dots,t_n>0$ of any size n. Moreover, if f is a nonnegative C^3 function on $[0,\infty)$ with $f(0)=f'(0)=f''(0)=0$, then $f(t)/t$ is operator convex if and only if $L_f(t_1,\dots,t_n)$ is conditionally positive definite for all $t_1,\dots,t_n>0$ of any size n. More recently, Uchiyama [**14**] extended, by a rather different method, the first result stated above in such a way that a real C^1 function f on $(0,\infty)$ is operator convex if and only if $L_f(t_1,\dots,t_n)$ is conditionally negative definite for all $t_1,\dots,t_n>0$ of any size n and $\lim_{t\to\infty} f(t)/t>-\infty$. Here it should be noted that the conditional positive definiteness of the Loewner matrices and the matrix/operator monotony were related in [**10**] and [**6**, Chapter XV] for a real function on a general open interval (see Remark 2.8 for more details).

In the present paper we consider the following conditions for a real C^1 function f on $(0,\infty)$ and for each integer $n\ge 1$:

(a)$_n$ f is matrix convex of order n on $(0,\infty)$;
(b)$_n$ $\liminf_{t\to\infty} f(t)/t>-\infty$ and $L_f(t_1,\dots,t_n)$ is conditionally negative definite for all $t_1,\dots,t_n\in(0,\infty)$;
(c)$_n$ $\limsup_{t\searrow 0} tf(t)\ge 0$ and $L_{tf(t)}(t_1,\dots,t_n)$ is conditionally positive definite for all $t_1,\dots,t_n\in(0,\infty)$.

We improve the proof in [**3**] without use of integral representation of operator convex functions and prove the implications (a)$_{2n+1}$ $\Rightarrow$ (b)$_n$, (b)$_{4n+1}$ $\Rightarrow$ (a)$_n$, (a)$_{n+1}$ $\Rightarrow$ (c)$_n$, and (c)$_{2n+1}$ $\Rightarrow$ (a)$_n$. In this way, the results in [**3**] (also [**14**]) are refined to those in the matrix level.

The paper is organized as follows. In Section 1 we prepare several implications among a number of conditions related to matrix monotone and convex functions, providing technical part of the proofs of our theorems. Some essential part of those implications are from [**13**]. In Section 2 we prove the above stated theorem (Theorem 2.1) characterizing matrix convex functions on $(0,\infty)$ in terms of the conditional negative or positive definiteness of the Loewner matrices. Similar characterizations of matrix monotone functions on $(0,\infty)$ are also obtained (Theorem 2.6). In Section 3 our theorems are exemplified with the power functions t^α on $(0,\infty)$. (An elementary treatment of the conditional positive and negative definiteness of the Loewner matrices for those functions is found in [**4**].) Finally in Section 4, we further obtain similar characterizations of matrix monotone functions on a finite interval (a,b) by utilizing an operator monotone bijection between (a,b) and $(0,\infty)$.

1. Definitions and lemmas.

For $n \in \boldsymbol{N}$ let $\boldsymbol{M}_n$ denote the set of all $n \times n$ complex matrices. Let f be a continuous real function on an interval J of the real line. It is said that f is *matrix monotone of order* n (n*-monotone* for short) on J if

$$A \geq B \quad \text{implies} \quad f(A) \geq f(B) \tag{1.1}$$

for Hermitian matrices A, B in $\boldsymbol{M}_n$ with $\sigma(A), \sigma(B) \subset J$, where $\sigma(A)$ stands for the spectrum (the eigenvalues) of A. It is said that f is *matrix convex of order* n (n*-convex* for short) on J if

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B) \tag{1.2}$$

for all Hermitian $A, B \in \boldsymbol{M}_n$ with $\sigma(A), \sigma(B) \subset J$ and for all $\lambda \in (0,1)$. Also, f is said to be n*-concave* on J if $-f$ is n-convex on J. Furthermore, it is said that f is *operator monotone* on J if (1.1) holds for self-adjoint operators A, B in $B(\mathscr{H})$ with $\sigma(A), \sigma(B) \subset J$, and *operator convex* on J if (1.2) holds for all self-adjoint $A, B \in B(\mathscr{H})$ with $\sigma(A), \sigma(B) \subset J$ and for all $\lambda \in (0,1)$, where $B(\mathscr{H})$ is the set of all bounded operators on an infinite-dimensional (separable) Hilbert space $\mathscr{H}$. As is well known, f is operator monotone (resp., operator convex) on J if and only if it is n-monotone (resp., n-convex) on J for all $n \in \boldsymbol{N}$.

For each $n \in \boldsymbol{N}$ let $\boldsymbol{C}_0^n$ denote the subspace of $\boldsymbol{C}^n$ consisting of all $x = (x_1, \dots, x_n)^t \in \boldsymbol{C}^n$ such that $\sum_{i=1}^n x_i = 0$. A Hermitian matrix A in $\boldsymbol{M}_n$ is said to be *conditionally positive definite* (*c.p.d.* for short) if $\langle x, Ax \rangle \geq 0$ for all $x \in \boldsymbol{C}_0^n$, and *conditionally negative definite* (*c.n.d.* for short) if $-A$ is c.p.d. Let f be a real C^1 (i.e., continuously differentiable) function f on an interval (a,b) with $-\infty \leq a < b \leq \infty$. The *divided difference* of f is defined by

$$f^{[1]}(s,t) := \begin{cases} \dfrac{f(s)-f(t)}{s-t} & \text{if } s \neq t, \\ f'(s) & \text{if } s = t, \end{cases}$$

which is a continuous function on $(a,b)^2$ (see [**6**, Chapter I] for details on divided differences). For each $t_1, \dots, t_n \in (a,b)$, the *Loewner matrix* $L_f(t_1, \dots, t_n)$ associated with f (for $t_1, \dots, t_n$) is defined to be the $n \times n$ matrix whose (i,j)-entry is $f^{[1]}(t_i, t_j)$, i.e.,

$$L_f(t_1, \dots, t_n) := \left[f^{[1]}(t_i, t_j) \right]_{i,j=1}^n .$$

In the fundamental paper [**12**], Karl Löwner (later Charles Loewner) proved that, for a real C^1 function f on (a,b) and for each $n \in \boldsymbol{N}$, f is n-monotone on (a,b) if and only if $L_f(t_1,\dots,t_n)$ is positive semidefinite for any choice of $t_1,\dots,t_n$ from (a,b).

Let f be a continuous real function on $[0,\infty)$. For each $n \in \boldsymbol{N}$ we consider the following conditions:

(i)$_n$ f is n-monotone on $[0,\infty)$;
(ii)$_n$ f is n-concave on $[0,\infty)$;
(iii)$_n$ f is n-convex on $[0,\infty)$ and $f(0) \le 0$;
(iv)$_n$ $f(X^*AX) \le X^*f(A)X$ for all $A, X \in \boldsymbol{M}_n$ with $A \ge 0$ and $\|X\| \le 1$;
(v)$_n$ $f(t)/t$ is n-monotone on $(0,\infty)$.

When f is C^1 on $(0,\infty)$, we further consider the following conditions:

(vi)$_n$ $L_f(t_1,\dots,t_n)$ is c.n.d. for all $t_1,\dots,t_n \in (0,\infty)$;
(vii)$_n$ $L_{tf(t)}(t_1,\dots,t_n)$ is c.p.d. for all $t_1,\dots,t_n \in (0,\infty)$.

For a continuous real function f on $[0,\infty)$ such that $f(t) > 0$ for all $t > 0$, the following conditions are also considered:

(viii)$_n$ $t/f(t)$ is n-monotone on $(0,\infty)$;
(ix)$_n$ $t^2/f(t)$ is n-monotone on $(0,\infty)$.

In the rest of this section we present lemmas on several relations among the above conditions, which will be used in the next section. But they may be of some independent interest.

