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Tight 9-designs on two concentric spheres

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Abstract. The main purpose of this paper is to show the nonexistence of tight Euclidean 9-designs on 2 concentric spheres in $\boldsymbol{R}^n$ if $n \geq 3$. This in turn implies the nonexistence of minimum cubature formulas of degree 9 (in the sense of Cools and Schmid) for any spherically symmetric integrals in $\boldsymbol{R}^n$ if $n \geq 3$.

1. Introduction.

The concept of Euclidean t-designs (X, w), a pair of finite set X in $\boldsymbol{R}^n$ and a positive weight function w on X, is due to Neumaier-Seidel [**19**], though similar concepts have been existed in statistics as rotatable designs [**11**] and in numerical analysis as cubature formulas for spherically symmetric integrals in $\boldsymbol{R}^n$ ([**12**], [**11**], etc.). There exist natural Fisher type lower bounds (Möller's bound) for the size of Euclidean t-designs. Those which attain one of such lower bounds are called tight Euclidean t-designs. These lower bounds are basically obtained as functions of t, n and the number p of spheres (whose centers are at the origin) which meet the finite set X. We have been working on the classification of tight Euclidean t-designs, in particular those with $p = 2$ (or p being small). In [**9**] and [**5**], we gave the complete classification of tight Euclidean 5- and 7-designs on 2 concentric spheres in $\boldsymbol{R}^n$. (Exactly speaking modulo the existence of tight spherical 4-designs for $t = 5$.) The main purpose of this paper is to show the nonexistence of tight Euclidean 9-designs on 2 concentric spheres in $\boldsymbol{R}^n$ if $n \geq 3$.

The theory of Euclidean t-designs has strong connections with the theory of cubature formulas for so called spherically symmetric integrals on $\boldsymbol{R}^n$. Here, we consider a pair $(\Omega, d\rho(\boldsymbol{x}))$ such that Ω is a spherically symmetric (or sometimes called radially symmetric) subset of $\boldsymbol{R}^n$ and a spherically symmetric (or radially symmetric) measure $d\rho(\boldsymbol{x})$ on Ω. (Here, a subset $\Omega \subset \boldsymbol{R}^n$ is called spherically symmetric if $\boldsymbol{x} \in \Omega$, then any elements having the same distance from the origin as $\boldsymbol{x}$ are also in Ω, and $d\rho(\boldsymbol{x})$ is spherically symmetric if it is invariant under the action of orthogonal transformations.) A cubature formula (X, w) of degree t for

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$(\Omega, d\rho(\boldsymbol{x}))$ is defined as follows.

X is a subset in Ω containing a finite number of points, w is a positive weight function of X, i.e., a map from X to $\boldsymbol{R}_{>0}$, and (X, w) satisfies the following condition:

$$\int_{\Omega} f(\boldsymbol{x}) d\rho(\boldsymbol{x}) = \sum_{\boldsymbol{x}\in X} w(\boldsymbol{x}) f(\boldsymbol{x})$$

for any polynomials $f(\boldsymbol{x})$ of degree at most t.

Natural lower bounds of the size $|X|$ of a cubature formula (X, w) of degree t for spherically symmetric $(\Omega, d\rho(\boldsymbol{x}))$ are known as Möller's lower bounds as follows ([**17**], [**18**]). (It seems that the result for even t was essentially known much older.)

1. If $t = 2e$, then

$$|X| \geq \dim(\mathscr{P}_e(\Omega)).$$

2. If $t = 2e + 1$, then

$$|X| \geq \begin{cases} 2\dim(\mathscr{P}_e^*(\Omega)) - 1 & \text{if } e \text{ is even and } \boldsymbol{0} \in X, \\ 2\dim(\mathscr{P}_e^*(\Omega)) & \text{otherwise.} \end{cases}$$

In above $\mathscr{P}_e(\boldsymbol{R}^n)$ is the vector space of polynomials of degree at most e and $\mathscr{P}_e(\Omega) = \{f|_\Omega \mid f \in \mathscr{P}_e(\boldsymbol{R}^n)\}$, and $\mathscr{P}_e^*(\boldsymbol{R}^n)$ is the vector space of polynomials whose terms are all of degrees with the same parity as e and at most e. Also $\mathscr{P}_e^*(\Omega) = \{f|_\Omega \mid f \in \mathscr{P}_e^*(\boldsymbol{R}^n)\}$.

It is called a minimal cubature formula of degree t, if it satisfies a Möller's lower bound. Finding and classifying minimal cubature formulas have been interested by many researchers in numerical analysis, and have been studied considerably (see [**12**], [**15**], [**16**], [**21**], etc.). As it was pointed out by Cools-Schmid [**12**], the problem has a special feature when $t = 4k + 1$. In this case, we can conclude that (1) $\boldsymbol{0} \in X$, (2) X is on $k + 1$ concentric spheres, including $S_1 = \{\boldsymbol{0}\}$.

Cools-Schmid [**12**] (cf. also [**20**]) gave a complete determination of minimal cubature formulas for $n = 2$ when $t = 4k+1$. The case of $t = 5$ for arbitrary n was solved by Hirao-Sawa [**15**] completely, in the effect that the existence of minimal cubature formula (for any spherically symmetric $(\Omega, d\rho(\boldsymbol{x}))$ in $\boldsymbol{R}^n$ is equivalent to the existence of tight spherical 4-design in $\boldsymbol{R}^n$. More recently, Hirao-Sawa [**15**] discusses the case of $t = 9$ for many specific classical $(\Omega, d\rho(\boldsymbol{x}))$. As a corollary of our main theorem: nonexistence of tight Euclidean 9-designs on 2 concentric spheres in $\boldsymbol{R}^n$ if $n \geq 3$, we obtain the nonexistence of minimum cubature formulas of degree 9 (in the sense of Cools and Schmid) for any spherically symmetric integrals in $\boldsymbol{R}^n$ if $n \geq 3$. So, we think that this means a usefulness of the concept of Euclidean t-design as a master class for all spherically symmetric cubature formulas. At the end, we add our hope to study the classification problems of tight Euclidean t-designs (for larger t) on 2 concentric spheres (or p concentric spheres with small p), and to study minimal cubature formulas with $t = 4k+1$ for $t \geq 13$, extending the method used in the present paper.

