J. Math. Soc. Japan
Vol. 62, No. 4 (2010) pp. 1273–1288
doi: 10.2969/jmsj/06241273

Some cases of four dimensional linear Noether's problem

By Aiichi YAMASAKI

(Received Aug. 19, 2008)
(Revised Sep. 29, 2009)

Abstract. The linear Noether's problem means the rationality problem for the fixed field of linear actions on the rational function field. This paper deals with a part of our study on the four dimensional linear Noether's problem. Apart from the main part of our study, which will be published in other papers, the results which require complicated calculations by a computer are published here as a separate paper. The problem is affirmative for all of 5 non-solvable subgroups and the largest and one of the second largest subgroups of $GL(4,\boldsymbol{Q})$. As relevant topics, we remark that $PSp(3,2)$ (the simple group of order 1451520) has a generic polynomial over $\boldsymbol{Q}$.

1. Introduction.

The linear Noether's problem is defined as the following problem.

PROBLEM 1. Let K be a field and G be a finite subgroup of $GL(n,K)$. If we define G-action on $K(x_1,\dots,x_n)$ by

$$\sigma(x_j)=\sum_{i=1}^{n}a_{ij}x_i\ (j=1,\dots n)\quad \text{for } \forall\sigma=(a_{ij})\in G,$$

then is the fixed field $K(x_1,\dots,x_n)^G$ rational over K?

In general, for mutually conjugate subgroups G_1 and G_2 of $GL(n,K)$, linear Noether's problems are the same by changing variables. So we have only to consider the conjugacy classes of finite subgroups of $GL(n,K)$ and we have 227 ones in the case of $n=4$, $K=\boldsymbol{Q}$. See [**1**].

Up to the present, we have the following result, including the result of this paper. See the end of this section.

THEOREM 1. *Four-dimensional linear Noether's problem is affirmative for all groups except for the following* 6 *groups. The problem is negative for* $(4,26,1)$

2000 *Mathematics Subject Classification.* Primary 12F20; Secondary 12F12.

Key Words and Phrases. linear Noether's problem, finite reflection groups, generic polynomial, inverse Galois theorem.

and $(4,33,2)$ *and is unsolved for* $(4,33,3)$, $(4,33,6)$, $(4,33,7)$ *and* $(4,33,11)$. *Here* $(4,26,1)$ *etc. is the classification number in the GAP code* [**1**].

The linear Noether's problem is important for constructing a generic polynomial, which is defined as follows.

Definition 1 ([**7**], [**2**]). A Galois extension of a field K with the Galois group G is called a G-extension of K. A polynomial $f(t_1,\dots,t_m;X)\in K(t_1,\dots,t_m)[X]$ is called a generic polynomial for G-extension over K if it satisfies the following conditions:

(1) $\mathrm{Spl}_{K(t_1,\dots,t_m)}f(t_1,\dots,t_m;X)$ is a G-extension of $K(t_1,\dots,t_m)$,
(2) for every G-extension of infinte fields L/M with $M\supseteq K$, $L=\mathrm{Spl}_M f(a_1,\dots,a_m;X)$ for some $a_1,\dots,a_m\in M$.

The explicit affirmative answer of linear Noether's problem for (K,G) produces a generic polynomial explicitly as follows. If linear Noether's problem is affirmative, that is, $K(x_1,\dots,x_n)^G$ is rational (= purely transcendental) over K, then we have $K(x_1,\dots,x_n)^G=K(t_1,\dots,t_n)$. This means that $K(x_1,\dots,x_n)$ is a G-extension of $K(t_1,\dots,t_n)$. Let $f(X)\in K(t_1,\dots,t_n)[X]$ be such that $K(x_1,\dots,x_n)=\mathrm{Spl}_{K(t_1,\dots,t_n)}f(X)$.

Theorem 2 (Kemper, Mattig [**6**]). *$f(X)=f(t_1,\dots,t_n;X)$ defined above is a generic polynomial for G-extension over K.*

Thus linear Noether's problem is an effective way to construct generic polynomials.

For $n=4$, $K=\boldsymbol{Q}$, it is known that the linear Noether's problem is negative for a group G which is isomorphic to the cyclic group C_8 (See [**4**]), while affirmative for other abelian groups [**10**].

So the results for non-abelian groups attract the attention of many mathematicians.

Theorem 3 (Rikuna [**14**], Plans [**13**]). *Linear Noether's problem for $n=4$ has affirmative answer if G belongs to the conjugacy class $(4,32,5)$ or $(4,32,11)$ in the GAP code, which is isomorphic to $SL(2,3)$ or $GL(2,3)$ respectively.*

Theorem 4 (Kang [**5**]). *Linear Noether's problem for $n=4$, $k=\boldsymbol{Q}$ has an affirmative answer if G is a non-abelian group of order 16.*

Theorem 5 (Kitayama [**8**]). *Linear Noether's problem for $n=4$, $K=\boldsymbol{Q}$ has an affirmative answer if G is a 2-group which is not isomorphic to C_8.*

H. Kitayama and the present author studied on other conjugacy classes of finite subgroups of $GL(4,\boldsymbol{Q})$, and obtained the results of Theorem 1. Among them, 7 complicated cases which require calculations by a computer are discussed in this paper. The remaining cases are discussed in [**8**], [**9**].

2. Non-solvable case.

In this section, we consider the case when G is a non-solvable finite subgroup of $GL(4,\boldsymbol{Q})$. We use the following notations.

$$c_2 := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad c_2' := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad m := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$c_4 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad c_5 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

There are five non-solvable subgroups, all of which contain $\mathfrak{A}_5$.

$$\langle c_5, \gamma \rangle \simeq \mathfrak{A}_5, \quad \langle c_5, mc_4 \rangle \simeq \mathfrak{S}_5, \quad \langle c_5, c_2c_2'mc_4 \rangle \simeq \mathfrak{S}_5,$$
$$\langle c_5, \gamma, c_2c_2' \rangle \simeq \mathfrak{A}_5 \times C_2, \quad \langle c_5, mc_4, c_2c_2' \rangle \simeq \mathfrak{S}_5 \times C_2.$$

In the GAP notation (or the notation in [**1**]), these groups are (4,31,3), (4,31,4), (4,31,5), (4,31,6), (4,31,7) respectively.

Let $K = \boldsymbol{Q}(x_1, x_2, x_3, x_4, x_5)$. The symmetric group $\mathfrak{S}_5$ acts naturally on K. It is well known that $K^{\mathfrak{S}_5} = \boldsymbol{Q}(s_1, s_2, s_3, s_4, s_5)$, where s_i is the i-th elementary symmetric polynomial.

