J. Math. Soc. Japan
Vol. 63, No. 3 (2011) pp. 801–814
doi: 10.2969/jmsj/06330801

# Homotopy self-equivalences of 4-manifolds with $\pi_1$-free second homotopy

By Mehmetcik Pamuk

(Received Feb. 23, 2010)

**Abstract.** We calculate the group of homotopy classes of homotopy self-equivalences of 4-manifolds with $\pi_1$-free second homotopy.

## 1. Introduction.

Let $M$ be a closed, connected, oriented 4-manifold with a fixed base point $x_0 \in M$. We want to study the group of homotopy classes of homotopy self-equivalences of $M$, preserving both the given orientation on $M$ and the base-point. Let $\mathrm{Aut}_\bullet(M)$ denote the group of homotopy classes of such homotopy self-equivalences.

Let us start by fixing our notation. The fundamental group $\pi_1(M, x_0)$ will be denoted by $\pi$, the higher homotopy groups $\pi_i(M, x_0)$ will be denoted by $\pi_i$. Let $\Lambda = \boldsymbol{Z}[\pi]$ denote the integral group ring of $\pi$. We will mean homology and cohomology with integral coefficients unless otherwise noted.

Let $B$ denote the 2-type of $M$, we may construct $B$ by adjoining cells of dimension at least 4 to kill the homotopy groups in dimensions $\geq 3$. The natural map $c\colon M \to B$ is given by the inclusion of $M$ into $B$. Hambleton and Kreck [**3**], defined a thickening $\mathrm{Aut}_\bullet(M, w_2)$ of $\mathrm{Aut}_\bullet(M)$ (see Section 3 for the definition) and established a commutative braid of exact sequences, valid for any closed, oriented smooth or topological 4-manifold. The authors defined

$$\mathrm{Isom}[\pi, \pi_2, k_M, c_*[M]] := \left\{\phi \in \mathrm{Aut}_\bullet(B) \mid \phi_*(c_*[M]) = c_*[M]\right\}$$

and obtained an explicit formula when the fundamental group is finite of odd order.

In this paper, we define an extension $\mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M]$ of $\mathrm{Isom}[\pi, \pi_2, k_M, c_*[M]]$ and use the braid in [**3**] to obtain an explicit formula when $\pi_2$ is a free $\Lambda$-module. Examples of such manifolds are obtained when $\pi \cong *_p\boldsymbol{Z}$ or when $M \cong X\sharp Y$, where $X$ is simply-connected and $\pi_2(Y) = 0$, for instance one may

---

2000 *Mathematics Subject Classification.* Primary 57N13; Secondary 55P10, 57R80.
*Key Words and Phrases.* homotopy self-equivalences, 4-manifolds, $\pi_1$-free second homotopy.

take $Y$ to be aspherical. Our main theorem is the following:

Theorem 1.1. *Let $M$ be a closed, oriented manifold of dimension 4. If $\pi_2$ is a free $\Lambda$-module of finite rank $r$, then*

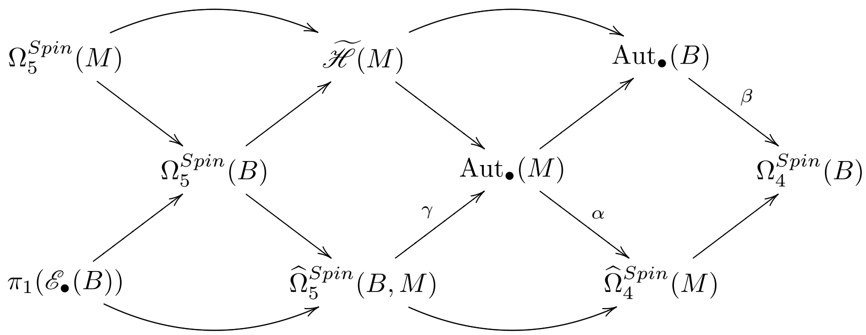
$$\mathrm{Aut}_\bullet(M, w_2) \cong \big(KH_2(M; \boldsymbol{Z}/2) \oplus H_3(M; \boldsymbol{Z}/2)\big) \rtimes \mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M]$$

*where $KH_2(M; \boldsymbol{Z}/2) := \ker(w_2\colon H_2(M; \boldsymbol{Z}/2) \to \boldsymbol{Z}/2)$.*

The author would like to thank Jonathan A. Hillman for taking the time to answer all his questions.

## 2. Spin case.

For simplicity we start with spin manifolds. Throughout this section let $M$ be a spin manifold. Hambleton and Kreck constructed a braid of exact sequences




that is commutative up to sign, the sub-diagrams are all strictly commutative except for the two composites ending in $\mathrm{Aut}_\bullet(M)$, and valid for any closed, oriented spin 4-manifold. Throughout this paper we always refer to [**3**] for the details of the definitions.

We will fix a lift $\nu_M\colon M \to BSpin$ of the classifying map for the stable normal bundle of $M$. The Abelian group $\Omega_n^{Spin}(M)$, with disjoint union as the group operation, denotes the singular bordism group of spin manifolds with a reference map to $M$. By imposing the requirement that the reference maps to $M$ must have degree zero, we obtain the modified bordism groups $\widehat{\Omega}_4^{Spin}(M)$.

