J. Math. Soc. Japan
Vol. 62, No. 4 (2010) pp. 1167–1204
doi: 10.2969/jmsj/06241167

Global well-posedness for the exterior initial-boundary value problem to the Kirchhoff equation

Dedicated to Professor Mitsuharu Ôtani on his 60th birthday

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(Received Feb. 10, 2009)
(Revised Aug. 27, 2009)

Abstract. The aim of this paper is to find a general class of data in which the global well-posedness for the exterior initial-boundary value problem to the Kirchhoff equation is assured. The result obtained in the present paper will be applied to the existence of scattering states. A class of weighted Sobolev spaces will be also presented in which the global well-posedness is assured. For this purpose, the method of generalized Fourier transforms is developed for some oscillatory integral associated with this equation. The crucial point is to obtain the resolvent expansion of the minus Laplacian around the origin in $\boldsymbol{C}$, and the differentiability of the generalized Fourier transforms.

1. Introduction and statement of results.

The Kirchhoff equation was proposed by Kirchhoff in 1883, as a model of the vibrating string with fixed ends. The global well-posedness on bounded domains was studied by some authors (see, e.g., [**25**], [**26**]). Up to the last decade many authors have investigated the global well-posedness for the Cauchy problem to the Kirchhoff equation with small data in Sobolev spaces (see [**2**], [**4**], [**5**], [**6**], [**7**], [**8**], [**11**], [**16**], [**18**], [**19**], [**28**], [**33**]). Greenberg and Hu studied this equation with small C_0^∞ data in one dimensional space ([**11**]). After them, the general space dimensional case was thoroughly investigated by D'Ancona and Spagnolo [**5**], [**6**], [**7**], [**8**] in a weighted Sobolev space, and then, Yamazaki found a more general class of initial data to ensure the global well-posedness ([**33**]).

The global well-posedness for the initial-boundary value problem to the Kirchhoff equation in exterior domains is also of interest. The study of this problem was initiated by Racke in 1995 (see [**27**]). He employed the generalized Fourier

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2000 *Mathematics Subject Classification.* Primary 35L05; Secondary 35L10.
Key Words and Phrases. Kirchhoff equation, generalized Fourier transform, scattering theory.
This research was supported by Grant-in-Aid for Scientific Research (C) (No. 21540198), Japan Society for the Promotion of Science.

transforms to get the small amplitude global solutions in a Sobolev space. Heiming improved his result (see [**12**], [**17**]). But then, they assumed that the supports of the generalized Fourier transform of data are away from the origin. The main object in the present paper is to remove this restrictive assumption and improve the regularity of the data (see Theorem 1.4), together with introducing more general class of data which ensures the global well-posedness (see Theorem 1.1). The cruicial tool in our argument is the asymptotic expansion of resolvent of $-\Delta$ around the origin in the complex plane (see Proposition 2.5). We should refer to the results of Yamazaki (see [**34**], [**35**]), who gave some sufficient conditions without any weight condition on data. That is, she assumed that the data belong to $W^{s,q}(\Omega) \times W^{s-1,q}(\Omega)$ for some $s > 2$ and $q \in (1, 2)$ depending on $n(\geq 3)$, where Ω is a non-trapping domain in $\boldsymbol{R}^n$ (see example 1.2). We have an advantage of considering the classes of Theorems 1.1 and 1.4 below; they are more useful in the scattering problem rather than the ones in [**12**], [**27**], [**34**], [**35**].

Once the global well-posedness is established, a problem will arise whether the scattering states exist or not. We will formulate this problem in Theorem 1.5. For the Cauchy problem to the Kirchhoff equation, the scattering operators were constructed by Yamazaki [**33**]. This result was recently extended to a wider class of the variable coefficients by Kajitani (see [**16**]).

To become more precise, let Ω be an arbitrary exterior domain in $\boldsymbol{R}^n$ $(n \geq 1)$ such that $\boldsymbol{R}^n \setminus \Omega$ is compact and its boundary $\partial\Omega$ is of C^∞. We consider the initial-boundary value problem to the Kirchhoff equation, for function $u = u(t, x)$:

$$\partial_t^2 u - \left(1 + \int_\Omega |\nabla u|^2 \, dx\right) \Delta u = 0, \quad t \neq 0, \quad x \in \Omega, \tag{1.1}$$

with the data

$$u(0, x) = f_0(x), \quad \partial_t u(0, x) = f_1(x), \quad x \in \Omega, \tag{1.2}$$

and the boundary condition

$$u(t, x) = 0, \quad t \in \boldsymbol{R}, \quad x \in \partial\Omega. \tag{1.3}$$

We shall introduce notation in order to state the results. For a non-negative integer m and real number κ, we define the weighted Sobolev space over a domain G in $\boldsymbol{R}^n$:

$$H^m_\kappa(G) = \left\{ f : \langle x \rangle^\kappa \partial_x^\alpha f \in L^2(G), \, |\alpha| \leq m \right\},$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$ and we put $L^2_\kappa(G) = H^0_\kappa(G)$. We define also the weighted

Sobolev space $H^\sigma_\kappa(G)$ of fractional order $\sigma \geq 0$ by the complex interpolation method:

$$H^\sigma_\kappa(G) = \big[L^2_\kappa(G), H^m_\kappa(G)\big]_\theta, \quad \sigma \leq m, \quad \sigma = \theta m \quad \text{with } 0 \leq \theta \leq 1,$$

where m is an integer. $H^\sigma(G)$ (or even $H^1_0(G)$) is the usual Sobolev space of order σ over G. Let A be a self-adjoint realization of $-\Delta$ on $L^2(\Omega)$ with the Dirichlet boundary condition in the exterior domain Ω, i.e.,

$$\begin{cases} \mathscr{D}(A) = H^2(\Omega) \cap H^1_0(\Omega), \\ Au = -\Delta u, \quad u \in C^\infty_0(\Omega). \end{cases} \tag{1.4}$$

Since A is the non-negative self-adjoint operator on $L^2(\Omega)$, we can define the square root $A^{1/2}$ of A. In what follows, we put $|D| = A^{1/2}$.

