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Surfaces carrying sufficiently many Dirichlet finite harmonic functions that are automatically bounded

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Abstract. It is shown that there exists a Riemann surface on which every Dirichlet finite harmonic function is automatically bounded and yet the linear dimension of the linear space of Dirichlet finite harmonic functions on it is infinite.

1. Introduction.

There are two important norms in the study of harmonic functions on open (i.e. noncompact) Riemann surfaces: one is the supremum norm

$$\|u; R\|_\infty := \sup_{z\in R} |u(z)| \tag{1.1}$$

of a harmonic function u on a Riemann surface R which plays a core role in the harmonic version of the normal family argument; the other is the Dirichlet seminorm $\sqrt{D(u;R)}$ given by the Dirichlet integral

$$D(u;R) := \int_R du \wedge *du = \int_R |\nabla u(z)|^2 dxdy \quad (z = x+iy) \tag{1.2}$$

of a harmonic function u taken over a Riemann surface R, which is repeatedly used in connection with the Dirichlet principle. The notation $H(R)$ indicates the linear space of harmonic functions u on a Riemann surface R. Two important main linear subspaces of $H(R)$ are, firstly,

$$HB(R) := \{u \in H(R) : \|u; R\|_\infty < \infty\}, \tag{1.3}$$

which forms a Banach space equipped with the supremum norm $\|\cdot\,; R\|_\infty$, and, secondly,

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$$HD(R) := \{u \in H(R) : D(u; R) < \infty\}, \tag{1.4}$$

which is a semi-Hilbert space equipped with the semi-inner product $D(\cdot\ ,\cdot\ ; R)$ given by the mutual Dirichlet integral

$$D(u, v; R) := \int_R du \wedge *dv = \int_R \nabla u(z) \cdot \nabla v(z) dxdy \tag{1.5}$$

of u and v in $HD(R)$. The symbol B in (1.3) suggests the boundedness and D in (1.4) the Dirichlet finiteness. It is often necessary to consider harmonic functions which are both bounded and Dirichlet finite, and for this reason we use the notation

$$HBD(R) := HB(R) \cap HD(R), \tag{1.6}$$

which also forms a Banach space equipped with the combined norm

$$\|\cdot\ ; R\|_\infty + \sqrt{D(\cdot\ ; R)}.$$

This is traditional notation in the classification theory of Riemann surfaces (cf. e.g. [**1**], [**11**], etc.). Usually the boundedness of a harmonic function u on R does not imply the Dirichlet finiteness of u and vice versa but we know a lot of instances in which the boundedness (Dirichlet finiteness, resp.) of harmonic functions u on a Riemann surface R implies the Dirichlet finiteness (boundedness, resp.) of u on R., i.e. $HB(R) \subset HD(R)$ ($HD(R) \subset HB(R)$, resp.), and $\dim HB(R) < \infty$ ($\dim HD(R) < \infty$, resp.) (cf. [**11**]), where e.g. $\dim HB(R)$ is the linear dimension of the linear space $HB(R)$, which is either a finite number in $\boldsymbol{N}$, the set of positive integers, or infinite ∞.

In the present paper we discuss Riemann surfaces R for which the inclusion relation $HD(R) \subset HB(R)$ is valid. The relation $HD(R) \subset HB(R)$ is equivalent to the relation

$$HD(R) = HBD(R). \tag{1.7}$$

Riemann surfaces R satisfying (1.7) are quite interesting and also important in view of the fact that the relation (1.7) serves to give significant recognition to e.g. the theory of fullsuperharmonic functions and also that of (Dirichlet finite harmonic) Bergmann kernels, which will be discussed elsewhere. We discuss in this paper the *problem* of whether the condition (1.7) always implies $\dim HD(R) < \infty$ or not. If we denote by $\mathscr{D}_D$ the set of $\dim HD(R)$ for R with (1.7), then it is well known as stated above (cf. [**11**]) that $\mathscr{D}_D \supset \boldsymbol{N}$ so that the problem is restated as to clarify whether $\mathscr{D}_D = \boldsymbol{N}$ or $\mathscr{D}_D = \boldsymbol{N} \cup \{\infty\}$.

We state a few background materials concerning (1.7). The Royden harmonic boundary $\delta_{\mathscr{R}} = \delta_{\mathscr{R}} R$ of R is the set of regular points in the sense of potential theory in the Royden boundary $\gamma_{\mathscr{R}} = \gamma_{\mathscr{R}} R = R^*_{\mathscr{R}} \setminus R$ of R with $R^*_{\mathscr{R}}$ the Royden compactification of R (cf. e.g. [**2**], [**11**], [**4**]). The Royden harmonic boundary $\delta_{\mathscr{R}}$ is compact and

$$\dim HD(R) = \#\delta_{\mathscr{R}} \quad \text{(the number of points in } \delta_{\mathscr{R}}\text{)},$$

which is either finite or infinite. We denote by $\mathrm{cap}(K)$ the capacity, or more precisely, the variational 2-capacity (cf. e.g. [**3**]), of a compact subset K of $\delta_{\mathscr{R}}$ (see [**7**]). Then we have the following characterization of (1.7) (see [**6**]).

THEOREM A. *A Riemann surface R satisfies* (1.7) *if and only if*

$$\inf_{\zeta \in \delta_{\mathscr{R}}} \mathrm{cap}(\{\zeta\}) > 0. \tag{1.8}$$

Looking at this result one might feel that each point in $\delta_{\mathscr{R}}$ is distributed quite sporadically and yet $\delta_{\mathscr{R}}$ is compact and thus $\#\delta_{\mathscr{R}} < \infty$ might be the case so that $\dim HD(R) = \#\delta_{\mathscr{R}} < \infty$. This is a motivation for that we are tempted to maintain that $\dim HD(R) < \infty$. On the other hand, however, we have the following result originally due to Virtanen and then Royden (cf. e.g. [**11**]).

THEOREM B. *For any Riemann surface R*

$$HD(R) = \overline{HBD(R)} \tag{1.9}$$

in the sense that for any $u \in HD(R)$ and for any positive number $\varepsilon > 0$ there is a $u_\varepsilon \in HBD(R)$ such that $D(u - u_\varepsilon; R) < \varepsilon$.

The relation (1.9) says that the subspace $HBD(R)$ may not be identical with $HD(R)$ but very close to this situation in the sense that $HBD(R)$ almost exhausts $HD(R)$. Therefore the state (1.7) is certainly a pathological phenomenon but might not be so virulent as to destroy the normal situation $\infty \in \mathscr{D}_D$. This is thus a reverse motivation for that we suspect the existence of R with (1.7) and yet $\dim HD(R) = \infty$. Our *problem*, whether a Riemann surface R satisfying (1.7) always has a finite dimensional linear space $HD(R)$ or not, is thus not only quite challenging one but also very far from being trivial in view of the above two seemingly conflicting observations. Nevertheless, although we needed some period of struggles, we come to the following conclusion, to give a proof to which is the central *purpose* of this paper.

The main theorem. *There exists a Riemann surface R such that the Dirichlet finiteness always guarantees the boundedness for harmonic functions on R, i.e. $HD(R) \subset HB(R)$ or equivalently* (1.7), *and yet there exist sufficiently many Dirichlet finite harmonic functions on R in the sense that* $\dim HD(R) = \infty$. *In short,* $\mathscr{D}_D = \boldsymbol{N} \cup \{\infty\}$.

We have already studied Riemann surfaces R on which the boundedness always implies the Dirichlet finiteness for harmonic functions on R, i.e. $HB(R) \subset HD(R)$ which is equivalent to

$$HB(R) = HBD(R). \tag{1.10}$$

A characterization of (1.10) in terms of the class of harmonic measures on R was given in [**8**] and as a counterpart to the present main theorem we have also shown the following result (see [**8**]).

Theorem C. *There exists a Riemann surface R such that* (1.10) *is satisfied and yet* $\dim HB(R) = \infty$.

If we denote by $\mathscr{D}_B$ the set of $\dim HB(R)$ for R with (1.10), then it is also well known (cf. e.g. [**11**]) that $\mathscr{D}_B \supset \boldsymbol{N}$ and thus the above theorem assures that $\mathscr{D}_B = \boldsymbol{N} \cup \{\infty\}$. Superficially Theorem C entirely resembles to the main theorem above but to derive it is in reality an easy and simple task compared with the case of the main theorem above. Actually (1.7) is quasiconformally invariant while (1.10) is not and a fortiori (1.7) and (1.10) should be understood to be essentially different in nature. Anyhow, observe that the simultaneous validity of both of (1.7) and (1.10) is equivalent to $HB(R) = HD(R)$. Related to this it is interesting to compare the present main theorem and Theorem C with the following beautiful result due to Masaoka [**5**] (for a simple elementary proof of it, see also [**6**]).

Theorem D. *The relation $HB(R) = HD(R)$ (i.e. the synchronous validity of both of* (1.7) *and* (1.10)) *is equivalent to the relation* $\dim HB(R) = \dim HD(R) < \infty$.