Maeda proved the rationality of $K^{\mathfrak{A}_5}$, giving the generators $\{F_i\}$ of $K^{\mathfrak{A}_5}$ [**11**].

According to Maeda's result, $K^{\mathfrak{A}_5} = \boldsymbol{Q}(F_1, F_2, F_3, F_4, F_5)$,

$$F_1 := \frac{\sum_{\sigma\in\mathfrak{S}_5} \sigma([12][13][14][15][23]^4[45]^4 x_1)}{\sum_{\sigma\in\mathfrak{S}_5} \sigma([12][13][14][15][23]^4[45]^4)}$$

$$F_2 := \frac{\sum_{\sigma\in\mathfrak{S}_5} \sigma([12]^3[13]^3[14]^3[15]^3[23]^{10}[45]^{10})}{\prod_{i<j}[ij]^2 \cdot \sum_{\sigma\in\mathfrak{S}_5} \sigma([12][13][14][15][23]^4[45]^4)}$$

$$F_3 := \frac{\sum_{\sigma\in\mathfrak{S}_5}\sigma([12]^3[13]^3[14]^3[15]^3[23]^{10}[45]^{10}x_1)}{\prod_{i<j}[ij]^2\cdot\sum_{\sigma\in\mathfrak{S}_5}\sigma([12][13][14][15][23]^4[45]^4)}$$

$$F_4 := \frac{\sum_{i=1}^{10}\mu_i([12]^2[13]^2[23]^2[45]^4)}{\prod_{i<j}[ij]}$$

$$F_5 := \frac{\sum_{i=1}^{10}\mu_i([12]^2[13]^2[23]^2[14]^4[24]^4[34]^4[15]^4[25]^4[35]^4)}{\prod_{i<j}[ij]^3}$$

where $[i\ j] := x_i - x_j$, $H := \langle(12),(13),(45)\rangle$, $\mathfrak{S}_5 = \sqcup_{i=1}^{10}\mu_i H$.

The rationality of $\boldsymbol{Q}(x_1,x_2,x_3,x_4)^G$ can be reduced to Maeda's result. The reduction is rather easy except for (4,31,5). As for (4,31,5), the reduction is complicated, and we need the concrete expression of the change of generators of $K^{\mathfrak{S}_5}$.

2.1. Change of generators of $K^{\mathfrak{S}_5}$.

Since F_1, F_2, F_3 are $\mathfrak{S}_5$-invariant and the transposition (1 2) maps F_i to $-F_i$ for $i = 4$ and 5, from Maeda's result we have $K^{\mathfrak{S}_5} = \boldsymbol{Q}(F_1, F_2, F_3, F_4^2, F_4F_5)$.

In this subsection, we will write each elementary symmetric polynomial s_i as a rational function of F_1, F_2, F_3, F_4^2 and F_4F_5.

For this purpose, we define the action of $GL(2,\boldsymbol{Q})$ on K and study the behavior of F_i under this action. $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in GL(2,\boldsymbol{Q})$ acts on K as $x_i \mapsto (ax_i+b)/(cx_i+d)$ $(1 \le i \le 5)$. This group $\mathscr{G}$ commutes with $\mathfrak{S}_5$, therefore it preserves $K^{\mathfrak{S}_5}$ and induces an action on $K^{\mathfrak{S}_5}$.

$\mathscr{G}$ is generated by the following $\sigma_\alpha, \tau_\lambda, *$ $(\alpha \in \boldsymbol{Q}, \lambda \in \boldsymbol{Q}^\times)$.

$$\sigma_\alpha : x_i \mapsto x_i + \alpha$$

$$\tau_\lambda : x_i \mapsto \lambda x_i$$

$$* : x_i \mapsto \frac{1}{x_i}.$$

F_2, F_4, F_5 are invariant under σ_α, while $F_1 \mapsto F_1+\alpha$ and $F_3/F_2 \mapsto F_3/F_2+\alpha$. If we define X_3 as $X_3 := F_1 - F_3/F_2$, then we have $K^{\mathfrak{S}_5} = \boldsymbol{Q}(F_1, F_2, X_3, F_4^2, F_4F_5)$ and F_2, X_3, F_4, F_5 are invariant under σ_α. Now we will write each s_i as a rational function of F_1, F_2, X_3, F_4, F_5.

The action of σ_α on s_i can be written as follows.

$$s_1 \mapsto s_1 + 5\alpha$$

$$s_2 \mapsto s_2 + 4\alpha s_1 + 10\alpha^2$$

$$s_3 \mapsto s_3 + 3\alpha s_2 + 6\alpha^2 s_1 + 10\alpha^3$$

$$
\begin{aligned}
s_4 &\mapsto s_4 + 2\alpha s_3 + 3\alpha^2 s_2 + 4\alpha^3 s_1 + 5\alpha^4 \\
s_5 &\mapsto s_5 + \alpha s_4 + \alpha^2 s_3 + \alpha^3 s_2 + \alpha^4 s_1 + \alpha^5.
\end{aligned}
$$

Differentiating s_i by α and substituting $\alpha = 0$, we get

$$
\frac{\partial s_1}{\partial F_1} = 5, \quad \frac{\partial s_2}{\partial F_1} = 4s_1, \quad \frac{\partial s_3}{\partial F_1} = 3s_2, \quad \frac{\partial s_4}{\partial F_1} = 2s_3, \quad \frac{\partial s_5}{\partial F_1} = s_4.
$$

Therefore we get

$$
\begin{aligned}
s_1 &= 5F_1 + g_1 \\
s_2 &= 10F_1^2 + 4g_1F_1 + g_2 \\
s_3 &= 10F_1^3 + 6g_1F_1^2 + 3g_2F_1 + g_3 \\
s_4 &= 5F_1^4 + 4g_1F_1^3 + 3g_2F_1^2 + 2g_3F_1 + g_4 \\
s_5 &= F_1^5 + g_1F_1^4 + g_2F_1^3 + g_3F_1^2 + g_4F_1 + g_5
\end{aligned}
$$

where each g_i is a rational function of F_2, X_3, F_4^2, F_4F_5.