For more information on spherical designs, Euclidean designs, please refer [**1**], [**6**], etc. Explicit examples of tight 4-, 5-, 7- designs on 2 concentric spheres are given in [**10**], [**9**], [**5**], etc.

The following is the main theorem of this paper.

THEOREM 1. *Let (X, w) be a tight 9-design on 2 concentric spheres in $\boldsymbol{R}^n$ of positive radii. Let $X = X_1 \cup X_2$. Then the following hold.*

1. *X is antipodal.*
2. *Let $\boldsymbol{x} \in X_1$, $\boldsymbol{y} \in X_2$. Then $\boldsymbol{x} \cdot \boldsymbol{y}/r_1 r_2$ is a zero of the Gegenbauer polynomial $Q_{4,n-1}(x)$ of degree 4. More explicitly, $Q_{4,n-1}(x) = (n(n+6)/24)((n+4)(n+2)x^4 - 6(n+2)x^2 + 3)$ (Here Gegenbauer polynomial $Q_{l,n-1}(x)$ of degree l is normalized so that $Q_{l,n-1}(1)$ is the dimension of the vector space of homogeneous harmonic polynomials of degree l.).*
3. *$n = 2$ and (X, w) must be similar to the following.*
   $Y = Y_1 \cup Y_2$, Y_1 and Y_2 are regular 8-gons given by

$$Y_1 = \left\{ r_1(\cos\theta_k, \sin\theta_k) \,\middle|\, \theta_k = \frac{2k\pi}{8},\ 0 \leq k \leq 7 \right\},$$

$$Y_2 = \left\{ r_2(\cos\theta_k, \sin\theta_k) \,\middle|\, \theta_k = \frac{(2k+1)\pi}{8},\ 0 \leq k \leq 7 \right\},$$

where r_1 and r_2 are any positive real number satisfying $r_1 \neq r_2$. The weight function is defined by $w(\boldsymbol{y}) = w_1$ on Y_1 and $w(\boldsymbol{y}) = (r_1^8/r_2^8)w_1$ on Y_2.

It is known that tight Euclidean $(2e+1)$-designs of $\boldsymbol{R}^n$ containing the origin exist only when e is an even integer and $p = e/2 + 1$ (see Proposition 2.4.5 in [**8**]). Hence Theorem 1 implies the followings.

COROLLARY 1. *Let (X, w) be a tight 9-design of $\boldsymbol{R}^n$ containing the origin. Then $n = 2$ and X is supported by 3 concentric spheres and $(X \backslash \{\boldsymbol{0}\}, w)$ is similar to the 9-design (Y, w) given in Theorem 1.*

COROLLARY 2. *If $n \geq 3$, then there is no cubature formula of degree 9* *for spherically symmetric subset and measure* $(\Omega, d\rho(\boldsymbol{x}))$ *in* $\boldsymbol{R}^n$*.* (For minimal cubature formulas for $n = 2$ see [**16**].)

2. Definition and basic facts on the Euclidean t-designs.

We use the following notation.

Let $\mathscr{P}(\boldsymbol{R}^n)$ be the vector space over real number field $\boldsymbol{R}$ consists of all the polynomials in n variables $x_1, x_2, \ldots, x_n$ with real valued coefficients. For $f \in \mathscr{P}(\boldsymbol{R}^n)$, $\deg(f)$ denotes the degree of the polynomial f. Let $\mathrm{Harm}(\boldsymbol{R}^n)$ the subspace of $\mathscr{P}(\boldsymbol{R}^n)$ consists of all the harmonic polynomials. For each nonnegative integer l, let $\mathrm{Hom}_l(\boldsymbol{R}^n) = \langle f \in \mathscr{P}(\boldsymbol{R}^n) \mid \deg(f) = l \rangle$. We use the following notation:

$$\mathrm{Harm}_l(\boldsymbol{R}^n) := \mathrm{Harm}(\boldsymbol{R}^n) \cap \mathrm{Hom}_l(\boldsymbol{R}^n), \quad \mathscr{P}_e(\boldsymbol{R}^n) := \oplus_{l=0}^{e} \mathrm{Hom}_l(\boldsymbol{R}^n),$$

$$\mathscr{P}_e^*(\boldsymbol{R}^n) := \oplus_{l=0}^{[e/2]} \mathrm{Hom}_{e-2l}(\boldsymbol{R}^n),$$

$$\mathscr{R}_{2(p-1)}(\boldsymbol{R}^n) := \langle \|\boldsymbol{x}\|^{2i} \mid 0 \le i \le p-1 \rangle \subset \mathscr{P}_{2(p-1)}(\boldsymbol{R}^n)$$

For a subset $Y \subset \boldsymbol{R}^n$, $\mathscr{P}(Y) = \{f|_Y \mid f \in \mathscr{P}(\boldsymbol{R}^n)\}$. $\mathscr{H}(Y)$, $\mathrm{Hom}_l(Y)$, $\mathrm{Harm}_l(Y)$, $\ldots$, etc., are defined in the same way.

Let (X, w) be a weighted finite set in $\boldsymbol{R}^n$ whose weight satisfies $w(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \in X$. Let $\{r_1, r_2, \ldots, r_p\}$ be the set $\{\|\boldsymbol{x}\| \mid \boldsymbol{x} \in X\}$ of the length of the vectors in X. Where for $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \boldsymbol{R}^n$, $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^n x_i y_i$ and $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$. Let S_i, $1 \le i \le p$, be the sphere of radius r_i centered at the origin. We say that X is supported by p concentric spheres, or the union of p concentric spheres $S = S_1 \cup S_2 \cup \cdots \cup S_p$.