Lemma 1.1. *Let f be a continuous real function on $[0,\infty)$. Then for every $n \in \boldsymbol{N}$ the following implications hold*:

$$(\mathrm{iii})_{n+1} \Longrightarrow (\mathrm{iv})_n \Longleftrightarrow (\mathrm{v})_n, \qquad (\mathrm{v})_{2n} \Longrightarrow (\mathrm{iii})_n.$$

Proof. (iii)$_{n+1} \Rightarrow$ (v)$_n$ was shown in [**13**, Theorem 2.2], and (iv)$_n \Leftrightarrow$ (v)$_n$ was in [**13**, Theorem 2.1] while the following proof is comparatively simpler. Indeed, (iv)$_n \Rightarrow$ (v)$_n$ is seen from the proof of [**7**, Theorem 2.4]. Conversely, suppose (v)$_n$, and let $A \in \boldsymbol{M}_n$ be positive semidefinite and $X \in \boldsymbol{M}_n$ with $\|X\| \le 1$. We may assume that $A > 0$, and we further assume that X is invertible. Take the polar decomposition $A^{1/2}X = U|A^{1/2}X|$ and set $B := |X^*A^{1/2}|^2$. Then we have $B \le A$ and $B^{1/2} = U|A^{1/2}X|U^* = A^{1/2}XU^*$, so $A^{-1/2}B^{1/2} = XU^*$. Since $B^{-1/2}f(B)B^{-1/2} \le A^{-1/2}f(A)A^{-1/2}$, we have

$$f(B) \le B^{1/2}A^{-1/2}f(A)A^{-1/2}B^{1/2} = UX^*f(A)XU^*$$

and $f(B) = Uf(X^*AX)U^*$. Therefore, $f(X^*AX) \le X^*f(A)X$. When X is not invertible, choose a sequence $\varepsilon_k \to 0$ such that $X_k := (1+|\varepsilon_k|)^{-1}(X+\varepsilon_k I)$ is invertible for any k, and take the limit of $f(X_k^*AX_k) \le X_k^*f(A)X_k$. The remaining $(\mathrm{v})_{2n} \Rightarrow (\mathrm{iii})_n$ is seen from the proof of [**7**, Theorems 2.1 and 2.4]. □

LEMMA 1.2. *Let f be a continuous real function on $[0,\infty)$. Then for every $n \in \boldsymbol{N}$ the implication*

$$(\mathrm{i})_{2n} \Longrightarrow (\mathrm{ii})_n$$

holds. Moreover, if $f(t) > 0$ for all $t > 0$, then for every $n \in \boldsymbol{N}$ the following hold:

$$(\mathrm{ii})_n \Longrightarrow (\mathrm{i})_n, \qquad (\mathrm{i})_{2n} \Longrightarrow (\mathrm{viii})_n.$$

PROOF. $(\mathrm{i})_{2n} \Rightarrow (\mathrm{ii})_n$ is seen from the proof of [**14**, Theorem 2.4]. Now assume that $f(t) > 0$ for all $t > 0$. Then $(\mathrm{ii})_n \Rightarrow (\mathrm{i})_n$ is seen from the proof of [**7**, Theorem 2.5]. Next, suppose $(\mathrm{i})_{2n}$. Since f is $2n$-monotone on $[0,\infty)$ with $-f \le 0$, the proof of [**7**, Theorem 2.5] shows that $-f$ satisfies $(\mathrm{iv})_n$ and hence $(\mathrm{v})_n$ by Lemma 1.1, so $-f(t)/t$ is n-monotone on $(0,\infty)$. Since $-t^{-1}$ is operator monotone on $(-\infty,0)$, it follows that $t/f(t) = -(-f(t)/t)^{-1}$ is n-monotone on $(0,\infty)$. Hence $(\mathrm{viii})_n$ follows. □

Let f be as in Lemma 1.2 such that $f(t) > 0$ for all $t > 0$. Since $(\mathrm{viii})_n$ is equivalent to the n-monotony of $-f(t)/t$ on $(0,\infty)$, we further have $(\mathrm{viii})_{2n} \Rightarrow (\mathrm{ii})_n$ and $(\mathrm{ii})_{n+1} \Rightarrow (\mathrm{viii})_n$ by applying Lemma 1.1 to $-f$, though not used in the rest of the paper.

LEMMA 1.3. *Let f be a continuous real function on $[0,\infty)$ such that $f(t) > 0$ for all $t > 0$. Then for every $n \in \boldsymbol{N}$ the following hold*:

$$(\mathrm{v})_{2n} \Longrightarrow (\mathrm{ix})_n, \qquad (\mathrm{ix})_{2n} \Longrightarrow (\mathrm{v})_n.$$

PROOF. Since $t^2/f(t) = t/(f(t)/t)$ and $f(t)/t = t/(t^2/f(t))$, the stated implications are immediately seen from $(\mathrm{i})_{2n} \Rightarrow (\mathrm{viii})_n$ of Lemma 1.2. □

LEMMA 1.4. *Let f be a real C^1 function on $[0,\infty)$ such that $f(t) > 0$ for all $t > 0$, $f(0) = 0$, and $f'(0) \ge 0$. Then for every $n \in \boldsymbol{N}$ the following implications hold*:

$$(\mathrm{vi})_{n+1} \Longrightarrow (\mathrm{ix})_n \Longrightarrow (\mathrm{vi})_n.$$

Proof. $(\mathrm{vi})_{n+1} \Rightarrow (\mathrm{ix})_n$. First, recall (see [**1**, p. 193] or [**6**, p. 134]) that if a Hermitian $(n+1)\times(n+1)$ matrix $[a_{ij}]_{i,j=1}^{n+1}$ is c.p.d., then the $n\times n$ matrix

$$\bigl[a_{ij} - a_{i,n+1} - a_{n+1,j} + a_{n+1,n+1}\bigr]_{i,j=1}^{n}$$

is positive semidefinite. Hence for every $t_1,\dots,t_n,t_{n+1} \in (0,\infty)$, assumption $(\mathrm{vi})_{n+1}$ implies that

$$\bigl[f^{[1]}(t_i,t_j) - f^{[1]}(t_i,t_{n+1}) - f^{[1]}(t_j,t_{n+1}) + f'(t_{n+1})\bigr]_{i,j=1}^{n} \le 0.$$

Since $f(0)=0$, letting $t_{n+1} \searrow 0$ yields that

$$\left[f^{[1]}(t_i,t_j) - \frac{f(t_i)}{t_i} - \frac{f(t_j)}{t_j} + f'(0)\right]_{i,j=1}^{n} \le 0.$$

Since

$$f^{[1]}(t_i,t_j) - \frac{f(t_i)}{t_i} - \frac{f(t_j)}{t_j} = -\frac{f(t_i)}{t_i}\cdot\left(\frac{t^2}{f(t)}\right)^{[1]}(t_i,t_j)\cdot\frac{f(t_j)}{t_j}, \tag{1.3}$$

we see that

$$\left[\frac{f(t_i)}{t_i}\cdot\left(\frac{t^2}{f(t)}\right)^{[1]}(t_i,t_j)\cdot\frac{f(t_j)}{t_j}\right]_{i,j=1}^{n} - f'(0)E_n \ge 0,$$

where E_n stands for the $n\times n$ matrix of all entries equal to 1. Since $f'(0)\ge 0$, we have $L_{t^2/f(t)}(t_1,\dots,t_n) \ge 0$, which yields $(\mathrm{ix})_n$ by Löwner's theorem.

$(\mathrm{ix})_n \Rightarrow (\mathrm{vi})_n$. For every $t_1,\dots,t_n \in (0,\infty)$, it follows from (1.3) that

$$L_f(t_1,\dots,t_n) = -\left[\frac{f(t_i)}{t_i}\cdot\left(\frac{t^2}{f(t)}\right)^{[1]}(t_i,t_j)\cdot\frac{f(t_j)}{t_j}\right]_{i,j=1}^{n} + \left[\frac{f(t_i)}{t_i} + \frac{f(t_j)}{t_j}\right]_{i,j=1}^{n}.$$

Since $L_{t^2/f(t)}(t_1,\dots,t_n) \ge 0$ by assumption $(\mathrm{ix})_n$, the above expression yields that $L_f(t_1,\dots,t_n)$ is c.n.d. □

The proof of the next lemma is a modification of the argument in [**10**, p. 428].

Lemma 1.5. *Let f be a continuous real function on $[0,\infty)$ with $f(0)=0$ such that f is C^1 on $(0,\infty)$ and $\lim_{t\searrow 0} tf'(t) = 0$. (This is the case if f is C^1 on*

$[0,\infty)$ *with* $f(0)=0$.) *Then for every* $n\in\boldsymbol{N}$ *the following implications hold*:

$$(\mathrm{vii})_{n+1} \Longrightarrow (\mathrm{v})_n \Longrightarrow (\mathrm{vii})_n.$$

Proof. $(\mathrm{vii})_{n+1}\Rightarrow(\mathrm{v})_n$. Set $g(t):=tf(t)$ for $t\in[0,\infty)$ and for each $\varepsilon>0$ define

$$g_\varepsilon(t):=g(t+\varepsilon)-g(\varepsilon)-g'(\varepsilon)t, \qquad t\in[0,\infty).$$

Then g_ε is C^1 on $[0,\infty)$ and $g_\varepsilon(0)=g_\varepsilon'(0)=0$. From assumption $(\mathrm{vii})_{n+1}$ it follows that $L_{g_\varepsilon}(t_1,\dots,t_n,t_{n+1})$ is c.p.d. for every $t_1,\dots,t_n,t_{n+1}\in(0,\infty)$. Hence similarly to the proof of Lemma 1.4 we have

$$\left[g_\varepsilon^{[1]}(t_i,t_j)-\frac{g_\varepsilon(t_i)}{t_i}-\frac{g_\varepsilon(t_j)}{t_j}\right]_{i,j=1}^n\ge 0$$

for every $t_1,\dots,t_n\in(0,\infty)$. Since

$$g_\varepsilon^{[1]}(t_i,t_j)-\frac{g_\varepsilon(t_i)}{t_i}-\frac{g_\varepsilon(t_j)}{t_j}=t_i\cdot\left(\frac{g_\varepsilon(t)}{t^2}\right)^{[1]}(t_i,t_j)\cdot t_j, \tag{1.4}$$

we see that $L_{g_\varepsilon(t)/t^2}(t_1,\dots,t_n)\ge 0$. Since $g(\varepsilon)\to 0$ and $g'(\varepsilon)=f(\varepsilon)+\varepsilon f'(\varepsilon)\to 0$ as $\varepsilon\searrow 0$ thanks to assumption on f, it follows that $g_\varepsilon(t)/t^2\to g(t)/t^2=f(t)/t$ as $\varepsilon\searrow 0$ for any $t>0$. Hence we have $L_{f(t)/t}(t_1,\dots,t_n)\ge 0$, which yields $(\mathrm{v})_n$.

