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Holonomic systems of Gegenbauer type polynomials of matrix arguments related with Siegel modular forms

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Abstract. Differential operators on Siegel modular forms which behave well under the restriction of the domain are essentially intertwining operators of the tensor product of holomorphic discrete series to its irreducible components. These are characterized by polynomials in the tensor of pluriharmonic polynomials with some invariance properties. We give a concrete study of such polynomials in the case of the restriction from Siegel upper half space of degree $2n$ to the product of degree n. These generalize the Gegenbauer polynomials which appear for $n = 1$. We also describe their radial parts parametrization and differential equations which they satisfy, and show that these differential equations give holonomic systems of rank 2^n.

1. Introduction.

Differential operators acting on holomorphic Siegel modular forms on the Siegel upper half space H_n of degree n which preserves automorphy under the restriction to a natural subdomain $H_{n_1} \times \cdots \times H_{n_r}$ of H_n are important objects. They are often applied to the concrete or theoretical calculation of special values of L functions. But apart from their importance in the applications to number theory, they are interesting objects as themselves since they are sources of interesting special functions. For example, the classical Gegenbauer polynomials are included in this category as we can see in [**7**]. A certain characterization of such holomorphic linear differential operators with constant coefficients are given in [**13**]. These operators are naturally regarded as polynomials of partial derivations of independent variables of the domain and polynomials appearing here are characterized by certain pluriharmonic polynomials with some invariance property. Böcherer also studied this kind of operators in slightly different context in [**3**]. See also [**14**], [**15**].

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In this paper, we treat the case when domains are $H_n \times H_n \subset H_{2n}$. After reviewing our motivation for Siegel modular forms in Section 2, we will study the above mentioned invariant pluriharmonic polynomials. These polynomials have essentially two properties. One is a certain invariance by $GL(n) \times GL(n)$ and $O(d)$ and the other is pluriharmonicity. In Section 3, we study generators of polynomials which satisfy the above invariance (cf. Proposition 3.1) and in Section 4, we study concrete conditions for pluriharmonicity described by certain differential equations (cf. Proposition 4.3). We also give an explicit way to construct such polynomials, and review some generating functions for small n. The usual Gegenbauer polynomials appear in this context when $n = 1$ as radial parts of the above polynomials, so we study the radial parts parametrization for general n in Section 5 and construct explicit families of holonomic systems of rank 2^n which have the radial parts of our polynomials as one of the solutions (cf. Theorem 5.3). This is a generalization of the usual Gegenbauer differential equations to general n. Natural inner products for our polynomials are given in Section 6. By some change of variables, our differential equations turn out to be equivalent to the known system in Muirhead [**21**]. Moreover, we show that our polynomial solutions become generalized hypergeometric polynomials of several variables. These are explained in Section 7 (cf. Theorem 7.5). In Appendix A, we see the connection between our polynomials and spherical functions in L^2 space on Grassmann manifolds. In Appendix B, we review some criterions for systems to be holonomic and complete the proof of the fact that our systems are holonomic.

Acknowledgements. Many parts of our paper are related with the contents of [**14**], [**15**], which have been done before but is still in preparation. Some parts of our results were influenced implicitly by the arguments in those papers. So the first named author would like to thank Don Zagier for long lasting collaboration there. We would also like to thank C. Bachoc for suggestion that our polynomials have something to do with polynomials in [**1**] and [**17**]. This suggestion leads us to write Appendix A.

2. Review on motivation: Siegel modular forms.

Although our motivation on Siegel modular forms has logically no relation to the content of this paper, we shortly review the theory since it would make the picture clearer. We denote by H_n the Siegel upper half space.

$$H_n = \left\{ Z = X + iY \in M_n(\boldsymbol{C});\ X = {}^tX,\ Y = {}^tY \in M_n(\boldsymbol{R}),\ Y > 0 \right\}$$

where $Y > 0$ means that Y is positive definite. We put $J_n = \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right)$. The symplectic group is defined as usual by

$$Sp(n,\boldsymbol{R}) = \left\{ g \in M_{2n}(\boldsymbol{R});\ gJ_n{}^t g = J_n \right\}.$$

Then $Sp(n,\boldsymbol{R})$ acts on H_n by $gZ = (AZ+B)(CZ+D)^{-1}$ for $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(n,\boldsymbol{R})$. Now we fix natural numbers d and ν. We assume that d is even for a while for the sake of simplicity, but this assumption is not essential. For any $g \in Sp(n,\boldsymbol{R})$ and any holomorphic functions $F(Z)$ of H_n, we put

$$(F|_{d/2}[g])(Z) = \det(CZ+D)^{-d/2} F(gZ).$$

In the same way, for any holomorphic functions $F(Z_1, Z_2)$ of $H_n \times H_n$ and $g_i = \left(\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right) \in Sp(n,\boldsymbol{R})$ $(i=1,2)$, we put

$$\begin{aligned} &(F|_{d/2+\nu}[(g_1,g_2)])(Z_1,Z_2) \\ &\quad = \det(C_1Z_1+D_1)^{-d/2-\nu} \det(C_2Z_2+D_2)^{-d/2-\nu} F(g_1Z_1, g_2Z_2). \end{aligned}$$

Now we put $\Delta = H_n \times H_n$ and embed Δ diagonally to H_{2n}. We embed $Sp(n,\boldsymbol{R}) \times Sp(n,\boldsymbol{R})$ into $Sp(2n,\boldsymbol{R})$ by

$$g = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}$$

for $g_i \in Sp(n,\boldsymbol{R})$ as above and denote this element by $g = \iota(g_1,g_2)$. We consider holomorphic homogenous differential operators $\boldsymbol{D}$ with constant coefficients acting on holomorphic functions $F(Z)$ on H_{2n} such that the relation

$$\mathrm{Res}_\Delta \left(\boldsymbol{D}(F|_{d/2}[\iota(g_1,g_2)]) \right) = (\mathrm{Res}_\Delta(\boldsymbol{D}F))|_{d/2+\nu}[(g_1,g_2)] \tag{1}$$

holds for any holomorphic functions F, where Res_Δ is the restriction map to Δ. For $Z = (z_{ij}) \in H_{2n}$, we put $\partial_Z = \left(((1+\delta_{ij})/2)(\partial/\partial z_{ij}) \right)_{1 \le i,j \le 2n}$, where δ_{ij} is Kronecker's delta. So for $\boldsymbol{D}$, there exists a polynomial $P_{\boldsymbol{D}}$ in components of $2n \times 2n$ symmetric matrix such that $\boldsymbol{D} = P_{\boldsymbol{D}}(\partial_Z)$. So we would like to characterize $P_{\boldsymbol{D}}$.

We consider a polynomial $P^*(X,Y)$ in components of two $n \times d$ matrices X and Y which satisfies the following three conditions.

(i) $P^*(AX, BY) = \det(AB)^\nu P^*(X,Y)$ for any A, $B \in GL(n, \boldsymbol{C})$.
(ii) $P^*(Xh, Yh) = P^*(X,Y)$ for any $h \in O(d)$.
(iii) $P^*(X,Y)$ are pluriharmonic for each X and Y:

$$\Delta_{ij}(X)P^* = \Delta_{ij}(Y)P^* = 0, \ (i,j = 1,\dots,n),$$

where we put $\Delta_{ij}(X) = \sum_{\mu=1}^{d}(\partial^2/\partial x_{i\mu}\partial x_{j\mu})$ and $\Delta_{ij}(Y) = \sum_{\mu=1}^{d}(\partial^2/\partial y_{i\mu}\partial y_{j\mu})$ for $X = (x_{ij})$, $Y = (y_{ij})$. Under the condition (i), the condition (iii) is equivalent to say that $P^*(X,Y)$ are harmonic for each X and Y. We assume that $d \ge n$. Then by the classical invariant theory, for each P^* which satisfies (ii), we have the unique polynomial P in components of $2n \times 2n$ symmetric matrix such that

$$P^*(X,Y) = P\begin{pmatrix} X^tX & X^tY \\ Y^tX & Y^tY \end{pmatrix}.$$

If we write P as $P = P(T)$ where T is a $2n \times 2n$ symmetric matrix, then by (i) we have $P\left(\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)T\left(\begin{smallmatrix} {}^tA & 0 \\ 0 & {}^tB \end{smallmatrix}\right)\right) = \det(AB)^\nu P(T)$ for any A, $B \in GL(n, \boldsymbol{C})$. We denote by $\boldsymbol{P}_{n,\nu}$ the set of all such polynomials P and we call ν an index of the polynomials $P \in \boldsymbol{P}_{n,\nu}$. The total degree of P as a polynomial is $n\nu$. The following theorem is a part of the main theorem of [**13**].

Theorem 2.1 ([**13**]). *We fix natural numbers n. For each $d \ge n$ and ν, a differential operator $P(\partial_Z)$ satisfies the condition* (1) *if and only if $P \in \boldsymbol{P}_{n,\nu}$ and $P\left(\begin{smallmatrix} X^tX & X^tY \\ Y^tX & Y^tY \end{smallmatrix}\right)$ is pluriharmonic for each $X \in M_{n,d}$ or $Y \in M_{n,d}$. Besides, for each $d \ge n$ and ν, such a differential operator exists uniquely up to constant.*

Here note that the space $\boldsymbol{P}_{n,\nu}$ does not depend on d but the harmonicity condition depends on d. We denote by $\mathscr{H}_{n,\nu,d}$ the one-dimensional subspace of $\boldsymbol{P}_{n,\nu}$ which satisfies the pluriharmonicity defined above.

3. Invariant polynomials of $GL(n) \times GL(n)$.

In this section, we give generators of $\boldsymbol{P}_{n,\nu}$. We denote by $\mathrm{Sym}_n(\boldsymbol{R})$ the set of $n \times n$ symmetric matrices with coefficients in $\boldsymbol{R}$. We can regard $\boldsymbol{P}_{n,\nu}$ as the set of polynomials $P(R,S,W)$ in the components of $(R,S,W) \in \mathrm{Sym}_n(\boldsymbol{R}) \times \mathrm{Sym}_n(\boldsymbol{R}) \times M_n(\boldsymbol{R})$ such that the following relation is satisfied for any A, $B \in GL(n, \boldsymbol{R})$.

$$P(AR\,{}^tA, BS\,{}^tB, AW\,{}^tB) = \det(AB)^\nu P(R,S,W). \tag{2}$$

Here in the (X,Y) coordinates in the last section, we have $R = X\,{}^tX$, $S = Y\,{}^tY$

and $W = X\,{}^tY$. The direct sum $\boldsymbol{P}_n = \bigoplus_{\nu=0}^{\infty} \boldsymbol{P}_{n,\nu}$ becomes a graded ring by the natural multiplication. We also define the graded subring of even indices by $\boldsymbol{P}_{n,even} = \bigoplus_{\nu=0}^{\infty} \boldsymbol{P}_{n,2\nu}$. In order to give generators of these graded rings, we introduce the following notation. For each $0 \leq \alpha \leq n$, we define polynomials P_α in $(R, S, W) \in \mathrm{Sym}_n(\boldsymbol{R}) \times \mathrm{Sym}_n(\boldsymbol{R}) \times M_n(\boldsymbol{R})$ by

$$\det \begin{pmatrix} xR & W \\ {}^tW & S \end{pmatrix} = \sum_{\alpha=0}^{n} P_\alpha(R, S, W) x^\alpha,$$

where x is an indeterminate. For example, $P_0(R, S, W) = (-1)^n \det(W)^2$ and $P_n(R, S, W) = \det(RS)$.

PROPOSITION 3.1. *The graded ring $\boldsymbol{P}_{n,even}$ is generated by the polynomials P_α $(0 \leq \alpha \leq n)$ and $\boldsymbol{P}_n = \boldsymbol{P}_{n,even} \oplus \det(W)\boldsymbol{P}_{n,even}$. The $n+1$ polynomials $\det(W)$, $P_1, \ldots, P_n$ are algebraically independent.*

PROOF. We take $P \in \boldsymbol{P}_{n,\nu}$. If $P(R, S, W) \in \boldsymbol{P}_{n,\nu}$, then the polynomial P is determined by its values at $R = S = 1_n$ and $W=$ diagonal matrices. Indeed, this polynomial is determined by its values on any non-empty open subset e.g. the open set consisting of (R, S, W) such that $R > 0$, $S > 0$ (positive definite symmetric matrices) and $W \in GL(n, \boldsymbol{R})$. For these R, S, W, we can take A, $B \in GL(n, \boldsymbol{R})$ so that $AR\,{}^tA = BS\,{}^tB = 1_n$. Now put $W_0 = AW\,{}^tB$. Since we assumed that $\det(W) \neq 0$, there exist orthogonal matrices h_1, h_2 such that $h_1 W_0 h_2 = D$ where D is the diagonal matrix with diagonal elements d_i $(1 \leq i \leq n)$ with $d_i \neq 0$. So by (2) we have $P(1_n, 1_n, D) = \det(h_1 h_2)^\nu P(1_n, 1_n, W_0) = \det(h_1 h_2 AB)^\nu P(R, S, W)$ and this shows that P is determined by $P(1_n, 1_n, D)$. Now, since $P(1_n, 1_n, V^{-1}DV) = P(1_n, 1_n, D)$ for any permutation matrix V, the polynomial $P(1_n, 1_n, D)$ is a polynomial in elementary symmetric polynomials of $d_1, \ldots, d_n$. For each i with $1 \leq i \leq n$, take a diagonal matrix ϵ_i such that (i, i)-component is -1 and that the other diagonal components are 1. Then we see $P(1_n, 1_n, \epsilon_i D) = (-1)^\nu P(1_n, 1_n, D)$. So, if ν is even, then $P(1_n, 1_n, D)$ is a polynomial in elementary symmetric polynomials of $d_1^2, \ldots, d_n^2$. If ν is odd, then P changes sign if we change d_i into $-d_i$ for i. This means that $P(1_n, 1_n, D)$ is divisible by $d_1 \cdots d_n$ and $P(1_n, 1_n, D)/(d_1 \cdots d_n)$ is a symmetric polynomial of $d_1^2, \ldots, d_n^2$.

Put $\det(x1_n - W_0\,{}^tW_0) = \sum_{\alpha=0}^{n} x^\alpha P'_\alpha(W_0)$. By the relation

$$\det\left(x1_n - W_0\,{}^tW_0\right) = \det(x1_n - D^2),$$

we see that $P(1_n, 1_n, W_0)$ is a polynomial in $P'_\alpha(W_0)$ when ν is even. When

ν is odd, we have $P(1_n, 1_n, D)/\det(D) = P(1_n, 1_n, W_0)\det(h_1h_2)^\nu/\det(D) = P(1_n, 1_n, W_0)/\det(W_0)$, so we see also that $P(1_n, 1_n, W_0)$ is $\det(W_0)$ times a polynomial of P'_α. Since we have $\det(x1_n - W_0\,{}^tW_0) = \det(x1_n - B\,{}^tWR^{-1}W\,{}^tB) = \det(x1_n - S^{-1}\,{}^tWR^{-1}W) = \det(x1_n - R^{-1}WS^{-1}\,{}^tW)$, and

$$\begin{vmatrix} R^{-1} & 0 \\ 0 & S^{-1} \end{vmatrix} \begin{vmatrix} xR & W \\ {}^tW & S \end{vmatrix} \begin{vmatrix} 1_n & 0 \\ -S^{-1}\,{}^tW & 1_n \end{vmatrix} = \begin{vmatrix} x1_n - R^{-1}WS^{-1}\,{}^tW & R^{-1}W \\ 0 & 1_n \end{vmatrix},$$

we get

$$\det(RS)\det\left(x1_n - W_0\,{}^tW_0\right) = \begin{vmatrix} xR & W \\ {}^tW & S \end{vmatrix}.$$

Hence, we get $P_\alpha(R,S,W) = P'_\alpha(W_0)\det(RS)$.