If a finite positive weighted set (X, w) is supported by p concentric spheres, then $\dim(\mathscr{R}_{2(p-1)}(X)) = p$ holds. For each l, we define an inner product $\langle -, \ - \rangle_l$ on $\mathscr{P}_{2(p-1)}(X)$ by $\langle f, g \rangle_l = \sum_{\boldsymbol{x} \in X} w(\boldsymbol{x}) \|\boldsymbol{x}\|^{2l} f(\boldsymbol{x}) g(\boldsymbol{x})$. Then $\langle -, \ - \rangle_l$ is positive definite for each l. For each l, we define polynomials $\{g_{l,j} \mid 0 \le j \le p-1\} \subset \mathscr{R}_{2(p-1)}(\boldsymbol{R}^n)$ so that $\{g_{l,j}|_X \mid 0 \le j \le p-1\}$ is an orthonormal basis of $\mathscr{R}_{2(p-1)}(X)$ with respect to $\langle -, \ - \rangle_l$. We define so that $g_{l,j}(\boldsymbol{x})$ is a polynomial of degree $2j$ and a linear combination of $\{\|\boldsymbol{x}\|^{2i} \mid 0 \le i \le j\}$. We abuse the notation and we identify $g_{l,j}(\boldsymbol{x}) = g_{l,j}(r_\nu)$ for $\boldsymbol{x} \in X_\nu$ $(1 \le \nu \le p)$.

Definition 1 ([**19**]). A weighted finite set (X, w) is a Euclidean t-design if

$$\sum_{i=1}^{p} \frac{w(X_i)}{|S_i|} \int_{S_i} f(\boldsymbol{x}) d\sigma_i(\boldsymbol{x}) = \sum_{\boldsymbol{x} \in X} w(\boldsymbol{x}) f(\boldsymbol{x})$$

holds for any $f \in \mathscr{P}_t(\boldsymbol{R}^n)$. In above, $w(X_i) = \sum_{\boldsymbol{x} \in X_i} w(\boldsymbol{x})$, $\int_{S_i} f(\boldsymbol{x}) d\sigma_i(\boldsymbol{x})$ is the usual surface integral of the sphere S_i of radius r_i, $|S_i|$ is the surface area of S_i.

THEOREM 2 ([**17**], [**18**], [**19**], [**14**], [**9**], [**8**], etc). *Let $X \subset \boldsymbol{R}^n$ be a Euclidean t-design supported by a union S of p concentric spheres. Then the following hold.*

1. *For $t = 2e$,*

$$|X| \geq \dim(\mathscr{P}_e(S)).$$

2. *For $t = 2e+1$,*

$$|X| \geq \begin{cases} 2\dim(\mathscr{P}_e^*(S)) - 1 & \text{for } e \text{ even and } \boldsymbol{0} \in X \\ 2\dim(\mathscr{P}_e^*(S)) & \text{otherwise.} \end{cases}$$

DEFINITION 2 (Tightness of designs). If an equality holds in one of the inequalities given in Theorem 2, then (X, w) is a tight t-design on p concentric spheres in $\boldsymbol{R}^n$. Moreover if $\mathscr{P}_e(S) = \mathscr{P}_e(\boldsymbol{R}^n)$ holds for $t = 2e$, or $\mathscr{P}_e^*(S) = \mathscr{P}_e^*(\boldsymbol{R}^n)$ holds for $t = 2e+1$, then (X, w) is a tight t-design of $\boldsymbol{R}^n$.

Möller [**18**] proved that a tight $(2e+1)$-design (X, w) on p concentric spheres is antipodal and the weight function is center symmetric if e is odd or e is even and $\boldsymbol{0} \in X$. For the case e is even and $\boldsymbol{0} \notin X$, Theorem 2.3.6 in [**8**] implies if we assume $p \leq (e/2)+1$, then X is antipodal and the weight function is center symmetric. Hence Lemma 1.10 in [**3**] and Lemma 1.7 in [**9**] implies that weight function of a tight t-design on p concentric spheres is constant on each X_i for $t = 2e$; $t = 2e+1$ and e odd; $t = 2e+1$, e even and $\boldsymbol{0} \in X$; $t = 2e+1$, e even, $\boldsymbol{0} \notin X$ and $p \leq (e/2)+1$;

PROPOSITION 1. *Let (X, w) be a positive weighted finite subset in $\boldsymbol{R}^n$. Assume $\boldsymbol{0} \notin X$ and the weight function is constant on each X_i $(1 \leq i \leq p)$. Then the following holds.*

$$\sum_{j=0}^{p-1} g_{l,j}(r_\nu) g_{l,j}(r_\mu) = \delta_{\nu,\mu} \frac{1}{|X_\nu| w_\nu r_\nu^{2l}}.$$

PROOF. Let M_l be the $p \times p$ matrix whose (ν, j) entry is defined by $\sqrt{|X_\nu| w_\nu} r_\nu^l g_{l,j}(r_\nu)$ for $1 \leq \nu \leq p$, $0 \leq j \leq p-1$. Then

$$\big({}^tM_lM_l\big)(j_1,j_2)=\sum_{\nu=1}^{p}M_{\nu,j_1}M_{\nu,j_2}=\sum_{\nu=1}^{p}|X_\nu|w_\nu r_\nu^{2l}g_{l,j_1}(r_\nu)g_{l,j_2}(r_\nu)$$
$$=\sum_{\nu=1}^{p}\sum_{\boldsymbol{x}\in X_\nu}w(\boldsymbol{x})\|\boldsymbol{x}\|^{2l}g_{l,j_1}(r_\nu)g_{l,j_2}(r_\nu)$$
$$=\sum_{\boldsymbol{x}\in X}w(\boldsymbol{x})\|\boldsymbol{x}\|^{2l}g_{l,j_1}(\boldsymbol{x})g_{l,j_2}(\boldsymbol{x})=\delta_{j_1,j_2} \tag{1}$$

Hence M_l is invertible and $M_l^{-1}={}^tM_l$. Hence we have $M_l\ {}^tM_l=I$.

$$\big(M_l\ {}^tM_l\big)(\nu,\mu)=r_\nu^l r_\mu^l\sqrt{|X_\nu||X_\mu|w_\nu w_\mu}\sum_{j=0}^{p-1}g_{l,j}(r_\nu)g_{l,j}(r_\mu)=\delta_{\nu,\mu} \tag{2}$$

Hence we must have

$$\sum_{j=0}^{p-1}g_{l,j}(r_\nu)g_{l,j}(r_\mu)=\delta_{\nu,\mu}\frac{1}{|X_\nu|w_\nu r_\nu^{2l}}$$

3. Proof of Theorem 1 (2).

Now we prove Theorem 1. Let (X,w) be a tight 9-design on 2 concentric spheres and $\mathbf{0}\notin X$. Let $X=X_1\cup X_2$. By assumption

$$|X|=2\dim(\mathscr{P}_4^*(S))=2\Bigg(\sum_{i=0}^{1}\binom{n+4-2i-1}{4-2i}\Bigg)=\frac{n(n+1)(n^2+5n+18)}{12}.$$

Then, as we mentioned in Section 2, X is antipodal and the weight function is constant on each X_i, $i=1,\ 2$. Let $w_i=w(\boldsymbol{x})$ for $\boldsymbol{x}\in X_i$.

