

Good filtrations and F -purity of invariant subrings

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Abstract. Let k be an algebraically closed field of positive characteristic, G a reductive group over k , and V a finite dimensional G -module. Let B be a Borel subgroup of G , and U its unipotent radical. We prove that if $S = \text{Sym } V$ has a good filtration, then S^U is F -pure.

Throughout this paper, p denotes a prime number. Let k be an algebraically closed field of characteristic p , and G a reductive group over k . Let B be a Borel subgroup of G , and U its unipotent radical. We fix a maximal torus T contained in B , and fix a base of the root system Σ of G so that B is negative. For any weight $\lambda \in X(T)$, we denote the induced module $\text{ind}_B^G(\lambda)$ by $\nabla_G(\lambda)$. We denote the set of dominant weights by X^+ . For $\lambda \in X^+$, we call $\nabla_G(\lambda)$ the dual Weyl module of highest weight λ . We say that a G -module W has a good filtration [1] if $H^1(G, W \otimes \nabla_G(\lambda)) = 0$ for any $\lambda \in X^+$.

Let V be a finite dimensional G -module, and $S = \text{Sym } V$. The objective of this paper is to prove the following.

THEOREM 1. *If S has a good filtration, then S^U is F -pure.*

For a commutative ring R and an R -linear map $\varphi : M \rightarrow N$ of R -modules, we say that φ is pure or R -pure if $1_W \otimes \varphi : W \otimes_R M \rightarrow W \otimes_R N$ is injective for any R -module W . We say that φ is a split mono if there is an R -linear map $\psi : N \rightarrow M$ such that $\psi\varphi = 1_M$. A split mono is pure. A pure map $M \rightarrow N$ is a split mono if M/N is finitely presented [8, (5.2)]. A ring homomorphism $\varphi : R \rightarrow R'$ is said to be pure (resp. split) if it is pure (resp. a split mono) as an R -linear map. A commutative ring R of characteristic p is said to be F -pure (resp. F -finite) if the Frobenius map $F_R : R \rightarrow R$ given by $a \mapsto a^p$ is pure (resp. finite) as a ring homomorphism. F -purity was defined by Hochster–Roberts in 1970’s [7], [8]. This notion is deeply connected with log-canonical singularity in characteristic zero [12]. An F_p -scheme X is said to be Frobenius split if $\mathcal{O}_{X^{(1)}} \rightarrow F_*\mathcal{O}_X$ splits as an $\mathcal{O}_{X^{(1)}}$ -linear map [10]. For the notation $X^{(1)}$, see [9, (I.9.2)]. For an F -finite

ring R of characteristic p , R is F -pure if and only if $\operatorname{Spec} R$ is Frobenius split. If a nonnegatively graded F -finite noetherian ring R of characteristic p is F -pure, then $\operatorname{Proj} R$ is Frobenius split.

An F -finite noetherian ring R of characteristic p is said to be strongly F -regular if for any nonzerodivisor a of R , the $R^{(e)}$ -linear map $aF^e : R^{(e)} \rightarrow R$ ($x \mapsto ax^{p^e}$) is $R^{(e)}$ -split [4]. A strongly F -regular F -finite ring is F -regular in the sense of Hochster–Huneke [5], and hence it is Cohen–Macaulay normal ([6, (4.2)] and [11, (0.10)]). Under the same assumption as in Theorem 1, S^G is strongly F -regular [3], and in particular, Cohen–Macaulay.

PROOF OF THEOREM 1. Let $\Gamma \rightarrow [G, G]$ be the universal covering. Then S has a good filtration as a Γ -module by [1, (3.1.3), (3.2.7)], and U is identified with the unipotent radical of a Borel subgroup of Γ . Thus without loss of generality, we may assume that G is semisimple and simply connected.

Let ρ denote the half sum of positive roots. Let St denote the first Steinberg module $\nabla_G((p-1)\rho)$. For a G -module W and $r \geq 0$, $W^{(r)}$ denotes the r th Frobenius twist of W [9, (I.9.10)]. Note that $W^{(r)} = W$ as an abelian group. $w \in W$, viewed as an element of $W^{(r)}$ is sometimes denoted by $w^{(r)}$. If R is a G -algebra, then $R^{(r)}$ is also a G -algebra, and the r th Frobenius map $F^r : R^{(r)} \rightarrow R$ is a G -algebra map [3].