First, we assume that ν is even. Since

$$\det(RS)^{\nu/2}P(1_n, 1_n, W_0) = \det(AB)^{-\nu}P(1_n, 1_n, W_0) = P(R,S,W),$$

we see that $P(R,S,W)$ is a linear combination of the following functions

$$\det(RS)^{\nu/2}\prod_{\alpha=0}^{n-1}P'_\alpha(AW\,{}^tB)^{e_\alpha} = \prod_{\alpha=0}^{n-1}P_\alpha(R,S,W)^{e_\alpha}\det(RS)^{\nu/2-\sum_{\alpha=0}^{n-1}e_\alpha}.$$

We will show that $\nu/2 - \sum_{\alpha=0}^{n-1}e_\alpha$ is non-negative. Now we consider the degree of this polynomial P. We write $R = (r_{ij})$, $S = (s_{ij})$, $W = (w_{ij})$ and put

$$P(R,S,W) = \sum_{\substack{1\le i_1\le i_2\le n \\ 1\le i_3\le i_4\le n \\ 1\le i_5, i_6\le n}} c_{i_1i_2i_3i_4i_5i_6}r_{i_1i_2}^{l_{i_1i_2}}s_{i_3i_4}^{m_{i_3i_4}}w_{i_5i_6}^{n_{i_5i_6}}.$$

For simplicity, we put $l_{ij} = l_{ji}$ and $m_{ij} = m_{ji}$. Taking diagonal matrices $A = \mathrm{diag}(a_1,\ldots,a_n)$, $B = \mathrm{diag}(b_1,\ldots,b_n)$, we get $P(AR^tA, BS^tB, AW^tB) = (\prod_{i=1}^n a_ib_i)^\nu P(R,S,W)$. This means that for a fixed i or j, we have $2l_{ii} + \sum_{i_2\ne i}l_{i,i_2} + \sum_{i_6=1}^n n_{i,i_6} = \nu$, or $2m_{jj} + \sum_{i_1\ne j}m_{i_1,j} + \sum_{i_5=1}^n n_{i_5,j} = \nu$. Hence if we denote by N_{11} the degree of $P(R,S,W)$ with respect to w_{11}, then $N_{11} \le \nu$. If we assume that ν is even, then we may write $P(1_n, 1_n, D) = P(1_n, 1_n, W_0) = \sum c(e_0,\ldots,e_{n-1})\prod_{\alpha=0}^{n-1}P'_\alpha(W_0)^{e_\alpha}$. Here $P'_\alpha(W_0)$ is the elementary symmetric polynomial of d_i^2. By Lemma 3.2 we shall see below, we see that the degree of $P(1_n, 1_n, W_0)$ with respect to d_1 is the maximum of $2\sum_{\alpha=0}^{n-1}e_\alpha$ for

$c(e_0, \ldots, e_{n-1}) \neq 0$. On the other hand, the degree of $P(1_n, 1_n, D) = P(1_n, 1_n, W_0)$ with respect to d_1 is at most $N_{11} \leq \nu$. So we have $2\sum_{\alpha=0}^{n-1} e_\alpha \leq \nu$.

Next, we assume that ν is odd. Then, we have

$$P(1_n, 1_n, W_0) = \det(W_0) p\bigl(P'_0(W_0), \ldots, P'_{n-1}(W_0)\bigr),$$

where p is a polynomial of n variables. Since $\det(W_0) = \det(AB)\det(W)$, we get

$$\begin{aligned} P(R, S, W) &= \det(W)\det(AB)^{-\nu+1} p\bigl(P'_0(W_0), \ldots, P'_{n-1}(W_0)\bigr) \\ &= \det(W)\det(RS)^{(\nu-1)/2} p\bigl(P'_0(W_0), \ldots, P'_{n-1}(W_0)\bigr). \end{aligned}$$

This last polynomial is a linear combination of monomials

$$\det(W)\det(RS)^{(\nu-1)/2-\sum_{\alpha=0}^{n-1} e_\alpha} \prod_{\alpha=0}^{n-1} P_\alpha(R, S, W)^{e_\alpha}.$$

Hence by the same argument as in the case of even ν, we have $(\nu-1)/2 \geq \sum_{\alpha=0}^{n-1} e_\alpha$.

Finally, the restriction of $P_0, \ldots, P_{n-1}$ to $(R, S, W) = (1_n, 1_n, D)$ is algebraically independent, and since $P_0, \ldots, P_n$ are homogeneous polynomials of the same degree, this also implies that $P_0, \ldots, P_n$ are algebraically independent. $\square$

Now we show the lemma we used above. Let $F(z_1, \ldots, z_n)$ be a polynomial. We write $F(z_1, \ldots, z_n) = \sum_\beta c_\beta z^\beta$ where β runs over $\beta = (\beta_1, \ldots, \beta_n) \in (\boldsymbol{Z}_{\geq 0})^n$ and $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$. We put $|\beta| = \beta_1 + \cdots + \beta_n$. For i with $1 \leq i \leq n$, we denote by s_i the elementary symmetric polynomial of independent variables $d_1, \ldots, d_n$ of degree i.

Lemma 3.2. *Notation being as above, assume that $F(s_1, \ldots, s_n)$ is of degree a with respect to d_1. Then the total degree of $F(z_1, \ldots, z_n)$ is a.*

Proof. Denote by b the maximum of $|\beta|$ such that $c_\beta \neq 0$. We write all such indices by $\beta^{(1)}, \ldots, \beta^{(r)}$. We show that $b = a$. For i with $1 \leq i \leq n-1$, we denote by σ_i the elementary symmetric polynomial of $d_2, \ldots, d_n$ of degree i. For simplicity, we put $\sigma_0 = 1$. Then we have $s_i = d_1\sigma_{i-1} + \sigma_i$. So the highest degree term with respect to d_1 in $s_1^{\beta_1^{(i)}} s_2^{\beta_2^{(i)}} \cdots s_n^{\beta_n^{(i)}}$ is given by $d_1^b \sigma_1^{\beta_2^{(i)}} \sigma_2^{\beta_3^{(i)}} \cdots \sigma_{n-1}^{\beta_n^{(i)}}$. If $\beta_l^{(i)} = \beta_l^{(j)}$ for all $l = 2, \ldots, n$, then since $|\beta^{(i)}| = |\beta^{(j)}| = b$, we have $\beta_1^{(i)} = \beta_1^{(j)}$ and so $\beta^{(i)} = \beta^{(j)}$. So for different i, the coefficient of d_1^b is different. Since $\sigma_1, \ldots, \sigma_{n-1}$ are algebraically independent, the coefficient of d_1^b in $F(s_1, \ldots, s_n)$ does not vanish. So we have $a = b$. $\square$

Remark 3.3. For any $P(R,S,W) \in \boldsymbol{P}_n$, we have $P(S,R,{}^tW) = P(R,S,W)$. By virtue of Proposition 3.1, this is proved by seeing that P_α and $\det(W)$ satisfy the same property. As for $\det(W)$, this is trivial. As for P_α, we have

$$\begin{pmatrix} xR & W \\ {}^tW & S \end{pmatrix} = \begin{pmatrix} 0 & x1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} xS & {}^tW \\ W & R \end{pmatrix} \begin{pmatrix} 0 & x^{-1}1_n \\ 1_n & 0 \end{pmatrix},$$

so by definition we have the result. We also have a direct proof without using Proposition 3.1 from the relation (2) but omit the details.

The space $\boldsymbol{P}_n$ is not invariant by $\Delta_{ij}(X)$ or $\Delta_{ij}(Y)$, and in order to describe the action of Laplacians $\Delta_{ij}(X)$ or $\Delta_{ij}(Y)$, we must study the structure of the images of these operators. For any $m\times m$ matrix V and integers i, j with $1 \le i,j \le m$, we denote by $V_{i,j}$ the (i,j)-cofactor of V, i.e., $(-1)^{i+j}$ times the determinant of the matrix which is obtained by removing the i-th row and the j-th column from V. For each integer $0 \le \beta \le n-1$, we define polynomials $\widehat{P}_\beta(R,S,W)$ by

$$\begin{pmatrix} xR & W \\ {}^tW & S \end{pmatrix}_{1,1} = \sum_{\beta=0}^{n-1} \widehat{P}_\beta(R,S,W)x^\beta.$$

Lemma 3.4. *The $2n+1$ polynomials P_α $(0 \le \alpha \le n)$ and $\widehat{P}_\beta$ $(0 \le \beta \le n-1)$ are algebraically independent. A fortiori, the polynomials $\widehat{P}_\beta$ $(0 \le \beta \le n-1)$ are linearly independent over the ring $\boldsymbol{P}_n$.*

Proof. We prove this by induction on n. Let $F(X_0,\dots,X_n,Y_0,\dots,Y_{n-1})$ be a non-zero polynomial of the smallest total degree such that $F(P_0,\dots,P_n,\widehat{P}_0,\dots,\widehat{P}_{n-1}) = 0$. We shall show that $F=0$ by induction. When $n=1$, if we put $R=(r)$, $S=(s)$ and $W=(w)$, we have $P_0=-w^2$, $P_1=rs$, $\widehat{P}_0=s$, which are algebraically independent. Hence we have $F=0$.

Now we assume that $n>1$ and that the claim is true up to $n-1$. We can write F as

$$\begin{aligned} F(X_0,\dots,X_n,Y_0,\dots,Y_{n-1}) &= F_1(X_0,\dots,X_{n-1},Y_0,\dots,Y_{n-2}) \\ &\quad + X_nF_2(X_0,\dots,X_{n-1},X_n,Y_0,\dots,Y_{n-2}) \\ &\quad + Y_{n-1}F_3(X_0,\dots,X_{n-1},X_n,Y_0,\dots,Y_{n-1}). \end{aligned}$$

First, we put $r_{in}=s_{in}=0$ for all $1\le i\le n$ and $w_{in}=w_{ni}=0$ for $i\ne n$, $w_{nn}=1$. Then we get $\widehat{P}_{n-1}=P_n=0$ and $-P_\alpha$ $(0\le\alpha\le n-1)$ and $-\widehat{P}_\beta$ $(0\le\beta\le n-2)$

becomes the corresponding polynomials for $n-1$ of the first $(n-1)\times(n-1)$ matrices of R, S, W. Hence, by induction hypothesis, we get $F_1 = 0$. Now, let us go back to the original polynomials F. Put $r_{ni} = 0$ for all $i \neq 1$. Then we get $\widehat{P}_{n-1} = 0$ and $P_n = -r_{n1}^2 R_{1,n;1,n}\det(S)$, where $R_{1,n;1,n}$ means a minor of R where the first and the n-th rows and the first and the n-th columns are removed. Since this is not zero, we see that $F_2(P_0,\dots,P_{n-1},\widehat{P}_0,\dots,\widehat{P}_{n-2}) = 0$ if $r_{ni} = 0$ for (at least) all $i \neq 1$. Now write $F_2 = F_4(X_0,\dots,X_{n-1},Y_0,\dots,Y_{n-2}) + X_n F_5(X_0,\dots,X_n,Y_0,\dots,Y_{n-2})$. If we put here $r_{ni} = 0$, $s_{ni} = 0$ for all $1 \le i \le n$ and $w_{in} = w_{ni} = 0$ $(i \neq n)$, $w_{nn} = 1$ in F_2, then $P_n = 0$ and by the same argument for F_1, we see that F_4 is identically zero and F_2 is a multiple of X_n. Repeating the same procedure several times, we see that F_2 is divisible by X_n^l for l which exceeds the degree of F_2, so we have $F_2 = 0$. Hence finally we get $F_3(P_0,\dots,P_n,\widehat{P}_0,\dots,\widehat{P}_{n-1}) = 0$, but since F_3 is a polynomial of smaller degree than F, we get a contradiction. $\square$

4. Invariant pluriharmonic polynomials.

4.1. Pluriharmonicity for *R*, *S*, *W*.

To get the one-dimensional subspace $\mathscr{H}_{n,\nu,d}$ of $\boldsymbol{P}_{n,\nu}$, we must investigate the action of $\Delta_{ij}(X)$ and $\Delta_{ij}(Y)$ on $\boldsymbol{P}_{n,\nu}$. But, for any $P(R,S,W) \in \boldsymbol{P}_{n,\nu}$, we have $P \in \mathscr{H}_{n,\nu,d}$ if and only if $\Delta_{11}(X)P(X^tX, Y^tY, X^tY) = 0$. This is proved by Remark 3.3 in the last section and the fact that P becomes $(\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau))^{\nu}P$ under the permutation of indices of $R = (r_{ij})$, $S = (s_{ij})$, $W = (w_{ij})$ as $r_{ij} \to r_{\sigma(i)\sigma(j)}$, $s_{ij} \to s_{\tau(i)\tau(j)}$, $w_{ij} \to w_{\sigma(i)\tau(j)}$ for any element σ, τ in the symmetric group S_n of n letters. For the sake of simplicity, we write $\Delta_{11} = \Delta_{11}(X)$ in the rest of this paper. It is a routine calculation to rewrite the operator Δ_{11} by the coordinate of R, S, W. If we denote by $\partial_{ij} = (1+\delta_{ij})(\partial/\partial t_{ij})$ for $T = \left(\begin{smallmatrix} R & W \\ {}^tW & S \end{smallmatrix}\right) = (t_{ij})$, the result is given by

$$\Delta_{11} = d\partial_{11} + \sum_{i,j=1}^{2n} t_{ij}\partial_{1i}\partial_{1j}.$$

(cf. [**14**].) As we explained, the coordinates of X, Y and those of R, S, W correspond bijectively under our assumption $d \ge n$. So we often use (X,Y) coordinates instead of (R,S,W) in our calculation. For functions $F(X,Y)$ and $G(X,Y)$, we define (F,G) by

$$\Delta_{11}(FG) = (\Delta_{11}F)G + (F,G) + F(\Delta_{11}G).$$

In the (X,Y) coordinates, we have

$$(F, G) = 2\sum_{j=1}^{d} \frac{\partial F}{\partial x_{1j}} \frac{\partial G}{\partial x_{1j}}. \tag{3}$$

Of course we have $(P_\alpha, P_\beta) = (P_\beta, P_\alpha)$.

Proposition 4.1.

(1) *For $0 \leq \alpha \leq n$, we have*

$$\Delta_{11} P_\alpha = 2(d - 2n + \alpha + 1)\widehat{P}_{\alpha-1} - 2(\alpha + 1)\widehat{P}_\alpha.$$

(2) *For $0 \leq \alpha \leq \beta \leq n$, we have*

$$\begin{aligned}(P_\alpha, P_\beta) = 8P_\alpha \widehat{P}_{\beta-1} &- 8\sum_{i=0}^{\alpha-1}\bigl(-P_{\alpha-i-1}\widehat{P}_{\beta+i} + P_{\beta+i+1}\widehat{P}_{\alpha-i-2}\bigr) \\ &+ 8\sum_{i=0}^{\alpha}\bigl(-P_{\alpha-i}\widehat{P}_{\beta+i} + P_{\beta+i+1}\widehat{P}_{\alpha-i-1}\bigr).\end{aligned}$$

Here we understand that $P_\alpha = 0$ for $\alpha < 0$ or $n < \alpha$ and that $\widehat{P}_\alpha = 0$ for $\alpha < 0$ or $n \leq \alpha$.

We prove this by using (X, Y) coordinates. We prepare the following notation and Lemma. We put

$$T(x) = \begin{pmatrix} xX{}^tX & X{}^tY \\ {}^tYX & Y{}^tY \end{pmatrix} = \begin{pmatrix} xR & W \\ {}^tW & S \end{pmatrix}$$

and denote the components of R, S, W by the same notation as before. We denote by $T(x)_{ij}$ the (i, j) cofactor of $T(x)$, that is, $(-1)^{i+j}$ times the determinant of the $(2n-1) \times (2n-1)$ matrix obtained by removing i-th row and j-th column. For any j and α with $1 \leq j \leq 2n$ and $0 \leq \alpha \leq n-1$, we define $\widehat{P}_\alpha^{(j)} = \widehat{P}_\alpha^{(j)}(X, Y)$ by

$$T(x)_{1j} = \sum_{\alpha=0}^{n-1} \widehat{P}_\alpha^{(j)} x^\alpha.$$

In particular, we have $\widehat{P}_\alpha = \widehat{P}_\alpha^{(1)}$.