$(\mathrm{v})_n\Rightarrow(\mathrm{vii})_n$. Let g be as above. For every $t_1,\dots,t_n\in(0,\infty)$, from (1.4) for g instead of g_ε we have

$$L_g(t_1,\dots,t_n)=\left[t_i\cdot\left(\frac{f(t)}{t}\right)^{[1]}(t_i,t_j)\cdot t_j\right]_{i,j=1}^n+\left[\frac{g(t_i)}{t_i}+\frac{g(t_j)}{t_j}\right]_{i,j=1}^n,$$

which is c.p.d. due to $(\mathrm{v})_n$. □

2. Functions on $(0,\infty)$.

The aim of this section is to relate the n-convexity and the n-monotony of a C^1 function on $(0,\infty)$ to the c.p.d. and the c.n.d. of the Loewner matrices associated with certain corresponding functions. The first theorem is concerned with n-convex functions on $(0,\infty)$.

Theorem 2.1. *Let f be a real C^1 function on $(0,\infty)$. For each $n\in\boldsymbol{N}$*

consider the following conditions:

(a)$_n$ *f is n-convex on* $(0,\infty)$;
(b)$_n$ $\liminf_{t\to\infty} f(t)/t > -\infty$ *and* $L_f(t_1,\dots,t_n)$ *is c.n.d. for all* $t_1,\dots,t_n \in (0,\infty)$;
(c)$_n$ $\limsup_{t\searrow 0} tf(t) \ge 0$ *and* $L_{tf(t)}(t_1,\dots,t_n)$ *is c.p.d. for all* $t_1,\dots,t_n \in (0,\infty)$.

Then for every $n \in \boldsymbol{N}$ the following implications hold:

$$(\mathrm{a})_{2n+1} \Longrightarrow (\mathrm{b})_n, \quad (\mathrm{b})_{4n+1} \Longrightarrow (\mathrm{a})_n, \quad (\mathrm{a})_{n+1} \Longrightarrow (\mathrm{c})_n, \quad (\mathrm{c})_{2n+1} \Longrightarrow (\mathrm{a})_n.$$

Proof. First, note that $\lim_{t\to\infty} f(t)/t > -\infty$ (the limit may be $+\infty$) and $\liminf_{t\searrow 0} tf(t) \ge 0$, slightly stronger than the boundary conditions in (b)$_n$ and (c)$_n$, are satisfied as long as f satisfies (a)$_1$, i.e., f is convex as a numerical function on $(0,\infty)$. When $\liminf_{t\to\infty} f(t)/t > -\infty$, for any $\varepsilon > 0$ it follows that

$$\inf_{t\in(0,\infty)} \frac{f(t+\varepsilon) - f(\varepsilon)}{t} > -\infty.$$

So one can choose a $\gamma_\varepsilon \in \boldsymbol{R}$ smaller than the above infimum and define

$$f_\varepsilon(t) := f(t+\varepsilon) - f(\varepsilon) - \gamma_\varepsilon t, \qquad t \in [0,\infty),$$

so that $f_\varepsilon(t) > 0$ for all $t \in (0,\infty)$, $f_\varepsilon(0) = 0$ and $f_\varepsilon'(0) > 0$. In the proof below, f_ε will be such a function chosen for each $\varepsilon > 0$.

(a)$_{2n+1}$ $\Rightarrow$ (b)$_n$. For any $\varepsilon > 0$, since (a)$_{2n+1}$ implies that f_ε is $(2n+1)$-convex on $[0,\infty)$, one can apply (iii)$_{2n+1}$ $\Rightarrow$ (v)$_{2n}$ $\Rightarrow$ (ix)$_n$ $\Rightarrow$ (vi)$_n$ of Lemmas 1.1, 1.3, and 1.4 to f_ε so that $L_{f_\varepsilon}(t_1,\dots,t_n)$ is c.n.d. for every $t_1,\dots,t_n \in (0,\infty)$. Since

$$L_{f_\varepsilon}(t_1,\dots,t_n) = L_f(t_1+\varepsilon,\dots,t_n+\varepsilon) - \gamma_\varepsilon E_n, \tag{2.1}$$

it follows that $L_f(t_1+\varepsilon,\dots,t_n+\varepsilon)$ is c.n.d. Hence (b)$_n$ holds since $\varepsilon > 0$ is arbitrary.

(b)$_{4n+1}$ $\Rightarrow$ (a)$_n$. For any $\varepsilon > 0$, thanks to (2.1) with $4n+1$ in place of n, it follows from (b)$_{4n+1}$ that (vi)$_{4n+1}$ is satisfied for f_ε. So one can apply (vi)$_{4n+1}$ $\Rightarrow$ (ix)$_{4n}$ $\Rightarrow$ (v)$_{2n}$ $\Rightarrow$ (iii)$_n$ of Lemmas 1.4, 1.3, and 1.1 to f_ε so that f_ε is n-convex on $[0,\infty)$. Hence $f(t+\varepsilon)$ is n-convex on $[0,\infty)$ so that (a)$_n$ follows since $\varepsilon > 0$ is arbitrary.

(a)$_{n+1}$ $\Rightarrow$ (c)$_n$. For any $\varepsilon > 0$, since f_ε is $(n+1)$-convex on $[0,\infty)$, we can apply (iii)$_{n+1}$ $\Rightarrow$ (v)$_n$ $\Rightarrow$ (vii)$_n$ of Lemmas 1.1 and 1.5 to f_ε, so $L_{tf_\varepsilon(t)}(t_1,\dots,t_n)$ is c.p.d. for every $t_1,\dots,t_n \in (0,\infty)$. Since

$$L_{tf_\varepsilon(t)}(t_1,\dots,t_n)=L_{tf(t+\varepsilon)}(t_1,\dots,t_n)-f(\varepsilon)E_n-\gamma_\varepsilon\bigl[t_i+t_j\bigr]_{i,j=1}^n,$$

we see that $L_{tf(t+\varepsilon)}(t_1,\dots,t_n)$ is c.p.d. Furthermore, since $tf(t+\varepsilon)\to tf(t)$ and

$$(tf(t+\varepsilon))'=f(t+\varepsilon)+tf'(t+\varepsilon)\longrightarrow f(t)+tf'(t)=(tf(t))'$$

as $\varepsilon\searrow 0$ for any $t>0$, it follows that $L_{tf(t)}(t_1,\dots,t_n)$ is c.p.d. Hence (c)$_n$ holds.

(c)$_{2n+1}$ $\Rightarrow$ (a)$_n$. Let $g(t):=tf(t)$ for $t\in(0,\infty)$. Since $\limsup_{t\searrow 0}g(t)\ge 0$ by assumption, one can choose a sequence $\varepsilon_k\searrow 0$ in such a way that $g(\varepsilon_k)>0$ for all k when $\limsup_{t\searrow 0}g(t)>0$, or else $\lim_{k\to\infty}g(\varepsilon_k)=0$ when $\limsup_{t\searrow 0}g(t)=0$. Define

$$g_k(t):=g(t+\varepsilon_k)-g(\varepsilon_k)-g'(\varepsilon_k)t,\qquad t\in[0,\infty).$$

Thanks to $\lim_{t\searrow 0}g_k(t)/t=0$, g_k is written as $g_k(t)=tf_k(t)$ with a continuous function f_k on $[0,\infty)$ with $f_k(0)=0$. Notice that f_k is obviously C^1 on $(0,\infty)$ and furthermore

$$\begin{aligned}tf_k'(t)&=g_k'(t)-\frac{g_k(t)}{t}\\&=(g'(t+\varepsilon_k)-g'(\varepsilon_k))-\biggl(\frac{g(t+\varepsilon_k)-g(\varepsilon_k)}{t}-g'(\varepsilon_k)\biggr)\\&\longrightarrow 0\quad\text{as } t\searrow 0.\end{aligned}$$