In order to state the global well-posedness for the problem (1.1)–(1.3), we introduce a class $Y_k(\Omega)$:

$$Y_k(\Omega) := \big\{(f,g) \in (H^{\frac{3}{2}}(\Omega) \cap H^1_0(\Omega)) \times H^{\frac{1}{2}}(\Omega) : |(f,g)|_{Y_k(\Omega)} < +\infty\big\}, \quad (k > 1),$$

with

$$\begin{aligned} |(f,g)|_{Y_k(\Omega)} = \sup_{\tau \in \boldsymbol{R}} (1+|\tau|)^k \Big\{ &\Big|\big(e^{i\tau|D|}|D|^{\frac{3}{2}} f, |D|^{\frac{3}{2}} f\big)_{L^2(\Omega)}\Big| \\ &+ \Big|\big(e^{i\tau|D|}|D|^{\frac{3}{2}} f, |D|^{\frac{1}{2}} g\big)_{L^2(\Omega)}\Big| + \Big|\big(e^{i\tau|D|}|D|^{\frac{1}{2}} g, |D|^{\frac{1}{2}} g\big)_{L^2(\Omega)}\Big| \Big\}, \end{aligned}$$

where $(f,g)_{L^2(\Omega)}$ denotes the $L^2(\Omega)$-inner product of f and g.

We are now in a position to state the results. The main result is as follows:

Theorem 1.1. *Let $n \geq 1$. If the data $f_0(x)$ and $f_1(x)$ satisfy $(f_0, f_1) \in Y_k(\Omega)$ for some $k > 1$ and*

$$\|\nabla f_0\|^2_{L^2(\Omega)} + \|f_1\|^2_{L^2(\Omega)} + |(f_0, f_1)|_{Y_k(\Omega)} \ll 1,$$

then the initial-boundary value problem (1.1)–(1.3) *admits a unique solution $u \in C(\boldsymbol{R}; H^{3/2}(\Omega) \cap H^1_0(\Omega)) \cap C^1(\boldsymbol{R}; H^{1/2}(\Omega))$.*

The class $Y_k(\boldsymbol{R}^n)$ in $\boldsymbol{R}^n$ is introduced in [**33**] (see also [**16**], [**28**]). The inclusions among the classes $Y_k(\Omega)$ are as follows:

$$Y_k(\Omega) \subset Y_l(\Omega) \quad \text{if } k > l > 1. \tag{1.5}$$

The definition of $Y_k(\Omega)$ is somewhat complicated, hence we give two examples of spaces contained in $Y_k(\Omega)$:

Example 1.2. Let $n \geq 4$ and $\boldsymbol{R}^n \setminus \Omega \subset \{x \in \boldsymbol{R}^n : |x| \leq r_0\}$ for some $r_0 > 0$. Assume that Ω is non-trapping in the sense that there exists $T_{r_0} > 0$ such that no geodesic of length T_{r_0} is completely contained in $\Omega \cap \{x \in \boldsymbol{R}^n : |x| \leq r_0\}$. Let $2(n-1)/(n-3) < p < \infty$ and $1/p + 1/q = 1$. Let M be an integer satisfying $M \geq (n+1)(1/2 - 1/p)$. Then it is proved in [**34**, Theorem 4] that

$$W_0^{2M,q}(\Omega) \times W_0^{2M-1,q}(\Omega) \subset Y_{k(n)}(\Omega),$$

where $k(n) = (n-1)(1/2 - 1/p) > 1$ and $W_0^{2M,q}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{W^{2M,q}(\Omega)}$. Notice that $2M > 2(n+1)/(n-1)$.

Example 1.3. For $\sigma \geq 0$ and $\varkappa \in \boldsymbol{R}$, let $H^\sigma_{\varkappa,0}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{H^\sigma_\varkappa(\Omega)}$. Then it will be proved in Lemma 3.2 that if $n \geq 3$ and $\boldsymbol{R}^n \setminus \Omega$ is star-shaped with respect to the origin, then the inclusion

$$H^{s_0+1}_{s(k),0}(\Omega) \times H^{s_0}_{s(k),0}(\Omega) \subset Y_k(\Omega)$$

holds for any $s_0 > (n+1)/2$, $s(k) > \max(n + 1/2, k + n/2)$ and $k \in (1, n]$. This inclusion can be proved by using the generalized Fourier transforms.

As a consequence of Theorem 1.1, example 1.3 and the inclusion (1.5), we have:

Theorem 1.4. *Let $\Omega, n, s_0, s(k)$ be as in Example* 1.3. *If the data $f_0(x)$ and $f_1(x)$ satisfy*

$$f_0(x) \in H^{s_0+1}_{s(k),0}(\Omega), \quad f_1(x) \in H^{s_0}_{s(k),0}(\Omega)$$

for some $k > 1$, and

$$\|f_0\|_{H^{s_0+1}_{s(k)}(\Omega)} + \|f_1\|_{H^{s_0}_{s(k)}(\Omega)} \ll 1,$$

then the initial-boundary value problem (1.1)–(1.3) *admits a unique solution $u \in \cap_{j=0,1,2} C^j(\boldsymbol{R}; H^{s_0+1-j}(\Omega))$.*

Let us make a few remarks to compare our results with what is known in [**12**], [**27**], [**34**], [**35**]. Theorem 1.1 improves the above results in the sense that any geometrical condition on Ω is not assumed, while Heiming [**12**] and Racke [**27**] did not require any geometrical condition on Ω, if the supports of the generalized Fourier transform of data are compact. Moreover, Theorem 1.4 removes the assumption that the supports of the generalized Fourier transform of the data are away from the origin, which is assumed in [**12**], [**27**]. Roughly speaking, this condition means that the integrals over Ω of the data with any polynomially weight vanish. The class in example 1.2 is based on the L^p-L^q decay estimates for wave equation. Finally, when $n = 3$, if the data belong to a subspace of $(W_0^{9,q}(\Omega))^2$ for some $q \in (1, 2)$, then the global well-posedness in $H^3(\Omega)$ is obtained by [**35**, Theorem 5]. We note that neither assumption on data of Theorem 1.4 and both Example 1.2 and the class of [**35**, Theorem 5] imply the other.

