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Geometric decompositions of 4-dimensional orbifold bundles

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Abstract. We consider geometric decompositions of aspherical 4-manifolds which fibre over 2-orbifolds. We show that no such manifold admits infinitely many fibrations over hyperbolic base orbifolds and that "most" Seifert fibred 4-manifolds over hyperbolic bases have a decomposition induced from a decomposition of the base.

Introduction.

An n-manifold M admits a *geometric decomposition* if it has a finite collection $\mathscr{S}$ of disjoint connected 2-sided hypersurfaces such that each component of $M - \cup_{S\in\mathscr{S}} S$ is geometric of finite volume, i.e., is homeomorphic to $\Gamma\backslash X$, for some geometry $\boldsymbol{X}$ and lattice Γ. We shall call the hypersurfaces S *cusps* and the components of $M - \cup_{S\in\mathscr{S}} S$ *pieces* of M. The decomposition is *proper* if the set of cusps is nonempty.

We shall consider the possible geometric decompositions of aspherical orbifold bundles in dimension 4. A closed 4-manifold E is an *orbifold bundle* if there is an orbifold fibration $p : E \to B$ over a 2-dimensional base orbifold, with regular fibre F an aspherical surface. (Here "surface" shall mean closed 2-orbifold without exceptional points.) Let $\pi = \pi_1(E)$, $\phi = \pi_1(F)$ and $\beta = \pi_1^{orb}(B)$, and let $\theta : \beta \to Out(\phi)$ be the characteristic homomorphism (or monodromy).

We show first that if $\chi(E) > 0$ then E admits only finitely many orbifold fibrations. On the other hand, if B is the torus, $\chi(F) < 0$ and π/π' has rank at least 3 there are fibrations with fibre of arbitrarily large genus. In Section 2 we give an example to show that a 4-manifold which is finitely covered by a product of closed surfaces need not be an orbifold bundle. In Section 3 we give necessary and sufficient conditions for an aspherical orbifold bundle E with hyperbolic fibre to have one of the geometries $\boldsymbol{H}^2 \times \boldsymbol{E}^2$, $\boldsymbol{H}^3 \times \boldsymbol{E}^1$ or $\boldsymbol{H}^2 \times \boldsymbol{H}^2$. (These are easy extensions of results on bundle spaces from [**5**].) The only other possible geometry

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is $\boldsymbol{H}^4$; whether there are any such examples seems to be a difficult question. In Section 4 we constrain the possible geometries of pieces of a given orbifold bundle. In Section 5 and Section 6 we introduce the notions of (algebraically) horizontal and vertical decompositions, and show that no decomposition can have both algebraically horizontal and algebraically vertical cusps.

The main results are in Section 7 and Section 8. Although a Seifert fibred 4-manifold with hyperbolic base is geometric if and only if the monodromy is finite, we shall show that most such manifolds have a vertical decomposition into pieces of type $\boldsymbol{H}^2 \times \boldsymbol{E}^2$. Decompositions of type $\boldsymbol{F}^4$ are much more restricted; the Seifert fibred 4-manifolds admitting such decompositions are orientable, have general fibre the torus and the orientable cover of the base is a surface.

If G is a group G', ζG and $\sqrt{G}$ shall denote the commutator subgroup, centre and Hirsch-Plotkin radical of G, respectively. (In all the cases considered here $\sqrt{G}$ is the maximal normal nilpotent subgroup of G, and in many cases $\sqrt{G}$ is abelian.) Let T, Kb and T_g be the torus, Klein bottle and closed orientable surface of genus g, respectively, and let T_o be the once-punctured torus and P be the thrice-punctured sphere (the "pair of pants").

1. Bounding the orbifold fibrations of a given 4-manifold.

The orbifold bundles with flat fibre ($\chi(F) = 0$) are precisely the Seifert bundles in 4 dimensions. Every torsion-free group which is virtually an extension of a surface group by Z^2 arises in this way, and two such Seifert bundles are isomorphic if and only if their group extensions are equivalent [**12**]. The extension is in turn determined by the group π, since $\chi(B) < 0$ implies that ϕ is the unique maximal solvable normal subgroup of π. (Note that in [**12**] "Seifert bundle" is used to mean a codimension 2 foliation with all leaves compact, in other words, what we call an orbifold bundle here. Vogt gives also a corresponding result for orbifold bundles with $\chi(F) < 0$, subject to an additional arithmetic hypothesis which implies that ϕ is a characteristic subgroup.) Thus if E is Seifert fibred over a hyperbolic base, the Seifert fibration is essentially unique. If however E is an $\boldsymbol{H}^2 \times \boldsymbol{E}^2$-manifold, it may fibre over the torus in infinitely many ways, with fibre of arbitrarily high genus!

If π is a torsion-free extension of an aspherical 2-orbifold group β by a PD_2-group ϕ with $\chi(\phi) < 0$ then the extension is realized by an orbifold bundle p, and the bundle is determined up to bundle isomorphism by the group extension [**12**]. If moreover the action $\theta : \beta \to Out(\phi)$ has infinite image and nontrivial kernel then ϕ is unique and so p is determined by π. (See Theorems 5.5 and 7.3 of [**4**] and Theorem 5.3 of [**12**].) If θ has finite image, there is at most one other such subgroup, and π is the group of an $\boldsymbol{H}^2 \times \boldsymbol{H}^2$-manifold. (See Theorem 7.3 of

[**4**].) We shall show that any orbifold bundle space E with $\chi(E)$ nonzero has only finitely many orbifold fibrations.

Simple Euler characteristic considerations sometimes limit the possible singularities of the base orbifold.

LEMMA 1. *Let π be a torsion-free group which has a normal PD_2-subgroup ϕ with quotient β a 2-orbifold group. If $g \in \beta$ has finite order n then n divides $\chi(\phi)$.*

PROOF. Let H be the preimage in π of the cyclic subgroup of β generated by g. Then H is torsion free and $[H : \phi] = n$. Therefore H is a PD_2-group, and $\chi(\phi) = n\chi(H)$. $\square$

In particular, β is torsion free if $\chi(\phi) = -1$, while the base orbifold B has no reflector curves or cone points of order 2 if $\chi(\phi)$ is odd.

If an orbifold B has a d-fold cover which is a surface S then $\chi^{orb}(B) = \chi(S)/d$, and if moreover S is aspherical then $\chi^{orb}(B) = \chi^{virt}(\pi_1^{orb}(B))$.