Since (c)$_{2n+1}$ implies that (vii)$_{2n+1}$ is satisfied for f_k, we can apply (vii)$_{2n+1}$ $\Rightarrow$ (v)$_{2n}$ $\Rightarrow$ (iii)$_n$ of Lemmas 1.5 and 1.1 to f_k so that f_k is n-convex on $[0,\infty)$. Writing

$$f_k(t)=\frac{(t+\varepsilon_k)f(t+\varepsilon_k)}{t}-\frac{g(\varepsilon_k)}{t}-g'(\varepsilon_k),\qquad t>0,$$

we see that

$$\tilde f_k(t):=\frac{(t+\varepsilon_k)f(t+\varepsilon_k)}{t}-\frac{g(\varepsilon_k)}{t}$$

is n-convex on $(0,\infty)$. When $g(\varepsilon_k)>0$ for all k,

$$\frac{(t+\varepsilon_k)f(t+\varepsilon_k)}{t}=\tilde f_k(t)+\frac{g(\varepsilon_k)}{t}$$

is n-convex on $(0,\infty)$ since $g(\varepsilon_k)/t$ is operator convex on $(0,\infty)$. Furthermore, notice that $\lim_{k\to\infty}(t+\varepsilon_k)f(t+\varepsilon_k)/t = f(t)$ for all $t>0$. Hence $(\mathrm{a})_n$ holds. On the other hand, when $\lim_{k\to\infty} g(\varepsilon_k)=0$, we have $\lim_{k\to\infty}\tilde f_k(t)=f(t)$ for all $t>0$, and hence $(\mathrm{a})_n$ holds as well. □

The equivalence of the following (a)–(c) immediately follows from Theorem 2.1, which extends [**3**, Theorems 1.1, 1.2, 1.4, and 1.5]. The equivalence between (a) and (b) was proved in [**14**, Theorem 3.1] by a different method.

Corollary 2.2. *Let f be a real C^1 function on $(0,\infty)$. Then the following conditions are equivalent*:

(a) *f is operator convex on $(0,\infty)$;*
(b) *$\liminf_{t\to\infty} f(t)/t > -\infty$ and $L_f(t_1,\dots,t_n)$ is c.n.d. for all $n\in\mathbf{N}$ and all $t_1,\dots,t_n\in(0,\infty)$;*
(c) *$\limsup_{t\searrow 0} tf(t)\ge 0$ and $L_{tf(t)}(t_1,\dots,t_n)$ is c.p.d. for all $n\in\mathbf{N}$ and all $t_1,\dots,t_n\in(0,\infty)$.*

Moreover, if the above conditions are satisfied, then $\lim_{t\to\infty} f(t)/t$ and $\lim_{t\searrow 0} tf(t)$ exist in $(-\infty,\infty]$ and $[0,\infty)$, respectively.

Proof. It remains to show the last assertion. Assume that f is operator convex on $(0,\infty)$. Then $\lim_{t\to\infty} f(t)/t>-\infty$ is obvious as noted at the beginning of the proof of Theorem 2.1. Consider the function $g(t):=f^{[1]}(t,1)$ on $(0,\infty)$. Then the characterization of operator convex functions due to Kraus [**11**] says that g is operator monotone function on $(0,\infty)$ and so $g(t+1)$ is operator monotone on $(-1,1)$. By Löwner's theorem [**12**] (or [**2**, V.4.5]) we have the integral representation

$$g(t+1)=g(1)+g'(1)\int_{[-1,1]}\frac{t}{1-\lambda t}\,d\mu(\lambda),\qquad t\in(-1,1)$$

with a probability measure μ on $[-1,1]$. Letting $\alpha:=\mu(\{-1\})$ we write

$$(t+1)g(t+1)=g(1)(t+1)+\alpha g'(1)t+g'(1)\int_{(-1,1]}\frac{t(t+1)}{1-\lambda t}\,d\mu(\lambda),\qquad t\in(-1,1).$$

Since $(t+1)/(1-\lambda t)\le 1$ for all $\lambda\in(-1,1]$ and $t\in(-1,0]$, the Lebesgue convergence theorem yields that

$$\lim_{t\searrow 0} tg(t)=\lim_{t\searrow -1}(t+1)g(t+1)=-\alpha g'(1),$$

from which $\lim_{t\searrow 0} tf(t) = \alpha g'(1) \in [0,\infty)$ immediately follows. $\square$

REMARK 2.3. Concerning the operator convex functions $g_\lambda(t) := t^2/(1-\lambda t)$ on $(-1,1)$ with $\lambda \in [-1,1]$ (see [**2**, p. 134]), it was shown in [**4**, Theorem 3.1] that $L_{g_\lambda}(t_1,\dots,t_n)$ is c.n.d. if $\lambda \in [-1,0]$ and c.p.d. if $\lambda \in [0,1]$ for every $t_1,\dots,t_n \in (-1,1)$ of any size n. By considering $g_\lambda|_{(0,1)}$ and $-g_\lambda|_{(0,1)}$ with $\lambda \in (0,1)$, we see that neither (a) $\Rightarrow$ (b) nor (b) $\Rightarrow$ (a) of Corollary 2.2 can be extended to functions on a finite open interval $(0,b)$.

REMARK 2.4. The conditions $\liminf_{t\to\infty} f(t)/t > -\infty$ and $\limsup_{t\searrow 0} tf(t) \ge 0$ are obviously satisfied if $f(t) \ge 0$ for all $t > 0$. We remark that these boundary conditions are essential in Theorem 2.1 and Corollary 2.2, as seen from the following discussions.

When $1 \le \alpha \le 2$, the function t^α is operator convex on $(0,\infty)$. Hence Corollary 2.2 implies that $L_{t^{\alpha+1}}(t_1,\dots,t_n)$ is c.p.d. and so $L_{-t^{\alpha+1}}(t_1,\dots,t_n)$ is c.n.d. for all $t_1,\dots,t_n \in (0,\infty)$, $n \in \boldsymbol{N}$. However, $-t^{\alpha+1}$ is not operator convex (even not convex as a numerical function) on $(0,\infty)$. Note that $\lim_{t\to\infty}(-t^{\alpha+1})/t = -\infty$.

When $-1 \le \alpha \le 0$, the function t^α is operator convex on $(0,\infty)$. Hence Corollary 2.2 implies that $L_{t^\alpha}(t_1,\dots,t_n)$ is c.n.d. and so $L_{-t^\alpha}(t_1,\dots,t_n)$ is c.p.d. for all $t_1,\dots,t_n \in (0,\infty)$, $n \in \boldsymbol{N}$. However, $-t^{\alpha-1}$ is not operator convex (even not convex as a numerical function) on $(0,\infty)$. Note that $\lim_{t\searrow 0} t(-t^{\alpha-1}) \le -1$.

A problem arising from Theorem 2.1 would be to determine the minimal number $\nu(n)$ (resp., $\pi(n)$) of $m \in \boldsymbol{N}$ such that $(\mathrm{b})_m \Rightarrow (\mathrm{a})_n$ (resp., $(\mathrm{c})_m \Rightarrow (\mathrm{a})_n$) for all real C^1 functions on $(0,\infty)$. The problem does not seem easy even for the case $n = 2$ while $3 \le \nu(2) \le 9$ and $3 \le \pi(2) \le 5$ (see Proposition 3.1 for $(\mathrm{b})_2 \not\Rightarrow (\mathrm{a})_2$ and $(\mathrm{c})_2 \not\Rightarrow (\mathrm{a})_2$). In the case $n = 1$, the c.n.d. condition of $(\mathrm{b})_1$ and the c.p.d. condition of $(\mathrm{c})_1$ are void but $(\mathrm{a})_1$ means that f is simply convex on $(0,\infty)$. Hence the next proposition shows that $\nu(1) = \pi(1) = 2$, which will be used in the proof of the next theorem.

PROPOSITION 2.5. *Let f be a real C^1 function on $(0,\infty)$. Then for conditions $(\mathrm{a})_1$, $(\mathrm{b})_2$, and $(\mathrm{c})_2$ of Theorem 2.1 the following hold*:

$$(\mathrm{b})_2 \Longrightarrow (\mathrm{a})_1, \qquad (\mathrm{c})_2 \Longrightarrow (\mathrm{a})_1.$$

PROOF. $(\mathrm{b})_2 \Rightarrow (\mathrm{a})_1$. The c.n.d. condition of $(\mathrm{b})_2$ is equivalent to the concavity of f' on $(0,\infty)$ (see [**6**, p. 137, Lemma 3]). Now suppose that f' is not non-decreasing; then $\lim_{t\to\infty} f'(t) = -\infty$ from concavity. Hence for any $K > 0$ an $a > 0$ can be chosen so that $f'(s) < -K$ for all $s > a$. For every $t > a$, since

$$\frac{f(t)-f(a)}{t-a}=f'(s)<-K \quad \text{for some } s\in(a,t),$$

we have

$$\limsup_{t\to\infty}\frac{f(t)}{t}=\limsup_{t\to\infty}\frac{f(t)-f(a)}{t-a}\le -K,$$

which implies that $\lim_{t\to\infty} f(t)/t=-\infty$, contradicting the assumption. Hence f' is non-decreasing, so f is convex on $(0,\infty)$.