Let $A(X_i)=\{\boldsymbol{x}\cdot\boldsymbol{y}/r_i^2 \mid \boldsymbol{x}\neq\boldsymbol{y}\in X_i\}$ for $i=1,2$. Let $A(X_1,X_2)=\{\boldsymbol{x}\cdot\boldsymbol{y}/r_1r_2 \mid \boldsymbol{x}\in X_1,\ \boldsymbol{y}\in X_2\}$. Then X_1 and X_2 are spherical 7-designs and $|A(X_1)|$, $|A(X_2)|\leq 5$ and $|A(X_1,X_2)|\leq 4$. Since X_1, X_2 are spherical 7-designs, $|X_1|$, $|X_2|\geq 1/3(n+2)(n+1)n$. We may assume $|X_1|\leq|X_2|$. Hence

$$\frac{1}{3}(n+2)(n+1)n\leq|X_1|\leq\frac{|X|}{2}\leq|X_2|\leq|X|-|X_1|$$
$$\leq\frac{1}{12}n(n+1)(n^2+n+10)$$

holds. If $n=2$, then we must have $|X_1|=|X_2|=8$ and X_1 and X_2 are spherical tight 7-designs. We can easily check that for any $A(X_1,X_2)=\{\cos(k\pi/8)\mid k=1,3,5,7\}=\{\sqrt{2\pm\sqrt{2}}/2,-\sqrt{2\pm\sqrt{2}}/2\}$. Hence $\gamma\in A(X_1,X_2)$ is a zero of Gegenbauer polynomial $Q_{4,1}(x)=16x^2-16x+2$.

In the following we assume $n\geq 3$, then

$$|X_2|\geq\frac{|X|}{2}=\frac{n(n+1)(n^2+5n+18)}{24}>\frac{1}{3}(n+2)(n+1)n$$

holds and X_2 is not a spherical tight 7-design. Hence X_2 is a 5-distance set, i.e., $|A(X_2)|=5$. Let X_i be an antipodal half of X_i^* for $i=1,2$. That is, $X_i=X_i^*\cup(-X_i^*)$, $X_i^*\cap(-X_i^*)=\emptyset$. Then $|A(X_i^*)|\leq 4$ for $i=1,2$, and $|A(X_1^*,X_2^*)|\leq 4$ hold.

Then equations (3.1) and (3.2) in the proof of Lemma 1.7 in [**9**] imply the following equations.

$\boldsymbol{x}\in X_1^*$

$$r_1^8 g_{4,0}(r_1)^2Q_4(1)+r_1^4Q_2(1)\sum_{j=0}^{1}g_{2,j}(r_1)^2+\sum_{j=0}^{1}g_{0,j}(r_1)^2=\frac{1}{w_1}\tag{3}$$

$\boldsymbol{x}\in X_2^*$

$$r_2^8 g_{4,0}(r_2)^2Q_4(1)+r_2^4Q_2(1)\sum_{j=0}^{1}g_{2,j}(r_2)^2+\sum_{j=0}^{1}g_{0,j}(r_2)^2=\frac{1}{w_2}\tag{4}$$

$\boldsymbol{x}\neq\boldsymbol{y}\in X_1^*$

$$r_1^8 g_{4,0}(r_1)^2Q_4\left(\frac{(\boldsymbol{x},\boldsymbol{y})}{r_1^2}\right)+r_1^4Q_2\left(\frac{(\boldsymbol{x},\boldsymbol{y})}{r_1^2}\right)\sum_{j=0}^{1}g_{2,j}(r_1)^2+\sum_{j=0}^{1}g_{0,j}(r_1)^2=0\tag{5}$$

$\boldsymbol{x}\neq\boldsymbol{y}\in X_2^*$

$$r_2^8 g_{4,0}(r_2)^2Q_4\left(\frac{(\boldsymbol{x},\boldsymbol{y})}{r_2^2}\right)+r_2^4Q_2\left(\frac{(\boldsymbol{x},\boldsymbol{y})}{r_2^2}\right)\sum_{j=0}^{1}g_{2,j}(r_2)^2+\sum_{j=0}^{1}g_{0,j}(r_2)^2=0\tag{6}$$

$\boldsymbol{x}\in X_1^*,\boldsymbol{y}\in X_2^*$

$$r_1^4 r_2^4 g_{4,0}(r_1) g_{4,0}(r_2) Q_4\left(\frac{(\boldsymbol{x},\boldsymbol{y})}{r_1 r_2}\right) + r_1^2 r_2^2 Q_2\left(\frac{(\boldsymbol{x},\boldsymbol{y})}{r_1 r_2}\right) \sum_{j=0}^{1} g_{2,j}(r_1) g_{2,j}(r_2)$$

$$+ \sum_{j=0}^{1} g_{0,j}(r_1) g_{0,j}(r_2) = 0 \tag{7}$$

In above $g_{l,j}$ are defined for antipodal half $X^* = X_1^* \cup X_2^*$ of X. Since X_i^* is any antipodal half of X_i for $i = 1, 2$, Proposition 1 implies

$$Q_{4,n-1}\left(\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{r_1 r_2}\right) = 0$$

holds for any $\boldsymbol{x} \in X_1$ and $\boldsymbol{y} \in X_2$. □

Proposition 2. *Notation and definition are given as above.* $|A(X_1, X_2)| = 4$ *holds and*

$$A(X_1, X_2) = \left\{ \pm\sqrt{\frac{3n+6+\sqrt{6(n+2)(n+1)}}{(n+4)(n+2)}}, \pm\sqrt{\frac{3n+6-\sqrt{6(n+2)(n+1)}}{(n+4)(n+2)}} \right\}$$