2. Fundamental surfaces.

Our purpose is to exhibit a Riemann surface W satisfying the following two properties: firstly, $HD(W) = HBD(W)$; secondly, $\dim HD(W) = \infty$. Recall that a harmonic function u on a Riemann surface R is essentially positive if $|u|$

admits a harmonic majorant on R and we denote by $HP(R)$ the linear space of essentially positive harmonic functions on R. We denote by $u \vee v$ ($u \wedge v$, resp.) the least harmonic majorant (the greatest harmonic minorant, resp.) of u and v in $HP(R)$. Using $u \vee v$ ($u \wedge v$, resp.) as join (meet, resp.) of u and v in $HP(R)$, the class $HP(R)$ forms a Riesz space (or a vector lattice in the older terms). Setting $u^+ =: u \vee 0$ and $u^- := -(u \wedge 0)$ for $u \in HP(R)$ we have the Jordan decomposition of $u \in HP(R)$: $u = u^+ - u^-$ with $u^\pm \in H(R)^+ := \{u \in H(R) : u \geqq 0 \text{ on } R\}$. The symbol P in $HP(R)$ thus suggests the term positive. Then $HB(R)$, $HD(R)$, and $HBD(R)$ are Riesz subspaces of the Riesz space $HP(R)$. We denote by

$$\mathscr{O}_{HX} := \{R : HX(R) = \boldsymbol{R}\}$$

the class of Riemann surfaces R such that the space $HX(R)$ reduces to the class $\boldsymbol{R}$ of constant functions for $X = P, B, D$, and BD. We say that R is *hyperbolic* (*parabolic*, resp.) if R carries (does not carry, resp.) the Green function on R and we use also the traditional notation

$$\mathscr{O}_G := \{R : R \text{ is parabolic}\}$$

for the class of parabolic (i.e. nonhyperbolic) Riemann surfaces R. Then we have the following table of inclusion relations:

$$\mathscr{O}_G < \mathscr{O}_{HP} < \mathscr{O}_{HB} < \mathscr{O}_{HD} = \mathscr{O}_{HBD}$$

(cf. e.g. [**11**]), where $A < B$ for two sets A and B indicates the strict inclusion relation among A and B so that $A \subset B$ and $A \neq B$.

To construct the above W we will make an essential use of the so called Sario-Tôki disc $\hat{\boldsymbol{D}}$, which was presented independently by Sario and Tôki (cf. e.g. [**1**], [**11**], [**12**]) for the purpose of showing the strict inclusion $\mathscr{O}_G < \mathscr{O}_{HP}$:

$$\hat{\boldsymbol{D}} \in \mathscr{O}_{HP} \setminus \mathscr{O}_G. \tag{2.1}$$

Of course the above (2.1) is the most important property of $\hat{\boldsymbol{D}}$ but we still need to know the structure of $\hat{\boldsymbol{D}}$ to a certain extent. First of all $\hat{\boldsymbol{D}}$ is a quotient space of the unit disc $\boldsymbol{D}$ in the complex plane $\boldsymbol{C}$ by a certain equivalence relation Q, which we will not specify here except for a few point we really need to know: $\hat{\boldsymbol{D}} = \boldsymbol{D}/Q$. For each $\hat{z} \in \hat{\boldsymbol{D}}$ we have $|z_1| = |z_2|$ for any two z_1 and z_2 in $\hat{z}$ and thus we can define

$$|\hat{z}| := |z| \quad (z \in \hat{z}), \tag{2.2}$$

which is called the absolute value of $\hat{z}$. We often identify the one point set $\{z\}$ consisting of a single point z with the point z itself. The set

$$D := \{\hat{z} \in \hat{\boldsymbol{D}} : \hat{z} = \{z\} = z\} \tag{2.3}$$

is an open subset of $\hat{\boldsymbol{D}}$ and $\hat{z} \mapsto z$ is an immersion of D into $\boldsymbol{D}$ which is a conformal mapping and therefore we can view $D \subset \boldsymbol{D} \cap \hat{\boldsymbol{D}}$. We can choose a strictly increasing sequence $(t_n)_{n\in\boldsymbol{N}\cup\{0\}} \subset (0,1)$ converging to 1 and

$$D_0 := \{|z| \leqq t_0\} \cup \left(\bigcup_{n\in\boldsymbol{N}} \{t_{2n-1} < |z| < t_{2n}\} \right) \subset D, \tag{2.4}$$

where, as before $\boldsymbol{N}$ is the set of positive integers $n = 1, 2, \ldots$. In particular, the origin $0 \in D \subset \hat{\boldsymbol{D}}$ will be referred to as the origin of $\hat{\boldsymbol{D}}$.

The mapping $\hat{z} \mapsto \log|\hat{z}|$ defines a negative harmonic function on $\hat{\boldsymbol{D}}\setminus\{0\}$ with the negative pole at $\hat{z} = 0$ and with the ideal boundary values $\lim_{|\hat{z}|\uparrow 1} \log|\hat{z}| = 0$. In view of this we see that $\hat{\boldsymbol{D}}$ carries the Green function (kernel) $g(\cdot\ ,\cdot\ ; \hat{\boldsymbol{D}})$ and in particular

$$g(\hat{z}) := g(\hat{z}, 0; \hat{\boldsymbol{D}}) = \log\left(\frac{1}{|\hat{z}|}\right) \tag{2.5}$$

on $\hat{\boldsymbol{D}} \setminus \{0\}$. Especially,

$$g(z) = \log\left(\frac{1}{|z|}\right) = \log\left(\frac{1}{r}\right) \quad (z = re^{i\theta})$$

on D will be repeatedly used. Thus, $\hat{\boldsymbol{D}}$ is hyperbolic, i.e. $\hat{\boldsymbol{D}} \notin \mathscr{O}_G$ which proves a trivial part of (2.1). The really important part of (2.1) is the relation $\hat{\boldsymbol{D}} \in \mathscr{O}_{HP}$ or $HP(\hat{\boldsymbol{D}}) = \boldsymbol{R}$ and for this part we refer the reader to an excellent explanation in [**1**], among others cited above. We denote by $\hat{\boldsymbol{D}}^* = (\hat{\boldsymbol{D}})^*$ the Wiener compactification of $\hat{\boldsymbol{D}}$, $\gamma\hat{\boldsymbol{D}} := \hat{\boldsymbol{D}}^* \setminus \hat{\boldsymbol{D}}$ the Wiener boundary of $\hat{\boldsymbol{D}}$, and $\delta\hat{\boldsymbol{D}}$ the Wiener harmonic boundary of $\hat{\boldsymbol{D}}$, which is the set of regular points in $\gamma\hat{\boldsymbol{D}}$ in the sense of potential theory and is a compact subset of $\gamma\hat{\boldsymbol{D}}$. By virtue of the relation $\hat{\boldsymbol{D}} \in \mathscr{O}_{HP}$ or $HP(\hat{\boldsymbol{D}}) = \boldsymbol{R}$, we see that $\delta\hat{\boldsymbol{D}}$ consists of a single point d, say:

$$\delta\hat{\boldsymbol{D}} = \{d\}. \tag{2.6}$$

Related to the set D_0 in (2.4) we consider the following disjoint union $T \subset \boldsymbol{R}$

of open intervals in the open interval $0 < t_1 < t < 1$ given by

$$T := \bigcup_{n \in \boldsymbol{N}} (t_{2n-1}, t_{2n}). \tag{2.7}$$

For each $r \in T$ and $\theta \in (0, \pi)$ the concentric circular arc

$$\Gamma(r, \theta) := \{re^{is} : -\theta < s < \theta\}$$

in $D_0 \subset D \cap \hat{\boldsymbol{D}}$ will be referred to as an *admissible arc* in $\hat{\boldsymbol{D}}$ with radius $r \in T$ and opening angle $2\theta \in (0, 2\pi)$. We will construct a sequence $(\gamma_n)_{n \in \boldsymbol{N}}$ of mutually disjoint admissible arcs $\gamma_n := \Gamma(\rho_n, \theta_n)$ not accumulating in $\hat{\boldsymbol{D}}$ with a specified condition, and then we paste $\hat{\boldsymbol{D}} \setminus \gamma_1$ to $\hat{\boldsymbol{D}} \setminus \gamma_1 \cup \gamma_2$ anticonformally along γ_1, the resulting surface to $\hat{\boldsymbol{D}} \setminus \gamma_2 \cup \gamma_3$ anticonformally along γ_2, the resulting surface to $\hat{\boldsymbol{D}} \setminus \gamma_3 \cup \gamma_4$ along $\gamma_3, \ldots$. Repeating this process until all $n \in \boldsymbol{N}$ are exhausted we can complete the construction of Riemann surface W. We then show $HD(W) = HBD(W)$ and $\dim HD(W) = \infty$. This is our program in the sequel to complete the proof of our main theorem.

3. The first step to the construction.

Before constructing a specified sequence of admissible arcs $\gamma_n := \Gamma(\rho_n, \theta_n)$ $(n \in \boldsymbol{N})$ in $D_0 \subset D \cup \hat{\boldsymbol{D}}$, we choose and then fix three kinds of sequences in $\boldsymbol{R}$. Firstly, we take a divergent sequence $(K_n)_{n \in \boldsymbol{N}} \subset \boldsymbol{R}^+ := \{\lambda \in \boldsymbol{R} : \lambda > 0\}$ such that

$$\sum_{n \in \boldsymbol{N}} \frac{1}{K_n} < \infty. \tag{3.1}$$

Secondly, we fix an arbitrary zero sequence $(k_n)_{n \in \boldsymbol{N}} \subset (0, 1)$ satisfying

$$0 < k_{n+1} < \frac{k_n}{1000} \quad (n \in \boldsymbol{N}), \tag{3.2}$$

which is entirely independent of $(K_n)_{n \in \boldsymbol{N}}$. Thirdly and finally, we choose a zero sequence $(\varepsilon_n)_{n \in \boldsymbol{N}}$ such that

$$0 < \varepsilon_{n+1} < \frac{\varepsilon_n}{1000} \quad (n \in \boldsymbol{N}) \tag{3.3}$$

and, moreover depending upon $(k_n)_{n \in \boldsymbol{N}}$ this time,

$$0 < \varepsilon_n < \frac{k_n}{1000} \quad (n \in \boldsymbol{N}). \tag{3.4}$$

The sequence of admissible arcs $\gamma_n := \Gamma(\rho_n, \theta_n)$ $(n \in \boldsymbol{N})$ will be determined inductively. The procedure of the induction is to define γ_n when two precedent admissible arcs γ_{n-1} and γ_{n-2} are already given for every $n \in \boldsymbol{N}$. For this reason we arbitrarily choose and then fix two admissible arcs γ_{-1} and γ_0 in advance before defining the required sequence $(\gamma_n)_{n\in\boldsymbol{N}}$. Namely, assume the main terms ρ_{-1}, ρ_0 and the subsidiary terms σ_{-2}, σ_{-1}, σ_0 are arbitrarily given all in T such that