Proposition 2.1. *The relevant spin bordism groups of $M$ are given as follows*:

$$\Omega_4^{Spin}(M) \cong \Omega_4^{Spin}(*) \oplus H_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2) \oplus H_4(M),$$
$$\Omega_5^{Spin}(M) \cong H_1(M) \oplus H_3(M;\boldsymbol{Z}/2) \oplus H_4(M;\boldsymbol{Z}/2).$$

PROOF. This follows from the Atiyah - Hirzebruch spectral sequence, whose $E^2$-term is $H_p(M;\Omega_q^{Spin}(*))$. The first differential $d_2\colon E^2_{p,q} \to E^2_{p-2,q+1}$ is given by the dual of $Sq^2$ (if $q=1$) or this composed with reduction mod 2 (if $q=0$), see [**8**, p. 751]. We substitute the values

$$\Omega_q^{Spin}(*) = \boldsymbol{Z}, \boldsymbol{Z}/2, \boldsymbol{Z}/2, 0, \boldsymbol{Z}, 0 \quad \text{for} \quad 0 \le q \le 5.$$

The differential for $(p,q) = (4,1)$ is dual to $Sq^2\colon H^2(M;\boldsymbol{Z}/2) \to H^4(M;\boldsymbol{Z}/2)$ which is zero, since $M$ is spin. We have a short exact sequence

$$0 \longrightarrow \Omega_4^{Spin}(*) \oplus H_2(M;\boldsymbol{Z}/2) \longrightarrow F_{3,1} \longrightarrow H_3\big(M;\Omega_1^{Spin}(*)\big) \longrightarrow 0$$

and $V \times S^1 \xrightarrow{f \circ p_1} F_{3,1}$ gives the splitting, where we consider an embedding $f\colon V \to M$ of a closed spin 3-manifold representing a generator of $H_3(M;\boldsymbol{Z}/2) \cong (\boldsymbol{Z}/2)^r$, and $S^1$ is equipped with the non-trivial spin structure. Therefore, $\Omega_4^{Spin}(M) \cong \Omega_4^{Spin}(*) \oplus H_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2) \oplus H_4(M)$. The result for $\Omega_5^{Spin}(M)$ follows by similar arguments. □

PROPOSITION 2.2. *The homology groups of $B$ are given by*

$$H_i(B) \cong \begin{cases} H_i(M) & \text{if } i = 0, 1 \text{ or } 2 \\ 0 & \text{if } i = 3 \text{ or } 5 \\ \boldsymbol{Z} \otimes_\Lambda \Gamma(\pi_2) & \text{if } i = 4 \end{cases}$$

*where $\Gamma$ denotes the Whitehead's quadratic functor* [**9**].

PROOF. The result follows from the the Serre spectral sequence of the fibration $\widetilde{B} \to B \to K(\pi, 1)$ and [**7**, Proposition 4.2]. □

PROPOSITION 2.3. *Let $\Omega_*^{Spin}(B)$ denote the singular bordism group of spin manifolds with a reference map to $B$. We have the following*:

$$\Omega_4^{Spin}(B) \subset \Omega_4^{Spin}(*) \oplus H_4(B) \quad \textit{and} \quad \Omega_5^{Spin}(B) \cong H_1(B).$$

PROOF. We use the same spectral sequence. First note that

$$\widetilde{B} = K(\pi_2, 2) = \coprod_{i,g} \{\boldsymbol{C}P^{\infty} \times \{g\} \mid g \in \pi, i = 1, 2, \ldots, r\}.$$

Then consider the following commutative diagram

$$\begin{array}{ccc} H^2(\widetilde{B}; \boldsymbol{Z}/2) & \xrightarrow{Sq^2} & H^4(\widetilde{B}; \boldsymbol{Z}/2) \\ \uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle p^*} \\ H^2(B; \boldsymbol{Z}/2) & \xrightarrow{Sq^2} & H^4(B; \boldsymbol{Z}/2) \end{array}$$

which implies that $Sq^2\colon H^2(B; \boldsymbol{Z}/2) \to H^4(B; \boldsymbol{Z}/2)$ is injective. Hence $d_2\colon H_4(B; \boldsymbol{Z}/2) \to H_2(B; \boldsymbol{Z}/2)$ is surjective. Therefore, on the line $p + q = 4$, the only groups which survive to $E^{\infty}$ are $\boldsymbol{Z}$ in the $(0, 4)$ position, and a subgroup of $H_4(B)$ in the $(4, 0)$ position.

For the line $p + q = 5$, consider the diagram

$$\begin{array}{ccccc} H^2(\widetilde{B}; \boldsymbol{Z}/2) & \xrightarrow{Sq^2} & H^4(\widetilde{B}; \boldsymbol{Z}/2) & \xrightarrow{Sq^2} & H^6(\widetilde{B}; \boldsymbol{Z}/2) \\ \uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle p^*} \\ H^2(B; \boldsymbol{Z}/2) & \xrightarrow{Sq^2} & H^4(B; \boldsymbol{Z}/2) & \xrightarrow{Sq^2} & H^6(B; \boldsymbol{Z}/2). \end{array}$$

Let $\alpha \in H^4(B; \boldsymbol{Z}/2)$ such that $Sq^2(\alpha) = 0$ and $p^*(\alpha) = \beta$. There exists $\lambda \in H^2(\widetilde{B}; \boldsymbol{Z}/2)$ such that $Sq^2(\lambda) = \beta$, since the above row is exact and $p^*$ is onto. Therefore the sequence

$$H^2(B; \boldsymbol{Z}/2) \xrightarrow{Sq^2} H^4(B; \boldsymbol{Z}/2) \xrightarrow{Sq^2} H^6(B; \boldsymbol{Z}/2)$$

is exact. By the surjectivity of $H_6(B; \boldsymbol{Z}) \to H_6(B; \boldsymbol{Z}/2)$, we can conclude that $d_2\colon H_6(B; \boldsymbol{Z}) \to H_4(B; \boldsymbol{Z}/2)$ is surjective onto the kernel of the differential $d_2\colon H_4(B; \boldsymbol{Z}/2) \to H_2(B; \boldsymbol{Z}/2)$. Thus the only group which survive to $E_{\infty}$ is $H_1(B) = H_1(M)$ in the $(1, 4)$ position. □

The map $\alpha\colon \operatorname{Aut}_{\bullet}(M) \to \Omega_4^{Spin}(M)$ is defined by $\alpha(f) = [M, f] - [M, \mathrm{id}]$. An element $(W, F)$ of $\widehat{\Omega}_5^{Spin}(B, M)$ is a 5-dimensional spin manifold with boundary $(W, \partial W)$, equipped with a reference map $F\colon W \to B$ such that $F|_{\partial W}$ factors through the classifying map $c\colon M \to B$ and that $F|_{\partial W}\colon \partial W \to M$ has degree zero.