Lemma 4.2. *For all $1 \leq i \leq n$, $0 \leq \alpha \leq n$, and $1 \leq k \leq d$, we have*

$$\sum_{j=1}^{n} \left(r_{ij} \widehat{P}^{(j)}_{\alpha-1} + w_{ij} \widehat{P}^{(j+n)}_{\alpha} \right) = \delta_{1i} P_\alpha, \tag{4}$$

$$\sum_{j=1}^{n} \left(w_{ji} \widehat{P}^{(j)}_{\alpha} + s_{ij} \widehat{P}^{(j+n)}_{\alpha} \right) = 0, \tag{5}$$

$$2 \sum_{j=1}^{n} \left(x_{jk} \widehat{P}^{(j)}_{\alpha-1} + y_{jk} \widehat{P}^{(n+j)}_{\alpha} \right) = \frac{\partial P_\alpha}{\partial x_{1k}}. \tag{6}$$

Proof. Expanding $\det(T(x))$ at the i-th row for $1 \le i \le n$, we have

$$\det(T(x)) = \sum_{j=1}^{n} \left(x r_{1j} T(x)_{1j} + w_{1j} T(x)_{1,j+n} \right),$$

so we obtain (4) for $i = 1$ by taking the coefficient of x^α. If $i \neq 1$, then the left-hand side of (4) is the coefficient of the determinant of the matrix obtained by replacing the first row of $T(x)$ by the i-th row, so the determinant is zero. The assertion (5) is obtained similarly by replacing the first row of $T(x)$ by $(i+n)$-th row. We show (6). The variable x_{1k} appears only in the first row and column of $T(x)$, so by differentiating each row, we have

$$\begin{aligned} \frac{\partial \det(T(x))}{\partial x_{1k}} &= x \sum_{j=1}^{n} (1 + \delta_{1j}) x_{jk} T(x)_{1j} + \sum_{j=1}^{n} y_{jk} T(x)_{1,n+j} \\ &\quad + x \sum_{j=2}^{n} x_{jk} T(x)_{j1} + \sum_{j=1}^{n} y_{jk} T(x)_{n+j,1} \\ &= 2 \left(x \sum_{j=1}^{n} x_{jk} T(x)_{1j} + \sum_{j=1}^{n} y_{jk} T(x)_{1,n+j} \right), \end{aligned} \tag{7}$$

since $T(x)$ is symmetric and $T(x)_{i1} = T(x)_{1i}$ for any i. Taking the coefficient of x^α, we have (6). □

We give a remark on a formula of the general determinant. Let m and n be natural numbers such that $m < n$. $V = (v_{ij})$ be an $n \times n$ matrix with components v_{ij}. For any j with $1 \le j \le n$, denote by $V(j)$ the matrix obtained by replacing v_{ij} by 0 for $m + 1 \le i \le n$. Then we have the formula

$$m\det(V) = \sum_{j=1}^{n} \det(V(j)). \tag{8}$$

We can prove this by induction on m. If $m = 1$, the assertion is true by the expansion of $\det(V)$ at the first row. Now we may assume that the assertion is true for $m-1$. Now take the expansion of $\det V(j)$ at the first row. Then the part which contains v_{1k} is from $\det V(k)$ given by $v_{1k} \times \widetilde{V}_{1k}$ where $\widetilde{V}_{ij}$ is the (i,j)-cofactor of V and given by $v_{1k} \times (m-1)\widetilde{V}_{1k}$ from $\sum_{j\neq k} \det(V(k))$ by the inductive assumption. So we prove the formula (8).

Proof of Proposition 4.1. We calculate $\Delta_{11}P_\alpha$ by using (8). Differentiating both sides of (7), for each k with $1 \le k \le d$, we have

$$\frac{\partial^2 \det(T(x))}{\partial x_{1k}^2} = 2\left(xT(x)_{11} + x\sum_{i=2}^{n} x_{ik}\frac{\partial T(x)_{1i}}{\partial x_{1k}} + \sum_{i=1}^{n} y_{ik}\frac{\partial T(x)_{1,n+i}}{\partial x_{1k}}\right). \tag{9}$$

The variable x_{1k} is only in the first column of $T(x)_{1i}$ and the derivatives of the first column is calculated by $\partial r_{j1}/\partial x_{1k} = x_{jk}$, $\partial w_{1j}/\partial x_{1k} = y_{jk}$. Since $\sum_{k=1}^{d} x_{ik}x_{jk} = r_{ij}$ and $\sum_{k=1}^{d} x_{ik}y_{jk} = w_{ij}$, the sum $\sum_{k=1}^{d} x_{ik}(\partial T(x)_{1i}/\partial x_{1k})$ is obtained by replacing the first column of $T(x)_{1i}$ by ${}^t(r_{2i}, r_{3i}, \dots, r_{ni}, w_{i1}, w_{i2}, \dots, w_{in})$, so this is $-T(x)_{11}$ for each i (including the signature). So the sum of the first two terms in the parenthesis of (9) over $k = 1$ to d is $x(d-n+1)T(x)_{11}$. Now the third term is similar but slightly different. The reason is that if we sum up over $k = 1$ to d for each i, then by the same calculation as before, we have a matrix similar to $-T(x)_{11}$, but this time the $(i+n)$ column is replaced by ${}^t(x\tilde{w}_i, \tilde{s}_i)$ where $\tilde{w} = (w_{2i}, w_{2i}, \dots, w_{ni})$ and $\tilde{s} = (s_{1i}, s_{2i}, \dots, s_{ni})$, and not by ${}^t(\tilde{w}_i, \tilde{s}_i)$. So if we divide this column into two parts as $(x-1){}^t(\tilde{w}_i, 0, \dots, 0) + {}^t(\tilde{w}_i, \tilde{s}_i)$, then the latter vector gives $-T(x)_{11}$. On the other hand, if we take $x(\partial T(x)_{11}/\partial x)$, then this is the sum of the determinant obtained by replacing i-th column of $T(x)_{11}$ by $x^t(r_{2i}, r_{3i}, \dots, r_{ni}, 0, \dots, 0)$. So by the formula (8), the sum over $i = 1$ to n of the part coming from ${}^t(\tilde{w}_i, 0, \dots, 0)$ is given by

$$-(n-1)T(x)_{11} + x\frac{\partial T(x)_{11}}{\partial x}.$$

Hence we have

$$\sum_{i=1}^{n}\sum_{k=1}^{d} y_{ik}\frac{\partial T(x)_{1,n+i}}{\partial x_{1k}} = -nT(x)_{11} + (x-1)\left(-(n-1)T(x)_{11} + x\frac{\partial T(x)_{11}}{\partial x}\right).$$

So $\Delta_{11}\det(T(x))$ is

$$\begin{aligned}
&2(d-n+1)xT(x)_{11} - 2nT(x)_{11} - 2(x-1)(n-1)T(x)_{11} + 2(x-1)x\frac{\partial T(x)_{11}}{\partial x}\\
&\quad = 2\sum_{\alpha=0}^{n-1}(d-2n+\alpha+2)x^{\alpha+1}\widehat{P}_\alpha - 2\sum_{\alpha=0}^{n-1}(\alpha+1)x^\alpha\widehat{P}_\alpha.
\end{aligned}$$

So we have (1) of Proposition 4.1.

Now we show (2) of Proposition 4.1. By Lemma 4.2, assuming that $\alpha \le \beta$, we have

$$\begin{aligned}
&\sum_{k=1}^{d}\frac{\partial P_\alpha}{\partial x_{1k}}\frac{\partial P_\beta}{\partial x_{1k}}\\
&\quad = 4\sum_{k=1}^{d}\left(\sum_{i=1}^{n}\left(x_{ik}\widehat{P}^{(i)}_{\alpha-1} + y_{ik}\widehat{P}^{(n+i)}_{\alpha}\right)\right)\left(\sum_{j=1}^{n}\left(x_{jk}\widehat{P}^{(j)}_{\beta-1} + y_{jk}\widehat{P}^{(n+j)}_{\beta}\right)\right)\\
&\quad = 4\sum_{i,j=1}^{n}\left(r_{ij}\widehat{P}^{(i)}_{\alpha-1}\widehat{P}^{(j)}_{\beta-1} + w_{ij}\widehat{P}^{(i)}_{\alpha-1}\widehat{P}^{(n+j)}_{\beta} + w_{ji}\widehat{P}^{(n+i)}_{\alpha}\widehat{P}^{(j)}_{\beta-1} + s_{ij}\widehat{P}^{(n+i)}_{\alpha}\widehat{P}^{(n+j)}_{\beta}\right)\\
&\quad = 4P_\alpha\widehat{P}_{\beta-1} + 4\sum_{i,j=1}^{n} w_{ij}P^{(i)}_{\alpha-1}P^{(n+j)}_{\beta} + 4\sum_{i,j=1}^{n} s_{ij}P^{(n+i)}_{\alpha}P^{(n+j)}_{\beta}\\
&\quad = 4P_\alpha\widehat{P}_{\beta-1} - 4\sum_{i,j=1}^{n} s_{ij}\widehat{P}^{(n+i)}_{\alpha-1}\widehat{P}^{(n+j)}_{\beta} + 4\sum_{i,j=1}^{n} s_{ij}\widehat{P}^{(n+i)}_{\alpha}\widehat{P}^{(n+j)}_{\beta}.
\end{aligned}$$

Now using Lemma 4.2 repeatedly, we have

$$\begin{aligned}
\sum_{i,j=1}^{n} s_{ij}\widehat{P}^{(n+i)}_{\alpha}\widehat{P}^{(n+j)}_{\beta} &= -\sum_{i,j=1}^{n} w_{ji}\widehat{P}^{(n+i)}_{\alpha}\widehat{P}^{(j)}_{\beta}\\
&= -P_\alpha\widehat{P}_\beta + \sum_{i,j=1}^{n} r_{ji}\widehat{P}^{(i)}_{\alpha-1}\widehat{P}^{(j)}_{\beta}\\
&= -P_\alpha\widehat{P}_\beta + P_{\beta+1}\widehat{P}_{\alpha-1} - \sum_{i,j=1}^{n} w_{ji}\widehat{P}^{(n+j)}_{\alpha-1}\widehat{P}^{(n+i)}_{\beta+1}\\
&= -P_\alpha\widehat{P}_\beta + P_{\beta+1}\widehat{P}_{\alpha-1} + \sum_{i,j=1}^{n} s_{ij}\widehat{P}^{(n+i)}_{\alpha-1}\widehat{P}^{(n+j)}_{\beta+1}.
\end{aligned}$$

Using this repeatedly, we have (2) of Proposition 4.1. □

Now we study the pluriharmonicity of the polynomial $P(R,S,W) \in \boldsymbol{P}_{n,\nu}$. We can rewrite the formula (2) of Proposition 4.1 for (P_α, P_β) as

$$(P_\alpha, P_\beta) = 8\sum_{\gamma=0}^{\alpha-2}(P_{\alpha+\beta-\gamma} - P_{\alpha+\beta-\gamma-1})\widehat{P}_\gamma + 8P_{\beta+1}\widehat{P}_{\alpha-1} + 8P_\alpha\widehat{P}_{\beta-1} + 8\sum_{\gamma=\beta}^{\alpha+\beta}(P_{\alpha+\beta-\gamma-1} - P_{\alpha+\beta-\gamma})\widehat{P}_\gamma. \tag{10}$$

For $b = (b_0, \ldots, b_n) \in \boldsymbol{Z}_{\geq 0}^{n+1}$, we write $P^b = \prod_{\alpha=0}^n P_\alpha^{b_\alpha}$. Now by easy induction with respect to b using the definition (3), we have

$$\Delta_{11}(P^b) = \sum_{\alpha=0}^n \Delta_{11}(P_\alpha)b_\alpha P^b P_\alpha^{-1} + \frac{1}{2}\sum_{\alpha=0}^n (P_\alpha, P_\alpha)b_\alpha(b_\alpha - 1)P^b P_\alpha^{-2} + \sum_{0\leq\alpha<\beta\leq n}(P_\alpha, P_\beta)b_\alpha b_\beta P^b P_\alpha^{-1}P_\beta^{-1}. \tag{11}$$

By (11) together with Lemma 3.4, we see that the image of the action of Δ_{11} on $\boldsymbol{C}[P_0, \ldots, P_n]$ is in the free module over $\boldsymbol{C}[P_0, \ldots, P_n]$ spanned by $\widehat{P}_0, \ldots, \widehat{P}_{n-1}$. So denoting P_α by x_α, there exist differential operators L_γ $(0 \leq \gamma \leq n-1)$ in $x_0, \ldots, x_n$ with $\boldsymbol{C}[x_0, \ldots, x_n]$ coefficients such that

$$\Delta_{11} f(x_0, \ldots, x_n) = \sum_{\gamma=0}^{n-1}(L_\gamma f(x_0, \ldots, x_n))\widehat{P}_\gamma.$$

Now we write down L_γ explicitly. The formula (11) reads

$$\Delta_{11}(P^b) = \sum_{\alpha=0}^n \Delta_{11}(P_\alpha)\frac{\partial P^b}{\partial x_\alpha} + \frac{1}{2}\sum_{\alpha=0}^n (P_\alpha, P_\alpha)\frac{\partial^2 P^b}{\partial x_\alpha^2} + \sum_{0\leq\alpha<\beta\leq n}(P_\alpha, P_\beta)\frac{\partial^2 P^b}{\partial x_\alpha \partial x_\beta}. \tag{12}$$

By (1) of Proposition 4.1, the first term of (12) is given by

$$\sum_{\gamma=0}^{n}\Big(-2(\gamma+1)\frac{\partial P^b}{\partial x_\gamma}+2(d-2n+\gamma+2)\frac{\partial P^b}{\partial x_{\gamma+1}}\Big)\widehat{P}_\gamma.$$

Now we rewrite the third term of (12). We fix γ and see the coefficient of $\widehat{P}_\gamma$ by using (10). Corresponding to the terms in (10), we must consider the coefficients of $\partial^2/\partial x_\alpha \partial x_\beta$ in the following four cases (i) $\gamma \le \alpha-2$, (ii) $\gamma=\alpha-1$, (iii) $\gamma=\beta-1$, and (iv) $\beta \le \gamma \le \alpha+\beta$. For each case, the contribution to L_γ is given by

$$\text{(i)}\quad 8\sum_{\gamma+2\le\alpha<\beta}(x_{\alpha+\beta-\gamma}-x_{\alpha+\beta-\gamma-1})\frac{\partial^2 P^b}{\partial x_\alpha \partial x_\beta},$$

$$\text{(ii)}\quad 8\sum_{\gamma+2\le\beta} x_{\beta+1}\frac{\partial^2 P^b}{\partial x_{\gamma+1}\partial x_\beta},$$

$$\text{(iii)}\quad 8\sum_{\alpha=0}^{\gamma} x_\alpha\frac{\partial^2 P^b}{\partial x_\alpha \partial x_{\gamma+1}},$$

$$\text{(iv)}\quad 8\sum_{\beta\le\gamma\le\alpha+\beta}(x_{\alpha+\beta-\gamma-1}-x_{\alpha+\beta-\gamma})\frac{\partial^2 P^b}{\partial x_\alpha \partial x_\beta},$$

where the sums are always taken over α or β or over both.

The second term of (12) is obtained similarly and we have

$$\begin{aligned}&4\sum_{i=\gamma+2}^{n}\big(x_{2i-\gamma}-x_{2i-\gamma-1}\big)\frac{\partial^2 P^b}{\partial x_i^2}\\ &\quad+4\big(x_{\gamma+1}+x_{\gamma+2}\big)\frac{\partial^2 P^b}{\partial x_{\gamma+1}^2}+4\sum_{j\le\gamma}\big(x_{2j-\gamma-1}-x_{2j-\gamma}\big)\frac{\partial^2 P^b}{\partial x_j^2}.\end{aligned}$$

As a whole, for $0\le\gamma\le n-1$, we have

$$\begin{aligned}L_\gamma &= -2(\gamma+1)\frac{\partial}{\partial x_\gamma}+2(d-2n+\gamma+2)\frac{\partial}{\partial x_{\gamma+1}}\\ &\quad+4\sum_{\gamma+1\le i,j} x_{i+j-\gamma}\frac{\partial^2}{\partial x_i\partial x_j}-4\sum_{\gamma+2\le i,j} x_{i+j-\gamma-1}\frac{\partial^2}{\partial x_i\partial x_j}\\ &\quad+4\sum_{i,j\le\gamma+1} x_{i+j-\gamma-1}\frac{\partial^2}{\partial x_i\partial x_j}-4\sum_{i,j\le\gamma} x_{i+j-\gamma}\frac{\partial^2}{\partial x_i\partial x_j}.\end{aligned}$$

Rewriting this we have

Proposition 4.3. *For $0 \le \gamma \le n-1$, we have*

$$L_\gamma = -2(\gamma+1)\frac{\partial}{\partial x_\gamma} + 2(d-2n+\gamma+2)\frac{\partial}{\partial x_{\gamma+1}} + 4\sum_{k=\gamma+2}^{n} x_k \sum_{\beta=\gamma+1}^{k-1} \frac{\partial^2}{\partial x_{k+\gamma-\beta}\partial x_\beta} - 4\sum_{k=0}^{\gamma} x_k \sum_{\beta=k}^{\gamma} \frac{\partial^2}{\partial x_{k+\gamma-\beta}\partial x_\beta} - 4\sum_{k=\gamma+3}^{n} x_k \sum_{\beta=\gamma+2}^{k-1} \frac{\partial^2}{\partial x_{k+\gamma+1-\beta}\partial x_\beta} + 4\sum_{k=0}^{\gamma+1} x_k \sum_{\beta=k}^{\gamma+1} \frac{\partial^2}{\partial x_{k+\gamma+1-\beta}\partial x_\beta}.$$

Here we regard that $\partial/\partial x_\alpha = 0$ and $x_\alpha = 0$ if $\alpha < 0$ or $n < \alpha$.