$(\mathrm{c})_2 \Rightarrow (\mathrm{a})_1$. Write $g(t):=tf(t)$ for $t\in(0,\infty)$. The c.p.d. condition of $(\mathrm{c})_2$ is equivalent to the convexity of g' on $(0,\infty)$. From this and the assumption $\limsup_{t\searrow 0} g(t)\ge 0$ it follows that $\lim_{t\searrow 0} g(t)$ exists and is in $[0,\infty)$. Hence we may assume that g is continuous on $[0,\infty)$ with $g(0)\ge 0$. Notice that

$$f(t)=\frac{g(t)}{t}=\frac{g(0)}{t}+\frac{1}{t}\int_0^t g'(s)\,ds=\frac{g(0)}{t}+\lim_{\varepsilon\searrow 0}\frac{1}{t}\int_0^t g'(s+\varepsilon)\,ds, \quad t>0.$$

Hence the conclusion follows from the fact [**5**] that if h is a continuous convex function on $[0,\infty)$, then the function $(1/t)\int_0^t h(s)\,ds$ is convex on $(0,\infty)$. For the convenience of the reader a short proof is given here. Indeed, such a function h can be approximated uniformly on each finite interval $[0,a]$ by functions of the form

$$\alpha t+\beta+\sum_{i=1}^k \alpha_i(t-\lambda_i)_+$$

with $\alpha,\beta\in\boldsymbol{R}$ and $\alpha_i,\lambda_i>0$, where $x_+:=\max\{x,0\}$ for $x\in\boldsymbol{R}$. Since the function $(1/t)\int_0^t (s-\lambda)_+\,ds=(t-\lambda)_+^2/2t$ is convex on $(0,\infty)$ for any $\lambda>0$, the assertion follows. □

Note that the converse of each implication of Proposition 2.5 is invalid. Indeed, for the second consider the function

$$f(t):=\begin{cases} t^2, & 0\le t\le 1,\\ 2t-1, & t\ge 1,\end{cases}$$

and the function t^3 for the first (see Proposition 3.1).

The next theorem is concerned with n-monotone functions on $(0,\infty)$.

THEOREM 2.6. *Let f be a real C^1 function on $(0,\infty)$. For each $n \in \boldsymbol{N}$ consider the following conditions*:

$(\mathrm{a})'_n$ *f is n-monotone on $(0,\infty)$;*
$(\mathrm{b})'_n$ $\limsup_{t\to\infty} f(t)/t < +\infty$, $\limsup_{t\to\infty} f(t) > -\infty$, *and $L_f(t_1,\dots,t_n)$ is c.p.d. for all $t_1,\dots,t_n \in (0,\infty)$;*
$(\mathrm{c})'_n$ $\liminf_{t\searrow 0} tf(t) \le 0$, $\limsup_{t\to\infty} f(t) > -\infty$, *and $L_{tf(t)}(t_1,\dots,t_n)$ is c.n.d. for all $t_1,\dots,t_n \in (0,\infty)$;*
$(\mathrm{d})'_n$ $\liminf_{t\searrow 0} tf(t) \le 0$, $\limsup_{t\searrow 0} t^2 f(t) \ge 0$, *and $L_{t^2 f(t)}(t_1,\dots,t_n)$ is c.p.d. for all $t_1,\dots,t_n \in (0,\infty)$.*

Then for every $n \in \boldsymbol{N}$ the following implications hold:

$$(\mathrm{a})'_n \Longrightarrow (\mathrm{b})'_n \ \text{if}\ n \ge 2, \quad (\mathrm{b})'_{4n+1} \Longrightarrow (\mathrm{a})'_n, \quad (\mathrm{a})'_{2n+2} \Longrightarrow (\mathrm{c})'_n, \quad (\mathrm{c})'_{2n+1} \Longrightarrow (\mathrm{a})'_n,$$

$$(\mathrm{a})'_n \Longrightarrow (\mathrm{d})'_n \ \text{if}\ n \ge 2, \quad (\mathrm{c})'_{2n+1} \Longrightarrow (\mathrm{d})'_n, \quad (\mathrm{d})'_{2n+1} \Longrightarrow (\mathrm{c})'_n.$$

PROOF. First, note that $\limsup_{t\searrow 0} tf(t) \le 0$ and $\lim_{t\to\infty} f(t) > -\infty$, slightly stronger than the corresponding conditions in $(\mathrm{b})'_n$–$(\mathrm{d})'_n$, are obvious as long as f satisfies $(\mathrm{a})'_1$, i.e., f is non-decreasing on $(0,\infty)$.

$(\mathrm{a})'_n \Rightarrow (\mathrm{b})'_n$ if $n \ge 2$. Suppose $(\mathrm{a})'_n$ with $n \ge 2$. The stated c.p.d. of L_f is a consequence of Löwner's theorem. Next, we show that $\lim_{t\to\infty} f(t)/t \in [0,\infty)$, slightly stronger than $\limsup_{t\to\infty} f(t)/t < +\infty$. By taking $f(t+1) - f(1) + 1$ we may assume that $f(t) > 0$ for all $t > 0$. Then it follows from $(\mathrm{i})_2 \Rightarrow (\mathrm{viii})_1$ of Lemma 1.2 that $t/f(t)$ is non-decreasing on $(0,\infty)$, so the conclusion follows.

$(\mathrm{b})'_{4n+1} \Rightarrow (\mathrm{a})'_n$. One can apply $(\mathrm{b})_{4n+1} \Rightarrow (\mathrm{a})_n$ of Theorem 2.1 to $-f$ to see that f is n-concave on $(0,\infty)$. Thanks to $\limsup_{t\to\infty} f(t) > -\infty$ this implies also that f is non-decreasing on $(0,\infty)$. For any $\varepsilon > 0$ let $f_\varepsilon(t) := f(t+\varepsilon) - f(\varepsilon) + 1$ for $t \ge 0$, and apply $(\mathrm{ii})_n \Rightarrow (\mathrm{i})_n$ of Lemma 1.2 to f_ε so that f_ε is n-monotone on $[0,\infty)$. Hence f is n-monotone on $(0,\infty)$ since $\varepsilon > 0$ is arbitrary.

$(\mathrm{a})'_{2n+2} \Rightarrow (\mathrm{c})'_n$. It follows from $(\mathrm{i})_{2n+2} \Rightarrow (\mathrm{ii})_{n+1}$ of Lemma 1.2 that f is $(n+1)$-concave on $(0,\infty)$. Now $(\mathrm{c})'_n$ is shown by applying $(\mathrm{a})_{n+1} \Rightarrow (\mathrm{c})_n$ of Theorem 2.1 to $-f$.

$(\mathrm{c})'_{2n+1} \Rightarrow (\mathrm{a})'_n$ is proved similarly to $(\mathrm{b})'_{4n+1} \Rightarrow (\mathrm{a})'_n$ above. Indeed, apply $(\mathrm{c})_{2n+1} \Rightarrow (\mathrm{a})_n$ of Theorem 2.1 to $-f$ and use Lemma 1.2 as above.

$(\mathrm{a})'_n \Rightarrow (\mathrm{d})'_n$ if $n \ge 2$. For any $\varepsilon > 0$, since $f(t+\varepsilon) = (tf(t+\varepsilon))/t$ is n-monotone on $(0,\infty)$, it follows from $(\mathrm{v})_2 \Rightarrow (\mathrm{iii})_1$ of Lemma 1.1 that $tf(t+\varepsilon)$ is convex on $[0,\infty)$. Letting $\varepsilon \searrow 0$ yields that $tf(t)$ is convex on $(0,\infty)$, from which we have $\liminf_{t\searrow 0} t^2 f(t) \ge 0$, slightly stronger than $\limsup_{t\searrow 0} t^2 f(t) \ge 0$. For each $\varepsilon > 0$ let $g_\varepsilon(t) := (t-\varepsilon)^2 f(t)$ for $t \in (0,\infty)$. Note that the second divided difference $g_\varepsilon^{[2]}(t,\varepsilon,\varepsilon)$ is nothing but $f(t)$, which is n-monotone on $(0,\infty)$

by assumption. Hence by [**6**, p. 139, Lemma 5] we see that $L_{g_\varepsilon}(t_1,\dots,t_n)$ is c.p.d. for all $t_1,\dots,t_n\in(0,\infty)$. Letting $\varepsilon\searrow 0$ yields the stated c.p.d. of $L_{t^2f(t)}$.

(c)$'_{2n+1}\Rightarrow$ (d)$'_n$. It was already shown that (c)$'_{2n+1}\Rightarrow$ (a)$'_n\Rightarrow$ (d)$'_n$ if $n\ge 2$. For $n=1$ the c.p.d. condition in (d)$'_1$ is void and the two boundary conditions hold since (c)$'_3\Rightarrow$ (a)$'_1$ (see the beginning of the proof).