Corollary 2.4. *The group* $\widehat{\Omega}_5^{Spin}(B,M)$ *is isomorphic to* $H_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2)$ *and it injects into* $\mathrm{Aut}_\bullet(M)$. *The image of* $\alpha$ *is equal to* $H_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2)$.

Proof. The map $\Omega_5^{Spin}(M) \to \Omega_5^{Spin}(B)$, which is composing with our reference map $c\colon M \to B$, maps the summand $H_1(M)$ isomorphically to $H_1(B)$ and $H_3(M;\boldsymbol{Z}/2) \oplus H_4(M;\boldsymbol{Z}/2)$ to zero. By the exactness of the braid the map $\Omega_5^{Spin}(B) \to \widehat{\Omega}_5^{Spin}(B,M)$ is zero. Therefore

$$\begin{aligned}\widehat{\Omega}_5^{Spin}(B,M)) &\cong \ker\big(\widehat{\Omega}_4^{Spin}(M) \to \Omega_4^{Spin}(B)\big)\\ &\cong H_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2).\end{aligned}$$

The map $\widehat{\Omega}_5^{Spin}(B,M) \to \widehat{\Omega}_4^{Spin}(M)$ is injective, so by the commutativity of the braid the map $\pi_1(\mathscr{E}_\bullet(B)) \to \widehat{\Omega}_5^{Spin}(B,M)$ is zero. Therefore $\gamma\colon \widehat{\Omega}_5^{Spin}(B,M) \to \mathrm{Aut}_\bullet(M)$ is injective.

The natural map $\Omega_4^{Spin}(M) \to H_0(M)$ sends a spin 4-manifold to its signature, it follows that $\alpha(f) \in H_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2)$. On the other hand, since both the map $\widehat{\Omega}_5^{Spin}(B,M) \to \widehat{\Omega}_4^{Spin}(M)$ and $\gamma$ are injective we have $H_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2) \subseteq \operatorname{im}\alpha$. □

Let $\mathrm{Isom}[\pi,\pi_2]$ be the subgroup of $\mathrm{Aut}(\pi) \times \mathrm{Aut}(\pi_2)$ consisting of all those pairs $(\chi,\psi)$ for which $\psi(\eta a) = \chi(\eta)\psi(a)$ for all $\eta \in \pi$, $a \in \pi_2$. We have a split exact sequence [**5**, p. 31]

$$0 \longrightarrow H^2(\pi;\pi_2) \longrightarrow \mathrm{Aut}_\bullet(B) \xrightarrow{(\pi_1,\pi_2)} \mathrm{Isom}[\pi,\pi_2] \longrightarrow 1.$$

In particular we have $\mathrm{Aut}_\bullet(B) = H^2(\pi;\pi_2) \rtimes \mathrm{Isom}[\pi,\pi_2]$. If $\pi_2$ is a free $\Lambda$-module, then $H^2(\pi;\pi_2) = 0$. Hence we have

$$\mathrm{Aut}_\bullet(B) \cong \mathrm{Isom}[\pi_1,\pi_2].$$

Hambleton and Kreck [**2**] defined the quadratic 2-type of $M$ as the quadruple $[\pi,\pi_2,k_M,s_M]$. The isometries of the quadratic 2-type of $M$, which is denoted by $\mathrm{Isom}[\pi,\pi_2,k_M,s_M]$, consists of all pairs of isomorphisms

$$\chi\colon \pi \to \pi \quad \text{and} \quad \psi\colon \pi_2 \to \pi_2,$$

such that $\psi(gx) = \chi(g)\psi(x)$ for all $g \in \pi$ and $x \in \pi_2$, which preserve the $k$-invariant and $s_M$, the intersection form of $M$ on $\pi_2$. Since $H^3(\pi;\pi_2) = 0$ we have $k_M = 0$. For notational ease we will drop it from the notation and write $\mathrm{Isom}[\pi, \pi_2, s_M]$ for the group of isometries of the quadratic 2-type. Finally note that when $\pi_2$ is a free $\Lambda$-module, $c_*[M]$ and $s_M$ uniquely determine each other (see [**7**, Proposition 4.3]).

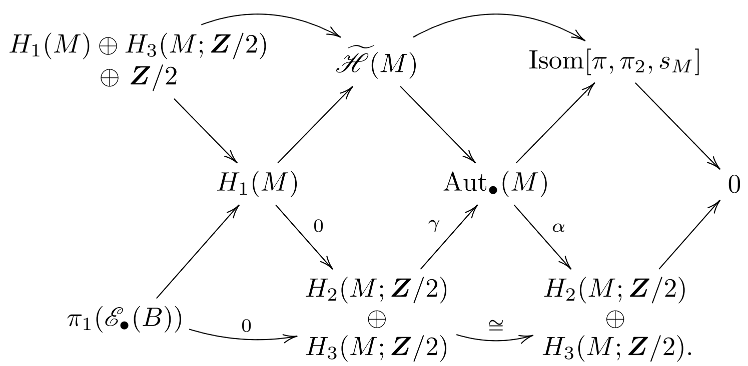
LEMMA 2.5. $\ker\left(\beta\colon \mathrm{Aut}_\bullet(B) \to \Omega_4^{Spin}(B)\right) = \mathrm{Isom}[\pi, \pi_2, s_M]$.