To consider the case when ν is odd, we need $\det(W)$, so we put $y_0 = \det(W)$. So we have $x_0 = (-1)^n y_0^2$. It is easy to rewrite the operators L_γ as a differential operator with respect to y_0, $x_1, \ldots, x_n$. The terms containing x_0 in L_γ is only of the following shape.

$$2\frac{\partial}{\partial x_0} + 4x_0\frac{\partial^2}{\partial x_0^2},$$

$$8x_0\frac{\partial^2}{\partial x_\alpha \partial x_0} \quad \alpha \ne 0.$$

The former appears only in L_0 and the latter appears in all L_γ. Anyway, the new operator is obtained by replacing the former by $(-1)^n(\partial^2/\partial y_0^2)$ and the latter by $4y_0(\partial^2/\partial y_0 \partial x_\alpha)$. We write this new operator by L_{γ,y_0} when we emphasize the expression depending on y_0. As for odd ν, by Proposition 3.1, we must consider a polynomial solution $y_0 F(x_0, \ldots, x_n)$. To calculate the action of the Laplacian to this in the coordinate y_0, $x_1, \ldots, x_n$, we need the following formulas which are easily proved.

$$\Delta_{11}(y_0) = 0,$$

$$(y_0, x_\alpha) = 4y_0\big(\widehat{P}_{\alpha-1} - \widehat{P}_\alpha\big) \quad (\alpha \ne 0),$$

$$(y_0, y_0) = -2(-1)^n \widehat{P}_0,$$

$$(y_0, x_0) = -4y_0 \widehat{P}_0.$$

Since we have

$$\begin{aligned}\Delta_{11}(y_0F) &= \Delta_{11}(y_0)F + (y_0, F) + y_0\Delta_{11}(F),\\ \Delta_{11}(F) &= \sum_{\gamma=0}^{n-1}(L_\gamma F)\widehat{P}_\gamma,\\ (y_0, F) &= \sum_{\alpha=0}^{n}(y_0, x_\alpha)\frac{\partial F}{\partial x_\alpha},\end{aligned}$$

we have

$$\Delta_{11}(y_0F) = y_0\sum_{\gamma=0}^{n-1}\bigl(\widetilde{L}_\gamma F\bigr)\widehat{P}_\gamma,$$

where

$$\widetilde{L}_\gamma = L_\gamma + 4\biggl(\frac{\partial}{\partial x_{\gamma+1}} - \frac{\partial}{\partial x_\gamma}\biggr).$$

From this, we can show that

$$\Delta_{11}(y_0F) = \sum_{\gamma=0}^{n-1}L_{\gamma,y_0}(y_0F)\widehat{P}_\gamma,$$

where L_{γ,y_0} is the same operator as in the case of even ν. So there is essentially no difference between even ν and odd ν. Hence we will often explain only the case of even ν, since the case of odd ν is treated similarly.

4.2. Generating our solutions.

4.2.1. Construction.

For small ν, it is not difficult to give an explicit polynomial in $\mathscr{H}_{n,\nu,d}$. For example, for $\nu = 1$ or 2, it is given by

$$\begin{aligned}P &= y_0, \text{ or}\\ P &= \sum_{\gamma=0}^{n}\binom{n}{\gamma}^{-1}\binom{d-n+1}{n-\gamma}x_\gamma,\end{aligned}$$

respectively. But for general ν, there is no such simple formula. In this section, we give some easy constructive method to obtain non-zero polynomial solutions $P \in \mathscr{H}_{n,\nu,d}$. First we consider the case when ν is even. We assume that $F(x_0, x_1, \ldots, x_n)$ is a homogeneous polynomial of degree m in $x_0, \ldots, x_n$ and that $L_\gamma F = 0$ for all $0 \le \gamma \le n-1$. For each γ with $0 \le \gamma \le n$, we put

$$F^{(\gamma)}(x_\gamma, x_{\gamma+1}, \ldots, x_n) = F(0, \ldots, 0, x_\gamma, \ldots, x_n).$$

In particular, $F^{(0)} = F$. We also write $F^{(\gamma)}$ as a polynomial of x_γ as follows.

$$F^{(\gamma)}(x_\gamma, x_{\gamma+1}, \ldots, x_n) = \sum_{\alpha=0}^{m} F_\alpha^{(\gamma+1)}(x_{\gamma+1}, \ldots, x_n) x_\gamma^\alpha.$$

So we have

$$F_0^{(\gamma)}(x_\gamma, \ldots, x_n) = F^{(\gamma)}(x_\gamma, \ldots, x_n) = F(0, \ldots, 0, x_\gamma, \ldots, x_n).$$

Since $F^{(n)} = F_0^{(n)}$ is a homogeneous polynomial in x_n of degree m, this is a constant multiple of x_n^m. Now we show how we can recover whole F from $F^{(n)} = x_n^m$. Since it is necessary that $(L_\gamma F)(0, \ldots, 0, x_\gamma, \ldots, x_n) = 0$, we study this condition first. For this, we can ignore the part of L_γ which contains the multiplication of x_k by $k < \gamma$. So in the fourth term in the expression of L_γ in Proposition 4.3, only the term $k = \beta = \gamma$ remains. This part is given by

$$-4x_\gamma \frac{\partial^2}{\partial x_\gamma^2}.$$

In the sixth term of L_γ, only the terms $(k, \beta) = (\gamma, \gamma)$, $(\gamma, \gamma+1)$, $(\gamma+1, \gamma+1)$ remain. This part is given by

$$8x_\gamma \frac{\partial^2}{\partial x_\gamma \partial x_{\gamma+1}} + 4x_{\gamma+1} \frac{\partial^2}{\partial x_{\gamma+1}^2}.$$

The other terms of L_γ contain only x_μ or derivatives at x_μ with $\mu \ge \gamma$, so we cannot omit. So we have

$$\begin{aligned}&(L_\gamma F)(0, \ldots, 0, x_\gamma, \ldots, x_n)\\ &= \Big(-2(\gamma+1)\frac{\partial}{\partial x_\gamma} + 2(d-2n+\gamma+2)\frac{\partial}{\partial x_{\gamma+1}} - 4x_\gamma \frac{\partial^2}{\partial x_\gamma^2}\end{aligned}$$

$$+ 8x_\gamma \frac{\partial^2}{\partial x_\gamma \partial x_{\gamma+1}} + 4x_{\gamma+1} \frac{\partial^2}{\partial x_{\gamma+1}^2} + 4 \sum_{k=\gamma+2}^{n} x_k \sum_{\beta=\gamma+1}^{k-1} \frac{\partial^2}{\partial x_{k+\gamma-\beta} \partial x_\beta}$$

$$- 4 \sum_{k=\gamma+3}^{n} x_k \sum_{\beta=\gamma+2}^{k-1} \frac{\partial^2}{\partial x_{k+\gamma+1-\beta} \partial x_\beta} \Bigg) F^{(\gamma)}(x_\gamma, \dots, x_n),$$

and this should be zero. This condition gives relations between $F_\alpha^{(\gamma+1)}$ and $F_{\alpha+1}^{(\gamma+1)}$. To describe this, we introduce the following differential operators for each pair of γ and α with $0 \le \gamma \le n-1$ and $0 \le \alpha \le m-1$.

$$N^{(\gamma+1)}(\alpha) = 2(d - 2n + \gamma + 4\alpha + 2) \frac{\partial}{\partial x_{\gamma+1}} + 4x_{\gamma+1} \frac{\partial^2}{\partial x_{\gamma+1}^2}$$
$$+ 4 \sum_{k=\gamma+2}^{n} x_k \sum_{\beta=\gamma+1}^{k-1} \frac{\partial^2}{\partial x_{k+\gamma-\beta} \partial x_\beta} - 4 \sum_{k=\gamma+3}^{n} x_k \sum_{\beta=\gamma+2}^{k-1} \frac{\partial^2}{\partial x_{k+\gamma+1-\beta} \partial x_\beta}.$$

Since we have

$$\left(- 2(\gamma+1) \frac{\partial}{\partial x_\gamma} - 4x_\gamma \frac{\partial^2}{\partial x_\gamma^2} \right) x_\gamma^\alpha F_\alpha^{(\gamma+1)} = -2\alpha(\gamma + 2\alpha - 1) x_\gamma^{\alpha-1} F_\alpha^{(\gamma+1)}$$

and

$$8x_\gamma \frac{\partial^2}{\partial x_\gamma \partial x_{\gamma+1}} \left(x_\gamma^\alpha F_\alpha^{(\gamma+1)} \right) = 8\alpha x_\gamma^\alpha \frac{\partial F_\alpha^{(\gamma+1)}}{\partial x_{\gamma+1}},$$

we have

$$\begin{aligned}
&(L_\gamma F)(0, \dots, 0, x_\gamma, \dots, x_n) \\
&\quad = - \sum_{\alpha=1}^{m} 2\alpha(\gamma + 2\alpha - 1) x_\gamma^{\alpha-1} F_\alpha^{(\gamma+1)} + \sum_{\alpha=0}^{m} x_\gamma^\alpha N^{(\gamma+1)}(\alpha) F_\alpha^{(\gamma+1)}.
\end{aligned}$$

For α with $0 \le \alpha \le m$, we write

$$N_{\gamma+1}(\alpha) = \frac{1}{2\alpha(\gamma + 2\alpha - 1)} N^{(\gamma+1)}(\alpha - 1)$$

and we put $N_{\gamma+1}(0) = 1$, i.e., the identity operator. Since $(L_\gamma F)(0, \dots, 0, x_\gamma, \dots, x_n) = 0$, for each α with $1 \le \alpha \le m$, we have

$$F_{\alpha}^{(\gamma+1)} = N_{\gamma+1}(\alpha) F_{\alpha-1}^{(\gamma+1)}.$$

So it is necessary that

$$F^{(\gamma)}(x_\gamma, \dots, x_n) = \sum_{\alpha=0}^{m} x_\gamma^\alpha N_{\gamma+1}(\alpha) N_{\gamma+1}(\alpha-1) \cdots N_{\gamma+1}(1) N_{\gamma+1}(0) F_0^{(\gamma+1)}.$$

We again introduce a notation. We write

$$N_{\gamma+1} = \sum_{\alpha=0}^{m} x_\gamma^\alpha N_{\gamma+1}(\alpha) \cdots N_{\gamma+1}(0).$$

Then we have $F^{(\gamma)} = N_{\gamma+1} F^{(\gamma+1)}$, so

$$F(x_0, \dots, x_n) = (N_1 N_2 \cdots N_n)\big(F_0^{(n)}\big) = (N_1 N_2 \cdots N_n)\big(c x_n^m\big),$$

where c is a constant. Since the right-hand side contains cx_n^m as a monomial, F is not identically zero unless $c = 0$. This is a formula for F in general.

Now we apply the same method for the solutions of the variable $y_0, x_1, \dots, x_n$. If we just change L_0 to L_{0,y_0} and consider the system $L_{0,y_0} F = 0$, $L_\gamma F = 0$ with $1 \le \gamma \le n-1$, then we have one problem. When ν is odd, then the solution is in $y_0 \boldsymbol{C}[x_0, \dots, x_n]$ and no monomial is independent of y_0, so the same method cannot apply. So instead of L_γ, we use $\widetilde{L}_\gamma$ with $1 \le \gamma \le n-1$ defined in the last section. Then the solution for $\nu = 2m+1$ is given by

$$y_0 \widetilde{N}_1 \cdots \widetilde{N}_n \big(c x_n^m\big),$$

where we put

$$\widetilde{N}_{\gamma+1} = \sum_{\alpha=0}^{m} x_\gamma^\alpha \widetilde{N}_{\gamma+1}(\alpha) \cdots \widetilde{N}_{\gamma+1}(0),$$

$$\widetilde{N}_{\gamma+1}(\mu) = \frac{1}{2\mu(\gamma+2\mu+1)} \Bigg(2(d-2n+\gamma+4\mu) \frac{\partial}{\partial x_{\gamma+1}} + 4x_{\gamma+1} \frac{\partial^2}{\partial x_{\gamma+1}^2} + 4 \sum_{k=\gamma+2}^{n} x_k \sum_{\beta=\gamma+1}^{k-1} \frac{\partial^2}{\partial x_{k+\gamma-\beta} \partial x_\beta} - 4 \sum_{k=\gamma+3}^{n} x_k \sum_{\beta=\gamma+2}^{k-1} \frac{\partial^2}{\partial x_{k+\gamma+1-\beta} \partial x_\beta} \Bigg)$$

for $\mu \geq 1$, and $\widetilde{N}_{\gamma+1}(0) = 1$.

The following lemma is obvious by the above consideration. This will be used later in Section 6.

LEMMA 4.4. *For any integer $\nu \geq 0$ and for any non-zero polynomial $F(y_0, x_1, \ldots, x_n) \in \boldsymbol{C}[y_0, x_1, \ldots, x_n]$, assume that $P(R, S, W) = F(\det(W), P_1, \ldots, P_n) \in \mathscr{H}_{n,\nu,d}$. Then $F(0, \ldots, 0, 1) \neq 0$.*

Sometimes we need explicit expressions of our polynomials to apply it to differential operators on Siegel modular forms, e.g. for calculation of special values of L functions (cf. [**20**]), and the above kind of concrete calculation would be useful.

4.2.2. Examples of generating functions.

In the previous section, we gave a concrete method to give solutions for each fixed degree ν up to constant. It is desirable to gather these for all ν and give a neat generating functions of the solutions. But here it is a problem how to choose each constant and the method in the previous section does not seem to work well for this problem. Generating functions are known for $n = 1$ and 2. We have no result for $n \geq 3$.

(1) When $n = 1$, it is the classical generating function of the Gegenbauer polynomials. Define P_ν by

$$\frac{1}{(1 - 2y_0u + x_1u^2)^{(d-2)/2}} = \sum_{\nu=0}^{\infty} P_\nu(y_0, x_1)u^\nu.$$

Then we have $0 \neq P_\nu \in \mathscr{H}_{1,\nu,d}$.

(2) When $n = 2$. This case has been given in [**13**]. Put

$$\Delta_0 = 1 - 2y_0u + x_2u^2,$$
$$R = \frac{\Delta_0 + \sqrt{\Delta_0^2 - 4(x_0 + x_1 + x_2)u^2}}{2}.$$

Define P_ν by

$$\frac{1}{R^{(d-5)/2}\sqrt{\Delta_0^2 - 4(x_0 + x_1 + x_2)u^2}} = \sum_{\nu=0}^{\infty} P_\nu(y_0, x_1, x_2)u^\nu.$$

Then we have $0 \neq P_\nu \in \mathscr{H}_{2,\nu,d}$.