LEMMA 2. *There is a $(G, S^{(1)})$ -linear splitting of the Frobenius map $1 \otimes F_S : St \otimes S^{(1)} \rightarrow St \otimes S$ given by $x \otimes s^{(1)} \mapsto x \otimes s^p$.*

PROOF. The same proof as that of (4) in the proof of [3, Theorem 6] works ($r = 1$ and $d = 0$ there). \square

Let $C := k[G]^U$, where U acts on $k[G]$ right regularly. Then $C = \bigoplus_{\lambda \in X^+} \nabla_G(\lambda)$, and it is an X^+ -graded G -algebra.

LEMMA 3. *There is a $(G, C^{(1)})$ -linear splitting of the Frobenius map $1 \otimes F_C : St \otimes C^{(1)} \rightarrow St \otimes C$ given by $x \otimes c^{(1)} \mapsto x \otimes c^p$.*

PROOF. The product $St \otimes \nabla_G(\lambda)^{(1)} \rightarrow \nabla_G(p(\lambda + \rho) - \rho)$ is nonzero, since for $x \in St \setminus 0$ and $y \in \nabla_G(\lambda) \setminus 0$, $xy^p \neq 0$, as C is an integral domain. Since $St \otimes \nabla_G(\lambda)^{(1)} \cong \nabla_G(p(\lambda + \rho) - \rho)$ [9, (II.3.19)] and $\operatorname{End}_G(\nabla_G(\mu)) \cong k$ for each $\mu \in X^+$, the product $St \otimes \nabla_G(\lambda)^{(1)} \rightarrow \nabla_G(p(\lambda + \rho) - \rho)$ is an isomorphism. This shows that the product $m : St \otimes C^{(1)} \rightarrow C_{(p-1)\rho+pX^+}$ is an isomorphism, where for a subset Λ of X^+ , $C_\Lambda := \bigoplus_{\lambda \in \Lambda} \nabla_G(\lambda)$.

Thus

$$St \otimes C \xrightarrow{\pi} St \otimes C_{pX^+} \xrightarrow{m'} C_{(p-1)\rho+pX^+} \xrightarrow{m^{-1}} St \otimes C^{(1)}$$

is the splitting of $1 \otimes F_C$, where π is the projection, and m' is the product. \square

LEMMA 4. *There exists some $(G, (S \otimes C)^{(1)})$ -linear splitting of the Frobenius map $1 \otimes F : St \otimes (S \otimes C)^{(1)} \rightarrow St \otimes S \otimes C$.*

PROOF. Follows immediately from Lemma 2 and Lemma 3. \square

PROOF OF THEOREM 1 (CONTINUED). Let $\Phi : St \otimes S \otimes C \rightarrow St \otimes (S \otimes C)^{(1)}$ be a $(G, (S \otimes C)^{(1)})$ -linear splitting of $1 \otimes F$ as in Lemma 4. Set $A := (S \otimes C)^G$, and consider the commutative diagram of $(G, A^{(1)})$ -modules

$$\begin{array}{ccccc} St \otimes A^{(1)} & \hookrightarrow & St \otimes (S \otimes C)^{(1)} & \xrightarrow{\text{id}} & St \otimes (S \otimes C)^{(1)} \\ \downarrow 1 \otimes F & & \downarrow 1 \otimes F & \nearrow \Phi & \\ St \otimes A & \hookrightarrow & St \otimes (S \otimes C) & & \end{array}$$

Applying the functor $\text{Hom}_G(St, ?)$ to this, we get a commutative diagram

$$\begin{array}{ccccc} A^{(1)} & \xrightarrow{\text{id}} & A^{(1)} & \xrightarrow{\text{id}} & A^{(1)} \\ \downarrow F & & \downarrow & \nearrow & \\ A & \longrightarrow & \text{Hom}_G(St, St \otimes (S \otimes C)) & & \end{array}$$

of $A^{(1)}$ -modules, see [3, Proposition 1, 5]. Thus $F : A^{(1)} \rightarrow A$ splits, and $A = (S \otimes C)^G$ is F -pure.

Finally, as in the proof of [2, (1.2)], $A = (S \otimes C)^G \cong S^U$ ($X = \text{Spec } S$ and $H = U$ there). Thus S^U is F -pure. \square

COROLLARY 5. *Under the same assumption as in Theorem 1, $\text{Proj } S^U$ is Frobenius split.*

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