PROOF. If $\phi \in \mathrm{Aut}_\bullet(B)$ and $c\colon M \to B$ is the classifying map, then $\beta(\phi) := [M, \phi \circ c] - [M, c]$. The natural map $\Omega_4^{Spin}(B) \to H_4(B)$ sends a bordism element to the image of its fundamental class. The image of $\beta(\phi)$ in $H_4(B)$ is zero when $\phi_*(c_*[M]) = c_*[M]$. Hence $\ker \beta$ is contained in the group of the isometries of the quadratic 2-type. On the other hand an element $\phi \in \mathrm{Isom}[\pi, \pi_2, s_M]$ will be $\phi \in \mathrm{Aut}_\bullet(B)$ such that $\phi_*(c_*[M]) = c_*[M]$, then clearly $\beta(\phi) = 0$. □

COROLLARY 2.6. *The images of* $\mathrm{Aut}_\bullet(M)$ *and* $\widetilde{\mathscr{H}}(M)$ *in* $\mathrm{Aut}_\bullet(B)$ *are precisely equal to* $\mathrm{Isom}[\pi, \pi_2, s_M]$.

PROOF. By obstruction theory for each $[f] \in \mathrm{Aut}_\bullet(M)$, we have a base-point preserving homotopy self-equivalence $\phi_f\colon B \to B$ such that $c \circ f = \phi_f \circ c$. All we have to show is $(\phi_f)_*(c_*[M]) = c_*[M]$. We have $(\phi_f)_*(c_*[M]) = (\phi_f \circ c)_*[M] = (c \circ f)_*[M] = c_*[M]$ since the fundamental class in $H_4(M)$ is preserved by an orientation preserving homotopy equivalence. We see that $\mathrm{im}(\mathrm{Aut}_\bullet(M) \to \mathrm{Aut}_\bullet(B))$ is contained in $\mathrm{Isom}[\pi, \pi_2, s_M]$. The other inclusion follows from [**1**, Corollary 3.3]. The result for the image of $\widetilde{\mathscr{H}}(M)$ follows by the exactness of the braid and the fact that $\ker(\beta) = \mathrm{Isom}[\pi, \pi_2, s_M]$. □

Here are the relevant terms of our braid diagram now:




THEOREM 2.7. *Let $M$ be a closed, oriented spin manifold of dimension 4. If $\pi_2$ is a free $\Lambda$-module of finite rank $r$, then*

$$\mathrm{Aut}_\bullet(M) \cong \big(H_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2)\big) \rtimes \mathrm{Isom}[\pi, \pi_2, s_M].$$

PROOF. From the braid diagram, we have

$$\ker\big(\widetilde{\mathscr{H}}(M) \to \mathrm{Isom}[\pi, \pi_2, s_M]\big) \cong H_1(M),$$

so $\mathrm{Isom}[\pi, \pi_2, s_M] \cong \widetilde{\mathscr{H}}(M)/H_1$. This gives the splitting of the short exact sequence

$$0 \to K_1 \to \mathrm{Aut}_\bullet(M) \to \mathrm{Isom}[\pi, \pi_2, s_M] \to 1$$

where $K_1 := \ker(\mathrm{Aut}_\bullet(M) \to \mathrm{Aut}_\bullet(B))$. Hence it follows that

$$\mathrm{Aut}_\bullet(M) \cong K_1 \rtimes \mathrm{Isom}[\pi, \pi_2, s_M].$$

We already know that $\gamma$ is injective (Corollary 2.4). By the commutativity of the braid to show that it is actually an injective *homomorphism*, it is enough to show that $\alpha$ is a homomorphism on the image of $\gamma$. Let $\gamma(W, F) = f$ and $\gamma(W', F') = g$. Note that $\alpha(f \circ g) = \alpha(f) + f_*(\alpha(g))$. We have to show that $f_*(\alpha(g)) = \alpha(g)$. By Corollary 2.4, $\alpha(g) \in H_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2)$ and any element $f$ in the image of $\gamma$ is trivial in $\mathrm{Aut}_\bullet(B)$. Since $H_3(M;\boldsymbol{Z}/2) \cong H^1(M;\boldsymbol{Z}/2)$ and $c$ induces isomorphisms on $H_2(M;\boldsymbol{Z}/2)$ and $H^1(M;\boldsymbol{Z}/2)$, $f$ acts as the identity on $H_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2)$. Now a diagram chase shows that $\gamma$ is a homomorphism. Therefore we have a short exact sequence of groups and homomorphisms

$$0 \to \big(H_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2)\big) \xrightarrow{\gamma} \mathrm{Aut}_\bullet(M) \to \mathrm{Isom}[\pi, \pi_2, s_M] \to 1.$$

Moreover, $K_1 = \operatorname{im}\gamma$ and $K_1$ is mapped isomorphically onto $H_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2)$ by the map $\alpha$. The conjugation action of $\mathrm{Isom}[\pi, \pi_2, s_M]$ on $K_1$ agrees with the induced action on homology under the identification $K_1 \cong H_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2)$ via $\alpha$ (see [**3**]). It follows that

$$\mathrm{Aut}_\bullet(M) \cong \big(H_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2)\big) \rtimes \mathrm{Isom}[\pi, \pi_2, s_M]. \qquad \square$$