Let us finalize this section by stating the existence of scattering states for the problem (1.1)–(1.3). The development of scattering problem in the classes of [**12**], [**27**], [**34**], [**35**] would be complicated. On the other hand, the advantage of the classes of Theorems 1.1 and 1.4 is to be able to discuss the scattering problem more easily. Thus we have the following:

THEOREM 1.5. *Let $k > 2$. For any solution $u(t, x)$ to (1.1)–(1.3) in Theorem 1.1, there exist unique solutions $u_\pm(t, x) \in C(\boldsymbol{R}; H^{3/2}(\Omega) \cap H_0^1(\Omega)) \cap C^1(\boldsymbol{R}; H^{1/2}(\Omega))$ of equations $\partial_t^2 u_\pm - c_\infty^2 \Delta u_\pm = 0$ with the Dirichlet boundary condition on $\partial\Omega$ such that*

$$\|\nabla u_\pm(t, \cdot) - \nabla u(t, \cdot)\|_{L^2(\Omega)} + \|\partial_t u_\pm(t, \cdot) - \partial_t u(t, \cdot)\|_{L^2(\Omega)} = O(|t|^{-(k-2)}) \quad (1.6)$$

as $t \to \pm\infty$, where the propagation speed c_∞ is uniquely determined by an equation

$$c_\infty = \sqrt{1 + \frac{1}{2}\left(\|\nabla u_\pm(0, \cdot)\|_{L^2(\Omega)}^2 + \frac{1}{c_\infty^2}\|\partial_t u_\pm(0, \cdot)\|_{L^2(\Omega)}^2\right)}. \quad (1.7)$$

As to the Cauchy problem, the condition $k > 2$ is sharp. For, when $1 < k \leq 2$, it is proved in [**21**] that there exists a solution of the Kirchhoff equation to the Cauchy problem in $\boldsymbol{R}^n$ which is never asymptotic to any free solution of the wave equation as $t \to \pm\infty$.

2. Resolvent estimates and the generalized Fourier transforms.

In this section we will review some results on the asymptotic behaviours of the resolvent of A and define the generalized Fourier transforms. We will apply these results to prove Example 1.3, or even Theorem 1.4. Consider the Helmholtz equation with a parameter $z \in \boldsymbol{C}$ in Ω:

$$\begin{cases} (-\Delta - z)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

It is well known that 0 is not eigenvalue of A, hence, the spectrum $\sigma(A)$ of A is absolutely continuous. Thus $\sigma(A)$ coincides with $[0, \infty)$. Therefore, $L^2(\Omega)$ is absolutely continuous space. We denote by $R(z) = (A - z)^{-1}$ the resolvent of A. We shall analyze the asymptotic behaviour of $R(z)$ as $|z| \to \infty$ and $z \to 0$. The basic ideas are simlar to those of Iwashita [**14**] who developed the cut-off technique to study the behaviour of resolvent $R(z)$ of the Stokes operator as $z \to 0$ (see also Iwashita and Shibata [**15**], and Tsutsumi [**30**]). The high energy part and its differentiability are rather well known (see Heiming [**12**], Mochizuki [**22**], [**23**], Racke [**27**] and Wilcox [**32**], and also Isozaki [**13**] who studied the Schrödinger operators with long range potentials). Based on these resolvent behaviours, we can obtain the differentiability properties of the generalized Fourier transforms.

In what follows we often use the following function space: For a domain G in $\boldsymbol{R}^n$ we define

$$\widehat{H}^2(G) = \text{the completion of } C_0^\infty(G) \text{ by } \sum_{|\alpha|=2} \|\partial_x^\alpha \cdot \|_{L^2(G)}.$$

We set

$$\Sigma_+ = \{z \in \boldsymbol{C} \setminus 0 : 0 < \arg z \le \pi\}, \quad \Sigma_- = \{z \in \boldsymbol{C} \setminus 0 : -\pi \le \arg z < 0\}, \tag{2.2}$$

and we denote by $\mathscr{B}(X, Y)$ the space of all bounded linear operators from X to Y.

2.1. The behaviour of the resolvent $R_0(z)$ around the origin.

Let us consider the Helmholtz equation in $\boldsymbol{R}^n$ with a parameter $z \in \boldsymbol{C}$:

$$(-\Delta - z)u = f \quad \text{in } \boldsymbol{R}^n. \tag{2.3}$$

When $f \in L^2(\boldsymbol{R}^n)$ and $z \in \boldsymbol{C}$, we can write the solution $u(x; z)$ to (2.3) as follows:

$$u(x; z) = (R_0(z)f)(x) = \mathscr{F}_0^{-1}\left[\frac{\widehat{f}(\xi)}{|\xi|^2 - z}\right](x),$$

where $\mathscr{F}_0 f = \widehat{f}$ and $\mathscr{F}_0^{-1} f$ stand for the Fourier transform and the inverse Fourier transform of f on $\boldsymbol{R}^n$, respectively.

The asymptotic behaviour of $R(z) = (A - z)^{-1}$ near the origin in $\boldsymbol{C}$ is based on the following two results. The first one is due to Lemma 2.2 of Murata [**24**] whose original version is stated in general elliptic operators (see also Theorems 1.2–1.4 of Iwashita and Shibata [**15**]). To this end, we introduce a function space: For $\sigma \geq 0$, and a non-negative integer k, $0 \leq \theta < 1$, a Banach space X, we say that an X-valued function f belongs to $o(\sigma, k+\theta; X)$ if f is holomorphic on $U_{\delta,\pm} = \{z \in \boldsymbol{C} : |\mathrm{Re} z| < \delta, 0 < \pm \mathrm{Im} z < 1\}$ for some $\delta > 0$, k-times differentiable on $\overline{U}_{\delta,\pm} \setminus \{0\}$ and satisfies

$$\left\|f^{(j)}(z)\right\|_X = o(z^{\sigma-j}) \quad \text{as } z \to 0 \text{ in } \overline{U}_{\delta,\pm}$$

for $j = 0, \ldots, k$, and when $k + \theta \leq \sigma + 1$, f satisfies the additional estimate

$$\left(\int_{-\delta}^{\delta-h} \left\|f^{(k)}(x+h+iy) - f^{(k)}(x+iy)\right\|_X^p dx\right)^{1/p} \leq Ch^\theta, \quad h > 0, \quad 0 \leq \pm y \leq 1,$$

where $p = \infty$ if $k + \theta \leq \sigma$ and $p = (k+\theta-\sigma)^{-1}$ if $\sigma < k + \theta \leq \sigma + 1$.