$$0 < \sigma_{-2} < \rho_{-1} < \sigma_{-1} < \rho_0 < \sigma_0 \tag{3.5}$$

and two angles θ_{-1}, θ_0 in $(0, \pi)$ are also arbitrarily given with

$$\theta_0 \leqq \pi k_0 := \frac{\pi k_1}{1000}.$$

Then γ_{-1} and γ_0 are given by

$$\gamma_j := \Gamma(\rho_j, \theta_j) \quad (j = -1, 0).$$

We start from this situation and we will construct $\gamma_1 := \Gamma(\rho_1, \theta_1)$ by choosing $\rho_1 \in (\sigma_0, 1)$ and $\theta_1 \in (0, \pi)$ in T suitably and then choose $\sigma_1 \in (\rho_1, 1) \cap T$. We now state how this task is accomplished as the first step. We take an arbitrary admissible arc $\gamma =: \Gamma(r, \theta)$ $(r \in (\sigma_0, 1) \cap T, \theta \in (0, \pi))$ and consider three kinds of functions u, v, w associated with γ and then determine $\rho_1 \in (\sigma_0, 1) \cap T$ and after ρ_1 is fixed we determine $\sigma_1 \in (\rho, 1) \cap T$ such that

$$\sigma_0 < \rho_1 < \sigma_1$$

all in T. This is hence the first step operation.

4. Fundamental functions in the first step.

We start by considering the solution $u \in H(\hat{\boldsymbol{D}} \setminus \gamma) \cap C(\hat{\boldsymbol{D}}^*)$ of the Dirichlet problem on the region $\hat{\boldsymbol{D}} \setminus \gamma$ with the boundary condition

$$\begin{cases} u \mid \gamma = 1 \\ u \mid \delta\hat{\boldsymbol{D}} = 0, \end{cases} \tag{4.1}$$

where $\gamma = \Gamma(r, \theta)$ $(r \in (\sigma_0, 1) \cap T, \theta \in (0, \pi))$. For the unique solvability of the above problem, see e.g. [**11**]. Observe that, for any $r \in (\sigma_0, 1) \cap T$, there exists a unique $\theta = \theta(r) \in (0, \pi)$ such that the above u satisfies

$$u(0) = k_1. \tag{4.2}$$

In fact, since the above solution u for (4.1) is determined by $\gamma = \Gamma(r, \theta)$, we denote it by $u = u(\cdot\,; r, \theta)$. The function of $\theta \in (0, \pi)$ given by $\theta \mapsto \varphi(\theta) := u(0; r, \theta)$ is strictly increasing continuous function on $(0, \pi)$ in view of the fact that the family $\{u; \gamma\}$ forms a normal family, and $\varphi(\theta) \downarrow 0$ as $\theta \downarrow 0$ and $\varphi(\theta) \uparrow 1$ as $\theta \uparrow \pi$. Hence by the intermediate value theorem for continuous functions we see the unique existence of $\theta(r) \in (0, \pi)$ with $\varphi(\theta(r)) = k_1$. Since $u(0; r, \theta(r)) = \varphi(\theta(r))$, (4.2) is established. Hereafter we denote by $u = u(\cdot\,; r)$ the function determined by (4.1) and (4.2) with $\gamma = \gamma(r) = \Gamma(r, \theta(r))$.

We denote by γ_0^+ and γ_0^- both sides of the cut γ_0 in $\hat{\boldsymbol{D}}$ and by giving suitable orientations to $\gamma_0^{\pm}$ we can view that $\gamma_0^+ + \gamma_0^-$ is an analytic Jordan curve which is the boundary of the partly bordered Riemann surface $\hat{\boldsymbol{D}} \setminus \gamma_0$. We often consider γ_0 for $\hat{\boldsymbol{D}} \setminus \gamma_0$ as an analytic Jordan curve $\gamma_0^+ + \gamma_0^-$ besides understanding γ_0 just the simple cut in $\hat{\boldsymbol{D}}$. We next consider the solution $v \in H(\hat{\boldsymbol{D}} \setminus \gamma_0 \cup \gamma(r)) \cap C(\overline{\hat{\boldsymbol{D}}^* \setminus \gamma_0})$ of the mixed boundary value problem on $\hat{\boldsymbol{D}} \setminus \gamma_0 \cup \gamma(r)$ with the boundary data

$$\begin{cases} v \mid \gamma(r) = 1 \\ *dv \mid \gamma_0 = 0 \\ v \mid \delta\hat{\boldsymbol{D}} = 0, \end{cases} \tag{4.3}$$

where we understand that $\overline{\hat{\boldsymbol{D}}^* \setminus \gamma_0}$ in the Carathéodory compactification of $\hat{\boldsymbol{D}}^* \setminus \gamma_0$ so that $\overline{\hat{\boldsymbol{D}}^* \setminus \gamma_0}$ is obtained by attaching the ideal boundary $\gamma\hat{\boldsymbol{D}}$ and the relative boundary $\gamma_0^+ + \gamma_0^-$ to $\hat{\boldsymbol{D}} \setminus \gamma_0$. For the unique solvability, again see [**11**]. Since the above function v is determined by $r \in (\sigma_0, 1) \cap T$, we denote it by $v = v(\cdot\,; r)$. We need to consider one more function $w = w(\cdot\,; r)$ which is the solution $w \in H(\boldsymbol{D} \setminus \gamma(r) \cup \gamma_0 \cup \gamma_{-1}) \cap C(\overline{\hat{\boldsymbol{D}}^* \setminus \gamma_0 \cup \gamma_{-1}})$ of the mixed boundary value problem on $\hat{\boldsymbol{D}} \setminus \gamma(r) \cup \gamma_0 \cup \gamma_{-1}$ with boundary data

$$\begin{cases} w \mid \gamma(r) = 1 \\ *dw \mid \gamma_{-1} \cup \gamma_0 = 0 \\ w \mid \delta\hat{\boldsymbol{D}} = 0. \end{cases} \tag{4.4}$$

For the unique existence of w like u and v, see [**11**].

Observe that the family $\{u(\cdot\,;r) : r \uparrow 1\}$ forms a normal family on $\hat{\boldsymbol{D}}$. Let f be an arbitrary limit function of directed subnet of the above family. Clearly $0 \leqq f \leqq 1$ on $\hat{\boldsymbol{D}}$ and therefore $f \in HP(\hat{\boldsymbol{D}})^+ = \boldsymbol{R}^+$. In view of (4.2), $f \equiv k_1$ on $\hat{\boldsymbol{D}}$. Thus we have seen that $u(\cdot\,;r) \to k_1$ as $r \uparrow 1$ locally uniformly on $\hat{\boldsymbol{D}}$. In particular, we see that

$$\begin{cases} \lim_{r\uparrow 1}\Big(\sup_{|\hat{z}|\leqq\sigma_0}\big|u(\hat{z};r)-k_1\big|\Big)=0 \\ \lim_{r\uparrow 1}\left(\sup_{t\in T,|te^{i\theta}|\leqq\sigma_0}\left|\frac{*du(te^{i\theta};r)}{d\theta}\right|\right)=0. \end{cases} \tag{4.5}$$

Next we consider the family $\{v(\cdot\,;r) : r \uparrow 1\}$ from the same view point as we took for $\{u(\cdot\,;r) : r \uparrow 1\}$, i.e. we wish to derive the relation for $v(\cdot\,;r)$ which is a counterpart to (4.5). We cannot conclude instantly the existence of the relation corresponding to (4.2), we need to make a detour as shown below. Note that we can understand that $D(u;\hat{\boldsymbol{D}}\setminus\gamma) = D(u;\hat{\boldsymbol{D}})$ for $u = u(\cdot\,;r)$ with $\gamma = \gamma(r)$. Similarly $D(v;\hat{\boldsymbol{D}}\setminus\gamma_0\cup\gamma) = D(v;\hat{\boldsymbol{D}})$ for $v = v(\cdot\,;r)$ with $\gamma = \gamma(r)$. For simplicity, we write $D(\cdot)$ for $D(\cdot\,;\hat{\boldsymbol{D}})$ and similarly $D(\cdot,\cdot)$ for the mutual Dirichlet integral $D(\cdot,\cdot\,;\hat{\boldsymbol{D}})$. Then

$$D(u-v) = D(u) + D(v) - 2D(u,v).$$

By virtue of the Stokes formula

$$\begin{aligned} D(u,v) &= \int_{\delta\hat{\boldsymbol{D}}+\gamma+\gamma_0} u * dv = \int_\gamma *dv \\ &= \int_\gamma v * dv = \int_{\delta\hat{\boldsymbol{D}}+\gamma+\gamma_0} v * dv = D(v). \end{aligned}$$