## 3. The non-spin case.

When $w_2(M) \neq 0$ the bordism groups must be modified. The class $w_2$ gives a fibration and we can form the pullback

$$\begin{array}{ccc} B\langle w_2\rangle & \xrightarrow{\ j\ } & B \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle w_2} \\ BSO & \xrightarrow{\ w\ } & K(\boldsymbol{Z}/2,2). \end{array}$$

The map $w = w_2(\gamma)$ pulls back the second Stiefel-Whitney class for the universal oriented vector bundle $\gamma$ over $BSO$. $B\langle w_2\rangle$ is called the normal 2-type of $M$ [**4**]. Let $\Omega_*(B\langle w_2\rangle)$ be bordism classes smooth manifolds equipped with a lift of the normal bundle. The spectral sequence used to compute $\Omega_*(B\langle w_2\rangle)$ has the same $E_2$-term as the one used above for $w_2 = 0$, but the differentials are twisted by $w_2$. In particular, $d_2$ is the dual of $Sq^2_w$, where $Sq^2_w(x) := Sq^2(x) + x \cup w_2$ (see [**8**, Section 2]).

There is a corresponding non-spin version of $\Omega_*^{Spin}(M)$, namely the bordism groups $\Omega_*(M\langle w_2\rangle)$. The $E_2$-term of the spectral sequence is unchanged from the spin case, but the differentials are twisted by $w_2$ with the above formula for $Sq^2_w$. We choose a particular representative for the map $w_2$ such that $w_2 = w \circ \nu_M$. Next we define a suitable "thickening" of $\mathrm{Aut}_\bullet(M)$ for the non-spin case:

DEFINITION 3.1 ([**3**]). Let $\mathrm{Aut}_\bullet(M, w_2)$ denote the set of equivalence classes of maps $\widehat{f}\colon M \to M\langle w_2\rangle$ such that (i) $f := j \circ \widehat{f}$ is a base-point and orientation preserving homotopy equivalence, and (ii) $\xi \circ \widehat{f} = \nu_M$.

There is a short exact sequence of groups [**3**]

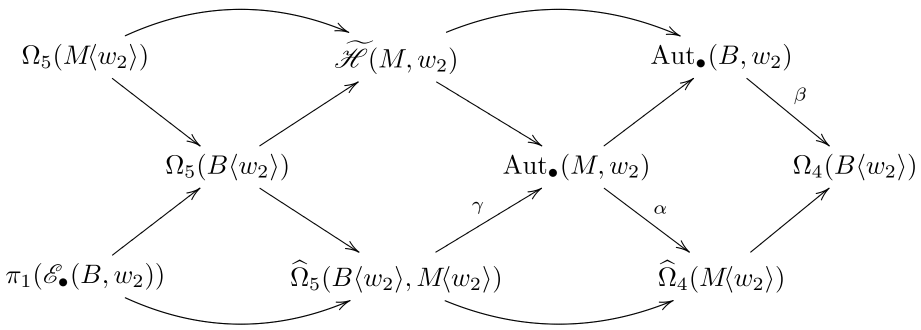
$$0 \longrightarrow H^1(M;\boldsymbol{Z}/2) \longrightarrow \mathrm{Aut}_\bullet(M, w_2) \longrightarrow \mathrm{Aut}_\bullet(M) \longrightarrow 1.$$

To define an analogous group $\mathrm{Aut}_\bullet(B, w_2)$ of self-equivalences, we should first state the following lemma from [**3**].

LEMMA 3.2. *Given a base-point preserving map $f\colon M \to B$, there is a unique extension (up to base-point preserving homotopy) $\phi_f\colon B \to B$ such that $\phi_f \circ c = f$. If $f$ is a 3-equivalence then $\phi_f$ is a homotopy equivalence. Moreover, if $w_2 \circ f = w_2$, then $w_2 \circ \phi_f = w_2$.*

DEFINITION 3.3 ([**3**]). Let $\mathrm{Aut}_\bullet(B, w_2)$ denote the set of equivalence classes of maps $\widehat{f}\colon M \to B\langle w_2\rangle$ such that (i) $f := j \circ \widehat{f}$ is a base-point preserving 3-equivalence, and (ii) $\xi \circ \widehat{f} = \nu_M$.

THEOREM 3.4 ([**3**]). *Let $M$ be a closed, oriented topological 4-manifold. Then there is a sign-commutative diagram of exact sequences*




*such that the two composites ending in* $\mathrm{Aut}_\bullet(M, w_2)$ *agree up to inversion, and the other sub-diagrams are strictly commutative.*

Proposition 3.5. *Let* $B\langle w_2\rangle$ *denote the normal* 2*-type of a* 4*-manifold* $M$ *with free fundamental group. Then we have*

$$\begin{aligned}
&\Omega_4(M\langle w_2\rangle) \cong \Omega_4^{Spin}(*) \oplus H_2(M; \boldsymbol{Z}/2) \oplus H_3(M; \boldsymbol{Z}/2) \oplus H_4(M)\\
&\Omega_5(M\langle w_2\rangle) \cong H_1(M) \oplus H_3(M; \boldsymbol{Z}/2) \oplus H_4(M; \boldsymbol{Z}/2)\\
&\Omega_4(B\langle w_2\rangle) \subset \Omega_4^{Spin}(*) \oplus \boldsymbol{Z}/2 \oplus H_4(B)\\
&\Omega_5(B\langle w_2\rangle) \cong H_1(M).
\end{aligned}$$

Proof. We only need to compute the $d_2$ differentials. Since $M$ is orientable, $w_2$ is also the second Wu class of $M$. We have $Sq_w^2(x) = 0$. Now, everything works exactly the same as in the spin case.