Proof. Theorem 1.4 and Theorem 1.5 in [**7**] imply that X has the structure of a coherent configuration. Since X is antipodal and $0 \notin A(X_1, X_2)$, either $|A(X_1, X_2)| = 2$ or $|A(X_1, X_2)| = 4$ holds. First assume $|A(X_1, X_2)| = 2$. Then $A(X_1, X_2) = \{\gamma, -\gamma\}$ with some $\gamma > 0$ satisfying $Q_{4,n-1}(\gamma) = 0$. Let $\gamma_1 = \gamma$ and $\gamma_2 = -\gamma$. Since X_2 is a 5-distance set let $A(X_2) = \{-1, \pm\beta_2, \pm\beta_4\}$ with real numbers $\beta_2 > \beta_4 > 0$. Let $\beta_0 = 1$, $\beta_1 = -1$, $\beta_3 = -\beta_2$, $\beta_5 = -\beta_4$. Then Proposition 3.2 (1) in [**7**] the following hold for any nonnegative integers l, k, j satisfying $l + k + 2j \le 9$

$$\sum_{u=2}^{5}\sum_{v=2}^{5} w_2 r_2^{l+k+2j} Q_{l,n-1}(\beta_u) Q_{k,n-1}(\beta_v) p_{\beta_u,\beta_v}^{\beta_0}$$

$$+ \sum_{u=1}^{2}\sum_{v=1}^{2} w_1 r_1^{l+k+2j} Q_{l,n-1}(\gamma_u) Q_{k,n-1}(\gamma_v) p_{\gamma_u,\gamma_v}^{\beta_0}$$

$$= \delta_{l,k} Q_{l,n-1}(1) \sum_{\nu=1}^{2} N_\nu w_\nu r_\nu^{2l+2j}$$
$$- w_2 r_2^{l+k+2j} \big((-1)^{l+k} + 1\big) Q_{l,n-1}(1) Q_{k,n-1}(1), \qquad (8)$$

$N_\nu = |X_\nu|$ for $\nu = 1, 2$ and $p^{\beta_0}_{\beta_u,\beta_v}$, $p^{\beta_0}_{\gamma_u,\gamma_v}$ denotes the corresponding intersection numbers. Since $Q_{4,n-1}(\gamma) = Q_{4,n-1}(-\gamma) = 0$, $p^{\alpha_0}_{\beta_u,\beta_v} = 0$, for any $2 \le u \ne v \le 5$, and $p^{\alpha_0}_{\gamma_u,\gamma_v} = 0$, for any $1 \le u \ne v \le 2$, $p^{\beta_0}_{\gamma_1,\gamma_1} = p^{\beta_0}_{\gamma_2,\gamma_2} = |X_1|/2$, $p^{\beta_0}_{\beta_3,\beta_3} = p^{\alpha_0}_{\beta_2,\beta_2}$, $p^{\beta_0}_{\beta_5,\beta_5} = p^{\alpha_0}_{\beta_4,\beta_4}$, equations for $(l,k,j) = (0,0,0), (1,0,0), (1,1,0), (2,1,1)$ imply

$$p^{\beta_0}_{\beta_2,\beta_2} = \frac{-w_2 r_2^2\big(n(N_2-2)\beta_4^2 - N_2 + 2n\big) - N_1 w_1 r_1^2\big(-1 + n\gamma_1^2\big)}{2n w_2 r_2^2\big(\beta_2^2 - \beta_4^2\big)}$$

and

$$p^{\beta_0}_{\beta_4,\beta_4} = \frac{w_2 r_2^2\big(n(N_2-2)\beta_2^2 - N_2 + 2n\big) + N_1 w_1 r_1^2\big(-1 + n\gamma_1^2\big)}{2n w_2 r_2^2\big(\beta_2^2 - \beta_4^2\big)}.$$

Then equation for $(l,k,j) = (1,1,1)$ implies

$$\big(r_1^2 - r_2^2\big)\big(-1 + n\gamma_1^2\big) r_2^2 w_1 N_1 n = 0.$$

Since γ_1 is a zero of $Q_{4,n-1}(x)$, this is a contradiction.

Since $n \ge 3$, we have $|X_2| \ge (1/2)|X| = (1/24)n(n+1)(n^2+5n+18) > (1/3)(n+2)(n+1)n$. We divide the proof of Theorem 1 into two cases I and II. In Case I, we assume X_1 is not a tight spherical 7-design, i.e. $|X_1| > (1/3)(n+2)(n+1)n$, and in Case II, we assume X_1 is a tight spherical 7-design, i.e. $|X_1| = (1/3)(n+2)(n+1)n$.

Case I: $|X_2| \ge |X_1| > (1/3)(n+2)(n+1)n$
In this case both X_1 and X_2 are antipodal spherical 7-designs and 5-distance sets.

$$A(X_1) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}, \quad \alpha_0 = 1,\ \alpha_1 = -1,\ \alpha_3 = -\alpha_2,\ \alpha_5 = -\alpha_4,$$
$$A(X_2) = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\}, \quad \beta_0 = 1,\ \beta_1 = -1,\ \beta_3 = -\beta_2,\ \beta_5 = -\beta_4,$$
$$A(X_1, X_2) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}, \qquad (9)$$

where

$$\gamma_1 = \sqrt{\frac{3n+6+\sqrt{6(n+2)(n+1)}}{(n+4)(n+2)}}, \quad \gamma_2 = -\gamma_1,$$

$$\gamma_3 = \sqrt{\frac{3n+6-\sqrt{6(n+2)(n+1)}}{(n+4)(n+2)}}, \quad \gamma_4 = -\gamma_3.$$

We may assume $\alpha_2 > \alpha_4 > 0$, $\beta_2 > \beta_4 > 0$. Then Proposition 9.1 and Theorem 9.2 in [**5**] imply the followings (see also [**2**], [**4**]).