Next we determine the X_3-dependence of g_i. F_2, F_4^2, F_4F_5 are invariant under τ_λ, and $X_3 \mapsto \lambda X_3$. Since τ_λ maps s_i to $\lambda^i s_i$, we can write $g_i = X_3^i h_i$ where h_i is a rational function of F_2, F_4^2, F_4F_5. Now we get

$$
\begin{aligned}
s_1 &= 5F_1 + X_3h_1 \\
s_2 &= 10F_1^2 + 4F_1X_3h_1 + X_3^2h_2 \\
s_3 &= 10F_1^3 + 6F_1^2X_3h_1 + 3F_1X_3^2h_2 + X_3^3h_3 \\
s_4 &= 5F_1^4 + 4F_1^3X_3h_1 + 3F_1^2X_3^2h_2 + 2F_1X_3^3h_3 + X_3^4h_4 \\
s_5 &= F_1^5 + F_1^4X_3h_1 + F_1^3X_3^2h_2 + F_1^2X_3^3h_3 + F_1X_3^4h_4 + X_3^5h_5.
\end{aligned}
$$

Next we determine the F_2-dependence of h_i. Let us see the action of $* : x_i \mapsto 1/x_i$. Since $[ij]^* = (x_i - x_j)^* = 1/x_i - 1/x_j = -[ij]/x_ix_j$, $\Delta = \prod_{i<j}[ij]$ is mapped to $\Delta^* = \Delta/s_5^4$. Since $([12][13][14][15][23]^4[45]^4)^* = x_1([12][13][14][15][23]^4[45]^4)/s_5^5$, we see that $F_1^* = 1/F_1$. Similarly we see that $(F_3/F_2)^* = 1/(F_3/F_2) = F_2/F_3$. Since $([12]^2[13]^2[23]^2[45]^4)^* = ([12]^2[13]^2[23]^2[45]^4)/s_5^4$, we see that $F_4^* = F_4$. Since $([12]^2[13]^2[23]^2[14]^4[24]^4[34]^4[15]^4[25]^4[35]^4)^* = ([12]^2[13]^2[23]^2[14]^4[24]^4[34]^4[15]^4[25]^4[35]^4)/s_5^{12}$, we see that $F_5^* = F_5$.

F_2^* is rather complicated. Since $([12]^3[13]^3[14]^3[15]^3[23]^{10}[45]^{10})^* =$

$x_1([12]^3[13]^3[14]^3[15]^3[23]^{10}[45]^{10})/s_5^{13}$, $F_2^* = F_3/F_1$. We define Z as $Z := F_3/F_1$ then $K^{\mathfrak{S}_5} = \boldsymbol{Q}(F_1, F_2, Z, F_4^2, F_4F_5)$ and $F_1^* = 1/F_1$, $F_4^* = F_4$, $F_5^* = F_5$, $F_2^* = Z$, $Z^* = F_2$. X_3 is written as $X_3 = F_1(1 - Z/F_2)$.

On the other hand $s_1^* = (x_1 + x_2 + x_3 + x_4 + x_5)^* = 1/x_1 + 1/x_2 + 1/x_3 + 1/x_4 + 1/x_5 = s_4/s_5$, similarly $s_2^* = s_3/s_5$, $s_3^* = s_2/s_5$, $s_4^* = s_1/s_5$, $s_5^* = 1/s_5$. (If we write $s_0 = 1$, $s_i^* = s_{5-i}/s_5$ $(0 \leq i \leq 5)$). Therefore $s_1 = s_4^*/s_5^*$, $s_2 = s_3^*/s_5^*$, $s_3 = s_2^*/s_5^*$, $s_4 = s_1^*/s_5^*$, $s_5 = 1/s_5^*$.

s_i/F_1^i can be viewed as a polynomial of degree i whose indeterminate is Z and coefficients are in $\boldsymbol{Q}(F_2, F_4^2, F_4F_5)$.

$$\frac{s_1}{F_1} = 5 + (F_2 - Z)\widetilde{h_1}$$
$$\frac{s_2}{F_1^2} = 10 + 4(F_2 - Z)\widetilde{h_1} + (F_2 - Z)^2\widetilde{h_2}$$
$$\frac{s_3}{F_1^3} = 10 + 6(F_2 - Z)\widetilde{h_1} + 3(F_2 - Z)^2\widetilde{h_2} + (F_2 - Z)^3\widetilde{h_3}$$
$$\frac{s_4}{F_1^4} = 5 + 4(F_2 - Z)\widetilde{h_1} + 3(F_2 - Z)^2\widetilde{h_2} + 2(F_2 - Z)^3\widetilde{h_3} + (F_2 - Z)^4\widetilde{h_4}$$
$$\frac{s_5}{F_1^5} = 1 + (F_2 - Z)\widetilde{h_1} + (F_2 - Z)^2\widetilde{h_2} + (F_2 - Z)^3\widetilde{h_3} + (F_2 - Z)^4\widetilde{h_4} + (F_2 - Z)^5\widetilde{h_5},$$

where $\widetilde{h_i} = h_i/F_2^i$. Since $s_i/F_1^i = F_1^{5-i}s_{5-i}^*/F_1^5 s_5^*$, we can write $\widetilde{h_1}, \ldots, \widetilde{h_5}$ successively as follows.

$$\widetilde{h_1} = \frac{\widetilde{h_1}^* - 2(F_2 - Z)\widetilde{h_2}^* + 3(F_2 - Z)^2\widetilde{h_3}^* - 4(F_2 - Z)^3\widetilde{h_4}^* + 5(F_2 - Z)^4\widetilde{h_5}^*}{1 - (F_2 - Z)\widetilde{h_1}^* + (F_2 - Z)^2\widetilde{h_2}^* - (F_2 - Z)^3\widetilde{h_3}^* + (F_2 - Z)^4\widetilde{h_4}^* - (F_2 - Z)^5\widetilde{h_5}^*}$$
$$\widetilde{h_2} = \frac{\widetilde{h_2}^* - 3(F_2 - Z)\widetilde{h_3}^* + 6(F_2 - Z)^2\widetilde{h_4}^* - 10(F_2 - Z)^3\widetilde{h_5}^*}{1 - (F_2 - Z)\widetilde{h_1}^* + (F_2 - Z)^2\widetilde{h_2}^* - (F_2 - Z)^3\widetilde{h_3}^* + (F_2 - Z)^4\widetilde{h_4}^* - (F_2 - Z)^5\widetilde{h_5}^*}$$
$$\widetilde{h_3} = \frac{\widetilde{h_3}^* - 4(F_2 - Z)\widetilde{h_4}^* + 10(F_2 - Z)^2\widetilde{h_5}^*}{1 - (F_2 - Z)\widetilde{h_1}^* + (F_2 - Z)^2\widetilde{h_2}^* - (F_2 - Z)^3\widetilde{h_3}^* + (F_2 - Z)^4\widetilde{h_4}^* - (F_2 - Z)^5\widetilde{h_5}^*}$$
$$\widetilde{h_4} = \frac{\widetilde{h_4}^* - 5(F_2 - Z)\widetilde{h_5}^*}{1 - (F_2 - Z)\widetilde{h_1}^* + (F_2 - Z)^2\widetilde{h_2}^* - (F_2 - Z)^3\widetilde{h_3}^* + (F_2 - Z)^4\widetilde{h_4}^* - (F_2 - Z)^5\widetilde{h_5}^*}$$
$$\widetilde{h_5} = \frac{\widetilde{h_5}^*}{1 - (F_2 - Z)\widetilde{h_1}^* + (F_2 - Z)^2\widetilde{h_2}^* - (F_2 - Z)^3\widetilde{h_3}^* + (F_2 - Z)^4\widetilde{h_4}^* - (F_2 - Z)^5\widetilde{h_5}^*}.$$