For the bordism groups of $B\langle w_2\rangle$, first consider the following commutative diagram

$$\begin{array}{ccc}
H^2(\widetilde{B}; \boldsymbol{Z}/2) & \xrightarrow{Sq_w^2} & H^4(\widetilde{B}; \boldsymbol{Z}/2)\\
{\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle p^*}\\
H^2(B; \boldsymbol{Z}/2) & \xrightarrow{Sq_w^2} & H^4(B; \boldsymbol{Z}/2).
\end{array}$$

By the commutativity of the diagram, we have

$$\begin{aligned}
\ker\big(Sq_w^2\colon H^2(B; \boldsymbol{Z}/2) \to H^4(B; \boldsymbol{Z}/2)\big) &\cong \langle w_2\rangle \cong \boldsymbol{Z}/2\\
&\cong \operatorname{coker}\big(d_2\colon H_4(B; \boldsymbol{Z}/2) \to H_2(B; \boldsymbol{Z}/2)\big).
\end{aligned}$$

Since all the other differentials are zero, this gives the $\boldsymbol{Z}/2$ in the $E^{\infty}_{2,2}$ position. To see that $H_1(B) \cong H_1(M)$ is the only group on the line $p+q=5$ which survives to $E_\infty$, we use the following commutative diagram

$$\begin{array}{ccccc} H^2(\widetilde{B};\boldsymbol{Z}/2) & \xrightarrow{Sq^2_w} & H^4(\widetilde{B};\boldsymbol{Z}/2) & \xrightarrow{Sq^2_w} & H^6(\widetilde{B};\boldsymbol{Z}/2) \\ \uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle p^*} \\ H^2(B;\boldsymbol{Z}/2) & \xrightarrow{Sq^2_w} & H^4(B;\boldsymbol{Z}/2) & \xrightarrow{Sq^2_w} & H^6(B;\boldsymbol{Z}/2). \end{array}$$

We are going to show that the bottom row is exact. Let $a \in H^2(B;\boldsymbol{Z}/2)$, then $Sq^2_w(a^2 + a \cup w_2) = 0$. Now, let $b \in H^4(B;\boldsymbol{Z}/2)$ such that $Sq^2_w(b) = 0$ and let $p^*(b) = y$, then $Sq^2_w(y) = 0$. There exists a $z \in H^2(\widetilde{B};\boldsymbol{Z}/2)$ such that $Sq^2_w(z) = y$. Then we also have a $c \in H^2(B;\boldsymbol{Z}/2)$ such that $p^*(c) = z$ and $Sq^2_w(c) = b$. Therefore the sequence

$$H^2(B;\boldsymbol{Z}/2) \xrightarrow{Sq^2_w} H^4(B;\boldsymbol{Z}/2) \xrightarrow{Sq^2_w} H^6(B;\boldsymbol{Z}/2)$$

is exact. Note also that $H_6(B) \to H_6(B;\boldsymbol{Z}/2)$ is surjective, hence $d_2\colon H_6(B) \to H_4(B;\boldsymbol{Z}/2)$ is onto the kernel of $d_2\colon H_4(B;\boldsymbol{Z}/2) \to H_2(B;\boldsymbol{Z}/2)$. □

Let $\widehat{c}\colon M \to B\langle w_2\rangle$ denote the map defined by the pair $(c\colon M \to B, \nu_M\colon M \to BSO)$. Consider the following diagram

$$\begin{array}{ccc} M\langle w_2\rangle & \xrightarrow{c\circ j} & B \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle w_2} \\ BSO & \xrightarrow{w} & K(\boldsymbol{Z}/2,2). \end{array}$$

We have $(w_2 \circ c)\circ j = w_2 \circ j$ and since the pullback satisfies the universal property, there exists a map $\overline{c}\colon M\langle w_2\rangle \to B\langle w_2\rangle$. Let $\widehat{\mathrm{id}}\colon M \to M\langle w_2\rangle$ denote the map defined by the pair $(\mathrm{id}_M\colon M \to M, \nu_M\colon M \to BSO)$. Given $[\widehat{f}] \in \mathrm{Aut}_\bullet(M, w_2)$, we define $\alpha\colon \mathrm{Aut}_\bullet(M,w_2) \to \widehat{\Omega}_4(M\langle w_2\rangle)$ by $\alpha(\widehat{f}) = [M,\widehat{f}] - [M,\widehat{\mathrm{id}}_M]$ where the modified bordism groups are defined by letting the degree of a reference map $\widehat{g}\colon N^4 \to\ Mw$ to be the ordinary degree of $g = j \circ \widehat{g}$. An element $(W,\widehat{F})$ of $\widehat{\Omega_5}(B\langle w_2\rangle, M\langle w_2\rangle)$ is a 5-dimensional manifold with boundary $(W,\partial W)$, equipped with a reference map $\widehat{F}\colon W \to B\langle w_2\rangle$ such that $\widehat{F}|_{\partial W}$ factors through $\overline{c}$.