5. The radial parts and system of differential equations.

In this section, we take the radial part of our system $\{L_\gamma\}$. If $P \in \boldsymbol{P}_{n,\nu}$, then we have $P(R,S,W) = \det(RS)^{\nu/2} P(1_n, 1_n, R^{-1/2}WS^{-1/2})$ for $\det(RS) \neq 0$. Since we have

$$\begin{vmatrix} xR & W \\ {}^tW & S \end{vmatrix} = \det(RS)\det(x1_n - R^{-1}WS^{-1}\,{}^tW),$$

we define variables ξ_α $(0 \le \alpha \le n)$ and λ_i $(1 \le i \le n)$ by

$$\det(x1_n - R^{-1}WS^{-1}\,{}^tW) = \sum_{\alpha=0}^{n} \xi_\alpha x^\alpha = \prod_{i=1}^{n}(x - \lambda_i^2).$$

Here the variables λ_α^2 are eigenvalues of $R^{-1/2}WS^{-1}\,{}^tWR^{-1/2}$. We have $x_\alpha = P_\alpha = \det(RS)\xi_\alpha = x_n\xi_\alpha$ for each α with $0 \le \alpha \le n$. In particular, we have $\xi_n = 1$. For any homogeneous polynomial $F(x_0,\dots,x_n) \in \boldsymbol{C}[x_0,\dots,x_n]$ of degree $m = \nu/2$ for even ν, we can write

$$F(x_0,\dots,x_n) = x_n^{\nu/2} F\left(\frac{x_0}{x_n},\dots,\frac{x_{n-1}}{x_n},1\right) = x_n^{\nu/2} F(\xi_0,\dots,\xi_{n-1},1).$$

If we put $G(\xi_0,\dots,\xi_{n-1}) = F(\xi_0,\dots,\xi_{n-1},1)$. Then we have

$$\begin{aligned}
\frac{\partial F}{\partial x_\alpha} &= x_n^{\nu/2-1}\frac{\partial G}{\partial \xi_\alpha} \quad \text{for } 0 \le \alpha \le n-1,\\
\frac{\partial F}{\partial x_n} &= \frac{\nu}{2}x_n^{\nu/2-1}G - x_n^{\nu/2-2}\sum_{\alpha=0}^{n-1} x_\alpha \frac{\partial G}{\partial \xi_\alpha}\\
&= x_n^{\nu/2-1}\left(\frac{\nu}{2}G - \sum_{\alpha=0}^{n-1}\xi_\alpha\frac{\partial G}{\partial \xi_\alpha}\right).
\end{aligned}$$

So for $0 \le \gamma \le n-1$, we may write $L_\gamma F = x_n^{\nu/2-1} M_\gamma G$ for some differential operator M_γ with respect to ξ_α. The derivatives with respect to x_n appears in L_{n-1}, but x_n appears only in coefficients for L_γ with $\gamma < n-1$, so by using the above relations between derivatives of x_α and ξ_α, we can show

$$
\begin{aligned}
M_\gamma = & -2(\gamma+1)\frac{\partial}{\partial \xi_\gamma} + 2(d-2n+\gamma+2)\frac{\partial}{\partial \xi_{\gamma+1}} \\
& + 4\sum_{k=\gamma+2}^{n} \xi_k \sum_{\beta=\gamma+1}^{k-1} \frac{\partial^2}{\partial \xi_{k+\gamma-\beta}\partial \xi_\beta} - 4\sum_{k=0}^{\gamma} \xi_k \sum_{\beta=k}^{\gamma} \frac{\partial^2}{\partial \xi_{k+\gamma-\beta}\partial \xi_\beta} \\
& - 4\sum_{k=\gamma+3}^{n} \xi_k \sum_{\beta=\gamma+2}^{k-1} \frac{\partial^2}{\partial \xi_{k+\gamma+1-\beta}\partial \xi_\beta} + 4\sum_{k=0}^{\gamma+1} \xi_k \sum_{\beta=k}^{\gamma+1} \frac{\partial^2}{\partial \xi_{k+\gamma+1-\beta}\partial \xi_\beta}
\end{aligned} \tag{13}
$$

for $\gamma < n-1$ and

$$
\begin{aligned}
M_{n-1} = & \, \nu(d-n+\nu-1) - 2(d-n+1)\sum_{\alpha=0}^{n-1} \xi_\alpha \frac{\partial}{\partial \xi_\alpha} - 4\sum_{\alpha,\beta=0}^{n-1} \xi_\alpha \xi_\beta \frac{\partial^2}{\partial \xi_\alpha \partial \xi_\beta} \\
& - 2n\frac{\partial}{\partial \xi_{n-1}} + 4\sum_{\alpha,\beta=0}^{n-1} \xi_{\alpha+\beta-n} \frac{\partial^2}{\partial \xi_\alpha \partial \xi_\beta} - 4\sum_{\alpha,\beta=0}^{n-1} \xi_{\alpha+\beta-n+1} \frac{\partial^2}{\partial \xi_\alpha \partial \xi_\beta}.
\end{aligned}
$$

These are differential operators which characterize the solution in $\mathscr{H}_{n,\nu,d}$ for even ν. But if we use variables λ_i instead of ξ_α, then $\det(W)/\sqrt{\det(RS)} = \prod_{i=1}^n \lambda_i$, so for $P \in \boldsymbol{P}_{n,\nu}$, $\det(RS)^{-\nu/2}P(R,S,W)$ is written by λ_i^2 (or ξ_α) and $\prod_{i=1}^n \lambda_i$, i.e. we may write $P(R,S,W) = \det(RS)^{\nu/2}Q(\lambda_1,\dots,\lambda_n)$ for some polynomial Q. Here if ν is even, then Q is a symmetric function with respect to $\lambda_1^2,\dots,\lambda_n^2$ and if ν is odd, then $Q/(\lambda_1\cdots\lambda_n)$ is so. In each case, put $Q_1 = Q$ or $Q_1 = Q/(\lambda_1\cdots\lambda_n)$, respectively. If $P \in H_{n,\nu,d}$ and $Q \neq 0$, then by Lemma 4.4, we have

$$
Q_1(0,\dots,0) \neq 0. \tag{14}
$$

Now we change variables from ξ_α to λ_i and give the expression of differential operators M_γ by λ_i. Here we use the same notation M_γ for λ_i as for ξ_α, so we have

$$
\Delta_{11}P = \det(RS)^{\nu/2-1}\sum_{\gamma=0}^{n-1}(M_\gamma Q)\widehat{P}_\gamma.
$$

Proposition 5.1. *For any $\nu \geq 0$ and $P \in \boldsymbol{P}_{n,\nu}$, we have*

$$
\Delta_{11}P(R,S,W) = \det(RS)^{\nu/2-1}\sum_{\gamma=0}^{n-1}(M_\gamma Q)\widehat{P}_\gamma,
$$

where

$$M_\gamma = \sum_{k=1}^{n} \left(\frac{(1-\lambda_k^2)\lambda_k^{2\gamma}}{\prod_{i\neq k}(\lambda_k^2-\lambda_i^2)} \right) \frac{\partial^2}{\partial \lambda_k^2}$$
$$+ \sum_{k=1}^{n} \left(\frac{\gamma\lambda_k^{2\gamma-1} - (d-2n+\gamma+1)\lambda_k^{2\gamma+1}}{\prod_{i\neq k}(\lambda_k^2-\lambda_i^2)} \right) \frac{\partial}{\partial \lambda_k} + \nu(\nu+d-n-1)\delta_{\gamma,n-1}.$$

The proof is obtained by routine calculations but fairly long and the most of the rest of this section is devoted to the proof of this proposition. We assume that ν is even for the sake of simplicity in the most part of the following calculation. The correction for odd ν is similar and easy, and the proof in that case will be omitted.

First of all, to express $\partial/\partial\xi_\alpha$ by $\partial/\partial\lambda_i$, for any j with $1 \le j \le n$, we define $\xi_\alpha^{(j)}$ by the following expansion.

$$\prod_{i\neq j}\left(x-\lambda_i^2\right) = \xi_{n-1}^{(j)}x^{n-1} + \xi_{n-2}^{(j)}x^{n-2} + \cdots + \xi_1^{(j)}x + \xi_0^{(j)}.$$

In particular, we have $\xi_{n-1}^{(j)} = 1$. Since

$$\frac{\partial}{\partial\lambda_j}\prod_{i=1}^{n}\left(x-\lambda_i^2\right) = -2\lambda_j\prod_{i\neq j}\left(x-\lambda_i^2\right)$$

for $0 \le \alpha \le n-1$, we have

$$\frac{\partial\xi_\alpha}{\partial\lambda_j} = -2\lambda_j\xi_\alpha^{(j)}$$

and

$$\frac{\partial}{\partial\lambda_j} = \sum_{\alpha=0}^{n-1}\left(-2\lambda_j\xi_\alpha^{(j)}\right)\frac{\partial}{\partial\xi_\alpha}.$$

Since

$$\sum_{\alpha=0}^{n-1}\lambda_l^{2\alpha}\xi_\alpha^{(j)} = \prod_{1\le i\le n, i\neq j}\left(\lambda_l^2-\lambda_i^2\right) = \delta_{lj}\prod_{1\le i\le n, i\neq j}\left(\lambda_j^2-\lambda_i^2\right),$$

where δ_{lj} is Kronecker's delta, the inverse matrix of $n \times n$ matrix $(-2\lambda_\alpha \xi_\alpha^{(j)})_{0\le\alpha\le n-1, 1\le j\le n}$ is easily obtained and we have

$$\begin{pmatrix} \dfrac{\partial}{\partial \xi_0} \\ \vdots \\ \dfrac{\partial}{\partial \xi_{n-1}} \end{pmatrix} = A \begin{pmatrix} \dfrac{\partial}{\partial \lambda_1} \\ \vdots \\ \dfrac{\partial}{\partial \lambda_n} \end{pmatrix},$$

where $A = (a_{ij})_{1\le i,j\le n}$ is given by

$$a_{ij} = -\frac{\lambda_j^{2i-3}}{2\prod_{l\neq j}\left(\lambda_j^2 - \lambda_l^2\right)}.$$

Now we must express the second order derivatives with respect to $\{\xi_\alpha\}_{0\le\alpha\le n-1}$ also by derivatives with respect to $\{\lambda_i\}_{1\le i\le n}$. To calculate this, we prepare several formulas.

Lemma 5.2. *For any i with $1 \le i \le n$, we have*

$$\sum_{k=0}^{n} \xi_k \lambda_i^{2k} = 0, \tag{15}$$

$$\sum_{k=0}^{n} k\xi_k \lambda_i^{2k-2} = \prod_{l\neq i}\left(\lambda_i^2 - \lambda_l^2\right), \tag{16}$$

$$\sum_{k=0}^{n} k(k-1)\xi_k \lambda_i^{2k-4} = 2\sum_{\substack{1\le m\le n \\ m\neq i}} \prod_{l\neq i,m}\left(\lambda_i^2 - \lambda_l^2\right). \tag{17}$$

Proof. (15) is trivial by the definition. Since we have

$$\frac{d}{dx}\prod_{j=1}^{n}\left(x - \lambda_j^2\right) = \sum_{m=1}^{n}\prod_{j\neq m}\left(x - \lambda_j^2\right) = \sum_{k=1}^{n} k\xi_k x^{k-1},$$

taking $x = \lambda_i^2$ we have (16). The assertion (17) is obtained by differentiating twice by x and putting $x = \lambda_i^2$. □

Proof of Proposition 5.1. Now for $\gamma < n-1$, we put

$$M(\gamma) = 4\sum_{k=\gamma+2}^{n} \xi_k \sum_{\beta=\gamma+1}^{k-1} \frac{\partial^2}{\partial\xi_{k+\gamma-\beta}\partial\xi_\beta} - 4\sum_{k=0}^{\gamma} \xi_k \sum_{\beta=k}^{\gamma} \frac{\partial^2}{\partial\xi_{k+\gamma-\beta}\partial\xi_\beta}. \tag{18}$$

Then the terms of the second order derivatives of M_γ in (13) with respect to variables ξ_α is $M(\gamma) - M(\gamma+1)$. We calculate $M(\gamma)$. We have

$$\frac{\partial^2}{\partial\xi_{k+\gamma-\beta}\partial\xi_\beta} = \sum_{i,j=1}^{n} \frac{\lambda_i^{2(k+\gamma-\beta)-1}\lambda_j^{2\beta-1}}{4\prod_{l\neq i}\left(\lambda_i^2-\lambda_l^2\right)\prod_{l\neq j}\left(\lambda_j^2-\lambda_l^2\right)} \frac{\partial^2}{\partial\lambda_i\partial\lambda_j}$$
$$+ \sum_{i=1}^{n} \frac{\lambda_i^{2(k+\gamma-\beta)-1}}{4\prod_{l\neq i}\left(\lambda_i^2-\lambda_l^2\right)} \frac{\partial}{\partial\lambda_i}\left(\frac{\lambda_j^{2\beta-1}}{\prod_{l\neq j}\left(\lambda_j^2-\lambda_l^2\right)}\right)\frac{\partial}{\partial\lambda_j}. \tag{19}$$

First we see the coefficient of $\partial^2/\partial\lambda_i\partial\lambda_j$ in $M(\gamma)$ for $i \neq j$, i.e., the term obtained by summation over the first term of (19) in $M(\gamma)$. We take the inner sum $\sum_{\beta=\gamma+1}^{k-1}$ and $\sum_{\beta=k}^{\gamma}$ of (18) first. Since only the term depending on β is essentially $\lambda_i^{2(k+\gamma-\beta)}\lambda_j^{2\beta}$, we have

$$\sum_{\beta=\gamma+1}^{k-1} \lambda_i^{2(k+\gamma-\beta)}\lambda_j^{2\beta} = \frac{\lambda_j^{2(\gamma+1)}\lambda_i^{2k} - \lambda_i^{2(\gamma+1)}\lambda_j^{2k}}{\lambda_i^2-\lambda_j^2},$$
$$\sum_{\beta=k}^{\gamma} \lambda_i^{2(k+\gamma-\beta)}\lambda_j^{2\beta} = \frac{\lambda_i^{2(\gamma+1)}\lambda_j^{2k} - \lambda_j^{2(\gamma+1)}\lambda_i^{2k}}{\lambda_i^2-\lambda_j^2}.$$

As for the summation of $\lambda_j^{2(\gamma+1)}\lambda_i^{2k}$ over k, by (15) we have

$$\sum_{k=0}^{\gamma} \xi_k\lambda_j^{2(\gamma+1)}\lambda_i^{2k} + \sum_{k=\gamma+2}^{n} \xi_k\lambda_j^{2(\gamma+1)}\lambda_i^{2k} = -\xi_{\gamma+1}(\lambda_i\lambda_j)^{2(\gamma+1)}.$$

The summation over $\lambda_i^{2(\gamma+1)}\lambda_j^{2k}$ is (-1) times the above, so since we assumed $i \neq j$, we have 0 as a total. Now let us see the coefficient of $\partial^2/\partial\lambda_i^2$. In this case we have $i = j$, so $\lambda_i^{2(k+\gamma-\beta)-1}\lambda_j^{2\beta-1} = \lambda_i^{2(k+\gamma-1)}$. Since this is independent of β, the summation from $\beta = \gamma+1$ to $k-1$ or from $\beta = k$ to γ is just a multiplication of $k-\gamma-1$ or $\gamma-k+1$. (Of course each occurs only when $k \geq \gamma+2$ or $k \leq \gamma$.) So we should take the following sum, which is simplified by (15), (16).

$$\begin{aligned}\sum_{\substack{0\le k\le n\\ k\ne\gamma+1}}(k-\gamma-1)\xi_k\lambda_i^{2(k+\gamma-1)} &= \sum_{k=0}^{n}(k-\gamma-1)\xi_k\lambda_i^{2(k+\gamma-1)}\\ &= \sum_{k=0}^{n}k\xi_k\lambda_i^{2k-2}\lambda_i^{2\gamma}\\ &= \lambda_i^{2\gamma}\prod_{l\ne i}\left(\lambda_i^2-\lambda_l^2\right).\end{aligned}$$

Taking the corresponding term of $M(\gamma)-M(\gamma+1)$, we have

$$\frac{\lambda_i^{2\gamma}\left(1-\lambda_i^2\right)}{\prod_{l\ne i}\left(\lambda_i^2-\lambda_l^2\right)}$$

as a coefficient of $\partial^2/\partial\lambda_i^2$. Now we calculate the coefficient of $\partial/\partial\lambda_j$ in $M(\gamma)$. If $i\ne j$, then the term depending on β in $\partial/\partial\lambda_i(\lambda_j^{2\beta-1}/\prod_{l\ne j}(\lambda_j^2-\lambda_l^2))$ is essentially $\lambda_j^{2\beta-1}$. By the same calculation for the coefficient of $\partial^2/\partial\lambda_i\partial\lambda_j$ for $i\ne j$, we see that the summation is zero for this term. So we may assume that $j=i$. Then we have

$$\begin{aligned}&\frac{\partial}{\partial\lambda_i}\left(\frac{\lambda_i^{2\beta-1}}{\prod_{l\ne i}\left(\lambda_i^2-\lambda_l^2\right)}\right)\\ &\quad=\frac{(2\beta-1)\lambda_i^{2\beta-2}}{\prod_{l\ne i}\left(\lambda_i^2-\lambda_l^2\right)}-\sum_{\substack{1\le m\le n\\ m\ne i}}\frac{2\lambda_i^{2\beta}}{\left(\lambda_i^2-\lambda_m^2\right)\prod_{l\ne i}\left(\lambda_i^2-\lambda_l^2\right)}.\end{aligned}\tag{20}$$

We have $(2\beta-1)\lambda_i^{2\beta-2}\lambda_i^{2(k+\gamma-\beta)-1}=(2\beta-1)\lambda_i^{2(k+\gamma)-3}$ and

$$\sum_{\beta=\gamma+1}^{k-1}(2\beta-1)=(k-1-\gamma)(k-1+\gamma)=-\sum_{\beta=k}^{\gamma}(2\beta-1).$$

This vanishes for $k=\gamma+1$. We have

$$\begin{aligned}&(k-1-\gamma)(k-1+\gamma)\lambda_i^{2(k+\gamma)-3}\\ &\quad=k(k-1)\lambda_i^{2(k-2)}\lambda_i^{2\gamma+1}-k\lambda_i^{2(k-1)}\lambda_i^{2\gamma-1}+(1-\gamma^2)\lambda_i^{2k}\lambda_i^{2\gamma-3}\end{aligned}$$

and the sum of this over $0\le k\le n$ is calculated by Lemma 5.2. So the contribution from the first term of (20) to the coefficient of $\partial/\partial\lambda_i$ in $M(\gamma)$ is given by

$$\frac{4}{4\prod_{l\neq i}\left(\lambda_i^2-\lambda_l^2\right)^2}\times\left(2\lambda_i^{2\gamma+1}\sum_{\substack{1\leq m\leq n\\ m\neq i}}\prod_{l\neq i,m}\left(\lambda_i^2-\lambda_l^2\right)-\lambda_i^{2\gamma-1}\prod_{l\neq i}\left(\lambda_i^2-\lambda_l^2\right)\right)$$

$$=\frac{2\lambda_i^{2\gamma+1}}{\prod_{l\neq i}\left(\lambda_i^2-\lambda_l^2\right)}\sum_{\substack{1\leq m\leq n\\ m\neq i}}\frac{1}{\lambda_i^2-\lambda_m^2}-\frac{\lambda_i^{2\gamma-1}}{\prod_{i\neq l}\left(\lambda_i^2-\lambda_l^2\right)}. \tag{21}$$