$\widetilde{h_i}^*$ is a rational function of Z whose coefficients are in $\boldsymbol{Q}(F_4^2, F_4F_5)$. Let D be the least common multiplier of the denominators of $\widetilde{h_i}^*$, and let $\widetilde{\widetilde{h_i}}^*$ be $D\widetilde{h_i}^*$.

$$\text{the denominator of } \widetilde{h_1} = D - (F_2 - Z)\widetilde{\widetilde{h_1}}^* + \cdots$$
$$\text{the numerator of } \widetilde{h_1} = \widetilde{\widetilde{h_1}}^* - 2(F_2 - Z)\widetilde{\widetilde{h_2}}^* + \cdots$$

Both D and $\widetilde{\widetilde{h_1}}^*$ are polynomials whose indeterminate is Z. If we substitute $Z = 0$, we get the constant term. Let α_i and α_0 be the constant term of $\widetilde{\widetilde{h_i}}^*$ and D respectively. As $\widetilde{h_i}$ does not depend on Z, we get

$$\begin{aligned}
\widetilde{h_1} &= \frac{\alpha_1 - 2F_2\alpha_2 + 3F_2^2\alpha_3 - 4F_2^3\alpha_4 + 5F_2^4\alpha_5}{\alpha_0 - F_2\alpha_1 + F_2^2\alpha_2 - F_2^3\alpha_3 + F_2^4\alpha_4 - F_2^5\alpha_5} \\
\widetilde{h_2} &= \frac{\alpha_2 - 3F_2\alpha_3 + 6F_2^2\alpha_4 - 10F_2^3\alpha_5}{\alpha_0 - F_2\alpha_1 + F_2^2\alpha_2 - F_2^3\alpha_3 + F_2^4\alpha_4 - F_2^5\alpha_5} \\
\widetilde{h_3} &= \frac{\alpha_3 - 4F_2\alpha_4 + 10F_2^2\alpha_5}{\alpha_0 - F_2\alpha_1 + F_2^2\alpha_2 - F_2^3\alpha_3 + F_2^4\alpha_4 - F_2^5\alpha_5} \\
\widetilde{h_4} &= \frac{\alpha_4 - 5F_2\alpha_5}{\alpha_0 - F_2\alpha_1 + F_2^2\alpha_2 - F_2^3\alpha_3 + F_2^4\alpha_4 - F_2^5\alpha_5} \\
\widetilde{h_5} &= \frac{\alpha_5}{\alpha_0 - F_2\alpha_1 + F_2^2\alpha_2 - F_2^3\alpha_3 + F_2^4\alpha_4 - F_2^5\alpha_5}.
\end{aligned}$$

Let D_h be the common denominator of $\widetilde{h_i}$, i.e. $D_h = \alpha_0 - F_2\alpha_1 + F_2^2\alpha_2 - F_2^3\alpha_3 + F_2^4\alpha_4 - F_2^5\alpha_5$, then we can write s_i as follows.

$$\begin{aligned}
\frac{s_1}{F_1} &= 5 + \frac{1}{D_h}(F_2 - Z)\varphi_1 \\
\frac{s_2}{F_1^2} &= 10 + \frac{1}{D_h}\{4(F_2 - Z)\varphi_1 + (F_2 - Z)^2\varphi_2\} \\
\frac{s_3}{F_1^3} &= 10 + \frac{1}{D_h}\{6(F_2 - Z)\varphi_1 + 3(F_2 - Z)^2\varphi_2 + (F_2 - Z)^3\varphi_3\} \\
\frac{s_4}{F_1^4} &= 5 + \frac{1}{D_h}\{4(F_2 - Z)\varphi_1 + 3(F_2 - Z)^2\varphi_2 + 2(F_2 - Z)^3\varphi_3 + (F_2 - Z)^4\varphi_4\} \\
\frac{s_5}{F_1^5} &= 1 + \frac{1}{D_h}\{(F_2 - Z)\varphi_1 + (F_2 - Z)^2\varphi_2 + (F_2 - Z)^3\varphi_3 \\
&\qquad + (F_2 - Z)^4\varphi_4 + (F_2 - Z)^5\varphi_5\}
\end{aligned}$$

where

$$\begin{aligned}
\varphi_1 &= \alpha_1 - 2F_2\alpha_2 + 3F_2^2\alpha_3 - 4F_2^3\alpha_4 + 5F_2^4\alpha_5 \\
\varphi_2 &= \alpha_2 - 3F_2\alpha_3 + 6F_2^2\alpha_4 - 10F_2^3\alpha_5 \\
\varphi_3 &= \alpha_3 - 4F_2\alpha_4 + 10F_2^2\alpha_5 \\
\varphi_4 &= \alpha_4 - 5F_2\alpha_5 \\
\varphi_5 &= \alpha_5.
\end{aligned}$$

α_i is a polynomial of F_4^2 and F_4F_5.