THEOREM 2. *Let π be a torsion-free group which has a normal PD_2-subgroup ϕ with quotient a hyperbolic 2-orbifold group. If $\chi(\pi) > 0$, the set of such subgroups is finite.*

PROOF. Let $\mathscr{B}$ be the set of normal PD_2-subgroups $\phi \lhd \pi$ such that π/ϕ is a hyperbolic orbifold group. If $\phi \in \mathscr{B}$ then $\chi(\phi)\chi^{orb}(\pi/\phi) = \chi(\pi) > 0$. The smallest value of $|\chi^{orb}(\beta)|$ for β a hyperbolic orbifold is $|\chi^{orb}(S^2(2,3,7))| = 1/42$. Hence $0 < |\chi(\phi)| \leq 42\chi(\pi)$. Similarly, $|\chi^{orb}(\pi/\phi)| \leq \chi(\pi)$, since $|\chi(\phi)| \geq 1$. Hence there are only finitely many possible isomorphism classes of quotients. For each $\phi \in \mathscr{B}$ let d_ϕ be the least index of a torsion-free normal subgroup in π/ϕ. Then $d = lcm\{d_\phi \mid \phi \in \mathscr{B}\}$ is finite.

Let $\hat{\pi}$ be the intersection of all subgroups of π of index dividing d. This is a characteristic subgroup of finite index. If $\phi \in \mathscr{B}$ then $\hat{\pi}/\phi \cap \hat{\pi}$ is a PD_2-group. There are finitely many such normal subgroups of $\hat{\pi}$ [**8**]. If ψ is another such group and $\phi \cap \hat{\pi} = \psi \cap \hat{\pi}$, the image of ψ in π/ϕ is a finite normal subgroup, and so is trivial. Thus $\psi \leq \phi$, and hence $\psi = \phi$. Therefore $\mathscr{B}$ is finite. $\square$

If $\pi/\phi = Z^2$ and π/π' has rank 2 then ϕ is the unique normal PD_2-subgroup with quotient Z^2. We may adapt the work of [**3**] to see that there are infinitely many such subgroups ϕ when the rank is at least 3.

THEOREM 3. *Let E be a 4-manifold with a fibration $p : E \to T$ with fibre F a closed hyperbolic surface. If $\beta_1(E) > 2$ then there are such fibrations with fibre genus arbitrarily large.*

PROOF. Let $\phi = \mathrm{Ker}(p_*)$, and let $q : \pi = \pi_1(E) \to Z$ be an epimorphism with $\phi < \kappa = \mathrm{Ker}(q)$. Then $\kappa \cong \phi \rtimes Z$, and κ is the group of an aspherical 3-manifold. Restriction maps $H^1(\pi; Z) = Hom(\pi, Z)$ onto the subgroup H^{fix} of $H^1(\kappa; Z) = Hom(\kappa, Z)$ of homomorphisms invariant under conjugation by elements of π. This subgroup has rank at least 2, since π/π' has rank at least 3.

We may construct distinct epimorphisms $p_n : \kappa \to Z$ in H^{fix} such that $p_n \to p_*|_\kappa$ in $(\boldsymbol{R} \otimes H^{fix})/\boldsymbol{R}^\times$. Let $\tilde{p}_n : \pi \to Z$ be a homomorphism with $\tilde{p}_n|_\kappa = p_n$. Then $q_n = (\tilde{p}_n, q) : \pi \to Z^2$ is an epimorphism with kernel $\mathrm{Ker}(q_n) = \mathrm{Ker}(p_n)$. As in Theorem 4.2 of [**3**], for n large $\mathrm{Ker}(p_n)$ is finitely generated, and hence is a PD_2-group, by the coherence of κ, and $|\chi(\mathrm{Ker}(p_n))| \to \infty$ as n increases. This proves the theorem, since q_n may be realized by a fibration $f_n : E \to T$. $\square$

If E is an orientable $\boldsymbol{H}^2 \times \boldsymbol{E}^2$-manifold then $\pi = \pi_1(E)$ is virtually $\phi \times Z^2$ and E fibres over T if and only if $\pi' \cap \sqrt{\pi} = 1$.

2. Virtual bundle spaces.

A closed 4-manifold is a *virtual bundle space* if it has a finite regular covering space which fibres over a surface. If a torsion-free group π is virtually an extension of one surface group by another, is it the group of an aspherical 4-manifold? We may assume that π has a normal subgroup G which is an extension of a PD_2-group G/K by a normal PD_2-subgroup K. If K is characteristic in G (and hence normal in π) then π is the group of an orbifold bundle, by Theorem 7.3 of [**4**].

If $\chi(\pi) > 0$ and π is virtually a product, it is realized by some $\boldsymbol{H}^2 \times \boldsymbol{H}^2$-manifold, by Theorem 5 of [**5**], and has an index-2 subgroup which is an orbifold bundle group, but need not itself be such a group. Let G be a PD_2-group with $\zeta G = 1$ and let $\lambda : G \to Z$ be an epimorphism. Choose $x \in \lambda^{-1}(1)$, and let $K = \lambda^{-1}(2Z)$, so $y = x^2$ is in K. Then

$$\pi = \left\langle K \times K, t \mid t(k,l)t^{-1} = (xlx^{-1}, k) \ \forall\ (k,l) \in K \times K, t^4 = (y,y) \right\rangle$$

is torsion-free and has no normal subgroup which is a PD_2-group. (This corrects the example after Theorem 5 of [**5**], which has torsion.)

Our question remains open when $\chi(\phi) < 0$ and $\theta : \beta \to Out(\phi)$ is injective (type III of [**7**]). There are examples of this type in Section 14 of Chapter V of [**1**] with at least two such normal PD_2-groups. Must π have a characteristic PD_2-subgroup? Must there be at most two normal PD_2-subgroups with quotient an orbifold group?

3. Geometries on orbifold bundles with hyperbolic fibre.

Suppose that F is hyperbolic ($\chi(F) < 0$). The next result extends Theorem 8 of [**5**]. (The assertion regarding decompositions has been modified, as the argument involving splitting π/ϕ over $C/C \cap \phi$ is wrong. See the example following the theorem.)