COROLLARY 3.6. *The group*

$$\widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle) \cong KH_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2)$$

*and it injects into* $\mathrm{Aut}_\bullet(M, w_2)$. *The image of* $\alpha$,

$$\operatorname{im}\alpha = KH_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2).$$

Proof. As in the proof of Corollary 2.4, $\Omega_5(M\langle w_2\rangle) \to \Omega_5(B\langle w_2\rangle)$ is onto and by the exactness of the braid $\Omega_5(B\langle w_2\rangle) \to \widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)$ is zero. Thus

$$\begin{aligned}\widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle) &\cong \ker\left(\widehat{\Omega}_4(M\langle w_2\rangle) \to \Omega_4(B\langle w_2\rangle)\right)\\ &\cong KH_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2).\end{aligned}$$

The map $\pi_1(\mathscr{E}_\bullet(B, w_2)) \to \widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle)$ is zero, by the commutativity of the braid. Therefore

$$\gamma\colon \widehat{\Omega}_5(B\langle w_2\rangle, M\langle w_2\rangle) \to \mathrm{Aut}_\bullet(M, w_2)$$

is injective. The natural map $\Omega_4(M\langle w_2\rangle) \to H_0(M)$ sends a 4-manifold to its signature. Since the class $w_2 \in H^2(M;\boldsymbol{Z}/2)$ is a characteristic element for the cup product form (mod 2), it is preserved by the induced map of a self-homotopy equivalence of $M$. Therefore, the image of $\mathrm{Aut}_\bullet(M, w_2)$ in $\Omega_4(M\langle w_2\rangle)$ lies in the subgroup $KH_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2)$. Since, the map $\gamma$ is injective we also have $KH_2(M;\boldsymbol{Z}/2) \oplus H_3(M;\boldsymbol{Z}/2) \subseteq \operatorname{im}\alpha$. □

Next, we are going to define a homomorphism

$$\widehat{j}\colon\ \mathrm{Aut}_\bullet(B, w_2) \to \mathrm{Aut}_\bullet(B).$$

For any $\widehat{f} \in \mathrm{Aut}_\bullet(B, w_2)$, $f := j \circ \widehat{f}\colon M \to B$ is a 3-equivalence. There is a unique homotopy equivalence $\phi_f\colon B \to B$ such that $\phi_f \circ c \simeq f$. We define

$$\widehat{j}(\widehat{f}) := \phi_f.$$

Let $\widehat{g}$ be another element of $\mathrm{Aut}_\bullet(B, w_2)$, then $\widehat{f} \bullet \widehat{g}$ is defined by the pair $(\phi_f \circ \phi_g \circ c, \nu_M)$. Therefore $\widehat{j}(\widehat{f} \bullet \widehat{g}) = \phi_f \circ \phi_g$. Let

$$\mathrm{Isom}^{\langle w_2\rangle}[\pi, \pi_2, s_M] := \left\{\widehat{f} \in \mathrm{Aut}_\bullet(B, w_2) \mid \phi_f \in \mathrm{Isom}[\pi, \pi_2, s_M]\right\}.$$

LEMMA 3.7 ([**6**]). *There is a short exact sequence of groups*

$$0 \longrightarrow H^1(M;\boldsymbol{Z}/2) \longrightarrow \mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,s_M] \xrightarrow{\widehat{j}} \mathrm{Isom}[\pi,\pi_2,s_M] \longrightarrow 1.$$

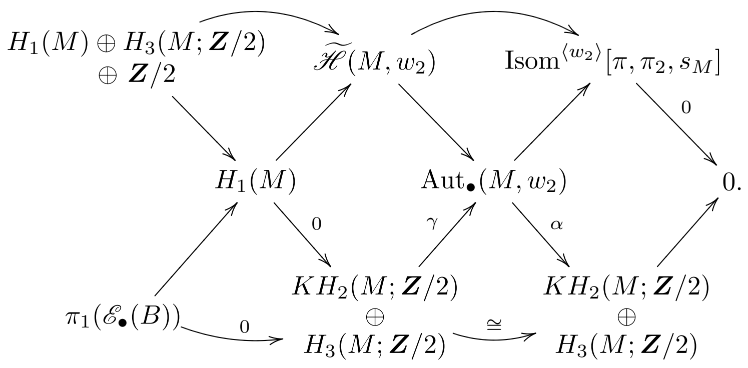
COROLLARY 3.8. *The image of* $\mathrm{Aut}_\bullet(M,w_2)$ *in* $\mathrm{Aut}_\bullet(B,w_2)$ *is precisely equal to* $\mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,s_M]$.

PROOF. Let $\widehat{f}\in \mathrm{Aut}_\bullet(M,w_2)$ and $\phi_{\widehat{f}}$ denote the image of $\widehat{f}$ in $\mathrm{Aut}_\bullet(B,w_2)$. Then $\widehat{j}(\phi_{\widehat{f}})=\phi_f$ satisfies $\phi_f\circ c=c\circ f$ and $\phi_f$ preserves $c_*[M]$. Hence $\phi_f\in \mathrm{Isom}[\pi,\pi_2,s_M]$. Now suppose that $\phi\in\mathrm{Isom}[\pi,\pi_2,s_M]$, then by [**1**, Corollary 3.3] there exists $f\in\mathrm{Aut}_\bullet(M)$ such that $\phi\circ f\simeq c\circ f$. We may assume that $\widehat{f}=(f,\nu_M)\in\mathrm{Aut}_\bullet(M,w_2)$ [**3**, Lemma 3.1]. Let $\phi_{\widehat{f}}\in\mathrm{Aut}_\bullet(B,w_2)$ denotes the image of $\widehat{f}$, we have $\widehat{j}(\phi_{\widehat{f}})=\phi$. □

LEMMA 3.9. $\ker(\beta\colon \mathrm{Aut}_\bullet(B,w_2)\to\Omega_4(B\langle w_2\rangle))=\mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,s_M]$ *and the image of* $\widetilde{\mathscr{H}}(M,w_2)$ *in* $\mathrm{Aut}_\bullet(B,w_2)$ *is equal to* $\mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,s_M]$.