As for the second term of (20), we have $\lambda_i^{2(k+\gamma-\beta)-1+2\beta}=\lambda_i^{2k+2\gamma-1}$, and the sum for $\beta=\gamma+1$ to $k-1$ or $\beta=k$ to γ is $\pm(k-\gamma-1)$. This vanishes for $k=\gamma+1$. So we should take the sum over $k=0$ to n. We have

$$\begin{aligned}2\sum_{k=0}^{n}(k-\gamma-1)\xi_k\lambda_i^{2k+2\gamma-1}&=2\sum_{k=0}^{n}k\xi_k\lambda_i^{2k-2}\lambda_i^{2\gamma+1}\\&=2\lambda_i^{2\gamma+1}\prod_{l\neq i}\left(\lambda_i^2-\lambda_l^2\right).\end{aligned}$$

So the term coming from this cancels with the first term of (21). Hence the coefficient of $\partial/\partial\lambda_i$ in $M(\gamma)-M(\gamma+1)$ is given by

$$\frac{\lambda_i^{2\gamma+1}-\lambda_i^{2\gamma-1}}{\prod_{l\neq i}\left(\lambda_i^2-\lambda_l^2\right)}.$$

For M_γ, we still have terms coming from the first order derivatives of ξ_γ and $\xi_{\gamma+1}$ in (13). The coefficient of $\partial/\partial\lambda_i$ is given directly by

$$\frac{(\gamma+1)\lambda_i^{2\gamma-1}-(d-2n+\gamma+2)\lambda_i^{2\gamma+1}}{\prod_{l\neq i}\left(\lambda_i^2-\lambda_l^2\right)}.$$

So taking the sum of all the above calculations, we obtained the assertion of Proposition 5.1 for M_γ with $\gamma<n-1$. The proof for the assertion for M_{n-1} is similarly obtained and omitted here. □

Now the term of the second order derivatives of M_γ with respect to λ_i variables consists only of second derivation of the same λ_k and there are no mixed terms, so it is natural to change M_γ to differential operators so that the second order term contains only derivation of λ_k for only one k. For that purpose, we define an invertible linear transform from M_γ $(0\leq\gamma\leq n-1)$ to a new system $\boldsymbol{D}_k$ $(1\leq k\leq n)$ as follows.

$$\boldsymbol{D}_k = \sum_{\gamma=0}^{n-1} \xi_\gamma^{(k)} M_\gamma.$$

We can show that these operators satisfy our demand. For that purpose we need the following formulas.

$$\begin{aligned}
\sum_{\gamma=0}^{n-1} \xi_\gamma^{(k)} \lambda_j^{2\gamma} &= \delta_{jk} \prod_{l\neq k} \big(\lambda_k^2 - \lambda_l^2\big), \\
\sum_{\gamma=0}^{n-1} \gamma \xi_\gamma^{(k)} \lambda_j^{2\gamma-2} &= \prod_{l\neq k,j} \big(\lambda_j^2 - \lambda_l^2\big) \quad \text{if } j \neq k, \\
\sum_{\gamma=0}^{n-1} \gamma \xi_\gamma^{(k)} \lambda_k^{2\gamma-2} &= \sum_{m\neq k} \prod_{l\neq k,m} \big(\lambda_k^2 - \lambda_l^2\big),
\end{aligned}$$

where δ_{jk} is Kronecker's delta. These relations are proved in the same way as in Lemma 5.2.

Using these relations and Proposition 5.1, we get the following theorem by an easy direct calculation.

THEOREM 5.3. *For each k with $1 \leq k \leq n$, we have*

$$\begin{aligned}
\boldsymbol{D}_k = \big(1-\lambda_k^2\big)\frac{\partial^2}{\partial\lambda_k^2} &+ \Bigg(-(d-2n+1)\lambda_k + \sum_{l\neq k} \frac{\lambda_k\big(1-\lambda_k^2\big)}{\lambda_k^2-\lambda_l^2} \Bigg) \frac{\partial}{\partial\lambda_k} \\
&+ \sum_{l\neq k} \frac{\big(1-\lambda_l^2\big)\lambda_l}{\big(\lambda_l^2-\lambda_k^2\big)} \frac{\partial}{\partial\lambda_l} + \nu(\nu+d-n-1).
\end{aligned}$$

Our polynomials $Q(\lambda_1, \ldots, \lambda_n)$ are solutions of the system

$$\boldsymbol{D}_k Q = 0, \qquad (1 \leq k \leq n).$$

When $n = 1$, this is nothing but the usual Gegenbauer differential equation.

6. Inner product.

We define a natural inner product for our spherical polynomials. Originally it comes from polynomials $P(R, S, W)$ on the domain $\mathfrak{D}_n$ where

$$\mathfrak{D}_n = \left\{ \begin{pmatrix} R & W \\ {}^tW & S \end{pmatrix} \in \mathrm{Sym}_{2n}(\boldsymbol{R}); \text{positive definite} \right\}$$

and now we can regard it as a polynomial $f(\lambda)$ where $\lambda = (\lambda_1, \ldots, \lambda_n)$. We define integrals for these two expressions. We put

$$I_1(P) = \int_{\mathfrak{D}_n} P(R,S,W) \begin{vmatrix} R & W \\ {}^tW & S \end{vmatrix}^{(d-2n-1)/2} dR\, dS\, dW,$$

$$I_2(f) = \int_{|\lambda_n| \le \lambda_{n-1} \le \cdots \le \lambda_1 < 1} f(\lambda) \times \prod_{1 \le j < k \le n} \left(\lambda_k^2 - \lambda_j^2\right) \prod_{i=1}^{n} \left(1 - \lambda_i^2\right)^{(d-2n-1)/2} d\lambda_1 \cdots d\lambda_n,$$

where $dR = \prod_{1 \le i \le j \le n} dr_{ij}$, $dS = \prod_{1 \le i \le j \le n} ds_{ij}$, $dW = \prod_{1 \le i,j \le n} dw_{ij}$ for $R = (r_{ij})$, $S = (s_{ij})$, $W = (w_{ij})$. Now for any polynomial $P \in \boldsymbol{P}_{n,\nu}$, put $f_P(\lambda_1, \ldots, \lambda_n) = P(1_n, 1_n, \Lambda)$ where Λ is the diagonal matrix whose diagonal entries are λ_i. Then we see

Theorem 6.1.

(1) *For $P \in \boldsymbol{P}_{n,\nu}$, $I_1(P)$ and $I_2(f_P)$ are equal up to constant depending only on n and d.*
(2) *For natural numbers μ, ν such that $\mu \ne \nu$, take $P_\mu \in \mathscr{H}_{n,\mu,d}$, $P_\nu \in \mathscr{H}_{n,\nu,d}$, and define f_{P_μ} and f_{P_ν} as above. Then we have*

$$I_1(P_\mu \overline{P_\nu}) = I_2(f_{P_\mu} \overline{f_{P_\nu}}) = 0$$

where $\overline{*}$ denotes the complex conjugation.

Proof. We give here only a sketch of the proof. For positive definite R and S, we have $P(R,S,W) = \det(RS)^{\nu/2} P(1_n, 1_n, R^{-1/2} W S^{-1/2})$. If we put $U = R^{-1/2} W S^{-1/2}$, then $\det \left(\begin{smallmatrix} R & W \\ {}^tW & S \end{smallmatrix} \right) = \det(RS) \det(1 - U\, {}^tU)$ and $I_1(P)$ becomes

$$\int_{1_n - U\,{}^tU > 0} \det(1_n - U\,{}^tU)^{(d-2n-1)/2} P(1_n, 1_n, U) dU$$

up to constant. We put $U = Ph$ ($P = (p_{ij})$ is upper triangular with positive diagonals and $h \in O(n)$) and $V = U\,{}^tU = P\,{}^tP$. We denote by λ_i^2 the eigenvalues of V. By the condition that $1_n - V > 0$, we may assume that $1 > |\lambda_1| \ge |\lambda_2| \ge$

$\cdots \geq |\lambda_n|$. Since $P(1_n, 1_n, h_1 U h_2) = P(1_n, 1_n, U)$ for any $h_1,\, h_2 \in SO(n)$, we may assume besides that

$$|\lambda_n| \leq \lambda_{n-1} \leq \cdots \leq \lambda_1 < 1.$$

We see easily that

$$\begin{aligned}
dU &= \prod_{i=1}^{n} p_{ii}^{n-i} dP dh, \\
dV &= 2^n \prod_{i=1}^{n} p_{ii}^{n-i+1} dP, \\
dU &= 2^{-n} \det(V)^{-1/2} dV dh, \\
dV &= \prod_{1\leq j<k\leq n} \big(\lambda_k^2 - \lambda_j^2\big) d\lambda_1^2 \cdots d\lambda_n^2\, dh,
\end{aligned}$$

where dh is a suitable measure of $SO(n)$ and dU, dV, dP are natural Lebesgue measures. The integral with respect to dh is a constant and does not matter. Since $\det(V)^{-1/2}|\lambda_1 \cdots \lambda_n| = 1$, we see that $I_1(P)$ and $I_2(f_P)$ are proportional and we prove (1). Now we define a measure for any function $F(X)$ of $X \in M_{n,d}(\boldsymbol{R})$ by

$$I_3(F) = \int_{M_{n,d}(\boldsymbol{R})} e^{-\mathrm{tr}(X\, {}^t X)} F(X) dX.$$

If F and G are pluriharmonic polynomials each of which belongs to a different irreducible representation space of $O(d)$, then $I_3(F\overline{G}) = 0$ (cf. [**19**]). Our P in question originally comes from a polynomial $P^*(X, Y)$ which is pluriharmonic with respect to each X or Y, and it is in the tensor product of pluriharmonic polynomials in the same representation space of $O(d)$. On the other hand, we can also see that $I_3(P^*(X,Y)) = I_1(P(R,S,W))$ up to constant. So (2) automatically follows from this. $\square$

7. Hypergeometric polynomials of several variables.

7.1. A second order differential operator.

In Theorem 5.3, we have written down the differential operators $\boldsymbol{D}_i$ in the coordinates $(\lambda_1, \ldots, \lambda_n)$. In this section, we express these operators in the new coordinates $(z_1, \ldots, z_n)$ in the relation $z_i = \lambda_i^2$ $(i = 1, \ldots, n)$. They turn out to be identified with the known differential operators.

Definition 7.1. Let a, b, c be complex parameters. We define linear partial differential operators in $(z_1, \dots, z_n)$ by

$$\boldsymbol{D}'_i(a,b,c) := z_i(1-z_i)\frac{\partial^2}{\partial z_i^2} + \left(c - \frac{1}{2}(n-1) - \left(a+b+1-\frac{n-1}{2}\right)z_i\right)\frac{\partial}{\partial z_i}$$
$$+ \frac{1}{2}\sum_{j(\neq i)} \frac{z_i(1-z_i)}{z_i - z_j}\frac{\partial}{\partial z_i} - \frac{1}{2}\sum_{j(\neq i)} \frac{z_j(1-z_j)}{z_i - z_j}\frac{\partial}{\partial z_j}.$$

Lemma 7.2. *For each $i = 1, \dots, n$, the differential operator $\boldsymbol{D}_i$ is equal to $4(\boldsymbol{D}'_i(a,b,c) - ab)$ under the change of coordinates $z_1 = \lambda_1^2, \dots, z_n = \lambda_n^2$, where the values of parameters are specified as*

$$a = -\frac{1}{2}\nu,\ b = \frac{1}{2}(\nu + d - n - 1), \quad \text{and} \quad c = \frac{1}{2}n.$$

In particular, the system of the differential equations $\boldsymbol{D}_1 Q = \cdots = \boldsymbol{D}_n Q = 0$ is equivalent to the system of differential equations $\boldsymbol{D}'_1 Q = \cdots = \boldsymbol{D}'_n Q = abQ$.

Proof. Under the change of variable $z_i = \lambda_i^2$, we have $\lambda_i(\partial/\partial\lambda_i) = 2z_i(\partial/\partial z_i)$, and $\partial^2/\partial\lambda_i^2 = 4z(\partial^2/\partial z_i^2) + 2(\partial/\partial z_i)$. □

7.2. Hypergeometric solutions.

In order to describe the special solution of this system of differential equations, we introduce, so-called, the hypergeometric functions ${}_2F_1$ with matrix argument, introduced by A. G. Constantine [**5**].

For $a \in \boldsymbol{C}$ and $k \in \boldsymbol{Z}_{\geq 0}$, we denote

$$(a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

For a partition $\kappa = (k_1, \dots, k_n)$ of k into not more than n parts, that is, $k_1 \geq k_2 \geq \cdots k_n \geq 0$ and $k = k_1 + k_2 + \cdots + k_n$, we set

$$(a)_\kappa = \prod_{i=1}^{n}\left(a - \frac{1}{2}(i-1)\right)_{k_i}.$$

We denote by $C_\kappa = C_\kappa(z_1, \dots, z_n)$ the zonal polynomial corresponding to the partition κ (see Section 7.3).

Definition 7.3. We define a series in $z = (z_1, \dots, z_n)$ by

$$ {}_2F_1(a,b;c;z) = \sum_{\kappa} \frac{(a)_\kappa (b)_\kappa}{(c)_\kappa} \frac{C_\kappa(z)}{(k_1 + \cdots + k_n)!}, $$

where $\kappa = (k_1, \ldots, k_n)$ runs over the partition into at most n parts.

Note that if a is a negative integer, then the above expression of ${}_2F_1(a,b;c;z)$ is a finite sum, and is a polynomial since $(a)_k = 0$ for all integers greater than $-a$.

The following is conjectured by Constantine [**5**] and is proved by R. J. Muirhead [**21**, Theorem 3.1].

PROPOSITION 7.4. *The function ${}_2F_1(a,b;c;z_1,\ldots,z_n)$ is the unique solution f of the system of the differential equations*

$$ \boldsymbol{D}'_1 f = \boldsymbol{D}'_2 f = \cdots = \boldsymbol{D}'_n f = abf $$

with the property

(a) *f is a holomorphic function near the origin $(z_1,\ldots,z_n) = (0,\ldots,0)$ and $f(0,\ldots,0) = 1$.*
(b) *$f(z_1,\ldots,z_n)$ is symmetric with respect to the variables $z_1,\ldots,z_n$.*

We know ([**10**, Section 6], [**9**, Section 7], [**2**, Theorem 4.1]) that the above function ${}_2F_1(a,b;c;z)$ is the hypergeometric function associated with the root system BC_n and with a degenerate spectral parameter $(-a,\ldots,-a)$.

We identify the polynomial Q defined in Section 5 with the hypergeometric function with matrix argument.

THEOREM 7.5. *Let d, n and ν be integers in Section 2.*

(1) *Suppose ν is even. Then*

$$ Q(\lambda_1,\ldots,\lambda_n) = {}_2F_1\left(-\frac{1}{2}\nu, \frac{1}{2}(\nu+d-n-1); \frac{n}{2}; \lambda_1^2,\ldots,\lambda_n^2 \right) $$

up to a constant multiple.

(2) *Suppose ν is odd. Then*

$$ Q(\lambda_1,\ldots,\lambda_n) = \lambda_1\lambda_2\cdots\lambda_n \cdot {}_2F_1\left(-\frac{\nu-1}{2}, \frac{1}{2}(\nu+d-n); \frac{n}{2}+1; \lambda_1^2,\ldots,\lambda_n^2 \right) $$

up to a constant multiple.

PROOF. We will appeal to the uniqueness criterion of Proposition 7.4.

(1) We have seen in (14) that the function Q is a polynomial in $z_1, \ldots, z_n$ at the origin, is symmetric, and satisfies the non-vanishing condition at the origin. By Theorem 5.3, the function Q satisfies the system of differential equations with the specified parameters. Hence Q is a multiple of ${}_2F_1(a, b; c; z_1, \ldots, z_n)$.