Theorem 4. *Let E be the total space of an aspherical F-bundle over a 2-orbifold B with $\chi^{orb}(B) = 0$ and $\chi(F) < 0$. Then*

(1) *E admits the geometry $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ if and only if θ has finite image;*
(2) *E admits the geometry $\boldsymbol{H}^3 \times \boldsymbol{E}^1$ if and only if* $\mathrm{Ker}(\theta)$ *has two ends and* $\mathrm{Im}(\theta)$ *contains the class of a pseudo-Anasov homeomorphism of F;*
(3) *otherwise E is not geometric.*

If $\mathrm{Ker}(\theta) \neq 1$ *then E has a finite covering space with a geometric decomposition.*

Proof. Note that flat 2-orbifold groups are 2-dimensional crystallographic groups. In particular, a non-trivial normal subgroup of $\pi_1^{orb}(B)$ must be infinite, and so the hypotheses on θ are preserved on passage to subgroups of finite index.

The arguments of Theorem 8 of [**5**] extend to this situation. The only point that needs explanation is in showing that the algebraic conditions of part (2) suffice. Suppose that $\mathrm{Ker}(\theta)$ has two ends and $\mathrm{Im}(\theta)$ contains the class of a pseudo-Anasov homeomorphism ψ. Since $\mathrm{Im}(\theta) \cong \beta/\mathrm{Ker}(\theta)$ is virtually Z, it has a normal subgroup generated by the image of ψ^k for some $k \geq 1$. Let N be the mapping torus of ψ^k and $\nu = \pi_1(N)$. Then N is an $\boldsymbol{H}^3$-manifold and π has a normal subgroup of finite index isomorphic to $\nu \times Z$. Hence $\sqrt{\pi} \cong Z$, since $\sqrt{\nu} = 1$ and π is torsion-free. Since $\sqrt{\phi} = 1$ the image of $\sqrt{\pi}$ in β is an infinite cyclic normal subgroup. Since β has an infinite cyclic normal subgroup, its holonomy group has exponent 2. Therefore it has at least one other independent infinite cyclic normal subgroup. Thus there is a homomorphism $\lambda : \pi \to Isom(\boldsymbol{E}^1)$ with $\lambda(\sqrt{\pi}) \cong Z$, and the construction of the cited theorem may be carried through. □

Let $X(3_1)$ and $X(4_1)$ be the exteriors of the trefoil and figure-eight knots. The longitudes and meridians of the knots determine homeomorphisms $\partial X(3_1) \cong \partial X(4_1) \cong S^1 \times S^1$. Let $N_1 = X(3_1) \times S^1$ and $N_2 = X(4_1) \times S^1$, and let $f : \partial N_2 \cong \partial N_1$ be the homeomorphism which preserves the longitudinal coordinate but swaps the other two coordinates. Then $E = N_1 \cup_f N_2$ fibres over T with fibre T_2 and injective monodromy. This manifold is not geometric, but is the union of an $\boldsymbol{H}^2 \times \boldsymbol{E}^2$-manifold $X(3_1) \times S^1$ with an $\boldsymbol{H}^3 \times \boldsymbol{E}^1$-manifold $X(4_1) \times S^1$.

If, instead, we identify the boundaries of N_1 and N_2 so that the meridians and longitudes match, we obtain a bundle with monodromy generated by a reducible self-homeomorphism of T_2 and $\mathrm{Ker}(\theta) \cong Z$.

If B is also hyperbolic then $\chi(E) > 0$ and $\pi_1(E)$ has no solvable subgroups of Hirsch length 3. No such bundle space admits the geometry $\boldsymbol{H}^2(\boldsymbol{C})$, by Corollary 13.7.2 of [**4**]. Hence the only possible geometries on E are $\boldsymbol{H}^2 \times \boldsymbol{H}^2$ and $\boldsymbol{H}^4$. There are no known examples of $\boldsymbol{H}^4$-manifolds which are also bundle spaces.

THEOREM 5. *Let E be an aspherical orbifold bundle space with $\chi(E) > 0$. Then the following are equivalent*:

(1) *E admits the geometry $\boldsymbol{H}^2 \times \boldsymbol{H}^2$*;
(2) *E is finitely covered by a cartesian product of surfaces*;
(3) *θ has finite image.*

If E is an $\boldsymbol{H}^4$-manifold then θ is injective.

PROOF. The argument of Theorem 9 of [**5**] applies almost without change. □

The PD_2^+-group of genus 4 embeds in the mapping class group of genus 2, and so there are such bundle spaces with θ injective [**10**]. Are there infinitely many such with given base and fibre? In [**2**] it is shown that for any given surfaces B and F there are at most finitely many bundles for which π has no abelian subgroup of rank > 1. (For such groups θ must be injective.)

Such bundle spaces need not be geometric. Let $B = F = T_2$. Then B retracts onto $S^1 \vee S^1$. Mapping one generator of $F(2)$ to the involution τ which swaps the summands of F and the other to $c\tau c^{-1}$ where c is a Dehn twist gives rise to a bundle with base and fibre of genus 2 and $\mathrm{Im}(\theta) \cong D_\infty = Z/2Z * Z/2Z$. Thus $\mathrm{Im}(\theta)$ is infinite, but θ is not injective.

4. The possible pieces of a proper decomposition.

If $\chi^{orb}(B) = \chi(F) = 0$ then E has geometry $\boldsymbol{E}^4$, $\boldsymbol{Nil}^3 \times \boldsymbol{E}^1$, $\boldsymbol{Nil}^4$ or $\boldsymbol{Sol}^3 \times \boldsymbol{E}^1$, and has no proper geometric decomposition. Thus we may assume henceforth that F or B is hyperbolic.

THEOREM 6. *If an aspherical orbifold bundle space E has a proper decomposition then either*

(1) *$\chi^{orb}(B) = 0$, the pieces are $\boldsymbol{H}^3 \times \boldsymbol{E}^1$- or $\boldsymbol{H}^2 \times \boldsymbol{E}^2$-manifolds and the cusps are flat*; *or*
(2) *$\chi(F) = 0$, the pieces are $\boldsymbol{H}^2 \times \boldsymbol{E}^2$-manifolds and the cusps are flat*; *or*
(3) *$\chi(F) = 0$, the pieces are $\boldsymbol{F}^4$-manifolds and the cusps are $\boldsymbol{Nil}^3$-manifolds*; *or*
(4) *$\chi(E) > 0$, the pieces are reducible $\boldsymbol{H}^2 \times \boldsymbol{H}^2$-manifolds and the cusps are $\boldsymbol{H}^2 \times \boldsymbol{E}^1$-manifolds.*

PROOF. This follows from Theorem 1 of [**5**], with the following observations. Firstly, nonuniform $\widetilde{\boldsymbol{SL}}\times\boldsymbol{E}^1$-manifolds are also $\boldsymbol{H}^2\times\boldsymbol{E}^2$-manifolds, and vice versa [**6**].