Lemma 2.1 (Lemma 2.2 of [**24**] (Murata)). *Let $n \geq 1$, $\varepsilon(n) = 0$ for n odd and $\varepsilon(n) = 1$ for n even, $\sigma > -1/2$ and $s > \max(\sigma+1, 2\sigma+2-n/2)$. Then in $\mathscr{B}(L^2_s(\boldsymbol{R}^n), H^2_{-s}(\boldsymbol{R}^n))$ one has the expansion*

$$R_0(z) = \sum_{j=0}^{[\sigma+1-n/2]} z^{n/2-1+j}(\log z)^{\varepsilon(n)} F_j + \sum_{j=0}^{[\sigma]} z^j G_j + o(z^\sigma) \tag{2.4}$$

as $z \to 0$ in $\Sigma_\pm$, where the convention is $\sum_{j=k}^{l} a_k = 0$ when $l < k$; and the remainder term belongs to $o\big(\sigma, d; \mathscr{B}(L^2_s(\boldsymbol{R}^n), H^{2([\sigma]+1)}_{-s}(\boldsymbol{R}^n))\big)$ for any d with $d < s - 1/2$. Here the operators F_j belong to $\mathscr{B}(L^2_r(\boldsymbol{R}^n), H^\nu_{-r}(\boldsymbol{R}^n))$ for any $r > n/2+2j$ and $\nu > 0$; and $G_j \in \mathscr{B}(L^2_r(\boldsymbol{R}^n), H^{2(j+1)}_{-r'}(\boldsymbol{R}^n))$ for any r and r' such that (i) *$r, r' > 2j+2-n/2$ and $r + r' > 2j+2$ when $j < n/2-1$;* (ii) *$r, r' > 2j+2-n/2$ when $j \geq n/2-1$.*

When $d > \sigma$ in (2.4), the singularity would appear in the derivatives of remainder term near $z = 0$. We note that Vainberg proved the asymptotic expansion (2.4) in a more stringent space $\mathscr{B}(L^2_{\mathrm{comp}}(\boldsymbol{R}^n), H^2_e(\boldsymbol{R}^n))$ than Lemma 2.1, where $L^2_{\mathrm{comp}}(\boldsymbol{R}^n)$ is the subspace of $L^2(\boldsymbol{R}^n)$-functions with compact supports, and the space $H^2_e(\boldsymbol{R}^n)$ is the subspace of $H^2(\boldsymbol{R}^n)$-functions with weight $e^{-|x|}$ (see [**31**, Theorem 10]).

The second result is concerning with the asymptotic behaviour of $R_0(0)f$ near infinity.

Lemma 2.2. *Let $n \geq 3$ and $s > n/2$. Put $u = R_0(0)f$. If $f \in L^2_s(\boldsymbol{R}^n)$, then $u \in \widehat{H}^2(\boldsymbol{R}^n) \cap H^1_{s-1}(\boldsymbol{R}^n)$ and*

$$\lim_{R\to\infty} \frac{1}{R^n} \int_{R<|x|<2R} |u(x)|^2 \, dx = 0. \tag{2.5}$$

Proof. We use an idea of [**14**, Lemma 2.2] that treated the stationary Stokes equation. The derivatives $\partial_x^\alpha u$, $|\alpha| = 2$, become

$$\partial_x^\alpha u(x) = i^{|\alpha|} \mathscr{F}_0^{-1} \left[\frac{\xi^\alpha}{|\xi|^2} \widehat{f}(\xi) \right](x).$$

Observing that the multipliers $\xi^\alpha/|\xi|^2$ are homogeneous of order 0, we conclude from the L^2-boundedness of the singular integral operators that $u \in \widehat{H}^2(\boldsymbol{R}^n)$. Inserting the cut-off function $\chi(\xi) \in C_0^\infty(\xi)$ equal to one for $|\xi| \leq 1/2$ and 0 for $|\xi| \geq 1$, we can write $u = u_1 + u_2$, where

$$u_1(x) = \mathscr{F}_0^{-1} \left[\frac{\chi(\xi)\widehat{f}(\xi)}{|\xi|^2} \right](x), \quad u_2(x) = \mathscr{F}_0^{-1} \left[\frac{(1-\chi(\xi))\widehat{f}(\xi)}{|\xi|^2} \right](x).$$

We claim that u_2 satisfies (2.5). In fact, let m be an integer with $m \leq s \leq m+1$. Then

$$\begin{cases} \|\langle x\rangle^m u_2\|_{L^2(R<|x|<2R)} \leq C_m \|\langle x\rangle^m f\|_{L^2(R<|x|<2R)}, \\ \|\langle x\rangle^{m+1} u_2\|_{L^2(R<|x|<2R)} \leq C_{m+1} \|\langle x\rangle^{m+1} f\|_{L^2(R<|x|<2R)}, \end{cases}$$

with certain constants C_m, C_{m+1} independent of R. Hence, interpolating these estimates we get

$$\|\langle x\rangle^s u_2\|_{L^2(R<|x|<2R)} \leq C_s \|\langle x\rangle^s f\|_{L^2(R<|x|<2R)}.$$

Similarly, we get $u_2 \in H^1_{s-1}(\boldsymbol{R}^n)$, since $f \in L^2_s(\boldsymbol{R}^n)$. Then u_2 satisfies (2.5), since

$$\begin{aligned} \frac{1}{R^n} \int_{R<|x|<2R} |u_2(x)|^2 \, dx &\leq R^{-2s-n} \int_{R<|x|<2R} \langle x\rangle^{2s} |u_2(x)|^2 \, dx \\ &\leq C_s R^{-2s-n} \int_{R<|x|<2R} \langle x\rangle^{2s} |f(x)|^2 \, dx \to 0 \end{aligned}$$

as $R \to \infty$.

In order to see that u_1 satisfies (2.5), we show that u_1 can be represented by a C^∞ kernel bounded by $C\langle x-y\rangle^{-(n-2)}$, $C>0$. Indeed, we note that $R_0(0)$ is a convolution operator with a kernel $G_0(x-y)$ defined by

$$G_0(x) = \frac{\Gamma(n/2)}{4(n-2)\pi^{n/2}} \cdot \frac{1}{|x|^{n-2}}. \tag{2.6}$$

Put

$$K(w) = \int_{\boldsymbol{R}^n} e^{iw\cdot\xi} \frac{\chi(\xi)}{|\xi|^2}\, d\xi.$$