- X_i^* $(1 \leq i \leq 2)$ has the structure of a strongly regular graphs.
- $(1-\alpha_2^2)/(\alpha_2^2-\alpha_4^2)$ and $(1-\beta_2^2)/(\beta_2^2-\beta_4^2)$ are integers.
- α_2, α_3, α_4, α_5 are the zeros of the following polynomial $a(x)$.

$$a(x) = (n+4)(n+2)(N_1-n^2-n)x^4 + (n+2)(n^3+6n^2+5n-6N_1)x^2 + 3N_1 - n^3 - 3n^2 - 2n.$$

- β_2, β_3, β_4, β_5 are the zeros of the following polynomial $b(x)$.

$$b(x) = (n+4)(n+2)(N_2-n^2-n)x^4 + (n+2)(n^3+6n^2+5n-6N_2)x^2 + 3N_2 - n^3 - 3n^2 - 2n.$$

- $n \geq 4$ and α_i, and β_i, $i = 2, 3, 4$, are rational numbers.

In above $N_i = |X_i|$ for $i = 1, 2$.

Hence we obtain

$$\alpha_2^2 = \frac{(n+2)\big(6N_1 - n(n+1)(n+5)\big) + \sqrt{(n+1)(n+2)D_1}}{2(n+4)(n+2)(N_1-n^2-n)} \tag{10}$$

$$\alpha_4^2 = \frac{(n+2)\big(6N_1 - n(n+1)(n+5)\big) - \sqrt{(n+1)(n+2)D_1}}{2(n+4)(n+2)(N_1-n^2-n)} \tag{11}$$

$$\beta_2^2 = \frac{(n+2)\big(6N_2 - n(n+1)(n+5)\big) + \sqrt{(n+1)(n+2)D_2}}{2(n+4)(n+2)(N_2-n^2-n)} \tag{12}$$

$$\beta_4^2 = \frac{(n+2)\big(6N_2 - n(n+1)(n+5)\big) - \sqrt{(n+1)(n+2)D_2}}{2(n+4)(n+2)(N_2-n^2-n)} \tag{13}$$

where $D_1 = n^2(n+1)(n+2)(n+3)^2 - 8n(n+1)(n+5)N_1 + 24N_1^2$, $D_2 = n^2(n+1)(n+2)(n+3)^2 - 8n(n+1)(n+5)N_1 + 24N_2^2$, $N_i = |X_i|$ $(1 \leq i \leq 2)$.

Next proposition is very important.

PROPOSITION 3. *Notation and definition are given as above. Assume* $n \geq 3$, *then* $\sqrt{6(n+1)(n+2)}$ *is an integer and* γ_i^2 $(1 \leq i \leq 4)$ *are rational numbers.*

PROOF. Theorem 1.4 and Theorem 1.5 in [**7**] imply that X has the structure of a coherent configuration. Let $\boldsymbol{x} \in X_1$ and $p_{\gamma_i,\gamma_i}^{\alpha_0} = |\{\boldsymbol{z} \in X_2 \mid \boldsymbol{x} \cdot \boldsymbol{z}/r_1 r_2 = \gamma_i\}|$. Using the equations given in Proposition 3.2 (1) in [**7**] the following hold for any nonnegative integers l, k, j satisfying $l + k + 2j \leq 9$

$$\begin{aligned} &\sum_{u=2}^{5}\sum_{v=2}^{5} w_1 r_1^{l+k+2j} Q_{l,n-1}(\alpha_u) Q_{k,n-1}(\alpha_v) p_{\alpha_u,\alpha_v}^{\alpha_0} \\ &+ \sum_{u=1}^{4}\sum_{v=1}^{4} w_2 r_2^{l+k+2j} Q_{l,n-1}(\gamma_u) Q_{k,n-1}(\gamma_v) p_{\gamma_u,\gamma_v}^{\alpha_0} \\ &\quad = \delta_{l,k} Q_{l,n-1}(1) \sum_{\nu=1}^{2} N_\nu w_\nu r_\nu^{2l+2j} \\ &\qquad - w_1 r_1^{l+k+2j}((-1)^{l+k}+1) Q_{l,n-1}(1) Q_{k,n-1}(1) \end{aligned} \tag{14}$$

Since $p_{\alpha_1,\alpha_1}^{\alpha_0} = 1$, $p_{\alpha_i,\alpha_j}^{\alpha_0} = 0$ for any $1 \leq i \neq j \leq 5$, and $p_{\gamma_i,\gamma_j}^{\alpha_0} = 0$ for any $1 \leq i \neq j \leq 4$, we have the followings.

$$\begin{aligned} p_{\gamma_1,\gamma_1}^{\alpha_0} = p_{\gamma_2,\gamma_2}^{\alpha_0} &= \frac{N_2(1-n\gamma_3^2)}{2n(\gamma_1^2-\gamma_3^2)}, \\ p_{\gamma_3,\gamma_3}^{\alpha_0} = p_{\gamma_4,\gamma_4}^{\alpha_0} &= \frac{N_2(n\gamma_1^2-1)}{2n(\gamma_1^2-\gamma_3^2)} \end{aligned} \tag{15}$$

Then $p_{\gamma_1,\gamma_1}^{\alpha_0} = (3n^2+3n-(n-2)\sqrt{6(n+1)(n+2)})N_2/12n(n+1)$. Hence $\sqrt{6(n+1)(n+2)}$ is an integer. This completes the proof.

Next, we express $(1-\alpha_2^2)/(\alpha_2^2-\alpha_4^2)$ and $(1-\beta_2^2)/(\beta_2^2-\beta_4^2)$ in terms of n and N_1, N_2. We have

$$\frac{1-\alpha_2^2}{\alpha_2^2-\alpha_4^2} = -\frac{1}{2} + F(n, N_1), \tag{16}$$

$$\frac{1-\beta_2^2}{\beta_2^2-\beta_4^2} = -\frac{1}{2} + F(n, N_2), \tag{17}$$

where

$$F(n,x) = \frac{(2x-n^2-3n)\sqrt{(n+1)(n+2)\bigl(n^2(n+1)(n+2)(n+3)^2 -8n(n+1)(n+5)x+24x^2\bigr)}}{2\bigl(n^2(n+1)(n+2)(n+3)^2-8n(n+1)(n+5)x+24x^2\bigr)} \tag{18}$$

We have

$$\frac{(n+2)(n+1)n}{3} < N_1 \leq \frac{1}{24}n(n+1)(n^2+5n+18) \leq N_2 \leq \frac{1}{12}n(n+1)(n^2+n+10).$$