Secondly, if $\chi(F)=0$ then $\sqrt{\phi}\cong Z^2$ is an abelian normal subgroup. Hence $\sqrt{\phi}$ is contained in the group of every cusp, and hence of every piece. Thus there can be no pieces of type $\boldsymbol{H}^3\times\boldsymbol{E}^1$.

Thirdly, if $\Gamma\backslash X$ is a piece of a geometric decomposition then $c.d.\Gamma=3$ and so $\phi\cap\Gamma\neq 1$. Hence if $\chi(F)<0$ we must have $\phi\cap\sqrt{\Gamma}=1$, and so $\phi\cap\Gamma$ centralizes $\sqrt{\Gamma}$. It follows that $\Gamma\backslash X$ cannot have type $\boldsymbol{F}^4$ if $\chi^{orb}(B)=0$.

Finally, if $\chi(E)\neq 0$ then π has no poly-Z subgroups of Hirsch length 3, and so we may eliminate pieces with geometry $\boldsymbol{H}^4$, $\boldsymbol{H}^2(\boldsymbol{C})$ or irreducible $\boldsymbol{H}^2\times\boldsymbol{H}^2$. Moreover, the inclusions of the cusps must be π_1-injective, since E is aspherical. Therefore no reducible $\boldsymbol{H}^2\times\boldsymbol{H}^2$ piece can be finitely covered by the product of two punctured surfaces, and so the cusps must be $\boldsymbol{H}^2\times\boldsymbol{E}^1$-manifolds. □

The conclusions of this theorem hold also for virtual bundle spaces.

Each of the first three possibilities may be realized by some closed 4-manifold which fibres over a surface. In the final case, when $\chi(E)>0$, any such manifold must be geometric, as we show next.

THEOREM 7. *Let M be an aspherical closed 4-manifold with a proper geometric decomposition with at least one piece which is a reducible $\boldsymbol{H}^2\times\boldsymbol{H}^2$-manifold. Then M is the total space of an orbifold bundle, and is itself a reducible $\boldsymbol{H}^2\times\boldsymbol{H}^2$-manifold.*

PROOF. Let $N=\Gamma\backslash\boldsymbol{H}^2\times\boldsymbol{H}^2$ be a piece of the decomposition which is a reducible $\boldsymbol{H}^2\times\boldsymbol{H}^2$-manifold. Then N is finitely covered by a product $F\times G$, where F and G are hyperbolic surfaces. One of the factors, G say, must be closed, since M is aspherical. Then $B=pr_2(\Gamma)\backslash\boldsymbol{H}^2$ is a closed $\boldsymbol{H}^2$-orbifold. Projection onto the second factor induces an orbifold bundle $p_N:N\to B$ with general fibre a closed surface and monodromy of finite order. The boundary components are $\boldsymbol{H}^2\times\boldsymbol{E}^1$-manifolds, and so have an essentially unique Seifert fibration. Hence the contiguous pieces are also reducible $\boldsymbol{H}^2\times\boldsymbol{H}^2$-manifolds, and are orbifold bundles over the same base B. Since M is connected, all pieces are of this type, and the projections p_N for the various pieces give rise to an orbifold bundle $p:M\to B$. The intersection of the kernels of the action of $\pi_1^{orb}(B)$ on the fundamental groups of the regular fibres of the pieces has finite index in $\pi_1^{orb}(B)$. Therefore M is homotopy equivalent to a reducible $\boldsymbol{H}^2\times\boldsymbol{H}^2$-manifold M_1, by Theorem 9.9 of [**4**]. We may arrange that the hypotheses of Theorem 4.1 of [**12**] hold, and so M is itself geometric. □

Conversely, every reducible $\boldsymbol{H}^2 \times \boldsymbol{H}^2$-manifold has a 2-fold covering space which is an orbifold bundle, by Corollary 9.8.1 of [**4**].

COROLLARY. *If an aspherical orbifold bundle space E with $\chi(E) > 0$ has a proper geometric decomposition then it is an $\boldsymbol{H}^2 \times \boldsymbol{H}^2$-manifold.* $\square$

5. Horizontal and vertical cusps.

A cusp S of a geometric decomposition of E is *horizontal* if it is transverse to all the fibres $F_b = p^{-1}(b)$. If the base is a surface then $p|_S$ is a submersion, and the leaf space of the foliation of S by the components of $S \cap F_b$ (for $b \in B$) is a surface which finitely covers B.

The following example shows that $S \cap F_b$ need not be connected. Let $M(\tau)$ be the mapping torus of the involution τ which swaps the summands of $F = T_2 = T\#T$. Then $E = M(\tau) \times S^1$ fibres over T with fibre F. Let C be a non-separating essential simple closed curve in T_o, and let $D = c \cup \tau(C)$. Then $M(\tau|_D) \times S^1$ is a cusp in E which meets each fibre in two circles.

A cusp S is *vertical* if it is a union of fibres. Thus $S = p^{-1}(A)$ for some 1-dimensional suborbifold $A \subset B$. Since S is connected, A must be either a circle S^1 or a reflector interval $\boldsymbol{I}$. Note that $\pi_1^{orb}(\boldsymbol{I})$ is the infinite dihedral group $D_\infty = Z/2Z * Z/2Z$.