Here we remark that $\delta\hat{\boldsymbol{D}}$ can be identified with the Royden harmonic boundary $\delta_{\mathscr{R}}\hat{\boldsymbol{D}}$ since both are just singleton. Then $*dv$ can be defined on $\delta\hat{\boldsymbol{D}}$ and the Stokes formula in the above form can be justified (cf. e.g. [**4**], [**7**]). Of course we can easily replace the above argument by the standard exhaustion method but it is only time-consuming. Anyway we see that

$$D(u-v) = D(u) - D(v) \geqq 0.$$

On the other hand, again by the Stokes formula, we see that

$$D(v) = D(u,v) = D(v,u) = \int_{\delta\hat{\boldsymbol{D}}+\gamma+\gamma_0} v * du = \int_\gamma *du + \int_{\gamma_0} v * du$$
$$= \int_{\delta\hat{\boldsymbol{D}}+\gamma} u * du + \int_{\gamma_0} v * du = D(u) + \int_{\gamma_0} v * du$$

and therefore

$$D(u-v) = -\int_{\gamma_0} v * du.$$

Combining the estimate

$$\left|\int_{\gamma_0} v * du\right| \leqq \int_{-\theta_0}^{\theta_0} \left|\frac{*du(r_0 e^{i\theta}; r)}{d\theta}\right| d\theta$$

with (4.5) we can now conclude that

$$\lim_{r\uparrow 1} D(u(\cdot\,; r) - v(\cdot\,; r)) = 0. \tag{4.6}$$

Recall that the Green function $g := G(\cdot\,, 0; \hat{\boldsymbol{D}})$ on $\hat{\boldsymbol{D}}$ with pole 0 is given by $g(\hat{z}) = \log(1/|\hat{z}|)$. Let $c := \{|z| = \varepsilon\}$ be a small circle with $0 < \varepsilon < t_0$. By the Stokes formula we have

$$\int_{\delta\hat{\boldsymbol{D}}+\gamma_0+\gamma+c} (g * d(u-v) - (u-v) * dg) = 0.$$

Since $*dg = (1/\varepsilon)\varepsilon d\theta$ on c, we deduce

$$\int_c (g * d(u-v) - (u-v) * dg) = \mathscr{O}(\varepsilon \log \varepsilon) - 2\pi(u(0) - v(0)),$$

where $\mathscr{O}(\cdot)$ is the Landau O. Again by (4.2): $u(0) = k_1$, on letting $\varepsilon \downarrow 0$ in the above, we see that

$$\int_{\delta\hat{\boldsymbol{D}}+\gamma_0+\gamma} (g * d(u-v) - (u-v) * dg) = 2\pi(k_1 - v(0)),$$

or equivalently

$$(D(v) - D(u)) \log\left(\frac{1}{r}\right) + \int_{\gamma_0} v * dg = 2\pi(k_1 - v(0)).$$

We denote by v^+ and v^- the continuous extension of v to γ_0^+ and γ_0^-. Then

$$\begin{aligned}\left|\int_{\gamma_0} v * dg\right| = \left|\int_{\gamma_0^+ + \gamma_0^-} v * dg\right| = \left|\int_{-\theta_0}^{\theta_0} \big(v^+(\rho_0 e^{i\theta}) - v^-(\rho_0 e^{i\theta})\big) d\theta\right| \\ \leqq 2\theta_0 \sup_{|\theta| \leqq \theta_0} \left|v^+(\rho_0 e^{i\theta}; r) - v^-(\rho_0 e^{i\theta}; r)\right|.\end{aligned}$$

By (4.6) we deduce

$$\limsup_{r \uparrow 1} |v(0; r) - k_1| \leqq \frac{\theta_0}{\pi} \limsup_{r \uparrow 1} \Big(\sup_{|\theta| \leqq \theta_0} \left|v^+(\rho_0 e^{i\theta}; r) - v^-(\rho_0 e^{i\theta}; r)\right|\Big). \qquad (4.7)$$

Since $|v(\cdot\ ; r)| \leqq 1$ on $\hat{\boldsymbol{D}} \setminus \gamma_0 \cup \gamma$, $\{v(\cdot\ ; r) : r \uparrow 1\}$ is a normal family on $\hat{\boldsymbol{D}} \setminus \gamma_0$. Choose any directed subnet $v(\cdot\ ; r_\iota)$ $(r_\iota \uparrow 1)$ converging to a limit f locally uniormly on $\hat{\boldsymbol{D}} \setminus \gamma_0$. By virtue of the condition $*dv(\cdot\ ; r_\iota) \mid \gamma_0 = 0$, we see that $v(\cdot\ ; r_\iota)$ $(r_\iota \uparrow 1)$ is locally uniformly convergent on $(\hat{\boldsymbol{D}} \setminus \gamma_0) \cup (\gamma_0^+ + \gamma_0^-)$ and we can conclude the existence of $*df|\gamma_0$ and in fact $*df \mid \gamma_0 = 0$. The boundedness $|f| \leqq 1$ then determines $f|\delta\hat{\boldsymbol{D}} =: a \in [0, 1]$. Since $*df|\gamma_0 = 0$, we can conclude that $f \equiv a$ on $\hat{\boldsymbol{D}}$ by the maximum principle. This shows that $v(\cdot\ ; r_\iota)$ converges to a uniformly on $|z| \leqq \sigma_0$ and a fortiori $v^\pm(\rho_0 e^{i\theta}; r\iota) \to a$ as $r_\iota \uparrow 1$ uniformly on $|\theta| \leqq \theta_0$. The relation (4.7) now assures that

$$\lim_{r \uparrow 1} |v(0; r) - k_1| = 0$$

so that, first of all, $a = k_1$ and then, finally, the family $\{v(\cdot\ ; r) : r \uparrow 1\}$ converges to k_1 locally uniformly on $(\hat{\boldsymbol{D}} \setminus \gamma_0) \cup (\gamma_0^+ + \gamma_0^-)$. Hence, as the counterpart to (4.5), we deduce

$$\begin{cases} \lim_{r \uparrow 1} \Big(\sup_{|\hat{z}| \leqq \sigma_0} |v(\hat{z}; r) - k_1|\Big) = 0 \\ \lim_{r \uparrow 1} \Big(\sup_{t \in T, |te^{i\theta}| \leqq \sigma_0} \left|\dfrac{*dv(te^{i\theta}; r)}{d\theta}\right|\Big) = 0. \end{cases} \qquad (4.8)$$

Finally we consider the family $\{w(\cdot\ ; r) : r \uparrow 1\}$. As we examined $v(\cdot\ ; r)$ $(r \uparrow 1)$ with the aid of $u(\cdot\ ; r)$ $(r \uparrow 1)$, we can repeat exactly the same procedure to examine

$w(\cdot\,;r)$ $(r \uparrow 1)$ with the aid of $v(\cdot\,;r)$ $(r \uparrow 1)$. Then we come to the conclusion that $\{w(\cdot\,;r) : r \uparrow 1\}$ converges to k_1 locally uniformly on $(\hat{\boldsymbol{D}} \setminus \gamma_0 \cup \gamma_{-1}) \cup (\gamma_0^+ + \gamma_0^- + \gamma_{-1}^+ + \gamma_{-1}^-)$ and obtain the following relation as the counterpart to (4.5) and also (4.8):

$$\begin{cases} \lim_{r\uparrow 1} \Big(\sup_{|\hat{z}| \leqq \sigma_0} |w(\hat{z};r) - k_1| \Big) = 0 \\ \lim_{r\uparrow 1} \left(\sup_{t\in T, |te^{i\theta}| \leqq \sigma_0} \left| \frac{*dw(te^{i\theta};r)}{d\theta} \right| \right) = 0. \end{cases} \tag{4.9}$$

Based upon the three conclusions (4.5), (4.8), and (4.9), we will determine $\rho_1 \in T$ as follows. First ρ_1 is required, at least, to satisfy

$$\max\left(\sigma_0, \exp\left(-\frac{k_1}{4K_1}\right)\right) < \rho_1 < 1. \tag{4.10}$$

Moreover ρ_1 is supposed to satisfy (4.14), (4.15), and (4.16) below as follows. Once ρ_1 is tentatively so determined as to satisfy (4.10), we set

$$\theta_1 := \theta(\rho_1) \in (0, \pi) \tag{4.11}$$

(cf. (4.2)) and then the most decisively

$$\gamma_1 := \Gamma(\rho_1, \theta_1) = \Gamma(\rho_1, \theta(\rho_1)) \tag{4.12}$$

and finally we set

$$u_1 := u(\cdot\,;\rho_1), \quad v_1 := v(\cdot\,;\rho_1), \quad w_1 := w(\cdot\,;\rho_1). \tag{4.13}$$

As a consequence of (4.5), we can moreover choose ρ_1 so close enough to 1 as to yield

$$\begin{cases} u_1(0) = k_1 \\ k_1 - \varepsilon_1 \leqq u_1(\hat{z}) \leqq k_1 + \varepsilon_1 \quad (|\hat{z}| \leqq \sigma_0) \\ \sup_{t\in T, |te^{i\theta}| \leqq \sigma_0} \left| \frac{*du_1(te^{i\theta})}{d\theta} \right| \leqq \varepsilon_1 \end{cases} \tag{4.14}$$

and similarly also by (4.8) we can and may make the relation

$$\begin{cases} k_1 - \varepsilon_1 \leqq v_1(\hat{z}) \leqq k_1 + \varepsilon_1 \quad (|\hat{z}| \leqq \sigma_0) \\ \sup\limits_{t \in T, |te^{i\theta}| \leqq \sigma_0} \left| \dfrac{*dv_1(te^{i\theta})}{d\theta} \right| \leqq \varepsilon_1 \end{cases} \tag{4.15}$$

valid and finally, based upon (4.9), we can assume, by taking ρ_1 enough close to 1, that the following relation holds:

$$\begin{cases} k_1 - \varepsilon_1 \leqq w_1(\hat{z}) \leqq k_1 + \varepsilon_1 \quad (|\hat{z}| \leqq \sigma_0) \\ \sup\limits_{t \in T, |te^{i\theta}| \leqq \sigma_0} \left| \dfrac{*dw_1(te^{i\theta})}{d\theta} \right| \leqq \varepsilon_1. \end{cases} \tag{4.16}$$

We have thus seen that we can find and then fix a ρ_1 satisfying the conditions (4.10), (4.14), (4.15), and (4.16).