(d)$'_{2n+1}\Rightarrow$ (c)$'_n$. Set $g(t):=tf(t)$ for $t>0$. Since (d)$'_{2n+1}$ implies that g satisfies (c)$_{2n+1}$ of Theorem 2.1, g is convex on $(0,\infty)$ by Proposition 2.5 (or by (c)$_{2n+1}\Rightarrow$ (a)$_n$ of Theorem 2.1), and so $\lim_{t\to\infty}f(t)>-\infty$. For each $\varepsilon>0$ choose a constant $\gamma_\varepsilon<g'(\varepsilon)$ and define

$$g_\varepsilon(t):=g(t+\varepsilon)-g(\varepsilon)-\gamma_\varepsilon t,\qquad t\in[0,\infty).$$

Since g_ε satisfies (vii)$_{2n+1}$, one can apply (vii)$_{2n+1}\Rightarrow$ (v)$_{2n}\Rightarrow$ (ix)$_n\Rightarrow$ (vi)$_n$ of Lemmas 1.5, 1.3, and 1.4 to g_ε so that $L_{g_\varepsilon}(t_1,\dots,t_n)$ is c.n.d. for all $t_1,\dots,t_n\in(0,\infty)$. This shows the asserted c.n.d. of $L_g=L_{tf(t)}$ by letting $\varepsilon\searrow 0$. $\square$

The equivalence of the following (a)$'$–(d)$'$ immediately follows from Theorem 2.6. In [**14**, Theorem 2.4], Uchiyama extended [**7**, Theorem 2.5] in such a way that a continuous function f on $(0,\infty)$ is operator monotone if and only if f is operator concave and $\lim_{t\to\infty}f(t)>-\infty$ (or $\limsup_{t\to\infty}f(t)>-\infty$). Due to this result, the equivalence of (a)$'$, (b)$'$, and (c)$'$ is also an immediate consequence of Corollary 2.2. Furthermore, the equivalence of (a)$'$, (c)$'$, and (d)$'$ extends [**3**, Theorems 1.1, 1.2, 1.4, and 1.5] as Corollary 2.2 does. The equivalence between (a)$'$ and (b)$'$ was proved in [**14**, Theorem 3.3] by a different method.

Corollary 2.7. *Let f be a real C^1 function on $(0,\infty)$. Then the following conditions are equivalent*:

(a)$'$ *f is operator monotone on $(0,\infty)$;*
(b)$'$ $\limsup_{t\to\infty}f(t)/t<+\infty$, $\limsup_{t\to\infty}f(t)>-\infty$, *and $L_f(t_1,\dots,t_n)$ is c.p.d. for all $n\in\boldsymbol{N}$ and all $t_1,\dots,t_n\in(0,\infty)$;*
(c)$'$ $\liminf_{t\searrow 0}tf(t)\le 0$, $\limsup_{t\to\infty}f(t)>-\infty$, *and $L_{tf(t)}(t_1,\dots,t_n)$ is c.n.d. for all $n\in\boldsymbol{N}$ and all $t_1,\dots,t_n\in(0,\infty)$;*
(d)$'$ $\liminf_{t\searrow 0}tf(t)\le 0$, $\limsup_{t\searrow 0}t^2f(t)\ge 0$, *and $L_{t^2f(t)}(t_1,\dots,t_n)$ is c.p.d. for all $n\in\boldsymbol{N}$ and all $t_1,\dots,t_n\in(0,\infty)$.*

Moreover, if the above conditions are satisfied, then $\lim_{t\to\infty}f(t)/t$, $\lim_{t\to\infty}f(t)$, *and* $\lim_{t\searrow 0}tf(t)$ *exist in* $[0,\infty)$, $(-\infty,\infty]$, *and* $(-\infty,0]$, *respectively, and* $\lim_{t\searrow 0}t^\alpha f(t)=0$ *for any* $\alpha>1$.

Proof. It remains to show the last assertion. Assume that f is operator monotone on $(0,\infty)$. The existence of $\lim_{t\to\infty}f(t)/t\in[0,\infty)$ and $\lim_{t\to\infty}f(t)\in$

$(-\infty, \infty]$ was seen in the proof of Theorem 2.6. Since (c)$'$ implies that $-f$ satisfies (c) of Corollary 2.2, the existence of $\lim_{t \searrow 0} tf(t) \in (-\infty, 0]$ follows from Corollary 2.2, so it is obvious that $\lim_{t \searrow 0} t^\alpha f(t) = 0$ if $\alpha > 1$. $\square$

Remark 2.8. In the proof of (a)$'_n \Rightarrow$ (d)$'_n$ (if $n \geq 2$) of Theorem 2.6 we used a result from [**6**, Chapter XV]. In this respect, the equivalence between (a)$'$ and (d)$'$ has a strong connection to [**10**, Theorem 10] and [**6**, p. 139, Theorem III], in which the following result was given: Let g be a C^1 function on an interval (a, b) and c any point in (a, b). Then $L_g(t_1, \dots, t_n)$ is c.p.d. for all $t_1, \dots, t_n \in (a, b)$ of any size n if and only if g is of the form

$$g(t) = g(c) + g'(c)(t - c) + (t - c)^2 f(t)$$

with an operator monotone function f on (a, b). This in particular says that a C^1 function f on (a, b) is operator monotone if and only if $L_{(t-c)^2 f(t)}(t_1, \dots, t_n)$ is c.p.d. for all $t_1, \dots, t_n \in (a, b)$, $n \in \boldsymbol{N}$. An essential difference between the last condition and (d)$'$ is that the point c is inside the domain of f for the former while it is the boundary point 0 of $(0, \infty)$ for the latter. So it does not seem easy to prove (a)$' \Leftrightarrow$ (d)$'$ based on the above result in [**10**], [**6**].

Remark 2.9. Consider operator monotone functions $f_\lambda(t) := t/(1 - \lambda t)$ on $(-1, 1)$ with $\lambda \in (-1, 1)$, so $tf_\lambda(t)$ is g_λ in Remark 2.3. By considering $f_\lambda|_{(0,1)}$ and $-f_\lambda|_{(0,1)}$ with $\lambda \in (0, 1)$, we see that neither (a)$' \Rightarrow$ (c)$'$ nor (c)$' \Rightarrow$ (a)$'$ of Corollary 2.7 can be extended to functions on a finite open interval $(0, b)$. Indeed, the right counterparts of Theorem 2.6 and Corollary 2.7 for functions on a finite interval (a, b) will be presented in Section 4 (see Theorem 4.1 and Corollary 4.2).

Remark 2.10. Any of boundary conditions as $t \searrow 0$ or $t \to \infty$ in Theorem 2.6 and Corollary 2.7 is essential. For instance, the functions t^3, t^{-1}, $-t$, and $-t^{-2}$ on $(0, \infty)$ are not 2-monotone; see Proposition 3.1 (1) for t^3 and $-t^{-2}$, and t^{-1} and $-t$ are even not increasing as a numerical function. By taking account of Proposition 3.1, the functions t^3 and $-t$ show that (b)$' \Rightarrow$ (a)$_2$ is not true without $\limsup_{t \to \infty} f(t)/t < +\infty$ and $\limsup_{t \to \infty} f(t) > -\infty$, respectively. Similarly, consider the functions t^{-1} and $-t$ to see that the two boundary conditions of (c)$'$ are essential for (c)$' \Rightarrow$ (a)$_2$, and the functions t^{-1} and $-t^{-2}$ for the two boundary conditions of (d)$'$.

3. Examples: power functions.

In this section we examine the conditions in Theorems 2.1 and 2.6 in the cases of lower orders $n = 2, 3$ for the power functions t^α on $(0, \infty)$. In fact, we sometimes used such examples of power functions in the preceding section, for instance, in

Remarks 2.4 and 2.10. Elementary discussions on the c.p.d. and c.n.d. properties of t^α based on the Cauchy matrix and the Schur product theorem are found in [**4**, Section 2].

Proposition 3.1. *Consider the power functions t^α on $(0,\infty)$, where $\alpha \in \boldsymbol{R}$. Then*:

(1) *t^α is 2-monotone if and only if $0 \le \alpha \le 1$, or equivalently, t^α is operator monotone. Moreover, $-t^\alpha$ is 2-monotone if and only if $-1 \le \alpha \le 0$.*
(2) *t^α is 2-convex if and only if either $-1 \le \alpha \le 0$ or $1 \le \alpha \le 2$, or equivalently, t^α is operator convex.*
(3) *$L_{t^\alpha}(t_1,t_2)$ is c.p.d. for all $t_1,t_2 \in (0,\infty)$ if and only if either $0 \le \alpha \le 1$ or $\alpha \ge 2$.*
(4) *$L_{t^\alpha}(t_1,t_2)$ is c.n.d. for all $t_1,t_2 \in (0,\infty)$ if and only if either $\alpha \le 0$ or $1 \le \alpha \le 2$.*
(5) *$L_{t^\alpha}(t_1,t_2,t_3)$ is c.p.d. for all $t_1,t_2,t_3 \in (0,\infty)$ if and only if either $0 \le \alpha \le 1$ or $2 \le \alpha \le 3$.*
(6) *$L_{t^\alpha}(t_1,t_2,t_3)$ is c.n.d. for all $t_1,t_2,t_3 \in (0,\infty)$ if and only if either $-1 \le \alpha \le 0$ or $1 \le \alpha \le 2$.*

Proof. For (1) and (2) see [**8**, Proposition 3.1]. Here note that $-t^\alpha$ is 2-monotone if and only if so is $t^{-\alpha} = -(-t^\alpha)^{-1}$. (3) and (4) are immediately seen from [**6**, p. 137, Lemma 3].