Then $K(w)$ is of class $C^\infty(\boldsymbol{R}^n)$ and we can write

$$u_1(x) = (K * f)(x) = \int_{\boldsymbol{R}^n} K(x-y) f(y)\, dy.$$

On the other hand, since $u_1(x) = u(x) - u_2(x)$ and $u(x) = (G_0 * f)(x)$, it follows that

$$u_1(x) = \int_{\boldsymbol{R}^n} G_0(x-y) f(y)\, dy - \int_{\boldsymbol{R}^n} G_1(x-y) f(y)\, dy, \tag{2.7}$$

where $G_1(x-y)$ is a C^∞ kernel of $u_2(x)$ and majorized by $C\langle x-y\rangle^{-k}$ for any $k \in \boldsymbol{N}$. The boundedness of $K(x-y)$ is obvious provided $n \geq 3$. Hence we conclude from (2.6)–(2.7) that

$$|K(x-y)| \leq C\langle x-y\rangle^{-(n-2)}.$$

As a by-product, we have $u_1 \in H^1_{s-1}(\boldsymbol{R}^n)$. Now we can estimate

$$\int_{R<|x|<2R} |u_1(x)|^2\, dx \leq C \int_{R<|x|<2R} \left| \int_{\boldsymbol{R}^n} \langle x-y\rangle^{-(n-2)} |f(y)|\, dy \right|^2 dx. \tag{2.8}$$

Putting $V(x) = \{y \in \boldsymbol{R}^n; |x|/2 \leq |y| \leq 2|x|\}$, we see that

$$\begin{aligned} &\int_{R<|x|<2R} \left| \int_{V(x)} \langle x-y\rangle^{-(n-2)} |f(y)|\, dy \right|^2 dx \\ &\quad \leq C\left(\int_{R<|x|<2R} \langle x\rangle^{-(2s-n)}\, dx \right) \|f\|^2_{L^2_s(\boldsymbol{R}^n)} \leq CR^{2(n-s)} \|f\|^2_{L^2_s(\boldsymbol{R}^n)}. \end{aligned} \tag{2.9}$$

By the assumption $s > n/2$, we have

$$2(n-s) < n. \tag{2.10}$$

On the other hand, we can estimate

$$\begin{aligned}
&\int_{R<|x|<2R} \left| \int_{\boldsymbol{R}^n \setminus V(x)} \langle x-y \rangle^{-(n-2)} |f(y)| \, dy \right|^2 dx \\
&\le \left(\int_{R<|x|<2R} \langle x \rangle^{-2(n-2)} \, dx \right) \left| \int_{\boldsymbol{R}^n} \langle y \rangle^{-2s} \langle y \rangle^{2s} |f(y)|^2 \, dy \right|^2 \\
&\le C R^{-2(n-2)+n} \|f\|^2_{L^2_s(\boldsymbol{R}^n)}.
\end{aligned} \tag{2.11}$$

Thus combining (2.8)–(2.11) with $n \ge 3$, we conclude that u_1 satisfies (2.5). Furthermore, we have $u \in \widehat{H}^2(\boldsymbol{R}^n) \cap H^1_{s-1}(\boldsymbol{R}^n)$. The proof of Lemma 2.2 is complete. □

2.2. Resolvent estimates in high frequency.

The resolvent estimates are thoroughly investigated by Mochizuki [**22**], [**23**] (see also Wilcox [**32**]). As a starting point, we have:

Lemma 2.3 (Limiting absorption principle (Mochizuki [**22**], [**23**])). *Let $n \ge 1$ and $s > 1/2$. Then, for any $\lambda > 0$ there exist strong limits* $\mathrm{s}-\lim_{\varepsilon \searrow 0} R(\lambda \pm i\varepsilon) = R(\lambda \pm i0)$ *in $\mathscr{B}(L^2_s(\Omega), H^2_{-s}(\Omega))$. The functions $u_\pm = R(\lambda \pm i0)f$ for $f \in L^2_s(\Omega)$ are the unique outgoing and incoming solutions to* (2.1), *respectively, in the sense that*

$$u_\pm \in L^2_{-s}(\Omega), \quad \nabla\big(e^{\pm i\sqrt{\lambda}|x|} u_\pm\big) \in L^2_{-s-1}(\Omega).$$

In addition to the above assumption, let us suppose $n \ge 3$, and that $\boldsymbol{R}^n \setminus \Omega$ is star-shaped with respect to the origin. Then, for any $a > 0$ there exists a constant $C = C(a) > 0$ such that

$$\|R(\lambda \pm i0)f\|_{L^2_{-s}(\Omega)} \le C\lambda^{-1/2} \|f\|_{L^2_s(\Omega)}, \quad f \in L^2_s(\Omega) \tag{2.12}$$

for all $\lambda > a$.

We remark that Mochizuki proved the estimate

$$\|R(\lambda \pm i0)f\|_{L^2_{-s}(\Omega)} \le C\|f\|_{L^2_s(\Omega)}, \quad f \in L^2_s(\Omega)$$

for any compact interval in $\lambda > 0$ in an arbitrary domain Ω (see [**23**], cf. [**32**]), and the upper bound is removed and the estimate (2.12) is obtained for $\lambda \gg 1$ under the assumption that $\boldsymbol{R}^n \setminus \Omega$ is star-shaped (see [**22**]). Combining these results, we get (2.12) for all $\lambda > a$.

Based on Lemma 2.3, Heiming established the differentiability property of the resolvent by employing the argument of Isozaki [**13**, Theorem 1.10] (see [**12**, Theorem 2.8], cf. [**17**]).

LEMMA 2.4 (Heiming [**12**]). *Let $n \geq 3$. Assume that $\boldsymbol{R}^n \setminus \Omega$ is star-shaped with respect to the origin. Then, for any $s > 1/2$, $a > 0$ and $N \in \boldsymbol{N}$, $R(\lambda \pm i0)$ are N-times strongly differentiable with respect to $\lambda > a$ in $\mathscr{B}(L^2_{s+N}(\Omega), L^2_{-s-N}(\Omega))$. Furthermore, the following estimate hold*:

$$\left\|\partial_\xi^\alpha R(|\xi|^2 \pm i0) f\right\|_{L^2_{-s-|\alpha|}(\Omega)} \leq C_\alpha |\xi|^{-1} \|f\|_{L^2_{s+|\alpha|}(\Omega)}, \quad f \in L^2_{s+|\alpha|}(\Omega)$$

for all $\xi \in \boldsymbol{R}^n$ with $|\xi| > a$ and any multi-index α.