In addition to three functions u_1, v_1, and w_1, we also consider one more function $p_1 := u_1/2$, only notationally new but essentially u_1 up to the multiplicative constant, i.e. p_1 is the solution in $HD(\hat{\boldsymbol{D}} \setminus \gamma_1) \cap C(\hat{\boldsymbol{D}}^*)$ of the Dirichlet problem on $\hat{\boldsymbol{D}} \setminus \gamma_1$ with the boundary data

$$\begin{cases} p_1 \mid \gamma_1 = \dfrac{1}{2} = \dfrac{2}{4} \\ p_1 \mid \delta\hat{\boldsymbol{D}} = 0. \end{cases} \tag{4.17}$$

Note that $\hat{\boldsymbol{D}}$ is *hyperbolically regular*, i.e. every level line of the Green function $G(\cdot\ , \zeta; \hat{\boldsymbol{D}})$ of $\hat{\boldsymbol{D}}$ is compact as is easily seen by looking at $G(\hat{z}, 0; \hat{\boldsymbol{D}}) = g(\hat{z}) = \log(1/|\hat{z}|)$. Hence every level line $\{p_1 = a\}$ $(0 < a < 1)$ of p_1 is compact and therefore

$$P_1 := \left\{ \hat{z} \in \hat{\boldsymbol{D}} : p_1(\hat{z}) > \frac{1}{4} \right\} \supset \gamma_1 \tag{4.18}$$

is relatively compact subregion of $\hat{\boldsymbol{D}}$ containing γ_1. We now determine arbitrarily and then fix a $\sigma_1 \in (\rho_1, 1) \cap T$ such that

$$\{|\hat{z}| < \sigma_1\} \supset \overline{P}_1. \tag{4.19}$$

Hence we have established a particular procedure to construct ρ_1 and σ_1 (and four functions u_1, v_1, w_1, p_1 and a region P_1) as follows when $\sigma_{-2} < \rho_{-1} < \sigma_{-1} < \rho_0 < \sigma_0$ (but only dummy ones) are given:

$$0 < \sigma_{-2} < \rho_{-1} < \sigma_{-1} < \rho_0 < \sigma_0 < \rho_1 < \sigma_1 < 1. \tag{4.20}$$

5. Completion of the inductive construction.

In the first step, starting from given admissible arcs γ_{-1} and γ_0, we determined the admissible arc $\gamma_1 := \Gamma(\rho_1, \theta_1 = \theta(\rho_1))$ and the associated fundamental functions u_1, v_1, w_1, p_1 and the region P_1 and the number σ_1. By exactly the same procedure as above, as the second step, starting from the admissible arcs γ_0 and γ_1, we can determine the admissible arc $\gamma_2 := \Gamma(\rho_2, \theta_2 = \theta(\rho_2))$ and the associated fundamental functions u_2, v_2, w_2, p_2 and the region P_2 and the number σ_2. Repeating this process until the construction in the $(n-1)^{th}$ step is over with σ_{n-1} and γ_{n-1} determined, as the n^{th} step construction, we can determine ρ_n as ρ_1 was determined in the case of the first step in Section 4. Namely, we can so choose $\rho_n \in T$ as to satisfy

$$\max\left(\sigma_{n-1}, \exp\left(-\frac{k_n}{4K_n}\right)\right) < \rho_n < 1 \tag{5.1}$$

and on taking the admissible arc

$$\gamma_n := \Gamma(\rho_n, \theta_n) = \Gamma(\rho_n, \theta(\rho_n)),$$

we consider the solution $u_n \in H(\hat{\boldsymbol{D}} \setminus \gamma_n) \cup C(\hat{\boldsymbol{D}}^*)$ of the Dirichlet problem on $\hat{\boldsymbol{D}} \setminus \gamma_n$ with the boundary data

$$\begin{cases} u_n \mid \gamma_n = 1 \\ u_n \mid \delta\hat{\boldsymbol{D}} = 0 \end{cases} \tag{5.2}$$

and the solution $v_n \in H(\hat{\boldsymbol{D}} \setminus \gamma_{n-1} \cup \gamma_n) \cap C(\overline{\hat{\boldsymbol{D}}^* \setminus \gamma_{n-1}})$ of the mixed boundary value problem on $\hat{\boldsymbol{D}} \setminus \gamma_{n-1} \cup \gamma_n$ with the boundary data

$$\begin{cases} v_n \mid \gamma_n = 1 \\ *dv_n \mid \gamma_{n-1} = 0 \\ v_n \mid \delta\hat{\boldsymbol{D}} = 0 \end{cases} \tag{5.3}$$

and one more solution $w_n \in H(\hat{\boldsymbol{D}} \setminus \gamma_{n-2} \cup \gamma_{n-1} \cup \gamma_n) \cup C(\overline{\hat{\boldsymbol{D}}^* \setminus \gamma_{n-2} \cup \gamma_{n-1}})$ of the mixed boundary value problem on $\hat{\boldsymbol{D}} \setminus \gamma_{n-2} \cup \gamma_{n-1} \cup \gamma_n$ with boundary data

$$\begin{cases} w_n \mid \gamma_n = 1 \\ *dw_n \mid \gamma_{n-2} \cup \gamma_{n-1} = 0 \\ w_n \mid \delta\hat{\boldsymbol{D}} = 0 \end{cases} \tag{5.4}$$

and in addition to (5.1) we choose ρ_n so close to 1 as to satisfy the following three relations (5.5)–(5.7) corresponding to the relations (4.14)–(4.16): for $u_n = u(\cdot\,;\rho_n)$

$$\begin{cases} u_n(0) = k_n \\ k_n - \varepsilon_n \leqq u_n(\hat{z}) \leqq k_n + \varepsilon_n \quad (|\hat{z}| \leqq \sigma_{n-1}) \\ \sup\limits_{t\in T,|te^{i\theta}|\leqq\sigma_{n-1}} \left|\dfrac{*du_n(te^{i\theta})}{d\theta}\right| \leqq \varepsilon_n \end{cases} \tag{5.5}$$

and for $v_n = v(\cdot\,;\rho_n)$

$$\begin{cases} k_n - \varepsilon_n \leqq v_n(\hat{z}) \leqq k_n + \varepsilon_n \quad (|\hat{z}| \leqq \sigma_{n-1}) \\ \sup\limits_{t\in T,|te^{i\theta}|\leqq\sigma_{n-1}} \left|\dfrac{*dv_n(te^{i\theta})}{d\theta}\right| \leqq \varepsilon_n \end{cases} \tag{5.6}$$

and finally for $w_n = w(\cdot\,;\rho_n)$

$$\begin{cases} k_n - \varepsilon_n \leqq w_n(\hat{z}) \leqq k_n + \varepsilon_n \quad (|\hat{z}| \leqq \sigma_{n-1}) \\ \sup\limits_{t\in T,|te^{i\theta}|\leqq\sigma_{n-1}} \left|\dfrac{*dw_n(te^{i\theta})}{d\theta}\right| \leqq \varepsilon_n. \end{cases} \tag{5.7}$$

After functions u_n, v_n, and w_n are thus defined, we define one more function p_n and a region $P_n \subset \hat{\boldsymbol{D}}$ and then a number $\sigma_n \in (\rho_n, 1) \cup T$ as will be described below. Let $p_n := u_n/2$ so that $p_n \in H(\hat{\boldsymbol{D}} \setminus \gamma_n) \cap C(\hat{\boldsymbol{D}}^*)$ is the solution of the Dirichlet problem on $\hat{\boldsymbol{D}} \setminus \gamma_n$ with boundary data

$$\begin{cases} p_n \mid \gamma_n = \dfrac{1}{2} = \dfrac{2}{4} \\ p_n \mid \delta\hat{\boldsymbol{D}} = 0. \end{cases} \tag{5.8}$$

Since $\hat{\boldsymbol{D}}$ is hyperbolically regular, level lines of p_n are all compact in $\hat{\boldsymbol{D}}$. Hence

$$P_n := \left\{ p_n > \frac{1}{4} \right\} \supset \gamma_n \tag{5.9}$$

is a relatively compact subregion of $\hat{\boldsymbol{D}}$ containing γ_n. Finally we can choose arbitrarily and then fix a $\sigma_n \in (\rho_n, 1) \cap T$ such that

$$\{|\hat{z}| < \sigma_n\} \supset \overline{P}_n. \tag{5.10}$$

Hence we see that

$$0 < \sigma_{n-1} < \rho_n < \sigma_n < 1 \quad (n \in \boldsymbol{N}). \tag{5.11}$$

Lastly, we evaluate $\theta_n = \theta(\rho_n)$ in $\gamma_n = \Gamma(\rho_n, \theta_n)$ in terms of k_n. Let ω be the harmonic measure function on the region $\{|\hat{z}| < \rho_n\}$ of the boundary arc $\gamma_n = \Gamma(\rho_n, \theta_n) \subset \{|\hat{z}| = \rho_n\}$, i.e. $\omega \in HB(\{|\hat{z}| < \rho_n\})$ with boundary values 1 on the interior of the arc γ_n and 0 on $\{|\hat{z}| = \rho_n\} \setminus \gamma_n$. Since $\omega \leqq u_n$ on $\{|\hat{z}| < \rho_n\}$, we see in particular

$$\omega(0) \leqq u_n(0) = k_n.$$

On the other hand the Green function $G(\hat{z}, 0)$ on $\{|\hat{z}| < \rho_n\}$ with its pole 0 is

$$G(\hat{z}, 0) = \log\left(\frac{\rho_n}{|\hat{z}|}\right) = \log \rho_n + g(\hat{z}) = \log \rho_n - \log |\hat{z}|.$$

By using the Poisson formula we see that

$$\omega(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \omega\big(\rho_n e^{i\theta}\big) \left[\frac{\partial}{\partial r} G\big(re^{i\theta}, 0\big)\right]_{r=\rho_n} \rho_n d\theta = \frac{1}{2\pi} \int_{-\theta_n}^{\theta_n} d\theta = \frac{\theta_n}{\pi}.$$

Hence we have

$$0 < \theta_n \leqq \pi k_n \quad (n \in \boldsymbol{N}) \tag{5.12}$$

and this is also true for $n = 0$ by our convention (cf. Section 3). This is a quite rough estimate but sufficient for our later purpose.