Since

$$F(n,x) = \frac{\left(1-\dfrac{n^2+3n}{2x}\right)}{\left(\dfrac{n^6+9n^5+29n^4+39n^3+18n^2}{2x^2}-\dfrac{4n(n^2+6n+5)}{x}+12\right)} \times\sqrt{6(n+2)(n+1)\left(\frac{n^6+9n^5+29n^4+39n^3+18n^2}{24x^2}-\frac{n(n^2+6n+5)}{3x}+1\right)},$$

we can observe that for $x > (1/24)n(n+1)(n^2+5n+18)$, $F(n,x) \approx \sqrt{6(n+2)(n+1)}/12$. More precisely we have the followings.

$$\frac{\partial F(n,x)}{\partial x} = \frac{(n-1)(n+4)(n+2)(n+1)(n^3+4n^2+3n-4x)n}{\sqrt{(n+2)(n+1)\bigl(n^2(n+2)(n+1)(n+3)^2 -8n(n+5)(n+1)x+24x^2\bigr)^3}} \tag{19}$$

Hence $F(n,x)$ decreases for $x \geq (1/4)n(n+1)(n+3)$.

$$F\left(n, \frac{1}{12}n(n+1)(n^2+n+10)\right) = \frac{\sqrt{6}(n^2+3n+8)}{12\sqrt{n^2-n+4}} > \frac{\sqrt{6(n+1)(n+2)}}{12} \tag{20}$$

$$F\left(n, \frac{1}{24}n(n+1)(n^2+5n+18)\right)$$
$$= \frac{\sqrt{6(n+2)}(n^2+7n+18)}{12\sqrt{n^3+5n^2+16n+36}} < 1 + \frac{\sqrt{6(n+1)(n+2)}}{12} \tag{21}$$

Hence

$$-\frac{1}{2} + \frac{\sqrt{6(n+1)(n+2)}}{12} < -\frac{1}{2} + F(n, N_2) < \frac{1}{2} + \frac{\sqrt{6(n+1)(n+2)}}{12}$$

holds. Since $\sqrt{6(n+1)(n+2)}$ is an integer, $\sqrt{6(n+1)(n+2)} = \sqrt{6^2k^2} = 6k$ with an integer $k > 0$. Hence

$$\frac{k-1}{2} < -\frac{1}{2} + F(n, N_2) < \frac{k+1}{2}$$

If k is an odd integer, then $-(1/2) + F(n, N_2)$ cannot be an integer. Hence k must be an even integer and we must have

$$-\frac{1}{2} + F(n, N_2) = \frac{k}{2} = \frac{\sqrt{6(n+2)(n+1)}}{12}. \tag{22}$$

It is known $n = 23$, 2399, 235223 satisfy this condition. Otherwise $n > 300000$. The equation (22) implies

$$N_2 = \frac{n}{36(2n^2+6n+1)} \times \Big\{9(n+3)(n+1)(n^2+6n+2)$$
$$+ (n-1)(n+4)(n+2)(n+1)\sqrt{6(n+1)(n+2)}$$
$$+ \varepsilon(n-1)\big(\sqrt{6}(n^2+3n-1) + 3\sqrt{(n+2)(n+1)}\big)$$
$$\times \sqrt{(n+4)(n+1)}\sqrt{(n+5)(n+1) - \sqrt{6(n+2)(n+1)}}\Big\} \tag{23}$$

where $\varepsilon = 1$ or -1. If $\varepsilon = -1$, then we have

$$N_2 < \frac{1}{24}n(n+1)(n^2+5n+18).$$

This contradicts the assumption. Hence we must have $\varepsilon = 1$. Then we must have

$$\begin{aligned}N_1 = \frac{n}{36(2n^2+6n+1)} &\times \Big\{3n(n+1)(2n^3+13n^2+40n+53)\\ &- (n-1)(n+4)(n+2)(n+1)\sqrt{6(n+1)(n+2)}\\ &- (n-1)\big(\sqrt{6}(n^2+3n-1)+3\sqrt{(n+2)(n+1)}\big)\\ &\quad \times \sqrt{(n+4)(n+1)}\sqrt{(n+5)(n+1)-\sqrt{6(n+1)(n+2)}}\Big\} \qquad (24)\end{aligned}$$

Since $n = 23,\ 2399$, and 235223 do not give integral value for N_2, we must have $n > 300000$. Solve $-(1/2)+F(n,x) = (\sqrt{6(n+2)(n+1)}/12)+2$ for x, then we must have $x = K_\varepsilon$ given below.

$$\begin{aligned}K_\varepsilon = \frac{n}{60(6n^2+18n-213)} &\times \Big\{45(n+1)(n^3+9n^2-28n-234)\\ &+ (n-1)(n+4)(n+2)(n+1)\sqrt{6(n+2)(n+1)}\\ &+ \varepsilon(n-1)\big(\sqrt{6}(n^2+3n-73)+15\sqrt{(n+2)(n+1)}\big)\\ &\times \sqrt{n^2+6n-67-5\sqrt{6(n+2)(n+1)}}\Big\} \qquad (25)\end{aligned}$$

where $\varepsilon = \pm 1$. Now we may assume $n > 300000$. Then we have

$$\begin{aligned}K_+(=K_{+1}) &> \frac{n}{60(6n^2+18n-213)}\\ &\quad \times (n-1)(n+4)(n+2)(n+1)\sqrt{6(n+2)(n+1)}\\ &> \frac{\sqrt{6}n^5(n-1)}{60(6n^2+18n-213)} > \frac{n(n+1)(n+3)}{4}. \qquad (26)\end{aligned}$$

Next compare K_+ and N_1.

$$\begin{aligned}N_1 - K_+ = \frac{n(n-1)}{180(2n^2+6n+1)(2n^2+6n-71)}\\ \times \Big\{15(n+2)(n+1)(4n^4+28n^3-76n^2-442n-351)\\ - 6(n+4)(n+2)(n+1)(2n^2+6n-59)\sqrt{6(n+2)(n+1)}\\ - (2n^2+6n+1)\big(\sqrt{6}(n^2+3n-73)+15\sqrt{(n+2)(n+1)}\big)\end{aligned}$$

$$\times \sqrt{(n+4)(n+1)}\sqrt{n^2+6n-67-5\sqrt{6(n+2)(n+1)}}$$
$$-5(2n^2+6n-71)\big(\sqrt{6}(n^2+3n-1)+3\sqrt{(n+2)(n+1)}\big)$$
$$\times \sqrt{(n+4)(n+1)}\sqrt{n^2+6n+5-\sqrt{6(n+2)(n+1)}}\Big\} \qquad (27)$$