2.3. Resolvent expansions aroud the origon.

We recall the definition of the sets $\Sigma_\pm$ in (2.2). Then the main result of this subsection is as follows:

PROPOSITION 2.5. *Let $n \geq 3$ and $s > n/2$. Then there exists an operator $\widetilde{R}(z) \in \mathscr{B}(H^m_s(\Omega), H^{m+2}_{-s}(\Omega))$ for any integer $m \geq 0$ such that $\widetilde{R}(z)$ depends meromorphically in $z \in \Sigma_\pm$ having the following properties*:

(i) *The set Λ of poles is discrete and countable.*
(ii) *$\widetilde{R}(z)f$ is a solution to (2.1) for $z \in \Sigma_\pm \setminus \Lambda$ and $f \in L^2_s(\Omega)$.*
(iii) *Let $\Sigma_\pm(\varepsilon) = \Sigma_\pm \cap \{z \in \boldsymbol{C} : |z| < \varepsilon\}$ for $\varepsilon > 0$. Then there exists $\varepsilon_0 > 0$ such that $\Sigma_\pm(\varepsilon_0) \cap \Lambda = \emptyset$ and in $\mathscr{B}(H^m_s(\Omega), H^{m+2}_{-s}(\Omega))$ one has the expansion*

$$\widetilde{R}(z) = z^{\frac{n}{2}-1} (\log z)^{\varepsilon(n)} F + G(z) + o(z^{\frac{n}{2}-1}), \tag{2.13}$$

as $z \to 0$ in $\Sigma_\pm(\varepsilon_0)$, where $\varepsilon(n) = 0$ for n odd and $\varepsilon(n) = 1$ for n even; the operator F belongs to $\mathscr{B}(H^m_s(\Omega), H^{m+\nu}_{-s}(\Omega))$ for any $\nu > 0$, $G(z)$ is a polynomial of z of degree $[n/2-1]$ and belongs to $\mathscr{B}(H^m_s(\Omega), H^{m+2}_{-s}(\Omega))$, and the remainder term belongs to $o\big(n/2-1, d; \mathscr{B}(H^m_s(\Omega), H^{m+2([n/2-1]+1)}_{-s}(\Omega))\big)$ for any $d < s - 1/2$.
(iv) *$\widetilde{R}(z) = R(z)$ on $H^m_s(\Omega)$ for all $z \in \Sigma_\pm(\varepsilon_0)$.*

Relating with Proposition 2.5 and referring the notation of the remark of

Lemma 2.1, we should mention the results of Iwashita and Shibata [**15**] and Tsutsumi [**30**] in which $\widetilde{R}(z)$ belong to $\mathscr{B}(L^2_{\mathrm{comp}}(\Omega), H^2_e(\Omega))$. Thus Proposition 2.5 improves the results of [**15**], [**30**], since $\mathscr{B}(L^2_{\mathrm{comp}}(\Omega), H^2_e(\Omega)) \subset \mathscr{B}(L^2_s(\Omega), H^2_{-s}(\Omega))$.

We often use notation $\Omega_d = \Omega \cap B_d(0)$, where $B_d(0)$ is the ball in $\boldsymbol{R}^n$ with radius d centered at the origin. We prepare the following:

Lemma 2.6. *Let $n \geq 3$. Suppose that $u \in \widehat{H}^2(\Omega) \cap H^1_{s'}(\Omega)$ for some $s' \in \boldsymbol{R}$ satisfies*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\lim_{R\to\infty} \frac{1}{R^n} \int_{R<|x|<2R} |u(x)|^2 \, dx = 0. \tag{2.14}$$

Then $u \equiv 0$ in Ω.

Proof. We claim that $u(x)$ is analytic in Ω and behaves like

$$\partial_x^\alpha u(x) = O(|x|^{-(n-2+|\alpha|)}), \quad |\alpha| \leq 1 \tag{2.15}$$

as $|x| \to \infty$. To see this, let us consider the extension of u to $\boldsymbol{R}^n$. We denote by $\tilde{u}$ such an extension. More precisely, we define $\tilde{u}$ to be $\tilde{u}(x) = \psi(x)u(x)$, where $\psi(x) \in C^\infty(\boldsymbol{R}^n)$ is equal to 0 in a domain $\mathscr{O} \Subset \boldsymbol{R}^n \setminus \Omega$ and one in Ω. Then $\tilde{u}$ satisfies (2.14).

We set $\tilde{f} = -\Delta\tilde{u}$. Then $\tilde{f} \in L^2_s(\boldsymbol{R}^n)$ and $\tilde{f} = 0$ in Ω. It is well known that Poisson equation has a unique solution in $\mathscr{S}'(\boldsymbol{R}^n)$ (=the space of all tempered distributions) up to an additive polynomial. Hence $\tilde{u}$ can be represented as

$$\tilde{u}(x) = \int_{\boldsymbol{R}^n} G_0(x-y)\tilde{f}(y)\, dy + \text{polynomials},$$

where

$$G_0(x) = \frac{\Gamma\left(\dfrac{n}{2}\right)}{4(n-2)\pi^{\frac{n}{2}}} \cdot \frac{1}{|x|^{n-2}}.$$

Thus, by using the asymptotic behaviour (2.14), we get

$$\tilde{u}(x) = \int_{\boldsymbol{R}^n} G_0(x-y)\tilde{f}(y)\,dy. \tag{2.16}$$

Since $\tilde{u} = u$ in Ω, the analyticity of u and asymptotics (2.15) follow from (2.16).

Finally, we prove that $u = 0$ in Ω. Integrating by parts, we have

$$\begin{aligned}\int_{\Omega_R} |\nabla u(x)|^2\,dx &= -\int_{\Omega_R} \Delta u(x)\overline{u(x)}\,dx + \int_{\partial\Omega_R} \frac{x}{|x|}\cdot\nabla u(x)\overline{u(x)}\,dS_R \\ &= \int_{|x|=R} \frac{x}{|x|}\cdot\nabla u(x)\overline{u(x)}\,dS_R. \end{aligned}\tag{2.17}$$

We observe from (2.15) that $\nabla u(x)\overline{u(x)} = O(|x|^{-2n+3})$ as $|x| \to \infty$. Then, letting $R \to \infty$ in (2.17), we have

$$\int_{\Omega} |\nabla u(x)|^2\,dx = 0,$$

which impiles that u is constant in Ω. Hence by using the condition that $u = 0$ on $\partial\Omega$, we conclude that $u = 0$ in Ω. The proof of Lemma 2.6 is complete. $\square$