The behaviors of F_i under the action of $GL(2,\boldsymbol{Q})$ are now used up, and we need direct calculations by a computer to determine the concrete expression of α_i. The results are as follows.

$$\begin{aligned}
\alpha_0 = {} & 23F_4^{10} + 16F_5F_4^9 + 8452F_4^8 - 6400F_5F_4^7 + \big(-6656F_5^2 + 461792\big)F_4^6 \\
& - 3518976F_5F_4^5 + \big(-55296F_5^2 - 43334016\big)F_4^4 \\
& + \big(786432F_5^3 - 218943488F_5\big)F_4^3 + \big(378937344F_5^2 + 14052788992\big)F_4^2 \\
& + \big(87031808F_5^3 + 23540633600F_5\big)F_4 \\
& + \big(-18874368F_5^4 + 32835076096F_5^2 + 1907825841152\big) \\
\alpha_1 = {} & 2F_4^{10} + 2168F_4^8 + 396352F_4^6 - 647168F_5F_4^5 + \big(-139264F_5^2 + 5049088\big)F_4^4 \\
& - 119472128F_5F_4^3 + \big(45809664F_5^2 + 2694478336\big)F_4^2 \\
& + \big(12582912F_5^3 + 904003584F_5\big)F_4 + \big(8465022976F_5^2 + 612411627520\big) \\
\alpha_2 = {} & 128F_4^8 + 65024F_4^6 - 24576F_5F_4^5 + 5769216F_4^4 - 14745600F_5F_4^3 \\
& + \big(655360F_5^2 + 260513792\big)F_4^2 - 805175296F_5F_4 \\
& + \big(696778752F_5^2 + 75769774080\big) \\
\alpha_3 = {} & 3072F_4^6 + 798720F_4^4 - 524288F_5F_4^3 + 35995648F_4^2 - 96468992F_5F_4 \\
& + \big(18874368F_5^2 + 4566876160\big) \\
\alpha_4 = {} & 32768F_4^4 + 3604480F_4^2 - 3145728F_5F_4 + 135004160 \\
\alpha_5 = {} & 131072F_4^2 + 1572864.
\end{aligned}$$

2.2. Affirmative answers for 5 non-solvable groups.

The set of all homogeneous rational functions of degree 0 is a subfield of $\boldsymbol{Q}(x_1, x_2, x_3, x_4, x_5)$. We denote it with $\boldsymbol{Q}(x_1, x_2, x_3, x_4, x_5)_0$. We define $y_i := x_i - (1/5)s_1$ and $z_i = y_i/s_1$, then $\boldsymbol{Q}(x_1, x_2, x_3, x_4, x_5)_0 = \boldsymbol{Q}(z_1, z_2, z_3, z_4)$ and $\sum_i z_i = 0$. $\boldsymbol{Q}(x_1, x_2, x_3, x_4, x_5)_0$ is purely transcendental of degree 4 over $\boldsymbol{Q}$. $\mathfrak{S}_5$ acts linearly on $\boldsymbol{Q}(z_1, z_2, z_3, z_4)$, where $(12), (12345)$ and $(13)(24) \in \mathfrak{S}_5$ corresponds to mc_4, c_5 and γ respectively.

From Maeda's result we get $\boldsymbol{Q}(z_1, z_2, z_3, z_4)^{\langle c_5, \gamma\rangle} = \boldsymbol{Q}(F_1F_2/F_3, F_2, F_4, F_5)$, $\boldsymbol{Q}(z_1, z_2, z_3, z_4)^{\langle c_5, mc_4\rangle} = \boldsymbol{Q}(F_1F_2/F_3, F_2, F_4^2, F_4F_5)$.

In order to determine the fixed fields of other 3 groups, we define a new action $\rho : x_i \mapsto (2/5)s_1 - x_i$ on $\boldsymbol{Q}(x_1, x_2, x_3, x_4, x_5)$. ρ maps $y_i \mapsto -y_i$, $s_1 \mapsto s_1$, $z_i \mapsto -z_i$. ρ acts linearly on $\boldsymbol{Q}(z_1, z_2, z_3, z_4)$ and it corresponds to $c_2c_2' \in GL(4, \boldsymbol{Q})$.

F_1F_2/F_3 and F_2 are invariant under $(ij) \in \mathfrak{S}_5$ while $(ij)F_4 = -F_4$ and $(ij)F_5 = -F_5$. F_2, F_4, F_5 are invariant under ρ, but the action of ρ on F_1F_2/F_3 is rather complicated.

Since $\rho x_i = \frac{2}{5}s_1 - x_i$, ρ maps

$$F_1 \mapsto \frac{2}{5}s_1 - F_1, \quad \frac{F_3}{F_2} \mapsto \frac{2}{5}s_1 - \frac{F_3}{F_2}.$$

Therefore

$$\frac{F_1F_2}{F_3} \mapsto \frac{\frac{2}{5}s_1 - F_1}{\frac{2}{5}s_1 - \frac{F_3}{F_2}} = \frac{\frac{2}{5}\frac{s_1}{F_1} - 1}{\frac{2}{5}\frac{s_1}{F_1} - \frac{F_3}{F_1F_2}},$$

$$\frac{F_3}{F_1F_2} \mapsto \frac{\frac{2}{5}\frac{s_1}{F_1} - \frac{F_3}{F_1F_2}}{\frac{2}{5}\frac{s_1}{F_1} - 1}.$$

As we have seen in Section 2.1, s_1/F_1 can be written as $s_1/F_1 = 5 + (1 - (F_3/F_1F_2))h_1 = 5 + h_1 - (F_3/F_1F_2)h_1$, $h_1 \in \boldsymbol{Q}(F_2, F_4^2, F_4F_5)$. Let $f = (2/5)h_1 + 1$ then $f \in \boldsymbol{Q}(F_2, F_4^2, F_4F_5)$ and f is invariant under $\mathfrak{S}_5$ and

$$\rho : \frac{F_3}{F_1F_2} \mapsto \frac{-\frac{fF_3}{F_1F_2} + f + 1}{-(f-1)\frac{F_3}{F_1F_2} + f}.$$

Let $X = F_3/F_1F_2 - f/(f-1)$ then

$$X \mapsto \frac{-\frac{fF_3}{F_1F_2} + f + 1}{-(f-1)\frac{F_3}{F_1F_2} + f} - \frac{f}{f-1} = \frac{f + 1 - \frac{f^2}{f-1}}{-(f-1)\frac{F_3}{F_1F_2} + f} = \frac{-\frac{1}{f-1}}{-(f-1)X} = \frac{\frac{1}{(f-1)^2}}{X},$$

so that letting $Y = (f-1)X$, we have

$$Y \mapsto (f-1)\frac{\frac{1}{(f-1)^2}}{X} = \frac{1}{X(f-1)} = \frac{1}{Y}.$$

From these actions of ρ, we obtain the following result.