(5) If $0 \le \alpha \le 1$ or $2 \le \alpha \le 3$, then $t^{\alpha-1}$ is operator convex on $(0,\infty)$ and Corollary 2.2 implies the c.p.d. condition here. For the converse, since the c.p.d. of order three implies that of order two, we must have $0 \le \alpha \le 1$ or $\alpha \ge 2$ from (3). Moreover, one can easily check that a 3×3 real matrix $\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$ is c.p.d. (resp., c.n.d.) if and only if

$$\begin{gathered} a+c \ge 2e \quad (\text{resp., } a+c \le 2e), \\ b+c \ge 2f \quad (\text{resp., } b+c \le 2f), \\ (c+d-e-f)^2 \le (a+c-2e)(b+c-2f). \end{gathered}$$

For the c.p.d. of $L_{t^\alpha}(x,y,1)$ the latter condition in the above is written as

$$\begin{aligned} &\left(\alpha + \frac{x^\alpha - y^\alpha}{x-y} - \frac{x^\alpha - 1}{x-1} - \frac{y^\alpha - 1}{y-1}\right)^2 \\ &\quad \le \left(\alpha(x^{\alpha-1}+1) - 2\,\frac{x^\alpha - 1}{x-1}\right)\left(\alpha(y^{\alpha-1}+1) - 2\,\frac{y^\alpha - 1}{y-1}\right). \end{aligned} \tag{3.1}$$

Multiplying $(x-y)^2(x-1)^2(y-1)^2$ to the both sides of (3.1) gives

$$\begin{aligned}&\bigl(\alpha(x-y)(x-1)(y-1)+x^\alpha(y-1)^2-(x-1)y^\alpha(y-1)\\ &\qquad\qquad+(x-y)(y-1)-(x-y)(x-1)(y^\alpha-1)\bigr)^2\\ &\le(x-y)^2(x-1)(y-1)F_\alpha(x)F_\alpha(y),\end{aligned}\tag{3.2}$$

where

$$F_\alpha(x):=(\alpha-2)x^\alpha+\alpha x-\alpha x^{\alpha-1}-(\alpha-2).$$

When $\alpha>2$, the left-hand side of (3.2) has the term $x^{2\alpha}$ of maximal degree for x with positive coefficient $(y-1)^4$, and the right-hand side has the term $x^{\alpha+3}$ of maximal degree for x with coefficient $(\alpha-2)(y-1)F_\alpha(y)$ which is positive for large $y>0$. Hence $2\alpha\le\alpha+3$ or $\alpha\le 3$ is necessary for (3.1) to hold for all $x,y>0$. So we must have $0\le\alpha\le1$ or $2\le\alpha\le3$.

(6) If $-1\le\alpha\le0$ or $1\le\alpha\le2$, then t^α is operator convex on $(0,\infty)$ and Corollary 2.2 implies the c.n.d. condition here. Conversely, since the c.n.d. condition here implies that of order 2 in (4), we must have $\alpha\le0$ or $1\le\alpha\le2$ from (4). Moreover, (3.2) holds in this case too. When $\alpha<0$, the left-hand side of (3.2) has the term $x^{2\alpha}$ of maximal degree for $1/x$ with positive coefficient $(y-1)^4$, and the right-hand side has the term $x^{\alpha-1}$ of maximal degree for $1/x$ with coefficient $\alpha y^2(y-1)F_\alpha(y)$ which is positive for small $y>0$. Hence $2\alpha\ge\alpha-1$ or $\alpha\ge-1$ must hold, so we have $-1\le\alpha\le0$ or $1\le\alpha\le2$. □

Concerning the conditions of Theorem 2.1 the above proposition shows that $(\mathrm{b})_2\Rightarrow(\mathrm{a})_2$, $(\mathrm{c})_2\Rightarrow(\mathrm{a})_2$, $(\mathrm{b})_2\Rightarrow(\mathrm{c})_2$, and $(\mathrm{c})_2\Rightarrow(\mathrm{b})_2$ are all invalid while $(\mathrm{a})_2$, $(\mathrm{b})_3$, and $(\mathrm{c})_3$ are equivalent for the power functions t^α. Moreover, concerning Theorem 2.6, we notice from the proposition that, restricted to the power functions t^α, conditions $(\mathrm{a})'_2$, $(\mathrm{b})'_2$, and $(\mathrm{c})'_2$ are equivalent but $(\mathrm{d})'_2$ is strictly weaker.

4. Functions on (a,b).

For a real C^1 function f on (a,∞) where $-\infty<a<\infty$, we have the same implications as in Theorem 2.6 with slight modifications of $(\mathrm{a})'_n$–$(\mathrm{d})'_n$ by applying the theorem to $f(t+a)$ on $(0,\infty)$. For example, $(\mathrm{a})'_n$ and $(\mathrm{c})'_n$ are modified as

$(\mathrm{a})'_n$ f is n-monotone on (a,∞),

$(\mathrm{c})'_n$ $\liminf_{t\searrow a}(t-a)f(t)\le0$, $\limsup_{t\to\infty}f(t)>-\infty$, and $L_{(t-a)f(t)}(t_1,\dots,t_n)$ is c.n.d. for all $t_1,\dots,t_n\in(a,\infty)$,

and $(\mathrm{b})'_n$ and $(\mathrm{d})'_n$ are similarly modified.

Moreover, for a real C^1 function f on $(-\infty, b)$ where $-\infty < b < \infty$, one can apply Theorem 2.6 to $-f(b-t)$ on $(0,\infty)$ so that the same implications as there hold for the following conditions:

(a)$''_n$ f is n-monotone on $(-\infty, b)$;
(b)$''_n$ $\limsup_{t\to-\infty} f(t)/t < +\infty$, $\liminf_{t\to-\infty} f(t) < +\infty$, and $L_f(t_1,\dots,t_n)$ is c.p.d. for all $t_1,\dots,t_n \in (-\infty, b)$;
(c)$''_n$ $\limsup_{t\nearrow b}(b-t)f(t) \ge 0$, $\liminf_{t\to-\infty} f(t) < +\infty$, and $L_{(b-t)f(t)}(t_1,\dots,t_n)$ is c.n.d. for all $t_1,\dots,t_n \in (-\infty, b)$;
(d)$''_n$ $\limsup_{t\nearrow b}(b-t)f(t) \ge 0$, $\liminf_{t\nearrow b}(b-t)^2 f(t) \le 0$, and $L_{(b-t)^2 f(t)}(t_1,\dots,t_n)$ is c.p.d. for all $t_1,\dots,t_n \in (-\infty, b)$.

The aim of this section is to prove the next theorem that is the counterpart of Theorem 2.6 for a real C^1 function on a finite open interval (a,b).

Theorem 4.1. *Let f be a real C^1 function on (a,b) where $-\infty < a < b < \infty$. For each $n \in \boldsymbol{N}$ consider the following conditions*:

$(\alpha)_n$ *f is n-monotone on (a,b);*
$(\beta)_n$ $\limsup_{t\nearrow b}(b-t)f(t) < +\infty$, $\limsup_{t\nearrow b} f(t) > -\infty$, *and* $L_{(b-t)^2 f(t)}(t_1,\dots,t_n)$ *is c.p.d. for all* $t_1,\dots,t_n \in (a,b)$;
$(\gamma)_n$ $\liminf_{t\searrow a}(t-a)f(t) \le 0$, $\limsup_{t\nearrow b} f(t) > -\infty$, *and* $L_{(t-a)(b-t)f(t)}(t_1,\dots,t_n)$ *is c.n.d. for all* $t_1,\dots,t_n \in (a,b)$;
$(\delta)_n$ $\liminf_{t\searrow a}(t-a)f(t) \le 0$, $\limsup_{t\searrow a}(t-a)^2 f(t) \ge 0$, *and* $L_{(t-a)^2 f(t)}(t_1,\dots,t_n)$ *is c.p.d. for all* $t_1,\dots,t_n \in (a,b)$.