The order of the formula in $\{\cdots\}$ in above equals $2(30-11\sqrt{6})n^6$. Hence $N_1 > K_+$ holds for any n sufficiently large, in particular for $n > 300000$. This means

$$-\frac{1}{2}+F(n,N_1) < \frac{\sqrt{6(n+2)(n+1)}}{12}+2$$

holds for any n sufficiently large. Since $N_2 > N_1$, we must have $\sqrt{6(n+2)(n+1)}/12 = -(1/2)+F(n,N_2) < -(1/2)+F(n,N_1)$. Hence we must have $-(1/2)+F(n,N_1) = \sqrt{6(n+2)(n+1)}/12+1$. Next solve for $F(n,x) = \sqrt{6(n+2)(n+1)}/12+1$ then we have $x = G_\varepsilon$ given below.

$$G_\varepsilon = \frac{n}{6n^2+18n-69}\times\Big\{27(n+1)(n^3+9n^2+4n-74)$$
$$+(n-1)(n+4)(n+2)(n+1)\sqrt{6(n+2)(n+1)}$$
$$+\varepsilon(n-1)\big(\sqrt{6}(n^2+3n-25)+9\sqrt{(n+2)(n+1)}\big)$$
$$\times\sqrt{(n+4)(n+1)\big(n^2+6n-19-3\sqrt{6(n+2)(n+1)}\big)}\Big\} \qquad (28)$$

where $\varepsilon = \pm 1$. Compare N_1 and $G_+(=G_{+1})$.

$$G_+ - N_1 = \frac{n(n-1)}{108(2n^2+6n+1)(2n^2+6n-23)}$$
$$\times\Big\{-9(n+2)(n+1)(4n^4+28n^3+20n^2-106n-111)$$
$$+4(n+4)(n+2)(n+1)(2n^2+6n-17)\sqrt{6(n+2)(n+1)}$$
$$+(2n^2+6n+1)\big(\sqrt{6}(n^2+3n-25)+9\sqrt{(n+2)(n+1)}\big)$$
$$\times\sqrt{(n+4)(n+1)\big(n^2+6n-19-3\sqrt{6(n+2)(n+1)}\big)}$$
$$+3(2n^2+6n-23)\big(\sqrt{6}(n^2+3n-1)+3\sqrt{(n+2)(n+1)}\big)$$
$$\times\sqrt{(n+4)(n+1)\big(n^2+6n+5-\sqrt{6(n+2)(n+1)}\big)}\Big\} \qquad (29)$$

The order of the formula in $\{\cdots\}$ given above equals $4(4\sqrt{6}-9)n^6$. Hence $G_+ > N_1$ holds for any n sufficiently large, in particular $n > 300000$. Since $F(n,x)$ decreases for $x \geq (n+3)(n+1)n/4$, we have

$$N_2 > G_+ > N_1 > K_+ > \frac{(n+3)(n+1)n}{4}.$$

Hence we must have

$$\begin{aligned}\frac{\sqrt{6(n+2)(n+1)}}{12} &= -\frac{1}{2} + F(n,N_2) < -\frac{1}{2} + F(n,G_+) \\ &= \frac{\sqrt{6(n+2)(n+1)}}{12} + 1 < -\frac{1}{2} + F(n,N_1) \\ &< \frac{\sqrt{6(n+2)(n+1)}}{12} + 2. \end{aligned} \tag{30}$$

Hence, $-(1/2) + F(n,N_1)$ cannot be an integer for any sufficiently large n, in particular for $n > 300000$.

Case II: $|X_2| > |X_1| = (1/3)(n+2)(n+1)n$
In this case we must have $|X_2| = (1/12)n(n+1)(n^2+n+10)$. Since X_1 is a tight spherical 7-design, X_1 is a 4-distance set. On the other hand X_2 is a 5-distance set. It is known that $A(X_1) = \{0, -1, \pm\sqrt{3/(n+4)}\}$, $\sqrt{(n+4)/3}$ is an integer. Let $\alpha_1 = -1$, $\alpha_2 = 0$, $\alpha_3 = \sqrt{3/(n+4)}$, $\alpha_4 = -\sqrt{3/(n+4)}$ and $\alpha_0 = 1$. By Proposition 2, we have $\gamma_1 = \sqrt{3n+6+\sqrt{6(n+2)(n+1)}}/\sqrt{(n+4)(n+2)}$, $\gamma_3 = \sqrt{3n+6-\sqrt{6(n+2)(n+1)}}/\sqrt{(n+4)(n+2)}$. Proposition 9.1 and Theorem 9.2 in [**5**] imply that (12) and (13) also hold in this case. Since $N_2 = |X_2| = (1/12)n(n+1)(n^2+n+10)$, we obtain $\beta_2 = \sqrt{(n+4)(n+2)(3n+\sqrt{6n^2-6n+24})}$ $/(n+4)(n+2)$ and $\beta_4 = \sqrt{(n+4)(n+2)(3n-\sqrt{6n^2-6n+24})}/(n+4)(n+2)$. Hence we have $(1-\beta_2^2)/(\beta_2^2-\beta_4^2) = -(1/2) + (n^2+3n+8)/2\sqrt{6n^2-6n+24}$. Therefore

$$-\frac{1}{2} + \frac{n^2+3n+8}{2\sqrt{6n^2-6n+24}}$$

is an integer. Then $24((n^2+3n+8)/2\sqrt{6n^2-6n+24})^2$ must be an integer. Since

$$24\left(\frac{n^2+3n+8}{2\sqrt{6n^2-6n+24}}\right)^2 = \frac{(n^2+3n+8)^2}{n^2-n+4} = n^2+7n+28+\frac{48(n-1)}{n^2-n+4}$$

there is no integer n satisfying the condition. This implies that for $n \geq 3$, there is no tight 9-design on two concentric spheres satisfying $N_1 = (n+2)(n+1)n/3$.

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