LEMMA 8. *Let S be a cusp and $\sigma = \pi_1(S)$. Then*

(1) *either $\phi < \sigma$ or $\sigma \cap \phi \cong Z$*;
(2) *$\phi < \sigma \Leftrightarrow p_*\sigma$ has two ends*;
(3) *$\sigma \cap \phi \cong Z \Leftrightarrow [\beta : p_*\sigma]$ is finite*;
(4) *if S is horizontal then $\sigma \cap \phi \cong Z$*;
(5) *if S is vertical then $\phi < \sigma$, $p_*\sigma \cong Z$ or D_∞ and β splits over $p_*\sigma$.*

PROOF. The groups $\sigma \cap \phi$ and $p_*\sigma$ are infinite, since $c.d.\sigma = 3$ while $c.d.\phi = v.c.d.\beta = 2$. If $[\phi : \sigma \cap \phi]$ is finite then $\phi \leq \sigma$, since ϕ is a normal subgroup of a free product (or HNN extension) with amalgamation over σ. If $[\phi : \sigma \cap \phi]$ is infinite then $\sigma \cap \phi$ is free.

Suppose first that S is flat or is a $\boldsymbol{Nil}^3$-manifold. Then σ is virtually poly-Z, of Hirsch length $h(\sigma) = 3$. Since $h(\sigma) = h(\sigma \cap \phi) + h(p_*\sigma)$, either $h(\sigma \cap \phi) = 1$ and $h(p_*\sigma) = 2$, in which case $\sigma \cap \phi \cong Z$ and $[\beta : p_*\sigma]$ is finite, or $h(\sigma \cap \phi) = 2$ and $h(p_*\sigma) = 1$, in which case $\phi < \sigma$ and $p_*\sigma$ has two ends.

Otherwise, S is an $\boldsymbol{H}^2 \times \boldsymbol{E}^1$-manifold, and $\sqrt{\phi} \cong Z$ is centralized by a subgroup of index at most 2 in σ. In this case F and B are hyperbolic, and so centralizers in ϕ are cyclic, while centralizers in β are finite or have two ends. Therefore if $\sqrt{\sigma} \cap \phi \neq 1$ then $\sqrt{\sigma} \cap \phi \cong Z$. Hence $p_*\sigma$ is virtually a PD_2-group, and so $[\beta : p_*\sigma]$ is finite. If $\sqrt{\sigma} \cap \phi = 1$ then $p_*\sigma$ has two ends, and $\sigma \cap \phi$ is a PD_2-group. Hence $[\phi : \sigma \cap \phi]$ is finite, and so $\phi < \sigma$.

The final two implications are clear. □

We shall say that S is *algebraically horizontal* if $\sigma \cap \phi \cong Z$, and that it is *algebraically vertical* if $\phi \leq \sigma$. It is clear from the lemma that these possibilities are disjoint and exhaustive. Are there cusps which are neither horizontal nor vertical (up to isotopy)?

LEMMA 9. *Let S be an algebraically horizontal cusp. Then the bundle projection induced over some finite covering of the base admits a section. In particular, π is virtually a semidirect product.*

PROOF. Since S is an algebraically horizontal cusp, it is flat (if $\chi(E) = 0$) or is an $\boldsymbol{H}^2 \times \boldsymbol{E}^1$-manifold (if $\chi(E) \neq 0$). After passing to a finite covering, if necessary, we may assume that B is a surface, $S \cong B \times S^1$ and $p|_S$ is homotopic to the projection $pr_1 : S \to B$. Since pr_1 has an obvious section, there is a section $s : B \to E$ with image contained in S, by the homotopy lifting property. It follows that π is a semidirect product. □

If π is a semidirect product then β is torsion-free, and so B is a surface. Let $B = B_1 \cup D^2$, where $B_1 = \vee^r S^1$. We may construct a partial section for p over B_1 which realizes a given splitting : $\beta \to \pi$ for p_*. It is then easy to extend this to a section over B.

If $\chi(\phi) < 0$ then $\zeta\phi = 1$ and so π is a semidirect product if and only if θ factors through $Aut(\phi)$. (A perhaps more transparent necessary condition is that π/ϕ' must be virtually a semidirect product.)

6. Horizontal and vertical decompositions.

A geometric decomposition of an orbifold bundle space E is horizontal or vertical if all the cusps are horizontal or vertical, respectively. It follows immediately from Theorem 6 that if E is an orbifold bundle with $\chi(E) = 0$ and $\chi(F) < 0$ then E has no vertical decomposition, while if $\chi(E) = 0$ and $\chi^{orb}(B) < 0$ it has no horizontal decomposition. Some bundle spaces (such as direct products $B \times F$) may admit both horizontal and vertical decompositions. However no decomposition can involve both types.

LEMMA 10. *No geometric decomposition of an aspherical bundle space E has both algebraically horizontal and algebraically vertical cusps.*

PROOF. If $\chi(E) = 0$ then either $\chi^{orb}(B) = 0$ and every cusp is algebraically horizontal or $\chi(F) = 0$ and every cusp is algebraically vertical. Suppose that $\chi(E) > 0$ and S is an algebraically horizontal cusp. Then S is an $\boldsymbol{H}^2 \times \boldsymbol{E}^1$-manifold. After passing to a finite covering, if necessary, we may assume that there is a section $s : B \to E$ with image contained in S, by Lemma 9. Clearly $s(B) \cap F_b = s(b)$, for all $b \in B$ and so $s_*[B] \bullet [F] \neq 0$. (Here it suffices to use $\boldsymbol{F}_2$ coefficients.) Thus it is not possible to homotope F off $s(B)$. In particular S must meet every algebraically vertical cusp. $\square$

A geometric decomposition of a manifold M determines a graph of groups $\mathscr{G}$ whose underlying graph has vertices the set of pieces and edges the set of cusps, and for which the vertex and edge groups are the fundamental groups of these spaces. There is an epimorphism $\pi\mathscr{G} \to \pi_1(M)$, by van Kampen's theorem, and this is an isomorphism if M is aspherical. If E is an orbifold bundle with a geometric decomposition in which the cusps are algebraically vertical then ϕ is a normal subgroup of each vertex and edge group, and the quotients determine another graph of groups $\overline{\mathscr{G}}$ with the same underlying graph. Clearly $\beta = \pi/\phi \cong \pi\overline{\mathscr{G}}$. The following lemma is a partial converse. I am grateful to Peter Scott for the argument.