$\boldsymbol{Q}(F_3/F_1F_2, F_2, F_4, F_5) = \boldsymbol{Q}(Y, F_2, F_4, F_5)$ and

$$\boldsymbol{Q}(z_1, z_2, z_3, z_4)^{\langle c_5, \gamma, c_2c_2' \rangle} = \boldsymbol{Q}\left(Y + \frac{1}{Y}, F_2, F_4, F_5\right),$$

$$\boldsymbol{Q}(z_1, z_2, z_3, z_4)^{\langle c_5, mc_4, c_2c_2' \rangle} = \boldsymbol{Q}\left(Y + \frac{1}{Y}, F_2, F_4^2, F_4F_5\right).$$

Because $\rho(12)$ maps $Y \mapsto 1/Y$, $F_2 \mapsto F_2$, $F_4 \mapsto -F_4$, $F_5 \mapsto -F_5$,

$$\boldsymbol{Q}(z_1, z_2, z_3, z_4)^{\langle c_5, c_2c_2'mc_4 \rangle} = \boldsymbol{Q}\left(Y + \frac{1}{Y}, F_2, \frac{F_4}{Y - 1/Y}, F_4F_5\right).$$

Note that for $G = \langle c_5, \gamma, c_2c_2' \rangle$ and $\langle c_5, mc_4, c_2c_2' \rangle$, the rationality of $\boldsymbol{Q}(z_1, z_2, z_3, z_4)^G$ is a direct result of Lüroth's Theorem if we are not required to give the concrete generator $Y + 1/Y$. As for $G = \langle c_5, c_2c_2'mc_4 \rangle$, the rationality is proved only after the above calculations.

3. Reflection and related groups.

In this section, we consider the case when G is a reflection group.

Definition 2. An element g of $GL(n, \boldsymbol{Q})$ is called a reflection if and only if the characteristic polynomial $\Phi_g(x)$ of g is $(x-1)^{n-1}(x+1)$. A finite subgroup G of $GL(n, \boldsymbol{Q})$ is called a reflection group if and only if G is generated by reflection elements.

If G is a reflection subgroup of $GL(n, \boldsymbol{Q})$, then linear Noether's problem is true for G ([**3**, Chapter 7]).

Theorem 6. *Suppose G is a reflection subgroup of $GL(n, \boldsymbol{Q})$, then there exist basic generators $f_1, \ldots, f_n$ such that $\boldsymbol{Q}(x_1, \ldots, x_n)^G = \boldsymbol{Q}(f_1, \ldots, f_n)$, each f_i is a polynomial over $\boldsymbol{Q}$, and $f_1, \ldots, f_n$ are algebraically independent over $\boldsymbol{Q}$.*

3.1. The largest subgroup G_{1152}.

We will apply this theorem to the linear Noether's problem for

$$G_{1152} = \left\langle \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \right\rangle.$$

G_{1152} is the largest subgroup of $GL(4, \boldsymbol{Q})$ with the order of $2^m 3^n$, and the gap code is $(4, 33, 16)$.

Since G_{1152} is a reflection group, $\boldsymbol{Q}(x_1, x_2, x_3, x_4)^{G_{1152}}$ has generators consisting of polynomials. The Hilbert polynomial of G_{1152} is

$$H_{G_{1152}}(X) = \frac{1}{(1 - X^2)(1 - X^6)(1 - X^8)(1 - X^{12})},$$

so the degrees of generators must be 2,6,8 and 12.

A calculation by computer gives the concrete basis as follows.

$$f_1 = \sum_i x_i^2,$$

$$f_2 = \sum_{i,j} x_i^4 x_j^2 - 3 \sum_{i,j,k} x_i^2 x_j^2 x_k^2,$$

$$f_3 = \sum_{i,j} x_i^4 x_j^4 - \sum_{i,j,k} x_i^4 x_j^2 x_k^2 + 6 x_1^2 x_2^2 x_3^2 x_4^2,$$

$$f_4 = 2 \sum_{i,j} x_i^6 x_j^6 - 3 \sum_{i,j,k} x_i^6 x_j^4 x_k^2 + 12 \sum_{i,j,k,l} x_i^6 x_j^2 x_k^2 x_l^2 + 12 \sum_{i,j,k} x_i^4 x_j^4 x_k^4 - 6 \sum_{i,j,k,l} x_i^4 x_j^4 x_k^2 x_l^2.$$

The conjugate elements of x_1 by the action of G_{1152} are $\pm x_1, \pm x_2, \pm x_3, \pm x_4$, $1/2(\pm x_1 \pm x_2 \pm x_3 \pm x_4)$. We make a polynomial of degree 24 over $\boldsymbol{Q}(x_1, x_2, x_3, x_4)^G$ whose roots are the conjugate elements of x_1, then the obtained polynomial is a generic polynomial of G_{1152}. The coefficients of odd degrees are 0.

The coefficient of X^{22} through X^0 is as follows.

$$-3f_1$$

$$\frac{15}{4} f_1^2$$

$$-\frac{21}{8} f_1^3 + \frac{1}{2} f_2$$

$$\frac{147}{128} f_1^4 - \frac{11}{16} f_2 f_1 - \frac{5}{8} f_3$$

$$-\frac{21}{64}f_1^5+\frac{3}{8}f_2f_1^2+\frac{3}{4}f_3f_1$$

$$\frac{63}{1024}f_1^6-\frac{25}{256}f_2f_1^3-\frac{149}{384}f_3f_1^2+\left(-\frac{13}{384}f_2^2+\frac{5}{32}f_4\right)$$

$$-\frac{15}{2048}f_1^7+\frac{5}{512}f_2f_1^4+\frac{89}{768}f_3f_1^3+\left(\frac{25}{768}f_2^2-\frac{19}{192}f_4\right)f_1-\frac{1}{12}f_3f_2$$

$$\frac{33}{65536}f_1^8+\frac{3}{4096}f_2f_1^5-\frac{137}{6144}f_3f_1^4+\left(-\frac{35}{3072}f_2^2+\frac{3}{128}f_4\right)f_1^2+\frac{15}{256}f_3f_2f_1-\frac{15}{256}f_3^2$$

$$-\frac{1}{65536}f_1^9-\frac{1}{4096}f_2f_1^6+\frac{17}{6144}f_3f_1^5+\left(\frac{5}{3072}f_2^2-\frac{11}{3456}f_4\right)f_1^3-\frac{1}{72}f_3f_2f_1^2$$
$$+\frac{13}{768}f_3^2f_1+\left(\frac{5}{6912}f_2^3+\frac{1}{192}f_4f_2\right)$$

$$\frac{1}{65536}f_2f_1^7-\frac{19}{98304}f_3f_1^6+\left(-\frac{5}{98304}f_2^2+\frac{25}{73728}f_4\right)f_1^4+\frac{7}{6144}f_3f_2f_1^3$$
$$-\frac{41}{18432}f_3^2f_1^2+\left(-\frac{13}{36864}f_2^3-\frac{17}{9216}f_4f_2\right)f_1+\left(\frac{37}{18432}f_3f_2^2+\frac{1}{512}f_4f_3\right)$$