6. An essential function.

We have constructed four function sequences $(u_n)_{n\in\boldsymbol{N}}$, $(v_n)_{n\in\boldsymbol{N}}$, $(w_n)_{n\in\boldsymbol{N}}$, and $(p_n)_{n\in\boldsymbol{N}}$. We now construct a sequence $(h_n)_{n\in\boldsymbol{N}}$ of important functions h_n which play essential roles below. Except $(p_n)_{n\in\boldsymbol{N}}$, three function sequences $(u_n)_{n\in\boldsymbol{N}}$, $(v_n)_{n\in\boldsymbol{N}}$, and $(w_n)_{n\in\boldsymbol{N}}$ will play only supporting roles to $(h_n)_{n\in\boldsymbol{N}}$. Anyway we define the solution $h_n \in H(\hat{\boldsymbol{D}} \setminus \gamma_{n-1} \cup \gamma_n \cup \gamma_{n+1}) \cap C(\overline{\hat{\boldsymbol{D}}^* \setminus \gamma_{n-1} \cup \gamma_{n+1}})$ of the mixed boundary value problem on $\hat{\boldsymbol{D}} \setminus \gamma_{n-1} \cup \gamma_n \cup \gamma_{n+1}$ with boundary data

$$\begin{cases} h_n \mid \gamma_n = 1 \\ *dh_n \mid \gamma_{n-1} \cup \gamma_{n+1} = 0 \\ h_n \mid \delta\hat{\boldsymbol{D}} = 0. \end{cases} \tag{6.1}$$

For the unique existence of h_n, as appeared repeatedly before, see e.g. [**11**]. By the comparison principle we see that

$$-w_{n+1} + v_n \leqq h_n \leqq w_{n+1} + v_n$$

on $\hat{\boldsymbol{D}} \setminus \gamma_{n-1} \cup \gamma_n \cup \gamma_{n+1}$ because the same is true on its essential boundary $\gamma_{n-1} \cup \gamma_n \cup \gamma_{n+1} \cup \delta\hat{\boldsymbol{D}}$. By (5.6) and (5.7) we deduce

$$k_n - (k_{n+1} + \varepsilon_n + \varepsilon_{n+1}) \leqq h_n(\hat{z}) \leqq k_n + (k_{n+1} + \varepsilon_n + \varepsilon_{n+1})$$

on $|\hat{z}| \leqq \sigma_{n-1}$. By the manner k_n and ε_n are given in (3.2)–(3.4), we have

$$\frac{k_n}{2} \leqq h_n(\hat{z}) \leqq \frac{3k_n}{2} \quad (|\hat{z}| \leqq \sigma_{n-1}). \tag{6.2}$$

In particular, we have $h_n(0) \geqq k_n/2$. We apply the Stokes formula to the differential form $g * dh_n - h_n * dg$ on $(\hat{\boldsymbol{D}} \setminus \overline{(c)}) \setminus \gamma_{n-1} \cup \gamma_n \cup \gamma_{n+1}$, where c is a small circle with radius $0 < \varepsilon < t_0$ and $\overline{(c)}$ the closed disc bounded by c. Then we have

$$\int_{\delta\hat{\boldsymbol{D}}+c+\gamma_{n-1}+\gamma_n+\gamma_{n+1}} (g * dh_n - h_n * dg) = 0.$$

Since $\int_c (g * dh_n - h_n * dg) = \mathscr{O}(\varepsilon \log \varepsilon) - 2\pi h_n(0) \to -2\pi h_n(0)$ as $\varepsilon \downarrow 0$, we see that

$$\int_{\delta\hat{\boldsymbol{D}}+\gamma_{n-1}+\gamma_n+\gamma_{n+1}} (g * dh_n - h_n * dg) = 2\pi h_n(0).$$

But we also have

$$\int_{\delta\hat{\boldsymbol{D}}+\gamma_{n-1}+\gamma_n+\gamma_{n+1}} g * dh_n$$
$$= \log\left(\frac{1}{\rho_n}\right)\int_{\gamma_n} *dh_n = \log\left(\frac{1}{\rho_n}\right)\int_{\gamma_n} h_n * dh_n$$
$$= \log\left(\frac{1}{\rho_n}\right)\int_{\delta\hat{\boldsymbol{D}}+\gamma_{n-1}+\gamma_n+\gamma_{n+1}} h_n * dh_n = \log\left(\frac{1}{\rho_n}\right)D(h_n;\hat{\boldsymbol{D}})$$

and

$$\int_{\delta\hat{\boldsymbol{D}}+\gamma_{n-1}+\gamma_n+\gamma_{n+1}} h_n * dg = \int_{\gamma_{n-1}+\gamma_{n+1}} h_n * dg.$$

Therefore we obtain

$$D(h_n;\hat{\boldsymbol{D}})\log\left(\frac{1}{\rho_n}\right) = 2\pi h_n(0) + \int_{\gamma_{n-1}+\gamma_{n+1}} h_n * dg. \tag{6.3}$$

Observe that $*dg(\rho_j e^{i\theta}) = d\theta$ on $\gamma_j = \gamma_j^+ + \gamma_j^-$ $(j = n \pm 1)$. On denoting by $h_n^\pm$ the continuous boundary values of h_n on $\gamma_{n-1} = \gamma_{n-1}^+ + \gamma_{n-1}^-$ we have

$$\int_{\gamma_{n-1}} h_n * dg = \int_{\gamma_{n-1}^+ + \gamma_{n-1}^-} h_n * dg = \int_{-\theta_{n-1}}^{\theta_{n-1}} \big(h_n^+(\rho_{n-1}e^{i\theta}) - h_n^-(\rho_{n-1}e^{i\theta})\big)d\theta.$$

By (6.2) and (5.12), we deduce

$$\left|\int_{\gamma_{n-1}} h_n * dg\right| \leqq \int_{-\theta_{n-1}}^{\theta_{n-1}} \frac{3}{2}k_n d\theta = 3k_n\theta_{n-1} \leqq 3\pi k_{n-1}\cdot k_n.$$

Though very rough but certainly $|h_n^+ - h_n^-| \leqq 2$ on $\gamma_{n+1} = \gamma_{n+1}^+ + \gamma_{n+1}^-$ and hence, by (5.12)

$$\left|\int_{\gamma_{n+1}} h_n * dg\right| \leqq \int_{-\theta_{n+1}}^{\theta_{n+1}} 2d\theta = 4\theta_{n+1} \leqq 4\pi k_{n+1}.$$

Using (3.2)–(3.4) we see that

$$\left| \int_{\gamma_{n-1}+\gamma_{n+1}} h_n * dg \right| \leqq \pi(3k_{n-1}k_n + 4k_{n+1}) \leqq \frac{k_n}{4}. \tag{6.4}$$

Since $h_n(0) \geqq k_n/2$, we conclude with (6.3) that

$$D(h_n; \hat{\boldsymbol{D}}) \log\left(\frac{1}{\rho_n}\right) \geqq \pi k_n - \frac{k_n}{4} = \frac{k_n}{4}$$

or equivalently we have

$$D(h_n; \hat{\boldsymbol{D}}) \geqq \frac{k_n}{4} \log\left(\frac{1}{\rho_n}\right).$$

By the manner ρ_n is chosen to satisfy (5.1) we finally conclude that

$$D(h_n; \hat{\boldsymbol{D}}) \geqq K_n \quad (n \in \boldsymbol{N}). \tag{6.5}$$

7. Construction of a surface.

We use the doubling process of two Riemann surfaces along a common slit. Concerning the welding (pasting) of two surfaces along a cut, we refer the reader to the splendid description in the monograph [**10**] of Oikawa (see also [**9**]).

Let X and Y be two Riemann surfaces and suppose there is a simply connected analytic Jordan region U contained both in X and also in Y. Let $\gamma \subset U$ be an analytic Jordan arc and γ^+ and γ^- be both sides of the cut γ. We view that $U \setminus \gamma$ is surrounded by two analytic Jordan curves ∂U and $\gamma^+ + \gamma^-$. We can also view that a concentric circular ring $A := \{a < |z| < b\}$ $(0 < a < b < 1)$ is a conformal representation of $U \setminus \gamma$ in which $\alpha := \{|z| = a\}$ ($\beta := \{|z| = b\}$, resp.) corresponds to $\gamma^+ + \gamma^-$ (∂U, resp.). We denote by Φ the restriction of the conformal structure of X to $U \setminus \gamma$ and Φ^* the restriction of the conformal structure of Y to $U \setminus \gamma$ and assume that Φ^* is the conjugate (i.e. reversed) conformal structure of Φ so that the identity mapping of $(U \setminus \gamma, \Phi)$ to $(U \setminus \gamma, \Phi^*)$ is anticonformal. We weld the bordered Riemann surface (A, Φ) which is the conformal representation of $(U \setminus \gamma, \Phi)$ to (A, Φ^*) which is the conformal representation of $(U \setminus \gamma, \Phi^*)$ by means of the identity mapping of the component α of the border ∂A, which gives rise to the welding of $X \setminus \gamma$ to $Y \setminus \gamma$ by the identity mapping of $\gamma^+ + \gamma^-$. In other words, we are considering the welding of $X \setminus \gamma$ to $Y \setminus \gamma$ induced by the double of $U \setminus \gamma = A$ about α. We say this process that we paste $X \setminus \gamma$ to $Y \setminus \gamma$ along γ *anticonformally* and the resulting surface is denoted by

$$(X \setminus \gamma) \uplus_\gamma (Y \setminus \gamma). \tag{7.1}$$

We now start the operations of constructing a Riemann surface W such that $HD(W) = HBD(W)$ and $\dim HD(W) = \infty$. We denote by Φ the original conformal structure of the Sario-Tôki disc $\hat{\boldsymbol{D}}$ and by Φ^* the conjugate conformal structure of Φ on $\hat{\boldsymbol{D}}$. We define a sequence of Riemann surfaces S_n by

$$\begin{cases} S_{2\nu-1} := (\hat{\boldsymbol{D}}, \Phi) & (\nu \in \boldsymbol{N}) \\ S_{2\nu} := (\hat{\boldsymbol{D}}, \Phi^*) & (\nu \in \boldsymbol{N}). \end{cases} \tag{7.2}$$

We take the sequence $(\gamma_n)_{n \in \boldsymbol{N}}$ of admissible arcs $\gamma_n = \Gamma(\rho_n, \theta_n)$ $(n \in \boldsymbol{N})$ on $\hat{\boldsymbol{D}}$ constructed in Sections 3–6. Then the required surface W is given by

$$W := \cdots \{\{\{(S_1 \setminus \gamma_1) \uplus_{\gamma_1} (S_2 \setminus \gamma_1 \cup \gamma_2)\} \uplus_{\gamma_2} (S_3 \setminus \gamma_2 \cup \gamma_3)\} \uplus_{\gamma_3} (S_4 \setminus \gamma_3 \cup \gamma_4)\} \cdots . \tag{7.3}$$

It may be impressive to call surfaces as above as *grafted surfaces*.