Proof of Proposition 2.5. The proof can be done along the idea of [**14**, Theorem 3.1]. We may prove the case $m = 0$ on acount of the elliptic regularity theorem. Let us introduce numbers b and d such that $d > b > r_0 + 3$ and fix them, where $r_0 > 0$ is chosen such that $\boldsymbol{R}^n \setminus \Omega \subset B_{r_0}(0) = \{x \in \boldsymbol{R}^n : |x| < r_0\}$. Put $\Omega_d = \Omega \cap B_d(0)$. We consider the boundary value problem to the Poisson equation in the bounded domain Ω_d:

$$\begin{cases} -\Delta u = f & \text{in } \Omega_d, \\ u = 0 & \text{on } \partial\Omega_d. \end{cases}\tag{2.18}$$

By the elliptic regularity theorem, for any $f \in L^2(\Omega_d)$, there exists a unique solution $u \in H^2(\Omega_d)$ to (2.18) such that

$$\|u\|_{H^2(\Omega_d)} \le C\|f\|_{L^2(\Omega_d)}.$$

Hence the mapping of $f \in L^2(\Omega_d)$ to the unique solution $u \in H^2(\Omega_d)$ determines an operator in $\mathscr{B}(L^2(\Omega_d), H^2(\Omega_d))$, which is denoted by L. Take C^∞-functions $\varphi(x)$ and $\chi(x)$ such that $\varphi(x) = 1$ for $|x| \ge b$ and equal to 0 for $|x| < b - 1$;

$\chi(x) = 1$ for $|x| \geq b - 2$ and equal to 0 for $|x| < b - 3$. For $f \in L^2_s(\Omega)$, let f_d be the restriction to Ω_d, and let $f_0 = f$ in Ω and equal to 0 in $\boldsymbol{R}^n \setminus \Omega$. Define the operator $R_1(z)$ by

$$R_1(z)f = \varphi R_0(z)(\chi f_0) + (1 - \varphi)Lf_d, \quad f \in L^2_s(\Omega). \tag{2.19}$$

Then we have $R_1(z) \in \mathscr{B}(L^2_s(\Omega), H^2_{-s}(\Omega))$, and $R_1(z)f$ satisfies $R_1(z)f \mid_{\partial\Omega} = 0$. The operator thus defined obeys

$$(-\Delta - z)R_1(z)f = f + S(z)f \quad \text{in } \Omega \tag{2.20}$$

for any $f \in L^2_s(\Omega)$, where $S(z)$ is defined

$$S(z)f = -\big\{2(\nabla\varphi) \cdot \nabla + \Delta\varphi\big\}\big\{R_0(z)(\chi f_0) - Lf_d\big\} - z(1 - \varphi)Lf_d. \tag{2.21}$$

The support of $S(z)f$ is contained in $\overline{\Omega}_d$ and $S(z) \in \mathscr{B}(L^2_s(\Omega), H^1_s(\Omega))$, and hence, $S(z)$ is a compact operator in $L^2_s(\Omega)$. $S(z)$ is holomorphic in $z \in \Sigma_\pm$, continuous in $\Sigma_\pm \cup \{0\}$, and has the same asymptotic expansion as that for $R_0(z)$ as $z \to 0$ in $\Sigma_\pm$.

Lemma 2.7. *Let the operator $S(z)$ be defined as* (2.21). *Then the inverse $(I+S(z))^{-1}$ of $I+S(z)$ exists as a $\mathscr{B}(L^2_s(\Omega), L^2_s(\Omega))$-valued meromorphic function of $z \in \Sigma_\pm$. The set Λ of poles is discrete and countable, and has no intersection with $\Sigma_\pm(\varepsilon_0)$ for some $\varepsilon_0 > 0$. In addition, $(I + S(z))^{-1}$ has the same type of expansion as* (2.4) *from Lemma* 2.1 *with $\sigma = n/2 - 1$.*

Proof. We can claim that $(I + S(0))^{-1} \in \mathscr{B}(L^2_s(\Omega), L^2_s(\Omega))$. Indeed, if we prove that $I+S(0)$ is injective, the conclusion follows from Fredholm's alternative, since the operator $S(0)$ is compact. Therefore, for the time being, we concentrate on proving the injectivity of $I + S(0)$. Let us assume that

$$(I + S(0))f = 0, \quad f \in L^2_s(\Omega).$$

Then it follows from (2.20) that

$$\begin{cases} \Delta R_1(0)f = 0 & \text{in } \Omega, \\ R_1(0)f = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.22}$$

We observe from (2.19) that $R_1(0)f = R_0(0)(\chi f_0)$ for $|x| \geq b$. Therefore, it follows from Lemma 2.2 that

$$\begin{cases} R_1(0)f \in \widehat{H}^2(\Omega) \cap H^1_{s'}(\Omega), \\ \lim_{R\to\infty} \dfrac{1}{R^n} \displaystyle\int_{R<|x|<2R} |R_1(0)f|^2 \, dx = 0. \end{cases}$$

Then we conclude from Lemma 2.6 that

$$R_1(0)f = 0 \quad \text{in } \Omega. \tag{2.23}$$

The equality (2.23) together with (2.19) imply that

$$R_0(0)(\chi f_0) = 0 \quad \text{for } |x| \geq b, \tag{2.24}$$

$$Lf_d = 0 \quad \text{for } x \in \Omega,\ |x| \leq b-1. \tag{2.25}$$

Since $R_0(0)(\chi f)$ satisfies the equation $-\Delta R_0(0)(\chi f_0) = \chi f_0$ in $\boldsymbol{R}^n$, it follows from (2.24) that $\chi f_0 = 0$ for $|x| \geq b$, i.e.,

$$f = 0 \quad \text{for } |x| \geq b. \tag{2.26}$$

Similarly, Lf_d satisfies the equation $-\Delta Lf_d = f_d$ in Ω_d, and hence, by using (2.25), we get

$$f = 0 \quad \text{for } x \in \Omega,\ |x| \leq b-1.$$

These imply that $\chi f_0 = f_0$ and

$$\begin{cases} -\Delta R_0(0)(\chi f_0) = f_0 & \text{in } \boldsymbol{R}^n, \\ R_0(0)(\chi f_0) = 0, & \text{on } |x| = d. \end{cases} \tag{2.27}$$

On the other hand, if we define

$$v = \begin{cases} Lf_d & \text{in } \Omega_d, \\ 0 & \text{in } \boldsymbol{R}^n \setminus \Omega, \end{cases}$$

then we see from the elliptic regularity theorem that $v \in H^2(B_d(0))$, and

$$\begin{cases} -\Delta v = f_0 & \text{in } B_d(0), \\ v = 0 & \text{on } |x| = d. \end{cases} \tag{2.28}$$

Hence it follows from (2.27)–(2.28) that $R_0(0)(\chi f_0) = v$ in $B_d(0)$, and hence,

$$R_0(0)(\chi f_0) = L f_d \quad \text{in } \Omega_d,$$

which implies that $R_1(0)f = Lf_d$ in Ω_d. By this relation and (2.23) we have

$$0 = -\Delta R_1(0) f = -\Delta L f_d = f_d \quad \text{in } \Omega_d,$$

i.e., $f = 0$ in Ω_d, which together with (2.26) shows $f = 0$ in Ω. This proves the injectivity of $I + S(0)$.