$$\frac{1}{196608}f_3f_1^7+\left(-\frac{1}{196608}f_2^2-\frac{1}{49152}f_4\right)f_1^5+\frac{1}{12288}f_3^2f_1^3$$
$$+\left(\frac{1}{24576}f_2^3+\frac{1}{6144}f_4f_2\right)f_1^2+\left(-\frac{5}{12288}f_3f_2^2-\frac{1}{3072}f_4f_3\right)f_1+\frac{1}{1536}f_3^2f_2$$

$$\frac{1}{1769472}f_4f_1^6-\frac{1}{589824}f_3f_2f_1^5+\frac{1}{589824}f_3^2f_1^4+\left(\frac{1}{1769472}f_2^3-\frac{1}{147456}f_4f_2\right)f_1^3$$
$$+\left(\frac{5}{294912}f_3f_2^2+\frac{1}{73728}f_4f_3\right)f_1^2-\frac{1}{18432}f_3^2f_2f_1$$
$$+\left(-\frac{1}{196608}f_2^4+\frac{1}{73728}f_4f_2^2+\left(\frac{1}{27648}f_3^3-\frac{1}{110592}f_4^2\right)\right).$$

3.2. The second largest group G_{576}.

The derived subgroup of G_{1152} is

$$G_{288}=\left\langle \begin{pmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}, \begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1&1&1&-1\\1&1&-1&1\\1&-1&1&1\\1&-1&-1&-1\end{pmatrix}\right\rangle,$$

and the gap code is $(4, 33, 13)$.

$G_{1152}/G_{288} \simeq C_2 \times C_2$ and we can choose representatives

$$E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ m' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ \gamma_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ c_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$G_{576a} = \langle G_{288}, m' \rangle$, $G_{576b} = \langle G_{288}, \gamma_2 \rangle$, $G_{576c} = \langle G_{288}, c_4 \rangle$ are groups of order 576 and the gap code is $(4, 33, 14)$, $(4, 33, 14)$, $(4, 33, 15)$ respectively.

G_{576a} and G_{576b} are conjugate in $GL(4, \boldsymbol{Q})$, while $G_{576c} = G_{1152} \cap SL(4, \boldsymbol{Q})$.

Since the Hilbert polynomials of $G_{288}, G_{576a}, G_{576c}$ are $H_{G_{288}}(X) = (1 + 2X^{12} + X^{24})/((1 - X^2)(1 - X^6)(1 - X^8)(1 - X^{12}))$, $H_{G_{576a}}(X) = H_{G_{576b}}(X) = (1 + X^{12})/((1 - X^2)(1 - X^6)(1 - X^8)(1 - X^{12}))$, $H_{G_{576c}}(X) = (1 + X^{24})/((1 - X^2)(1 - X^6)(1 - X^8)(1 - X^{12}))$, we see that

$$\begin{aligned} \boldsymbol{Q}[x_1, x_2, x_3, x_4]^{G_{288}} &= \boldsymbol{Q}[f_1, f_2, f_3, f_4] \oplus f_a \boldsymbol{Q}[f_1, f_2, f_3, f_4] \\ &\qquad \oplus f_b \boldsymbol{Q}[f_1, f_2, f_3, f_4] \oplus f_c \boldsymbol{Q}[f_1, f_2, f_3, f_4], \\ \boldsymbol{Q}[x_1, x_2, x_3, x_4]^{G_{576a}} &= \boldsymbol{Q}[f_1, f_2, f_3, f_4] \oplus f_a \boldsymbol{Q}[f_1, f_2, f_3, f_4], \\ \boldsymbol{Q}[x_1, x_2, x_3, x_4]^{G_{576b}} &= \boldsymbol{Q}[f_1, f_2, f_3, f_4] \oplus f_b \boldsymbol{Q}[f_1, f_2, f_3, f_4], \\ \boldsymbol{Q}[x_1, x_2, x_3, x_4]^{G_{576c}} &= \boldsymbol{Q}[f_1, f_2, f_3, f_4] \oplus f_c \boldsymbol{Q}[f_1, f_2, f_3, f_4] \end{aligned}$$

where the degrees of f_a, f_b, f_c are $12, 12, 24$ respectively. By a computer calculations, we can determine them concretely. The results are as follows.

$$\begin{aligned} f_a &= \sum_{\sigma \in \mathfrak{S}_4} \operatorname{sgn}(\sigma) x_{\sigma(1)}^6 x_{\sigma(2)}^4 x_{\sigma(3)}^2 = \prod_{1 \le i < j \le 4} \left(x_i^2 - x_j^2 \right), \\ f_b &= \sum_{i,j,k,l} x_i^9 x_j x_k x_l - 4 \sum_{i,j,k,l} x_i^7 x_j^3 x_k x_l + 6 \sum_{i,j,k,l} x_i^5 x_j^5 x_k x_l \\ &\quad + 4 \sum_{i,j,k,l} x_i^5 x_j^3 x_k^3 x_l - 40 x_1^3 x_2^3 x_3^3 x_4^3 \\ &= x_1 x_2 x_3 x_4 \prod_{i=0,1} \prod_{j=0,1} \prod_{k=0,1} \left\{ x_1 + (-1)^i x_2 + (-1)^j x_3 + (-1)^k x_3 \right\}, \end{aligned}$$

$$\begin{aligned}
f_c = f_a f_b = &\sum_{\sigma\in\mathfrak{S}_4} \mathrm{sgn}(\sigma) x_{\sigma(1)}^{15} x_{\sigma(2)}^{5} x_{\sigma(3)}^{3} x_{\sigma(4)} - 5 \sum_{\sigma\in\mathfrak{S}_4} \mathrm{sgn}(\sigma) x_{\sigma(1)}^{13} x_{\sigma(2)}^{7} x_{\sigma(3)}^{3} x_{\sigma(4)} \\
&+ 10 \sum_{\sigma\in\mathfrak{S}_4} \mathrm{sgn}(\sigma) x_{\sigma(1)}^{11} x_{\sigma(2)}^{9} x_{\sigma(3)}^{3} x_{\sigma(4)} \\
&+ 3 \sum_{\sigma\in\mathfrak{S}_4} \mathrm{sgn}(\sigma) x_{\sigma(1)}^{11} x_{\sigma(2)}^{7} x_{\sigma(3)}^{5} x_{\sigma(4)} \\
&- 55 \sum_{\sigma\in\mathfrak{S}_4} \mathrm{sgn}(\sigma) x_{\sigma(1)}^{9} x_{\sigma(2)}^{7} x_{\sigma(3)}^{5} x_{\sigma(4)}^{3}.
\end{aligned}$$

They are invariant under G_{288} and the actions of m', γ_2, c_4 are as follows.

$$\begin{aligned}
m' &: f_a \mapsto f_a, f_b \mapsto -f_b, f_c \mapsto -f_c, \\
\gamma_2 &: f_a \mapsto -f_a, f_b \mapsto f_b, f_c \mapsto -f_c, \\
c_4 &: f_a \mapsto -f_a, f_b \mapsto -f_b, f_c \mapsto f_c.
\end{aligned}$$

From the relation $f_a^2 = -f_4^2/27 + 4f_3^3/27$, we see that the linear Noether's problem for G_{576a} is positive, and the transcendental basis is $f_1, f_2, f_4/f_3, f_a/f_3$.