(2) We will consider the function $f(z_1, \ldots, z_n) = Q(\lambda_1, \ldots, \lambda_n)/(\lambda_1 \cdots \lambda_n)$. We have seen in (14) that the function f is a polynomial in $z_1, \ldots, z_n$ at the origin, is symmetric, and satisfies the non-vanishing condition at the origin. By Theorem 5.3, the function Q satisfies the system of differential equations $\boldsymbol{D}'_k(a, b, c)Q = abQ$ with the parameters $a = -\nu/2$, $b = (\nu + d - n - 1)/2$, $c = n/2$. Now we use the relation

$$\big(\boldsymbol{D}'_k(a, b, c) - ab\big) \circ \sqrt{z_1 z_2 \cdots z_n} = \sqrt{z_1 z_2 \cdots z_n} \circ \big(\boldsymbol{D}'_k(a', b', c') - a'b'\big),$$

where $a' = a + (1/2)$, $b' = b + (1/2)$, $c' = c + 1$. This relation shows that f satisfies the system of differential equations $\boldsymbol{D}'_k(a', b', c')f = a'b'f$. Hence Q is a multiple of ${}_2F_1(a', b'; c'; z_1, \ldots, z_n)$. $\square$

This is an explicit formula of the polynomials which we are interested in.

We give a remark on the system of differential equations appearing in Proposition 7.4. As is mentioned, the function ${}_2F_1(a, b; c; z)$, which is annihilated by the differential operators $\boldsymbol{D}'_k - ab$ ($k = 1, \ldots, n$), is the hypergeometric function associated with the root system BC_n and with a degenerate spectral parameter $(-a, \ldots, -a)$. But for other functions annihilated by all the differential operators $\boldsymbol{D}'_k - ab$ ($k = 1, \ldots, n$), we do not know they would satisfy the system of hypergeometric differential equations associated with the root system BC_n and with a degenerate spectral parameter $(-a, \ldots, -a)$. The D-module counter part is also a question; It is suggested that the left ideal of the ring $\mathscr{D}$ of differential operators generated by $\boldsymbol{D}'_k - ab$ ($k = 1, \ldots, n$) would be contained in the left ideal of the commuting differential operators corresponding to the generalized hypergeometric systems of type BC_n with the parameter $(-a, \ldots, -a)$. Note that the rank of the generalized hypergeometric systems of type BC_n is the order of the Weyl group $W(B_n)$ of type B_n, which is $2^n n!$. The generalized hypergeometric system associated with the root system is irreducible for generic parameters. We show in Appendix B that the system given by $\boldsymbol{D}'_k - ab$ ($k = 1, \ldots, n$) is holonomic of rank 2^n. We will expect that there exists the subsystem of rank 2^n in the generalized hypergeometric system of type BC_n with a degenerate parameter $(-a, -a, \ldots, -a) \in \boldsymbol{C}^n$ with $a \neq 0$, and such a system is given by the operators $\boldsymbol{D}'_k$. This expectation is compatible with the fact that the number of orbits of the Weyl group $W(B_n)$ through $(-a, -a, \ldots, -a) \in \boldsymbol{C}^n$ with $a \neq 0$ is 2^n.

7.3. Appendix: zonal polynomial.

We recall the definition of zonal polynomials. The monomial symmetric function $m_\kappa = m_\kappa(z_1, \dots, z_n)$ is by definition the sum of distinct permutations of a monomial $z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$. We introduce a lexicographic order $\leq$ on the set of partitions of k. That is, two partitions κ and κ' of k has a relation $\kappa' < \kappa$ if and only if there exists a natural number i such that $k_1' = k_1, \dots, k_{i-1}' = k_{i-1}$ and $k_i' < k_i$. For example, $(1, \dots, 1) \leq \kappa \leq (k)$ for any partition κ. We denote by $C_\kappa(z) = C_\kappa(z_1, \dots, z_n)$ the zonal polynomial corresponding to the partition κ. This polynomial has the following properties (see, e.g., [**16**]):

(i) $C_\kappa(z)$ is a homogeneous symmetric polynomial of degree $k(= k_1 + \cdots + k_n)$.

(ii) $C_\kappa(z)$ is a linear combination of monomial symmetric functions $m_{\kappa'}$ with $\kappa' \leq \kappa$. The coefficient of m_κ in C_κ is non-zero.

(iii) $C_\kappa(z)$ satisfies the differential equation

$$\left(\sum_{i=1}^n z_i^2 \frac{\partial^2}{\partial z_i^2} + \sum_{i=1}^n \sum_{j(\neq i)} \frac{z_i^2}{z_i - z_j} \frac{\partial}{\partial z_i}\right) C_\kappa(z)$$
$$= \left(k(n-1) + \sum_{i=1}^n k_i(k_i - i)\right) C_\kappa(z).$$

(iv) We have the following expression in the generating function

$$(z_1 + \cdots + z_n)^k = \sum_\kappa C_\kappa(z_1, \dots, z_n).$$

Note that the conditions (i) (ii) (iii) define C_κ up to a constant multiple, and the condition (iv) gives a normalization of this constant multiple. Note that the zonal polynomial is a zonal spherical function on $GL(n)/O(n)$ with a parameter κ.

8. Appendix A: Spherical polynomials on symmetric spaces.

In this section we give a summary on pluriharmonic polynomials and zonal spherical functions on Grassmann manifolds.

We assume that $d > 2n$, and we put $GL(n) = GL(n, \boldsymbol{R})$, $O(n) = O(n, \boldsymbol{R})$, and $M_{n,d} = M_{n,d}(\boldsymbol{R})$ for short.

8.1. Irreducible representations of $GL(n)$.

Each irreducible (finite-dimensional) polynomial representation ρ of $GL(n)$ corresponds to a partition $(f_1, f_2, \dots, f_n)$ of length at most n, where $f_1 \geq f_2 \geq \cdots \geq f_n$ are non-negative integers. A partition is often identified with the Young diagram.

Lemma 8.1. *For an irreducible polynomial representation ρ of $GL(n)$, the followings are equivalent.*

- *The restriction of ρ to the subgroup $SO(n)$ contains the trivial representation of $SO(n)$. In such a case, the multiplicity of the trivial representation is always one.*
- *ρ or $\det^{-1}\otimes\rho$ (or equivalently $\det\otimes\rho$) arises in $\boldsymbol{C}[M(n)]^{SO(n)}$, where $\det$ is the determinant representation of $GL(n)$. In such a case, the multiplicity of ρ on $\boldsymbol{C}[M(n)]^{SO(n)}$ is always one.*
- *ρ or $\rho\otimes\mathrm{sgn}$ arises in $\boldsymbol{C}[\mathrm{Sym}_n]$. In such a case, the multiplicity of such a representation in $\boldsymbol{C}[\mathrm{Sym}_n]$ is always one.*
- *The partition corresponding to ρ satisfies the condition that $f_i - f_j$ is even for any $1 \le i < j \le n$.*

Here $\boldsymbol{C}[M(n)]^{SO(n)}$ is defined to be the space of polynomials f on $M(n)$ such that $f(xk) = f(x)$ for all $k \in SO(n)$, $x \in M(n)$. The action $L(g)$ of $g \in GL(n)$ is given by the left translation $(L(g)f)(x) = f(g^{-1}x)$ for $x \in M(n)$. Let Sym_n be the set of symmetric matrices of size n and $\boldsymbol{C}[\mathrm{Sym}_n]$ the space of the polynomials $P(X)$ on Sym_n. The action of $g \in GL(n)$ on P is given by $P(X) \mapsto P(gX{}^tg)$. The proof of Lemma 8.1 is easily obtained by using [**8**, p. 257, Theorem 5.2.9] and the Frobenius reciprocity.

We denote by Ψ the set of all irreducible polynomial representations of $GL(n)$, and by $\Psi^0 \subset \Psi$ the subset consisting of the representations with the properties in Lemma 8.1.

8.2. The space of pluriharmonic polynomials.

Recall that $\mathscr{H}_{n,d}$ is defined to be the space of pluriharmonic polynomials $P(X)$ in $M_{n,d}$. The group $GL(n) \times O(d)$ acts on $\mathscr{H}_{n,d}$ by $P({}^tAXh)$ for $(A,h) \in GL(n)\times O(d)$, $X \in M_{n,d}$. Now we consider a representation $\rho\otimes\lambda$ of $GL(n)\times O(d)$ realized in $\mathscr{H}_{n,d}$. Let Σ be the set of all irreducible representations of $O(d)$, and Σ^1 the set of irreducible representations of $O(d)$ which arises in $\mathscr{H}_{n,d}$. If an irreducible representation $\rho\otimes\lambda$ of $GL(n)\times O(d)$ is realized in $\mathscr{H}_{n,d}$, we put $\tau(\lambda) = \rho$. Kashiwara and Vergne [**19**] shows that τ gives an injective map from Σ^1 to the set of irreducible polynomial representations of $GL(n)$. We denote its image by Ψ^1. The map τ gives a bijective correspondence between Σ^1 and Ψ^1. We define $\Psi^2 := \Psi^0\cap\Psi^1$, and $\Sigma^2 := \tau^{-1}(\Psi^2)$. We denote by $\mathscr{H}_{n,d}^{SO(n)}$ the space of pluriharmonic polynomials which are left invariant by $SO(n)$. Since $SO(n)$-fixed vector in each irreducible representation of $GL(n)$ is at most one-dimensional (Lemma 8.1), the space $\mathscr{H}_{n,d}^{SO(n)}$ is a direct sum of irreducible subrepresentations of $O(d)$ in Σ^2 with multiplicity-free.

$$\begin{array}{ccccccc} \Sigma & \supset & \Sigma^1 & \underset{\sim}{\xrightarrow{\tau}} & \Psi^1 & \subset & \Psi \\ & & \cup & & \cup & & \cup \\ & & \Sigma^2 & \underset{\sim}{\xrightarrow{\tau}} & \Psi^2 & \subset & \Psi^0 \end{array}$$

We employ the standard parametrization of the irreducible representations of $O(d)$. By the explicit description of the map τ given in Theorem 6.9 and Theorem 6.13 of Kashiwara and Vergne [**19**], we can read off the set Σ^2. The conclusion is

$$\begin{aligned} \Sigma^1 &= \text{parameters with depth at most } n \\ &= \bigl\{\bigl(f_1,\dots,f_n,0,\dots,0;(-1)^{f_1+\cdots+f_n}\bigr) \mid f_1 \geq \cdots \geq f_n \geq 0\bigr\}, \\ \Sigma^2 &= \text{parameters in } \Sigma^1 \text{ with the `even' condition} \\ &= \bigl\{\bigl(f_1,\dots,f_n,0,\dots,0;(-1)^{f_1+\cdots+f_n}\bigr) \in \Sigma^1 \mid f_i - f_j \in 2\boldsymbol{Z}\ (1 \leq i < j \leq n)\bigr\} \end{aligned}$$

under our assumption $d > 2n$. We also have $\Psi = \Psi_1$ and $\Psi_2 = \Psi_0$.

8.3. Grassmann manifolds.

We consider the oriented Grassmann manifold $\mathscr{G}^\circ_{d,n}$ consisting of n-dimensional oriented subspaces in the d-dimensional fixed real vector space. We denote by $L^2(\mathscr{G}_{d,n})$ the space of square integrable functions on the oriented Grassmannian manifold $\mathscr{G}^\circ_{d,n}$. The orthogonal group $O(d)$ acts on $L^2(\mathscr{G}^\circ_{d,n})$ by the right regular representation. Let $L^2(\mathscr{G}^\circ_{d,n})_{O(d)}$ be the set of $O(d)$-finite vectors in $L^2(\mathscr{G}^\circ_{d,n})$. Every element in $L^2(\mathscr{G}^\circ_{d,n})_{O(d)}$ is a real analytic function on $\mathscr{G}^\circ_{d,n}$, and $L^2(\mathscr{G}^\circ_{d,n})_{O(d)}$ is a dense subspace of $L^2(\mathscr{G}^\circ_{d,n})$. The representation of $O(d)$ on $L^2(\mathscr{G}^\circ_{d,n})$ (resp. $L^2(\mathscr{G}^\circ_{d,n})_{O(d)}$) is decomposed into a Hilbert direct sum (resp. an algebraic direct sum) of irreducible representations of $O(d)$ with multiplicity-free, and the set of the irreducible representations of $O(d)$ arising there is Σ^2. See, e.g., in page 546 of [**8**].

We identify $M_{n,d}$ with the set of n vectors in $\boldsymbol{R}^d$, where $\boldsymbol{R}^d$ is considered to be the set of row vectors. We denote by $M'_{n,d}$ the open dense subset of $M_{n,d}$ consisting of n linearly independent vectors in $\boldsymbol{R}^d$, and by $M''_{n,d}$ the compact subset of $M'_{n,d}$ consisting of n orthonormal vectors in $\boldsymbol{R}^d$. The natural inclusion $M''_{n,d} \subset M'_{n,d} \subset M_{n,d}$ is compatible with the natural action of $O(d)$ from the right. The group $GL(n)$ acts on $M'_{n,d}$ from the left, and the subgroup $O(n)$ acts on the subset $M''_{n,d}$. The action of $O(d)$ is transitive on $M''_{n,d}$ so that $M''_{n,d} \cong O(d-n)\backslash O(d)$. Using these actions, we have

$$\begin{aligned}\mathscr{G}^\circ_{d,n} &\cong SO(n)\backslash M''_{n,d} \\ &\cong (SO(n)\times O(d-n))\backslash O(d) \\ &\cong (SO(n)\times SO(d-n))\backslash SO(d) \\ &\cong GL(n)_+\backslash M'_{n,d} \qquad (22)\end{aligned}$$

as $O(d)$-homogeneous manifold. Here $GL(n)_+ = GL(n,\boldsymbol{R})_+ := \{g \in GL(n) \mid \det(g) > 0\}$ is the identity component of $GL(n)$.

8.4. Relation between pluriharmonic polynomials and Grassmann manifolds.

The restriction of a polynomial on $M_{n,d}$ to $M''_{n,d}$ induces the map from $\mathscr{H}_{n,d}$ to the space of functions on $M''_{n,d}$. Since this map is $O(n)\times O(d)$-equivariant, $\mathscr{H}_{n,d}^{SO(n)}$ is mapped to the $SO(n)$-invariant functions on $M''_{n,d}$. By the isomorphism (22), we obtain an $O(d)$-equivariant map

$$\mathscr{H}_{n,d}^{SO(n)} \to L^2(\mathscr{G}^\circ_{d,n})_{O(d)}.$$

Since both sides have the same irreducible decomposition as $O(d)$-modules, we conclude that this is an isomorphism.

8.5. Zonal spherical functions on Grassmann manifolds.

First we recall the zonal spherical functions on $\mathscr{G}^\circ_{d,n}$. For each irreducible subrepresentation V of $O(d)$ on $L^2(\mathscr{G}^\circ_{d,n})$, we have the unique function $f(g)$ of $g \in O(d)$ up to a constant multiple which is bi-$(SO(n)\times O(d-n))$-invariant. This is also considered to be a function $f(x)$ in V of $x \in \mathscr{G}_{d,n}$ such that $f(xh) = f(x)$ (for all $h \in SO(n)\times O(d-n)$). This function is usually called the zonal spherical function.

Now we explain the standard idea of doubling the variables. Let us consider the diagonal action of $O(d)$ on the product $\mathscr{G}^\circ_{d,n}\times\mathscr{G}^\circ_{d,n}$ by $(x,y)\mapsto(xh,yh)$ for $h \in O(d)$. A natural isomorphism

$$\begin{aligned}&(\mathscr{G}_{d,n}\times\mathscr{G}_{d,n})/O(d) \\ &\quad\cong (((SO(n)\times O(d-n))\backslash O(d))\times((SO(n)\times O(d-n))\backslash O(d)))/O(d) \\ &\quad\cong (SO(n)\times O(d-n))\backslash O(d)/(SO(n)\times O(d-n)) \\ &\quad\cong \mathscr{G}^\circ_{d,n}/(SO(n)\times O(d-n))\end{aligned}$$

induces the isomorphism $L^2(\mathscr{G}^\circ_{d,n}\times\mathscr{G}^\circ_{d,n})^{O(d)} \cong L^2(\mathscr{G}^\circ_{d,n})^{SO(n)\times O(d-n)}$. In this manner, a zonal spherical function is considered to be a function in $L^2(\mathscr{G}^\circ_{d,n} \times \mathscr{G}^\circ_{d,n})^{O(d)}$. If we take an orthonormal basis $\{f_i \mid i = 1, \ldots, \dim V\}$ of an irreducible subrepresentation $(\lambda, V) \in \Sigma^2$ of $O(d)$ in $L^2(\mathscr{G}^\circ_{d,n})$, then $f(x, y) = \sum_{i=1}^{\dim V} f_i(x) f_i(y)$ is the zonal spherical function under this identification.