We denote by W^* the Wiener compactification of W and by $\gamma W = W^* \setminus W$ the Wiener boundary of W and the Wiener harmonic boundary δW is the set of regular points in γW with respect to the Dirichlet problem, and δW is a compact subset of γW. The Wiener harmonic boundary δS_j of each S_j is a singleton so that $\delta S_j = \{d_j\}$ $(j \in \boldsymbol{N})$. Then let

$$\hat{\delta} W := \cup_{j \in \boldsymbol{N}} \delta S_j = \{d_1, d_2, \ldots, d_j, \ldots\}, \tag{7.4}$$

which is seen to be a subset of δW:

$$\hat{\delta} W \subset \delta W. \tag{7.5}$$

Since $\hat{\delta} W$ is in general not compact but actually noncompact in the present W since $\#\hat{\delta} W = \infty$ while δW is compact, $\hat{\delta} W$ is always a proper subset of δW (see (7.4) above). But $\hat{\delta} W$ almost exhausts δW in a sense, which is crystallized by the following important result. At this point we need to recall the quasiboundedness for harmonic functions on a Riemann surface R. A harmonic function u on a Riemann surface R is quasibounded if

$$u = \lim_{m,n \in \boldsymbol{N}, m,n \to \infty} (u \wedge m) \vee (-n)$$

on R. The totality of quasibounded harmonic functions on R is denoted by

$HB'(R)$. It is entirely trivial that $HB(R) \subset HB'(R) \subset HP(R)$ but it is slightly less trivial that $HD(R) \subset HB'(R)$ (cf. e.g. [**11**]). Then we can state an important role played by the set $\hat{\delta}W$.

THE UNICITY PRINCIPLE. *If $u \in HB'(W)$ satisfies $u|\hat{\delta}W = 0$, then $u \equiv 0$ on W.*

PROOF. Since $HB'(W)$ is a Riesz subspace (i.e. vector sublattice) of $HP(W)$, the positive part u^+ and the negative part u^- of the Jordan decomposition $u = u^+ - u^-$ of $u \in HB'(W)$ also belong to $HB'(W)$. Let

$$\max(u, 0) = u^+ - s$$

be the Riesz decomposition of the subharmonic function $\max(u, 0)$ dominated by $u^+ + u^-$ into the harmonic part u^+ and the potential part s on W. Since $s \mid \delta W = 0$ (cf. e.g. [**2**], [**11**]) and $\hat{\delta}W \subset \delta W$, we see that $s \mid \hat{\delta}W = 0$. Clearly $\max(u, 0) \mid \hat{\delta}W = 0$ along with $u \mid \hat{\delta}W = 0$ and therefore $u^+ \mid \hat{\delta}W = 0$. Similarly $u^- \mid \hat{\delta}W = 0$. Hence we only have to show that if $u \in HB'(W)^+$ satisfies $u \mid \hat{\delta}W = 0$ then $u \equiv 0$ on W. Clearly $0 \leqq u \wedge n \leqq u$ on W and hence $u \wedge n \mid \hat{\delta}W = 0$ and $u = \lim_{n \in \boldsymbol{N}, n \to \infty} u \wedge n$ locally uniformly on W. Thus we really have to prove is that if $u \in H(W)$ with $0 \leqq u \leqq 1$ on W satisfies $u|\hat{\delta}W = 0$, then $u \equiv 0$ on W.

Let W_n ($n \geqq 2$) be the subsurface of W given by

$$W_n := \{ \cdots \{\{\{(S_1 \setminus \gamma_1) \uplus_{\gamma_1} (S_2 \setminus \gamma_1 \cup \gamma_2)\} \uplus_{\gamma_2} (S_3 \setminus \gamma_2 \cup \gamma_3)\} \cdots \} \uplus_{\gamma_{n-1}} (S_n \setminus \gamma_{n-1} \cup \gamma_n)\}$$

so that $\partial W_n = \gamma_n = \gamma_n^+ + \gamma_n^-$. We take the solution $s_n \in H(S_n \setminus \gamma_{n-1} \cup \gamma_n) \cap C(S_n^*)$ of the Dirichlet problem on $S_n \setminus \gamma_{n-1} \cup \gamma_n$ with the boundary data

$$\begin{cases} s_n \mid \gamma_n = 1 \\ s_n \mid \gamma_{n-1} = k_n \\ s_n \mid \delta S_n = 0, \end{cases} \tag{7.6}$$

where topologically $S_n^* = \hat{\boldsymbol{D}}^*$ and thus $\delta S_n = \delta \hat{\boldsymbol{D}} = \{d_n\}$. For the unique existence of s_n, cf. [**11**]. Comparing the boundary values of s_n with those of u_n, we see that $-\varepsilon_n \leqq s_n - u_n \leqq \varepsilon_n$, and hence by (5.5) we have $k_n - 2\varepsilon_n \leqq s_n \leqq k_n + 2\varepsilon_n$ on $P_{n-1} \subset \{|\hat{z}| < \sigma_{n-1}\}$. Considering $s_n + p_{n-1}$ on $P_{n-1} \setminus \gamma_{n-1}$, we see by (3.2)–(3.4) that

$$\begin{cases} (s_n + p_{n-1}) \mid \gamma_{n-1} = k_n + \dfrac{1}{2} \\ (s_n + p_{n-1}) \mid \partial P_{n-1} \leqq k_n + 2\varepsilon_n + \dfrac{1}{4} < k_n + 2 \cdot 10^{-6} + \dfrac{1}{4} < k_n + \dfrac{1}{2}. \end{cases} \tag{7.7}$$

We also consider the solution $q_{n-1} \in H(W_{n-1}) \cap C((W_{n-1} \cup \gamma_{n-1})^*)$ of the Dirichlet problem on W_{n-1} with the boundary data

$$\begin{cases} q_{n-1} \mid \gamma_{n-1} = k_n + \dfrac{1}{2} \\ q_{n-1} \mid \delta S_1 \cup \delta S_2 \cup \cdots \cup \delta S_{n-1} = 0. \end{cases} \tag{7.8}$$

For the unique existence of q_{n-1}, cf. [**11**]. We then set

$$s := \begin{cases} s_n + p_{n-1} & (\text{on } S_n \setminus \gamma_n) \\ q_{n-1} & (\text{on } W_{n-1}). \end{cases} \tag{7.9}$$

In view of (7.7) and (5.8) we see that s is superharmonic on W_n. By the manner k_n is chosen in (3.2), we infer that

$$s \mid \gamma_{n-1} = k_n + \frac{1}{2} \leqq \frac{2}{3} \quad (n \geqq 2). \tag{7.10}$$

A fortiori, since $0 \leqq u \leqq s_n$ on $\partial W_n = \gamma_n$, we have $0 \leqq u \leqq 2/3$ on $\partial W_{n-1} = \gamma_{n-1}$. We repeat the same discussion for W_{n-1} and $(3/2)u$ as was done for W_n and u and derive $0 \leqq u \leqq (2/3)^2$ on $\partial W_{n-2} = \gamma_{n-2}$. After $(n-1)$ repetitions of these procedures we arrive at $0 \leqq u \leqq (2/3)^{n-1}$ on $\partial W_1 = \gamma_1$. By the maximum principle, we obtain

$$0 \leqq u \mid W_1 \leqq \left(\frac{2}{3}\right)^{n-1}. \tag{7.11}$$

Since $n \geqq 2$ was arbitrarily chosen, we conclude $u \equiv 0$ on W_1 by $n \uparrow \infty$ in (7.11). Hence we can finally maintain $u \equiv 0$ on W. □

The meaning of the above result is that only a part $\hat{\delta} W$ of δW is already playing important roles originally played by δW. We state here some of these roles including inevitable ones for the later use. First we ask how large $\hat{\delta} W$ is quantitatively. As a reference point $o \in W$ of W we take the origin 0 in the first component $S_1 \setminus \gamma_1$, i.e. $0 \in S_1 \setminus \gamma_1$ and $o = 0 \in S_1 \setminus \gamma_1$. Let hm be the harmonic measure on δW with the reference point o. Then we can prove quite easily that the unicity principle above for $\hat{\delta} W$ is equivalent to

$$\mathrm{hm}(\delta W \setminus \hat{\delta} W) = 0, \tag{7.12}$$

which is also equivalent to the existence of an $e \in HB'(W)^+$ such that

$$e|\delta W \setminus \hat{\delta} W = +\infty. \tag{7.13}$$

The last function e above is a very powerful and useful tool to show a certain property valid for δW is already valid for $\hat{\delta} W$. As an example we prove the following fact: if $u|\hat{\delta} W \geqq 0$ for a $u \in HB'(W)$, then $u \geqq 0$ on W. In fact, for an arbitrary $\varepsilon > 0$ we see that $(u + \varepsilon e)|\delta W \geqq 0$. It is a standard knowledge that $(u + \varepsilon e)|\delta W \geqq 0$ implies $u + \varepsilon e \geqq 0$ on W. On making $\varepsilon \downarrow 0$, we conclude that $u \geqq 0$ on W, which was to be shown.