Then for every $n \in \boldsymbol{N}$ the following implications hold:

$$(\alpha)_n \Longrightarrow (\beta)_n \ \text{if}\ n \ge 2, \quad (\beta)_{4n+1} \Longrightarrow (\alpha)_n, \quad (\alpha)_{2n+2} \Longrightarrow (\gamma)_n,$$
$$(\gamma)_{2n+1} \Longrightarrow (\alpha)_n, \quad (\alpha)_n \Longrightarrow (\delta)_n \ \text{if}\ n \ge 2, \quad (\gamma)_{2n+1} \Longrightarrow (\delta)_n, \quad (\delta)_{2n+1} \Longrightarrow (\gamma)_n.$$

Proof. Define a bijective function $\psi : (a,b) \to (0,\infty)$ by

$$\psi(t) := \frac{t-a}{b-t} = -1 + \frac{b-a}{b-t}, \qquad t \in (a,b),$$

and hence

$$\psi^{-1}(x) = \frac{bx+a}{x+1} = b - \frac{b-a}{x+1}, \qquad x \in (0,\infty).$$

Furthermore, define a C^1 function $\tilde{f}$ on $(0,\infty)$ by $\tilde{f}(x) := f(\psi^{-1}(x))$ for $x \in (0,\infty)$. The theorem immediately follows from Theorem 2.6 once we show that $(\alpha)_n$, $(\beta)_n$,

$(\gamma)_n$, and $(\delta)_n$ are equivalent, respectively, to $(\mathrm{a})'_n$, $(\mathrm{b})'_n$, $(\mathrm{c})'_n$, and $(\mathrm{d})'_n$ for $\tilde f$. First, the equivalence of $(\alpha)_n$ to $(\mathrm{a})'_n$ for $\tilde f$ is immediate since both ψ on (a,b) and ψ^{-1} on $(0,\infty)$ are operator monotone. The following equalities are easy to check:

$$\begin{aligned}
\limsup_{x\to\infty}\frac{\tilde f(x)}{x} &= \frac{1}{b-a}\limsup_{t\nearrow b}(b-t)f(t),\\
\limsup_{x\to\infty}\tilde f(x) &= \limsup_{t\nearrow b} f(t),\\
\liminf_{x\searrow 0} x\tilde f(x) &= \frac{1}{b-a}\liminf_{t\searrow a}(t-a)f(t),\\
\limsup_{x\searrow 0} x^2\tilde f(x) &= \frac{1}{(b-a)^2}\limsup_{t\searrow a}(t-a)^2 f(t).
\end{aligned}$$

Next, let $t_1,\dots,t_n\in(a,b)$ be arbitrary and let $x_i := \psi(t_i)$ for $i=1,\dots,n$. By direct computations we have

$$\begin{aligned}
&\tilde f^{[1]}(x_i,x_j)\\
&\quad= \frac{f(t_i)-f(t_j)}{\psi(t_i)-\psi(t_j)}\\
&\quad= \frac{1}{b-a}(b-t_i)f^{[1]}(t_i,t_j)(b-t_j)\\
&\quad= \frac{1}{b-a}\bigl\{((b-t)^2 f(t))^{[1]}(t_i,t_j)+(b-t_i)f(t_i)+(b-t_j)f(t_j)\bigr\},\\
&\bigl(x\tilde f(x)\bigr)^{[1]}(x_i,x_j)\\
&\quad= \frac{\psi(t_i)f(t_i)-\psi(t_j)f(t_j)}{\psi(t_i)-\psi(t_j)}\\
&\quad= \frac{1}{b-a}(b-t_i)\left(\frac{t-a}{b-t}\,f(t)\right)^{[1]}(t_i,t_j)(b-t_j)\\
&\quad= \frac{1}{b-a}\bigl\{\bigl((t-a)(b-t)f(t)\bigr)^{[1]}(t_i,t_j)+(t_i-a)f(t_i)+(t_j-a)f(t_j)\bigr\},\\
&\bigl(x^2\tilde f(x)\bigr)^{[1]}(x_i,x_j)\\
&\quad= \frac{\psi(t_i)^2 f(t_i)-\psi(t_j)^2 f(t_j)}{\psi(t_i)-\psi(t_j)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{b-a}(b-t_i)\left(\left(\frac{t-a}{b-t}\right)^2 f(t)\right)^{[1]}(t_i,t_j)(b-t_j) \\
&= \frac{1}{b-a}\left\{\left((t-a)^2 f(t)\right)^{[1]}(t_i,t_j) + \frac{(t_i-a)^2}{b-t_i}f(t_i) + \frac{(t_j-a)^2}{b-t_j}f(t_j)\right\}.
\end{aligned}
$$

It is seen from the above equalities that $(\beta)_n$, $(\gamma)_n$, and $(\delta)_n$ are equivalent, respectively, to $(\mathrm{b})'_n$, $(\mathrm{c})'_n$, and $(\mathrm{d})'_n$ for $\tilde{f}$. □

Corollary 4.2. *Let f be a real C^1 function on (a,b) where $-\infty < a < b < \infty$. Then the following conditions are equivalent*:

(α) *f is operator monotone on (a,b);*
(β) $\limsup_{t\nearrow b}(b-t)f(t) < +\infty$, $\limsup_{t\nearrow b} f(t) > -\infty$, *and* $L_{(b-t)^2f(t)}(t_1,\dots,t_n)$ *is c.p.d. for all $n \in \boldsymbol{N}$ and all $t_1,\dots,t_n \in (a,b)$;*
(γ) $\liminf_{t\searrow a}(t-a)f(t) \le 0$, $\limsup_{t\nearrow b} f(t) > -\infty$, *and* $L_{(t-a)(b-t)f(t)}(t_1,\dots,t_n)$ *is c.n.d. for all $n \in \boldsymbol{N}$ and all $t_1,\dots,t_n \in (a,b)$;*
(δ) $\liminf_{t\searrow a}(t-a)f(t) \le 0$, $\limsup_{t\searrow a}(t-a)^2 f(t) \ge 0$, *and* $L_{(t-a)^2f(t)}(t_1,\dots,t_n)$ *is c.p.d. for all $n \in \boldsymbol{N}$ and all $t_1,\dots,t_n \in (a,b)$.*

Remark 4.3. Let f be a C^1 function on a finite interval (a,b) and c be an arbitrary point in (a,b). As mentioned in Remark 2.8, it is known by [**10**], [**6**] that f is operator monotone on (a,b) if and only if $L_{(t-c)^2f(t)}(t_1,\dots,t_n)$ is c.p.d. for all $t_1,\dots,t_n \in (a,b)$, $n \in \boldsymbol{N}$. By letting $c \nearrow b$ and $c \searrow a$ it follows that (α) implies the c.p.d. conditions in (β) and (δ). Corollary 4.2 says that the c.p.d. of $L_{(t-c)^2f(t)}$ for the boundary point $c=b$ or $c=a$ with additional boundary conditions conversely implies the c.p.d. of $L_{(t-c)^2f(t)}$ for all $c \in (a,b)$. On the other hand, it is known (see [**9**, Corollary 2.7.8] and [**14**, Lemma 2.1]) that f is operator convex on (a,b) if and only if $f^{[1]}(c,\cdot)$ is opertor monotone on (a,b) for some $c \in (a,b)$. So one can also obtain characterizations of the operator convexity of f by applying Corollary 4.2 to $f^{[1]}(c,\cdot)$ when f is assumed to be C^2 on (a,b). However, such characterizations are not so immediate to the function f as those in Corollary 2.2 for f on $(0,\infty)$.

Remark 4.4. Let f_λ, $\lambda \in [-1,1]$, be operator monotone functions on $(-1,1)$ given in Remark 2.9, which are kernel functions in Löwner's integral representation for operator monotone functions on $(-1,1)$. Theorem 4.1 says that $L_{(1-t)^2f_\lambda(t)}(t_1,\dots,t_n)$ and $L_{(t+1)^2f_\lambda(t)}(t_1,\dots,t_n)$ are c.p.d. and $L_{(1-t^2)f_\lambda}(t_1,\dots,t_n)$ is c.n.d. for every $t_1,\dots,t_n \in (-1,1)$. Indeed, these can be directly checked by the following expressions for $\lambda \in [-1,1] \setminus \{0\}$:

$$\big((1-t)^2 f_\lambda(t)\big)^{[1]}(t_i,t_j) = -\frac{t_i+t_j}{\lambda} + \frac{2\lambda-1}{\lambda^2} + \frac{(\lambda^{-1}-1)^2}{(1-\lambda t_i)(1-\lambda t_j)},$$

$$\big((t+1)^2 f_\lambda(t)\big)^{[1]}(t_i,t_j) = -\frac{t_i+t_j}{\lambda} - \frac{2\lambda+1}{\lambda^2} + \frac{(\lambda^{-1}+1)^2}{(1-\lambda t_i)(1-\lambda t_j)},$$

$$\big((1-t^2) f_\lambda(t)\big)^{[1]}(t_i,t_j) = \frac{t_i+t_j}{\lambda} + \frac{1}{\lambda^2} - \frac{\lambda^{-2}-1}{(1-\lambda t_i)(1-\lambda t_j)}.$$

(The similar properties for $f_0(t) = t$ are also easy to check.) Moreover, if f is operator monotone on $(-1,1)$, then the boundary conditions as $t \nearrow 1$ or $t \searrow -1$ in (β)–(δ) are shown by Löwner's integral representation. Since an operator monotone functions on (a,b) is transformed into that on $(-1,1)$ by an affine function, the argument here supplies another (direct) proof of the implications from (α) to (β)–(δ) in Corollary 4.2. So the converse implications of these are of actual substance in the corollary.

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