Now we explain the relation between the polynomial $P^*(X, Y)$ in Section 2 and the zonal spherical function $f(x, y) = \sum_{i=1}^{\dim V} f_i(x) f_i(y)$. Take a lift $P_i^* \in \mathscr{H}^{SO(n)}_{n,d}$ of f_i under the identification $\mathscr{H}^{SO(n)}_{n,d} \cong L^2(\mathscr{G}^\circ_{d,n})$, explained in 8.4. We consider $P^*(X, Y) := \sum_{i=1}^{\dim V} P_i^*(X) P_i^*(Y)$. Then $P^*(X, Y)$ satisfies the following three conditions:

(i)$'$ The action of $GL(n) \times GL(n)$ on the linear span of $P^*(aX, bY)$ $(a, b \in GL(n))$ is $\rho \otimes \rho$, where $\rho = \tau(\lambda)$ is the irreducible representation of $GL(n)$.
(ii) $P^*(Xh, Yh) = P^*(X, Y)$ $(h \in O(d))$.
(iii) $P^*(X, Y)$ is pluriharmonic with respect to each X or Y.

Conversely, the restriction of P^* with these properties (i)$'$ (ii) (iii) to $\mathscr{G}_{d,n} \times \mathscr{G}_{d,n}$ gives a zonal spherical function associated with an irreducible representation $(\lambda, V) \in \Sigma^2$. Such a polynomial seems to be essentially a generalized Jacobi polynomial defined in [**17**].

We now consider the special case that the representation ρ of $GL(n)$ is one-dimensional; $\rho(A) = (\det A)^\nu$ for some non-negative integer ν. In this case the condition (i)' is rephrased as

(i) $P^*(AX, BY) = (\det AB)^\nu P^*(X, Y)$ for all $A, B \in GL(n)$,

which is the same as (i) in Section 2. The corresponding parameter of λ such that $\rho = \tau(\lambda)$ is given by $\lambda = (\underbrace{\nu, \cdots, \nu}_{n}, \underbrace{0, \ldots, 0}_{[d/2]-n}; (-1)^{n\nu})$.

9. Appendix B: Holonomic D-modules.

The purpose of this section is to give a proof of the following theorem:

THEOREM 9.1. *Let $\boldsymbol{D}_k$ be the operators given in Theorem* 5.3. *For each complex parameters d and ν, the system*

$$\boldsymbol{D}_k Q = 0 \qquad (1 \leq k \leq n)$$

is holonomic of rank 2^n.

We summarize the general terminology and the fact in D-modules. These are given in the standard textbook, e.g., [**12**], [**18**].

Let X be an n-dimensional complex manifold. In this paper, we may assume that X is an open subset of $\boldsymbol{C}^n$. We denote by T^*X the cotangent bundle of X, and by $(z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_n)$ the coordinates on T^*X.

Let $\mathscr{O}_X$ be the sheaf of the ring of holomorphic functions on X, $\mathscr{D} = \mathscr{D}_X$ the sheaf of the ring of (linear) differential operators with holomorphic coefficients on X, $\mathscr{O}_{T^*X}$ the sheaf of the ring of holomorphic functions on T^*X. For a differential operator $\boldsymbol{D} \in \mathscr{D}$, we denote by $\sigma(\boldsymbol{D}) \in \mathscr{O}_{T^*X}$ the principal symbol of $\boldsymbol{D}$.

Example 9.2 ([**12**, Example 2.2.6]). Let $\mathscr{I}$ be a left ideal of $\mathscr{D}$. We denote by $\sigma(\mathscr{I})$ the ideal of $\mathscr{O}_{T^*X}$ generated by $\{\sigma(\boldsymbol{D}) \mid \boldsymbol{D} \in \mathscr{I}\}$. The characteristic variety of the left D-module $\mathscr{D}/\mathscr{I}$ is equal to the common zeros of the ideal $\sigma(\mathscr{I})$;

$$\mathrm{Ch}(\mathscr{D}/\mathscr{I}) = \{(z, \zeta) \in T^*X \mid f(z, \zeta) = 0, \text{ for all } f \in \sigma(\mathscr{I})\}.$$

It is known that the dimension of a non-empty characteristic variety is at least $n = \dim X$. A left D-module $\mathscr{D}/\mathscr{I}$ is called *holonomic* if the dimension of the characteristic variety $\mathrm{Ch}(\mathscr{D}/\mathscr{I})$ is at most $n = \dim X$. For an irreducible component V of the characteristic variety $\mathrm{Ch}(\mathscr{D}/\mathscr{I})$, the multiplicity of $\mathscr{D}/\mathscr{I}$ along V is defined to be the multiplicity of $\mathscr{O}_{T^*X}/\sigma(\mathscr{I})$ along V; $\mathrm{mult}_V(\mathscr{D}/\mathscr{I}) := \mathrm{mult}_V(\mathscr{O}_{T^*X}/\sigma(\mathscr{I}))$.

The zero section of the tangent bundle T^*X is denoted by $T^*_X X$; $T^*_X X = \{(z, \zeta) \mid \zeta = 0\}$.

Lemma 9.3 ([**12**, Example 2.2.4, Proposition 2.2.5]). *The following conditions on $\mathscr{I}$ are equivalent.*

- (i) *The characteristic variety* $\mathrm{Ch}(\mathscr{D}/\mathscr{I}) = T^*_X X$*, and the multiplicity* $r = \mathrm{mult}_{T^*_X X}(\mathscr{D}/\mathscr{I})$.
- (ii) *The $\mathscr{O}_X$-module $\mathscr{O}_{T^*X}/\sigma(\mathscr{I})$ is locally free of rank r.*
- (iii) *The left D-module $\mathscr{D}/\mathscr{I}$ is an integrable connection of rank r.*
- (iv) *The space* $\mathrm{Hom}_{\mathscr{D}}(\mathscr{D}/\mathscr{I}, \mathscr{O}_X)$ *of solutions forms a vector bundle of rank r over X.*

Moreover, such a D-module $\mathscr{D}/\mathscr{I}$ is holonomic on X.

Note that as for the condition (iv), the sheaf of holomorphic solutions is given by

$$\begin{aligned}\mathrm{Hom}_{\mathscr{D}}\big(\mathscr{D}/\mathscr{I}, \mathscr{O}_X\big) &\cong \big\{f \in \mathscr{O}_X \mid \boldsymbol{D}f = 0 \text{ for all } \boldsymbol{D} \in \mathscr{I}\big\} \\ &= \big\{f \in \mathscr{O}_X \mid \boldsymbol{D}_1 f = \cdots = \boldsymbol{D}_N f = 0\big\}\end{aligned}$$

if $\mathscr{I}$ is generated by $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_N$.

The following fact is a direct consequence from the definition.

LEMMA 9.4. *Let $\mathscr{I}$ be a left ideal of $\mathscr{D}$ generated by $\boldsymbol{D}_i$ with $i = 1, 2, \ldots, N$.*

(1) *The ideal generated by $\sigma(\boldsymbol{D}_i)$ with $i = 1, 2, \ldots, N$ is contained in $\sigma(\mathscr{I})$.*
(2) *The characteristic variety $\mathrm{Ch}(\mathscr{D}/\mathscr{I})$ is contained in the common zeros of $\sigma(\boldsymbol{D}_1), \ldots, \sigma(\boldsymbol{D}_N)$.*
(3) *If the dimension of such common zeros is at most $n(= \dim X)$, then the D-module $\mathscr{D}/\mathscr{I}$ is holonomic on X.*

We give an example of Lemma 9.4(3).

EXAMPLE 9.5. Let $\boldsymbol{D}_i \in \mathscr{D}$ $(1 \leq i \leq n)$ be the differential operators with holomorphic coefficients on X such that $\sigma(\boldsymbol{D}_i) = \zeta_i^2$ $(1 \leq i \leq n)$. Let $\mathscr{I}$ be the ideal of $\mathscr{D}$ generated by $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_n$. Then the left D-module $\mathscr{D}/\mathscr{I}$ is holonomic.

In general, the inclusion (1) in Lemma 9.4 could be strict. The set $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_N \in \mathscr{I}$ is called an *involutive system of generators* if the symbols $\sigma(\boldsymbol{D}_1), \ldots, \sigma(\boldsymbol{D}_N)$ generate $\sigma(\mathscr{I})$ over $\mathscr{O}_{T^*X}$. We give a sufficient condition to be an involutive system.

PROPOSITION 9.6 ([**18**, Proposition 2.12]). *Let $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_N \in \mathscr{D}$ be differential operators of order $m_1, \ldots, m_N$, respectively. Let $\mathscr{I} = \mathscr{D}\boldsymbol{D}_1 + \cdots + \mathscr{D}\boldsymbol{D}_N$ be the left ideal of $\mathscr{D}$ generated by $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_N$. Let Y be the common zeros of the symbols $\sigma(\boldsymbol{D}_1), \ldots, \sigma(\boldsymbol{D}_N)$. Assume the following* (a) *and* (b):

(a) *The codimension of Y in T^*X is N.*
(b) *There exist differential operators $G_{ijk} \in \mathscr{D}$ of order $\leq m_i + m_j - m_k - 1$ such that $[\boldsymbol{D}_i, \boldsymbol{D}_j] = \sum_{k=1}^N G_{ijk}\boldsymbol{D}_k$ for all $i, j = 1, \ldots, N$.*

Then $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_N$ is an involutive system of generators and $\mathrm{Ch}(\mathscr{D}/\mathscr{I}) = Y$.

Now we consider the case when the number N of generators is equal to the dimension n of the manifold X.

PROPOSITION 9.7. *Suppose $\boldsymbol{D}_i \in \mathscr{D}$ $(1 \leq i \leq n)$ be the differential operators with holomorphic coefficients on X which satisfy the condition* (b) *in Proposition* 9.6 *and the following condition*:

(a') *The common zeros of the symbol $\sigma(\boldsymbol{D}_i)$ $(1 \leq i \leq n)$ is the zero section $\{(z, \zeta) \in T^*X \mid \zeta = 0\}$.*

Then the space of solutions of the system of differential equations

$$\boldsymbol{D}_1 f = \cdots = \boldsymbol{D}_n f = 0$$

forms a vector bundle over X of rank r, where r is given by the multiplicity:

$r = \mathrm{mult}_{T^*_X X}(\mathscr{O}_{T^*X}/(\sigma(\boldsymbol{D}_1), \ldots, \sigma(\boldsymbol{D}_n)))$.

Proof. We apply Proposition 9.6 for $N = n$. The condition (a') implies the condition (a). Then $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_n$ is an involutive system of generators. Let $\mathscr{I}$ be a left ideal of $\mathscr{D}$ generated by $\boldsymbol{D}_i$ with $i = 1, 2, \ldots, n$. Then $\sigma(\mathscr{I}) = (\sigma(\boldsymbol{D}_1), \ldots, \sigma(\boldsymbol{D}_n))$ and $\mathrm{Ch}(\mathscr{D}/\mathscr{I}) = T^*_X X$ by the condition (a). Finally, $r = \mathrm{mult}_{T^*_X X}(\mathscr{D}/\mathscr{I}) = \mathrm{mult}_{T^*_X X}(\mathscr{O}_{T^*X}/\sigma(\mathscr{I})) = \mathrm{mult}_{T^*_X X}(\mathscr{O}_{T^*X}/(\sigma(\boldsymbol{D}_1), \ldots, \sigma(\boldsymbol{D}_n)))$. Hence we see that the condition (i) in Lemma 9.3 is verified, and the conclusion of this Proposition is the condition (iv) in Lemma 9.3. □

Note that only the condition (a') is sufficient for the D-module $\mathscr{D}/\mathscr{I}$ to be a vector bundle because of Lemma 9.4(2). In order to obtain an exact formula of its rank r, we need an extra condition such as the condition (b).

Remark 9.8. The multiplicity r given in Proposition 9.7 seems to be equal to the product of the orders of $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_n$, that is, the product of the homogeneous degrees in ζ of $\sigma(\boldsymbol{D}_1), \ldots, \sigma(\boldsymbol{D}_n)$.

We show the following formula for the commutators.

Lemma 9.9. *Let $\boldsymbol{D}_k$ be the operators given in Theorem* 5.3. *Then we have*

$$[\boldsymbol{D}_k, \boldsymbol{D}_l] = \frac{2\lambda_k^2\lambda_l^2 - \lambda_k^2 - \lambda_l^2}{\left(\lambda_k^2 - \lambda_l^2\right)^2}(\boldsymbol{D}_k - \boldsymbol{D}_l).$$

Proof. Since the proof is obtained by a straight forward calculation, we omit it here. □

Proof of Theorem 9.1. We will apply Proposition 9.7. Let X be the set $\{(\lambda_1, \ldots, \lambda_n) \in \boldsymbol{C}^n \mid \lambda_i \neq \lambda_j (1 \leq i < j \leq n)\}$ and $m_1 = \cdots = m_n = 2$. Then Lemma 9.9 shows that $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_n$ satisfies the condition (b) in Proposition 9.7. Since the symbol $\sigma(\boldsymbol{D}_k) = \zeta_k^2$, then we see that the condition (a') in Proposition 9.7 is also satisfied. We compute the multiplicity as

$$\begin{aligned} r &= \mathrm{mult}_X \left(\mathscr{O}_X \otimes (\boldsymbol{C}[\zeta_1, \ldots, \zeta_n]/(\zeta_1^2, \ldots, \zeta_n^2))\right) \\ &= \dim_{\boldsymbol{C}} \left(\boldsymbol{C}[\zeta_1, \ldots, \zeta_n]/(\zeta_1^2, \ldots, \zeta_n^2)\right) = 2^n. \end{aligned}$$

Then the system is holonomic on X of rank 2^n. □

References

[1] C. Bachoc, R. Coulangeon and G. Nebe, Designs in Grassmannian spaces and lattices, J. Algebraic Combin., **16** (2002), 5–19.

[2] R. J. Beerends and E. M. Opdam, Certain hypergeometric series related to the root system *BC*, Trans. Amer. Math. Soc., **339** (1993), 581–609.

[3] S. Böcherer, Über die Fourier Jacobi-Entwicklung Siegelscher Eisensteinreihen, II (German), Math. Z., **189** (1985), 81–110.

[4] S. Böcherer, T. Satoh and T. Yamazaki, On the pullback of a differential operator and its application to vector valued Eisenstein series, Comment. Math. Univ. St. Paul., **41** (1992), 1–22.

[5] A. G. Constantine, Some non-central distribution problems in multivariate analysis, Ann. Math. Statist., **34** (1963), 1270–1285.

[6] W. Eholzer and T. Ibukiyama, Rankin-Cohen type differential operators for Siegel modular forms, Internat. J. Math., **9** (1998), 443–463.

[7] M. Eichler and D. Zagier, The theory of Jacobi forms, Progr. Math., **55**, Birkhäuser, Boston, Inc., Boston MA, 1985, v+148 pp.

[8] R. Goodman and N. R. Wallach, Representations and invariants of the classical groups, Encyclopedia of Mathematics and its Application, **68**, Cambridge University Press, Cambridge, 1998, xvi+685 pp.

[9] G. J. Heckman, Root systems and hypergeometric functions, II, Compositio Math., **64** (1987), 353–374.

[10] G. J. Heckman and E. M. Opdam, Root systems and hypergeometric functions, I, Compositio Math., **64** (1987), 329–352.

[11] G. Heckman and H. Schlichtkrull, Harmonic Analysis and Special Functions on Symmetric Spaces, (ed. S. Helgason), Perspectives in Mathematics, **16**, Academic Press Inc., San Diego, CA, 1994, xii+225 pp.

[12] R. Hotta, K. Takeuchi and T. Tanisaki, $\mathscr{D}$-Modules, Perverse Sheaves, and Representation Theory, Progr. Math., **236**, Birkhäuser Boston Inc., Boston, MA, 2008, xi+ 407 pp.

[13] T. Ibukiyama, On differential operators on automorphic forms and invariant pluriharmonic polynomials, Comment. Math. Univ. St. Paul., **48** (1999), 103–118.

[14] T. Ibukiyama and D. Zagier, Higher spherical polynomials, in preparation.

[15] T. Ibukiyama and D. Zagier, Higher spherical functions, in preparation.

[16] A. T. James, Zonal polynomials of the real positive definite symmetric matrices, Ann. of Math. (2), **74** (1961), 456–469.

[17] A. T. James and A. G. Constantine, Generalized Jacobi polynomials as spherical functions of the Grassmann manifold, Proc. London Math. Soc. (3), **29** (1974), 174–192.

[18] M. Kashiwara, Algebraic Analysis, Iwanami Shoten, Tokyo, 2000, 276 pp (Japanese).

[19] M. Kashiwara and M. Vergne, On the Segal-Shale-Weil representations and harmonic polynomials, Invent. Math., **44** (1978), 1–47.

[20] H. Katsurada, Exact standard zeta values of Siegel modular forms, Experiment. Math., **19** (2010), 65–77.

[21] R. Muirhead, Systems of partial differential equations for hypergeometric functions of matrix argument, Ann. Math. Statist., **41** (1970), 991–1001.

[22] M. Sato, M. Kashiwara, T. Kimura and T. Oshima, Micro-local analysis of prehomogeneous vector spaces, Invent. Math., **62** (1980), 117–179.

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