For each $i \in \boldsymbol{N}$ there is a unique solution $e_i \in HBD(W) \subset HB'(W)$ of the Dirichlet problem on W with the boundary data

$$\begin{cases} e_i \mid \delta S_i = e_i(d_i) = 1 \\ e_i \mid \delta W \setminus \delta S_i = 0. \end{cases} \tag{7.14}$$

For the unique existence of e_i, again see [**11**]. The unicity principle for $\hat{\delta} W$ assures that the condition (7.14) is equivalent to the weaker condition

$$e_i(d_j) = \delta_{ij} \text{ (the Kronecker delta)} \quad (i, j \in \boldsymbol{N}). \tag{7.15}$$

Therefore we see that

$$\mathrm{hm}(\delta S_i) = \mathrm{hm}(\{d_i\}) = e_i(o) \quad (i \in \boldsymbol{N}) \tag{7.16}$$

and, by (7.12), we have $\mathrm{hm}(\hat{\delta} W) = \mathrm{hm}(\delta W) = 1$ and a fortiori

$$\sum_{i \in \boldsymbol{N}} e_i(o) = 1. \tag{7.17}$$

Properties for $\hat{\delta} W$ stated below are all derived off hand from the corresponding ones for δW by using the function e in (7.13). First of all, we state the maximum principle for $\hat{\delta} W$: for $u \in HB'(W)$ we have

$$\begin{cases} \sup\limits_W u = \sup\limits_{\hat{\delta} W} u \\ \inf\limits_W u = \inf\limits_{\hat{\delta} W} u. \end{cases} \tag{7.18}$$

Next, the representation theorem: for any $u \in HB'(W)$, we have

$$u = \sum_{j \in \boldsymbol{N}} u(d_j) e_j \tag{7.19}$$

on W, and the convergence in (7.19) is of local uniform one on W. As the converse to this, we have the following solvability of the Dirichlet problem: a sequence $(a_j)_{j \in \boldsymbol{N}} \subset \boldsymbol{R}$ satisfies

$$\sum_{j \in \boldsymbol{N}} |a_j| e_j(o) < \infty \tag{7.20}$$

or, equivalently, $(a_j)_{j \in \boldsymbol{N}} \in L^1(\boldsymbol{N}, \mathrm{hm})$, if and only if

$$u := \sum_{j \in \boldsymbol{N}} a_j e_j \in HB'(W), \tag{7.21}$$

We are now in the stage that we can give a proof to

$$\dim HD(W) = \infty, \tag{7.22}$$

In fact, choose arbitrarily m mutually distinct functions e_{j_i} $(1 \leqq i \leqq m)$ from the family $\{e_j : j \in \boldsymbol{N}\}$ and suppose their linear combination vanishes on W:

$$\sum_{1 \leqq i \leqq m} \lambda_i e_{j_i} = 0$$

on W. By considering this at the point d_{j_i}, we obtain $\lambda_i = 0$ $(i = 1, 2, \ldots, m)$. We have thus seen that any finite subset of $\{e_j : j \in \boldsymbol{N}\}$ is linearly independent so that (7.22) is deduced.

8. Completion of the proof.

In this last section we show that the essential property $HD(W) \subset HB(W)$, or equivalently $HD(W) = HBD(W)$, is valid for the Riemann surface W given by

(7.3). At the final part of the preceding section 7 we have shown very easily the fact (7.22): $\dim HD(W)=\infty$. Thus the proof of our main theorem is complete if $HD(W)=HBD(W)$ is established, which is in reality very far from being trivial compared with that of (7.22). Let

$$R_n := (S_n \setminus \gamma_{n-1}\cup\gamma_n\cup\gamma_{n+1}) \uplus_{\gamma_n} (S_{n+1} \setminus \gamma_{n-1}\cup\gamma_n\cup\gamma_{n+1}) \quad (n\in \boldsymbol{N}). \tag{8.1}$$

We consider the symmetric selfmapping j_n of R_n induced by the identity mapping of $\hat{\boldsymbol{D}}$. Take the solution $f_n \in H(R_n)\cap C(\overline{R}_n)$ of the mixed boundary value problem on R_n with the boundary data

$$\begin{cases} f_n \mid \delta S_n = 0 \\ f_n \mid \delta S_{n+1} = 1 \\ *df_n \mid \partial R_n = 0, \end{cases}$$

where $\overline{R}_n$ is the closure of R_n in the Wiener compactification W^* of W. For the unique existence of f_n, see again [**11**]. The function $f_n + f_n\circ j_n \in H(R_n)$ takes the value 1 on the harmonic boundary δR_n of R_n and therefore, by the maximum principle, we deduce

$$f_n + f_n\circ j_n \equiv 1$$

on R_n. Since $j_n = id.$ (identity) on γ_n, we have $f_n = f_n\circ j_n$ on γ_n and hence $f_n|\gamma_n = 1/2$. Thus we see that $f_n = (1/2)h_n$ on $(S_n\setminus\gamma_{n-1}\cup\gamma_n\cup\gamma_{n+1})$. Therefore, if we define $\hat{h}_n$ by

$$\left(\frac{1}{2}\right)\hat{h}_n := \begin{cases} \left(\frac{1}{2}\right)h_n & (\text{on } S_n\setminus\gamma_{n-1}\cup\gamma_{n+1}) \\ 1-\left(\frac{1}{2}\right)h_n\circ j_n & (\text{on } S_{n+1}\setminus\gamma_{n-1}\cup\gamma_{n+1}), \end{cases} \tag{8.2}$$

then $(1/2)\hat{h}_n = f_n \in H(R_n)$ and a fortiori $\hat{h}_n \in H(R_n)$.

Choose an arbitrary $u\in HD(W)$. We wish to show that $u\in HB(W)$ or equivalently $u\in HBD(W)$. For the purpose we set

$$u(d_j) =: a_j \quad (j\in\boldsymbol{N}). \tag{8.3}$$

By the maximum principle for $\hat{\delta}W=\{d_1,d_2,\ldots,d_j,\ldots\}$, we see the equivalence of $u \in HBD(W)$ and $(a_j)_{j\in\boldsymbol{N}} \in l^\infty$. Hence we only have to show the boundedness of the sequence $(a_j)_{j\in\boldsymbol{N}}$. Observe that

$$\sum_{n\in\boldsymbol{N}} D(u; R_n) \leqq 2D(u; W). \tag{8.4}$$

Fix an arbitrary $n \in \boldsymbol{N}$. If $a_n \neq a_{n+1}$, then

$$v := \frac{u - a_n}{a_{n+1} - a_n}$$

belongs to $HBD(R_n)$ and $u = (a_{n+1} - a_n)v + a_n$ on R_n. Hence

$$D(u; R_n) = (a_{n+1} - a_n)^2 D(v; R_n).$$

By the boundary conditions $v - \hat{h}_n/2 = 0$ on the harmonic boundary $\delta S_n \cup \delta S_{n+1}$ of R_n and in particular $*d\hat{h}_n = 0$ on ∂R_n, the Stokes formula yields

$$D\left(v - \frac{\hat{h}_n}{2}, \frac{\hat{h}_n}{2}; R_n\right) = \int_{\delta S_n \cup \delta S_{n+1} + \partial R_n} \left(v - \frac{\hat{h}_n}{2}\right) * \frac{d\hat{h}_n}{2} = 0,$$

or $D(v, \hat{h}_n/2; R_n) = D(\hat{h}_n/2; R_n)$, and therefore we deduce

$$D\left(v - \frac{\hat{h}_n}{2}; R_n\right) = D(v; R_n) - D\left(\frac{\hat{h}_n}{2}; R_n\right) \geqq 0.$$

Hence by (6.5) we see that

$$D(v; R_n) \geqq D\left(\frac{\hat{h}_n}{2}; R_n\right) = 2D\left(\frac{h_n}{2}; \hat{\boldsymbol{D}}\right) \geqq \frac{K_n}{2}.$$

Therefore we obtain

$$D(u; R_n) \geqq \frac{K_n(a_{n+1} - a_n)^2}{2}. \tag{8.5}$$

We have deduced the above inequality (8.5) under the assumption $a_n \neq a_{n+1}$ but this inequality is trivially true if $a_n = a_{n+1}$. Thus (8.5) is always valid for every $n \in \boldsymbol{N}$. This with (8.4) we obtain

$$\sum_{n\in\boldsymbol{N}} K_n(a_{n+1}-a_n)^2 \leqq 4D(u;W). \tag{8.6}$$

Fix an arbitrary $m \in \boldsymbol{N}$. Then the Schwarz inequality implies that

$$\begin{aligned}
|a_m - a_1| &= \left| \sum_{1\leqq n<m} (a_{n+1}-a_n) \right| \leqq \sum_{1\leqq n<m} |a_{n+1}-a_n| \\
&= \sum_{1\leqq n<m} K_n^{1/2}|a_{n+1}-a_n| \cdot K_n^{-1/2} \\
&\leqq \sqrt{\sum_{1\leqq n<m} K_n(a_{n+1}-a_n)^2} \cdot \sqrt{\sum_{1\leqq n<m} 1/K_n}.
\end{aligned}$$

We have

$$\sqrt{\sum_{n\in\boldsymbol{N}} 1/K_n} =: K < \infty$$

by (3.1). Then (8.6) assures that

$$|a_m - a_1| \leqq 2\sqrt{D(u;W)}K$$

and a fortiori

$$|a_m| \leqq |a_1| + 2K\sqrt{D(u;W)} \quad (m \in \boldsymbol{N}),$$

or $(a_m)_{m\in\boldsymbol{N}} \in l^\infty$, which was to be shown. □

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