Lemma 11. *Let $\mathscr{G}$ be a finite graph of groups and $f : \beta \to \pi\mathscr{G}$ a homomorphism. Then B has a corresponding decomposition along a 1-dimensional suborbifold.*

Proof. Let M be a finite regular covering of B which is a closed surface, and let $H = Aut(M/B)$. Then there is a β-equivariant map from $\widetilde{M} = \widetilde{B}$ to a β-tree $\mathscr{T}$ corresponding to the splitting. This induces a H-equivariant map from M to $\pi_1(M)\backslash\mathscr{T}$. Using Stallings' method of "binding ties", we may construct a H-equivariant homotopy of this map to one for which the preimage of each edge of $\pi_1(M)\backslash\mathscr{T}$ is a single closed curve in M. This projects to a 1-orbifold in B which induces the given splitting. $\square$

If C is a 1-submanifold of T_g such that all the complementary components have $\chi < 0$ then $\beta_0(C) \leq 3g - 3$, with equality if and only if the complementary components are all copies of P. It follows easily that any bundle space admits only finitely many vertical geometric decompositions, up to bundle equivalence induced by a self-homeomorphism of the base.

Let π be the group with presentation

$$\langle a, b, c, d, e, f, x, y \mid [a,b][c,d][e,f] = 1,\ xax^{-1} = ab,\ ycy^{-1} = cd,$$
$$[x,y] = e,\ xc = cx,\ ya = ay,\ x, y \leftrightharpoons b, d, e, f \rangle.$$

Then π is the group of an orientable 4-manifold E which fibres over T with fibre of genus 3. Since θ is injective, E is not geometric. The cusps in any geometric decomposition of a bundle space with flat base are infrasolvmanifolds, and so cannot be algebraically vertical. Since no subgroup of finite index in $\beta = Z^2$ admits, a section E has no algebraically horizontal cusps, and hence E has no geometric decomposition.

7. $\boldsymbol{H}^2 \times \boldsymbol{E}^2$-decompositions of Seifert fibred 4-manifolds.

All Seifert fibred 4-manifolds over flat bases are geometric, of type $\boldsymbol{E}^4$, $\boldsymbol{Nil}^4$, $\boldsymbol{Nil}^3 \times \boldsymbol{E}^1$ or $\boldsymbol{Sol}^3 \times \boldsymbol{E}^1$. If the base is hyperbolic then the 4-manifold is geometric if and only if the monodromy has finite order. (See [**9**], [**11**] or Chapter 9 of the 2007 revision of [**4**].) No such Seifert fibred 4-manifold has a horizontal decomposition. However "most" have vertical decompositions, as we show next.

We must ensure that the base has a suitable decomposition into hyperbolic pieces. A 2-orbifold with nonempty boundary is hyperbolic unless it is a disc with one cone point ($D(p)$) or with two cone points of order 2 ($D(2,2)$), or is the quotient of one of these by reflection across a diameter through the cone point(s) ($\overline{D(p)}$ or $\overline{D(2,2)}$) or separating the cone points ($\overline{D}(2)$).

Lemma 12. *Let E be Seifert fibred over an aspherical base B. If $g \in \pi$ has image $p_*g \in \beta$ of finite order n then $\theta(p_*g) = 1$ unless $n = 2$ and $\det(\theta(p_*g)) = -1$.*

Proof. If $g \neq 1$ and p_*g has order n then g^n is a nontrivial element of ϕ which is fixed by $\theta(p_*g)$. Since $\theta(p_*g)$ has determinant 1, both eigenvalues are 1, and since it has finite order, $\theta(p_*g) = I$. □

Theorem 13. *Let E be Seifert fibred over a hyperbolic base B. Then E is an $\boldsymbol{H}^2 \times \boldsymbol{E}^2$-manifold if $F = Kb$, and otherwise has a vertical decomposition of type $\boldsymbol{H}^2 \times \boldsymbol{E}^2$, unless B or its orientable cover has a cone point of order 2 at which the action reverses the orientation of the fibre.*

Proof. If $F = Kb$ then $Out(\phi)$ is finite. Hence E is an $\boldsymbol{H}^2 \times \boldsymbol{E}^2$-manifold. Moreover all pieces in any proper geometric decomposition are of type $\boldsymbol{H}^2 \times \boldsymbol{E}^2$.

If $F = T$ then $Out(\phi) = GL(2, \boldsymbol{Z})$ is virtually free. Therefore $\mathrm{Im}(\theta) = \pi\mathscr{G}$ where $\mathscr{G}$ is a finite graph of finite groups. If $\mathrm{Im}(\theta)$ is finite then E is an $\boldsymbol{H}^2 \times \boldsymbol{E}^2$-manifold. Otherwise B has a proper decomposition along a 1-dimensional suborbifold into pieces $\{B_1, \dots, B_n\}$ on which θ has finite image, by Lemma 11. We may clearly assume this decomposition is minimal, and that the orientable cover of B has no cone point of order 2, at which the action has eigenvalues $\{1, -1\}$. If there were a piece $B_i \cong D(p)$ or $\overline{D(p)}$, then adjoining B_i to the contiguous piece B_j would merely add a relation to $\pi_1^{orb}(B_j)$, and so $\theta(\pi_1^{orb}(B_i \cup B_j))$ would still be finite. Similarly, if there were a piece $B_k \cong D(2,2)$, $\overline{D(2,2)}$ or $\overline{D}(2)$ then we may annex B_k to its neighbour while retaining this finiteness since the action at each cone point of the orientable cover is trivial. Thus we may assume that there are no such pieces in the decomposition of B. Hence each piece B_i is hyperbolic. The corresponding pieces of E are $\boldsymbol{H}^2 \times \boldsymbol{E}^2$-manifolds. $\square$

The argument simplifies if $\mathrm{Im}(\theta) \leq SL(2,\boldsymbol{Z})$. For then θ factors through the fundamental group of the (possibly bounded) surface underlying the base orbifold B, by Lemma 12.