PROOF. In the non-spin case, the map $\beta\colon \mathrm{Aut}_\bullet(B,w_2)\to\Omega_4(B\langle w_2\rangle)$ is defined by $\beta(\widehat{f})=[M,\widehat{f}]-[M,\widehat{c}]$. Let $\widehat{f}\in\mathrm{Aut}_\bullet(B,w_2)$ and suppose first that $\widehat{f}\in\ker\beta$, then $(j\circ\widehat{f})_*[M]=c_*[M]$. But since $(j\circ\widehat{f})$ is a 3-equivalence, there exists $\phi\in\mathrm{Aut}_\bullet(B)$ with $\phi\circ c=j\circ\widehat{f}$. So, $\phi_*(c_*[M])=c_*[M]$ which means $\widehat{j}(\widehat{f})=\phi\in\mathrm{Isom}[\pi,\pi_2,s_M]$. Therefore $\ker(\beta)\subseteq\mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,s_M]$. It is easy to see the other inclusion from the commutativity of the braid. The result about the image of $\widetilde{\mathscr{H}}(M,w_2)$ follows from the exactness of the braid [**3**, Lemma 2.7] and the fact that $\ker(\beta)=\mathrm{Isom}^{\langle w_2\rangle}[\pi,\pi_2,s_M]$. □

The relevant terms of our braid are now:



The proof of Theorem 1.1. We have a split short exact sequence

$$0 \longrightarrow \widehat{K_1} \longrightarrow \mathrm{Aut}_\bullet(M, w_2) \longrightarrow \mathrm{Isom}^{\langle w_2 \rangle}[\pi, \pi_2, s_M] \longrightarrow 1$$

where $\widehat{K_1} = \ker(\mathrm{Aut}_\bullet(M, w_2) \to \mathrm{Aut}_\bullet(B, w_2))$. Any element $\widehat{f}$ will act as identity on $\mathrm{im}(\alpha) = KH_2(M; \boldsymbol{Z}/2) \oplus H_3(M; \boldsymbol{Z}/2)$, so $\lambda$ is a homomorphism. Also $\widehat{K_1} \cong KH_2(M; \boldsymbol{Z}/2) \oplus H_3(M; \boldsymbol{Z}/2)$ and the rest of the proof follows as in the spin case. □

Remark 3.10. We have

$$H_2(M; \boldsymbol{Z}/2) \cong H_0\big(\pi; H_2(\widetilde{M}; \boldsymbol{Z}/2)\big) \cong (\pi_2 \otimes \boldsymbol{Z}/2) \otimes_\Lambda \boldsymbol{Z}.$$

Therefore any element of $H_2(M; \boldsymbol{Z}/2)$ can be represented by a map $S^2 \to M$. Let $0 \neq x \in KH_2(M; \boldsymbol{Z}/2)$ and $\alpha\colon S^2 \to M$ corresponds to $x$ via the above isomorphism. Choose an embedding $D^4 \hookrightarrow M$ and shrink $\partial D^4$ to a point, to get a map $M \to M \vee S^4$. Now let $\eta\colon S^3 \to S^2$ be the Hopf map, $S\eta\colon S^4 \to S^3$ its suspension and $\eta^2\colon S^4 \to S^2$ the composition $\eta^2 = \eta \circ S\eta$. Let $f$ be the composite map

$$M \longrightarrow M \vee S^4 \xrightarrow{\mathrm{id} \vee \eta^2} M \vee S^2 \xrightarrow{\mathrm{id} \vee \alpha} M$$

$f$ induces identities on $\pi_1$ and on $H_i(\widetilde{M})$, so $f$ is homologous to the $\mathrm{id}_M$, and hence it is a homotopy equivalence, but it is not homotopic to the identity, for $\gamma$ is injective.

To realize $H_3(M; \boldsymbol{Z}/2)$ as homotopy equivalences, first observe that $H_3(M) \cong H_3(\widetilde{M}) \otimes_\Lambda \boldsymbol{Z}$ and reduction mod 2 is onto, so by Hurewicz theorem for any element of $H_3(M; \boldsymbol{Z}/2)$ there exists a map $\beta\colon S^3 \to M$. Now the following composite map

$$M \longrightarrow M \vee S^4 \xrightarrow{\mathrm{id} \vee S\eta} M \vee S^3 \xrightarrow{\mathrm{id} \vee \beta} M$$

is again a homotopy-equivalence.

## References


[1] H. J. Baues and B. Bleile, Poincare duality complexes in dimension four, Algebr. Geom. Topol., **8** (2008), 2355–2389.

[2] I. Hambleton and M. Kreck, On the classification of topological 4-manifolds with finite fundamental group, Math. Ann., **280** (1988), 85–104.

[3] I. Hambleton and M. Kreck, Homotopy self-equivalences of 4-manifolds, Math. Z., **248** (2004), 147–172.

[4] M. Kreck, Surgery and duality, Ann. of Math. (2), **149** (1999), 707–754.

[5] J. M. Møller, Self-homotopy equivalences of group cohomology spaces, J. Pure Appl. Algebra, **73** (1991), 23–37.

[6] M. Pamuk, Homotopy self-equivalences of 4-manifolds, Ph. D. Thesis, McMaster University, 2008.

[7] F. Spaggiari, Four-manifolds with $\pi_1$-free second homotopy, Manuscripta Math., **111** (2003), 303–320.

[8] P. Teichner, On the signature of four-manifolds with universal covering spin, Math. Ann., **295** (1993), 745–759.

[9] J. H. C. Whitehead, A Certain exact sequence, Ann. of Math., **52** (1950), 51–110.

Mehmetcik Pamuk
Department of Mathematics
Middle East Technical University
Ankara 06531, Turkey
E-mail: mpamuk@metu.edu.tr