Thus the linear Noether's problems are affirmative for G_{1152} and G_{576a}, the largest and one of the second largest finite subgroups of $GL(4, \boldsymbol{Q})$.

Appendix. Higher dimensional reflection groups.

G_{1152} is known as the reflection group of type $\mathscr{F}_4$. There are three more irreducible reflection subgroups of $GL(4, \boldsymbol{Q})$, which are $\mathscr{A}_4, \mathscr{B}_4$ and $\mathscr{D}_4$.

$\mathscr{A}_4 \sim \langle c_5, mc_4 \rangle$ and

$$\mathscr{B}_4 \sim \left\langle \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} -1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \right\rangle$$

$$\mathscr{D}_4 \sim \left\langle \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} -1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix} \right\rangle.$$

In higher dimensions, there exist three reflection subgroups of $GL(n, \boldsymbol{Q})$ of exceptional type, namely $\mathscr{E}_6, \mathscr{E}_7$ and $\mathscr{E}_8$.

$$\mathscr{E}_6 = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix} \right\rangle.$$

$\mathscr{E}_6 \cap SL(6, \boldsymbol{Q}) \simeq PSp(2,3)$ and $\mathscr{E}_6 \simeq \mathrm{Aut} PSp(2,3)$, where $PSp(2,3)$ is a non-Abelian simple group of order 25920.

$$\mathscr{E}_7 = \left\langle \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & 1/2 & 1/2 & 0 & 0 & -1/2 \\ 0 & 1/2 & -1/2 & 1/2 & 0 & 0 & -1/2 \\ 0 & 1/2 & 1/2 & -1/2 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1/2 & -1/2 & -1/2 & 0 & 0 & -1/2 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\rangle.$$

$\mathscr{E}_7 \cap SL(7, \boldsymbol{Q}) \simeq PSp(3,2)$ and $\mathscr{E}_7 \simeq PSp(3,2) \times C_2$, where $PSp(3,2)$ is a non-Abelian simple group of order 1451520.

From [**4**, Chapters 1 and 5], the following theorem holds.

THEOREM 7. *Suppose that a finite subgroup G of $GL(n, \boldsymbol{Q})$ is a semi-direct product of N and H, $G \simeq N \rtimes H$. If the linear Noether's problem for G is affirmative, then there exists a generic polynomial of H.*

From this theorem, we see that there exists a generic polynomial of $PSp(3,2)$. Finite non-Abelian simple groups which are known to have a generic polynomial were $\mathfrak{A}_5$ and $PSL(2,7)$ [**11**], [**12**]. $PSp(3,2)$ shall be the 3rd simple group that reveals to have a generic polynomial.

$\mathscr{E}_8 = \langle \mathscr{D}_8, \mathscr{F}_4 \rangle$ where we embed $\mathscr{F}_4$ diagonally into $GL(8, \boldsymbol{Q})$. $\mathscr{E}_8 \cap SL(8, \boldsymbol{Q})$ is not simple, but it is a central extension of index 2 over a simple group isomorphic to $O_8(1,2)$.

Hopefully, these expressions of $\mathscr{E}_6$ and $\mathscr{E}_8$ may provide a clue to find generic polynomial of $PSp(2,3)$ and $O_8(1,2)$.

References

[1] H. Brown, R. Bulow, J. Neubuser, H. Wondratschek and H. Zassenhaus, Crystallographic groups of four-dimensional space, John Wiley, New York, 1978.

[2] F. R. DeMeyer, Generic polynomials, J. Algebra, **84** (1983), 441–448.

[3] L. C. Grove and C. T. Benson, Finite Reflection Groups, 2nd ed., Springer-Verlag, 1985.

[4] C. Jensen, A. Ledet and N. Yui, Generic polynomials, constructive aspects of the inverse Galois problem, Cambridge, 2003.

[5] M. Kang, Noether's problem for dihedral 2-groups II, Pacific J. Math., **222** (2005), 301–316.

[6] G. Kemper and E. Mattig, Generic polynomials with few parameters, J. Symbolic. Comput., **30** (2000), 843–857.

[7] G. Kemper, Generic polynomials are descent-generic, Manuscripta Math., **105** (2001), 139–141.

[8] H. Kitayama, Linear Noether's problem for 2-groups, J. Algebra, **324** (2010), 591–597.

[9] H. Kitayama and A. Yamasaki, Four-dimensional linear Noether's problem, J. Math. Kyoto Univ., **49** (2009), 359–380.

[10] H. W. Lenstra, Jr., Rational functions invariant under a finite abelian group, Invent. Math., **25** (1974), 299–325.

[11] T. Maeda, Noether's problem for $\mathfrak{A}_5$, J. Algebra, **125** (1989), 418–430.

[12] J. F. Mestre, Correspondances compatibles avec une relation binaire, relèvement d'extensions de groupe de Galois $L_3(2)$ et problème de Noether pour $L_3(2)$, arXiv.org, 2005.

[13] B. Plans, Noether's problem for $GL(2,3)$, Manuscripta Math., **124** (2007), 481–487.

[14] Y. Rikuna, The existence of generic polynomial for $SL(2,3)$ over $\boldsymbol{Q}$, preprint.

Aiichi YAMASAKI
Faculty of Science
Kyoto University
Kyoto 606-8502, Japan
E-mail: tagenkan@hyper.cx

Note added in proof. In Theorem 1 of the present paper, Noether's problem remains open for four groups. Recently, M. Kang and J. Zhou gave an affirmative answer for these groups as well (*The rationality problem for finite subgroups of* $GL_4(\boldsymbol{Q})$, arXiv: 1006.1156v1).