The condition on cone points of order 2 can be weakened, but some such condition is needed, as the following example shows. Let π be the group with presentation

$$\langle a,b,v,w,x,y,z \mid ab=ba,\ v^2=w^2=x^2=a,\ vbv^{-1}=wbw^{-1}=b^{-1},$$
$$xb=bx,\ y^2=z^2=ab,\ yay^{-1}=zaz^{-1}=ab^2,$$
$$yby^{-1}=zbz^{-1}=b^{-1},\ vwx=yz\rangle.$$

The images of a and b generate a normal subgroup $A \cong Z^2$, with quotient $\beta = \pi/A \cong \pi_1^{orb}(B)$, where B is the hyperbolic 2-orbifold $S^2(2,2,2,2,2)$. Let G,H,J,K,L,P be the subgroups generated by $\{A,v\}$, $\{A,w\}$, $\{A,x\}$, $\{A,y\}$, $\{A,z\}$ and $\{A,vwx\}$, respectively. Then

$$\pi \cong (G *_A H *_A K) *_P (K *_A L).$$

These subgroups are each torsion-free, and hence so is π. Therefore $\pi = \pi_1(E)$, where E is a Seifert manifold with base B and regular fibre T, by Theorem 7.4 of [**4**]. The action θ of $\pi_1^{orb}(B)$ on $\sqrt{\pi} = A$ is generated by $\theta(u) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\theta(x) = \left(\begin{smallmatrix} 1 & 0 \\ 2 & -1 \end{smallmatrix}\right)$. Hence $\mathrm{Im}(\theta) \cong Z/2Z * Z/2Z$ is infinite and has infinite index in $GL(2,\boldsymbol{Z})$. Thus E is not geometric and has no pieces of type $\boldsymbol{F}^4$. On the other hand B has no proper decomposition into hyperbolic pieces, and so E has no geometric decomposition at all. Are there any other such examples?

Most $\boldsymbol{H}^2 \times \boldsymbol{E}^2$-manifolds also admit proper vertical decompositions. The exceptions have base orbifold with no proper geometric decomposition. (These have orientable cover $S^2(2,2,2,2,2)$ or $S^2(2,2,2,s)$, for some $s > 2$, or $S^2(p,q,r)$, for some $\{p,q,r\}$ such that $(1/p)+(1/q)+(1/r) < 1$.)

There is an analogous result for orbifold bundles E with $\chi(E) > 0$. If the base B has at most one cone point of order 2 then E has a vertical decomposition (with pieces reducible $\boldsymbol{H}^2 \times \boldsymbol{H}^2$-manifolds) if and only if θ factors through a virtually free group. If $D(2,2)$ is a suborbifold of B, there are orbifold bundles with base B which have no vertical decomposition.

8. $\boldsymbol{F}^4$-decompositions of Seifert fibred 4-manifolds.

The Seifert fibred 4-manifolds with decompositions of type $\boldsymbol{F}^4$ are much more restricted.

We shall first recall the salient properties of the geometry $\boldsymbol{F}^4$. It has model space $\boldsymbol{R}^2 \times \boldsymbol{H}^2$, and isometry group the semidirect product $Isom(\boldsymbol{F}^4) = \boldsymbol{R}^2 \times_\alpha SL^{\pm}(2,\boldsymbol{R})$, where $SL^{\pm}(2,\boldsymbol{R})$ is the subgroup of $GL(2,\boldsymbol{R})$ consisting of matrices of determinant ± 1, and α is the natural action of $SL^{\pm}(2,\boldsymbol{R})$ on $\boldsymbol{R}^2$. This group acts on $\boldsymbol{R}^2 \times \boldsymbol{H}^2$ as follows: if $u \in \boldsymbol{R}^2$ and $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\boldsymbol{R})$ then $u(v,z) = (u+v,z)$ and $A(v,z) = (Av,(az+b)/(cz+d))$ for all $(v,z) \in \boldsymbol{R}^2 \times \boldsymbol{H}^2$. The matrix $D = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ acts via $D(w,z) = (Dw,-\bar{z})$ for all $(v,z) \in \boldsymbol{R}^2 \times \boldsymbol{H}^2$.

If Γ is an $\boldsymbol{F}^4$-lattice then $\sqrt{\Gamma} = \Gamma \cap \boldsymbol{R}^2 \cong \boldsymbol{Z}^2$. The quotient $\Gamma/\sqrt{\Gamma}$ is a subgroup of $SL^{\pm}(2,\boldsymbol{R})$ which preserves the lattice $\sqrt{\Gamma}$, and so is conjugate into $GL(2,\boldsymbol{Z})$. Let $\Gamma_0 = \Gamma \cap Isom(\boldsymbol{F}^4)_0$ be the intersection with the identity component of the isometry group. Then $\Gamma_0/\sqrt{\Gamma}$ acts discretely with finite covolume on $\boldsymbol{H}^2$ through its image in $PSL(2,\boldsymbol{R})$. All $\boldsymbol{F}^4$-manifolds are orientable. (See [**13**] for more details.)

Lemma 14. *Let Γ be an $\boldsymbol{F}^4$-lattice. Then $\Gamma_0/\sqrt{\Gamma}$ is torsion-free.*

Proof. Suppose that $g \in \Gamma_0$ has image $[g]$ in $SL(2,\boldsymbol{R})$. If $[g]^n = 1$ then $g^n \in \sqrt{\Gamma}$ is an eigenvector for $[g]$, and so the eigenvalues are ± 1. Since $\det[g] = 1$ it follows that $[g] = \pm I_2$. Since Γ is torsion-free, we must have $[g] = 1$. $\square$

Thus if the base orbifold is orientable, it is a surface. If it is not orientable, it may have reflector curves. For example, let $\{e,f\}$ be the standard basis for $\boldsymbol{Z}^2$, and let Γ be the subgroup of $Isom(\boldsymbol{F}^4)$ generated by $\boldsymbol{Z}^2 \rtimes SL(2,\boldsymbol{Z})'$ and $\delta = ((1/2)e, D)$. Then Γ is an $\boldsymbol{F}^4$-lattice, $\Gamma_0 = \boldsymbol{Z}^2 \rtimes SL(2,\boldsymbol{Z})'$ and $\Gamma/\sqrt{\Gamma} \cong F(2) \rtimes Z/2Z$ has 2-torsion. The base orbifold is the quotient of T_o by a reflection, since $SL(2,\boldsymbol{Z})' \backslash \boldsymbol{H}^2 \cong T_o$. (We could use any torsion-free subgroup of finite index in $SL(2,\boldsymbol{Z})$ which is normalized by D, instead of $SL(2,\boldsymbol{Z})'$, to obtain further examples.)

Theorem 15. *Let E be Seifert fibred over a hyperbolic base B. Suppose that E has a decomposition of type $\boldsymbol{F}^4$. Then*

(1) *E is orientable*;
(2) *$d = [GL(2,\boldsymbol{Z}) : \mathrm{Im}(\theta)]$ is finite*;
(3) *$24\chi^{orb}(B)$ is a nonzero integer which is divisible by d*;
(4) *the orientable cover of B is a surface*;
(5) *if $B = T_g$ and $\mathrm{Im}(\theta) \cong F(r)$ then $r \le g$, $r-1$ divides $2(g-1)$ and the number of pieces of the decomposition is at most $(2(g-1))/(r-1)$.*

Proof. The geometric pieces of E are orientable and their boundary components are Nil^3-manifolds. Self-homeomorphisms of Nil^3-manifolds are orientation-preserving, by Theorem 8.5 of [**4**]. Therefore E must be orientable.

Let $\beta \cong \overline{\mathscr{G}}$ be the representation of β as a graph of groups induced by the decomposition of E. The edge groups have two ends and the vertex groups are the groups of non-compact hyperbolic orbifolds of finite area, and so are virtually free. The action θ embeds each such vertex group G_v as a subgroup of finite index d_v in $GL(2, \boldsymbol{Z})$. Hence

$$\chi^{virt}(G_v) = d_v \chi^{virt}(GL(2, \boldsymbol{Z})) = \frac{1}{24} d_v \chi(SL(2, \boldsymbol{Z})') = -\frac{1}{24} d_v.$$

Hence $24\chi^{orb}(B) = 24\chi^{virt}(\beta) = 24\Sigma\chi^{virt}(G_v) = -\Sigma d_v$. Since $d = [GL(2, \boldsymbol{Z}) : \mathrm{Im}(\theta)]$ divides d_v for each vertex v, it divides $24\chi^{orb}(B)$.

The orientation preserving subgroups of the vertex groups are torsion-free, by Lemma 14, and so the orientable cover of B is nonsingular.

Suppose that $B = T_g$. If $f : \beta \to F(r)$ is a surjection then $H^1(f; \boldsymbol{F}_2)$ is injective, and the image is self-annihilating under the cup product pairing into $H^2(\beta; \boldsymbol{F}_2)$. Therefore $r \leq g$. The remaining assertions follow easily from the equation

$$2(g-1) = |\chi(B)| = \Sigma|\chi(G_v)| = (\Sigma[F(r) : \theta(G_v)])(r-1). \qquad \square$$

In particular, every such manifold also has vertical decompositions of type $\boldsymbol{H}^2 \times \boldsymbol{E}^2$. If the necessary conditions of the final part of Theorem 15 hold, must E have such a decomposition?

If $r \leq g$ then $\pi_1(T_g)$ maps onto $F(r)$, and so there are T-bundles over T_g with $\mathrm{Im}(\theta) \cong F(r)$ and of finite index in $SL(2, \boldsymbol{Z})$. However if $g = 5$ and $r = 4$ no such bundle has a decomposition of type $\boldsymbol{F}^4$, since $r - 1 = 3$ does not divide $2(g-1) = 8$.

If $B = T_g$ and $\mathrm{Im}(\theta) \cong F(g)$ any such decomposition of E is induced by a partition $B = B_1 \cup B_2$, with $\chi(B_1) = \chi(B_2) = 1 - g$. Since B_1 and B_2 are each orientable and have a common boundary, they are homeomorphic. The number of boundary components lies between 1 and $g+1$, and is congruent to $g + 1$ *mod* (2). Hence there are $[g/2] + 1$ essentially distinct partitions of the base into two homeomorphic pieces. The two possibilities with $g = 2$ are illustrated by Examples 1 and 2 of Section 3 of [**5**]. We may use these to construct examples with base of genus $g > 2$ and $\mathrm{Im}(\theta) \cong F(2)$, by pulling back such examples over suitable $Z/(g-1)Z$-coverings of the base T_2.

In general, $\mathrm{Im}(\theta)$ may have torsion. Let $X = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$, $U = -X$ and

$V = -Y$. Then $\{X, Y\}$ and $\{U, V\}$ each freely generate subgroups of finite index in $SL(2, \boldsymbol{Z})$, and $UV = XY$. The image of each of these subgroups in $PSL(2, \boldsymbol{Z})$ is the level-2 congruence group $\Gamma_2 \cong F(2)$, which acts on $\boldsymbol{H}^2$ with quotient $\Gamma_2 \backslash \boldsymbol{H}^2 \cong intP$. (See Example 2 of Section 3 of [**5**].) In fact $\pi_1(P) = \langle x, y, z \mid z = xy \rangle$, where x, y, z correspond to the boundary components. The standard integer lattice $\boldsymbol{Z}^2$ and the matrices X and Y (or U and V) together generate an $\boldsymbol{F}^4$-lattice. Let E_+ and E_- be the corresponding T-bundle spaces over P, with $\mathrm{Im}(\theta)$ generated by $\{X, Y\}$ and $\{U, V\}$, respectively, and whose interiors are $\boldsymbol{F}^4$-manifolds. We may then assemble two copies of E_+ and two copies of E_- into a T-bundle over T_3, by first doubling E_+ along the boundary components corresponding to x and y and E_- along the sides corresponding to u and v, and then identifying the remaining boundary components. The monodromy of the resulting bundle is generated by $\{X, Y, U, V\}$ and hence by $\{X, Y, -I_2\}$, and so has nontrivial torsion.

Since T_2 retracts onto $S^1 \vee S^1$ we may define a T-bundle with base T_2 and $\mathrm{Im}(\theta) = SL(2, \boldsymbol{Z})$ by mapping one generator of $F(2)$ to the involution $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and the other to $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. This 4-manifold surely has no decomposition of type $\boldsymbol{F}^4$.

Manifolds with $\boldsymbol{F}^4$-decompositions need not be Seifert fibred. There is an $\boldsymbol{F}^4$-manifold which is Seifert fibred over the once-punctured torus T_o, corresponding to the isomorphisms $\pi_1(T_o) \cong F(2) \cong SL(2, \boldsymbol{Z})'$. On passing to a suitable 2-fold covering of the base, we obtain a manifold which is Seifert fibred over the twice-punctured torus, with homeomorphic cusps. On passing to a further 2-fold covering, we obtain an $\boldsymbol{F}^4$-manifold X which is Seifert fibred over the twice-punctured pretzel surface B, and such that the two cusps are homeomorphic to the same Nil-coset space. Such a coset space admits self-homeomorphisms which do not preserve the fibration over S^1 induced by the Seifert fibration of X over B. The closed 4-manifold obtained by identifying the cusps via such a homeomorphism is not Seifert fibred.

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