Put $M = \|(I + S(0))^{-1}\|_{\mathscr{B}(L^2_s(\Omega))}$. By the continuity of $S(z)$ in $z \in \Sigma_\pm$, there exists $\varepsilon_0 > 0$ such that $\|S(0) - S(z)\|_{\mathscr{B}(L^2_s(\Omega))} < 1/2M$ for any $z \in \Sigma_\pm(\varepsilon_0)$. Thus the inverse $(I + S(z))^{-1}$ is obtained as a Neumann series expansion: For $z \in \Sigma_\pm(\varepsilon_0)$,

$$(I + S(z))^{-1} = (I + S(0))^{-1} \sum_{j=0}^{\infty} \left[(S(0) - S(z))(I + S(0))^{-1}\right]^j. \tag{2.29}$$

Since $S(z)$ is holomorphic in $\Sigma_\pm$, applying analytic Fredholm's alternative, we conclude from [**9**, Lemma 13] that $(I + S(z))^{-1}$ exists in $\Sigma_\pm$ as a meromorphic function, and the set Λ of the poles is discrete and countable in $\Sigma_\pm$. The expansion follows from Lemma 2.1 with $\sigma = n/2 - 1$ and (2.29). The proof of Lemma 2.7 is complete. □

Completion of the proof of Proposition 2.5. Define

$$\widetilde{R}(z) = R_1(z)(I + S(z))^{-1}. \tag{2.30}$$

Then the assertions (i)–(iii) of Proposition 2.5 are an immediate consequence of Lemma 2.1, Lemma 2.7 and (2.30). The assertion (iv) follows from the fact that the resolvent set of A contains $\Sigma_\pm(\varepsilon_0)$. The proof of Proposition 2.5 is now finished. □

2.4. Generalized Fourier transforms.

Following Wilcox [**32**], let us define the generalized Fourier transforms in an arbitrary exterior domain. The existence of the limits $R(|\xi|^2 \pm i0)$ is assured by Lemma 2.3. Introducing a function $j(x) \in C^\infty(\boldsymbol{R}^n)$ vanishing in a neighbourhood of $\boldsymbol{R}^n \setminus \Omega$ and equal to one for large $|x|$, let us define the generalized Fourier transform as follows:

$$(\mathscr{F}_{\pm}f)(\xi) = \lim_{R\to\infty}(2\pi)^{-n/2}\int_{\Omega_R}\overline{\psi_{\pm}(x,\xi)}f(x)\,dx \quad \text{in } L^2(\boldsymbol{R}^n),$$

where we put

$$\psi_{\pm}(x,\xi) = j(x)e^{ix\cdot\xi} + \bigl[R(|\xi|^2 \pm i0)M_\xi(\cdot)\bigr](x) \quad \text{with} \quad M_\xi(x) = (A-|\xi|^2)(j(x)e^{ix\cdot\xi}).$$

Notice that we can write formally

$$M_\xi(x) = -(\Delta j(x) + 2i\xi\cdot\nabla j(x))e^{ix\cdot\xi}, \tag{2.31}$$

hence, $\operatorname{supp} M_\xi(\cdot) \subset B_{r_0+1}(0)\setminus B_{r_0}(0)$ for any fixed $\xi \in \boldsymbol{R}^n$. The kernel $\psi_{\pm}(x,\xi)$ is called eigenfunction of the operator A with eigenvalue $|\xi|^2$ in the sense that, formally, $(A-|\xi|^2)\psi_{\pm}(x,\xi) = 0$, but $\psi(x,\xi) \notin L^2(\Omega)$. Similarly, the inverse transform is defined by

$$\bigl(\mathscr{F}_{\pm}^{*}g\bigr)(x) = \lim_{R\to\infty}(2\pi)^{-n/2}\int_{B_R(0)}\psi_{\pm}(x,\xi)g(\xi)\,d\xi \quad \text{in } L^2(\Omega).$$

We treat $\mathscr{F}_{+}f$ only and drop the subscript $+$, since $\mathscr{F}_{-}f$ can be dealt with by essentially the same method. The transform $\mathscr{F}f$ thus defined obeys the following properties (see, e.g., Shenk II [**29**, Theorem 1 and Corollary 5.1]):

- $\mathscr{F}$ is the unitary mapping

  $$\mathscr{F} : L^2(\Omega) \to L^2(\boldsymbol{R}^n).$$

  Hence

  $$\mathscr{F}\mathscr{F}^{*} = I.$$

- $\mathscr{F}$ is fulfilled with the generalized Parseval equality:

  $$(\mathscr{F}f, \mathscr{F}g)_{L^2(\boldsymbol{R}^n)} = (f,g)_{L^2(\Omega)}, \quad f,g \in L^2(\Omega). \tag{2.32}$$

- $\mathscr{F}$ diagonalizes the operator A in the sense that

  $$\mathscr{F}(\varphi(A)f)(\xi) = \varphi(|\xi|^2)(\mathscr{F}f)(\xi), \tag{2.33}$$

  where $\varphi(A)$ is the operator defined by the spectral reperesentation theorem for self-adjoint operators.

The following lemma is concerning with the differentiability properties of the generalized Fourier transform $(\mathscr{F}f)(\xi)$.

LEMMA 2.8. *Let $n \geq 3$ and ε_0 be the number as in Proposition 2.5. Then the following estimates hold:*

(i) (*High frequency estimates*). *Assume that $\boldsymbol{R}^n \backslash \Omega$ is star-shaped with respect to the origin. Let $s > 1/2$. If $f \in L^2_{s+|\alpha|}(\Omega)$ for some multi-index α, then*

$$\left|\partial_\xi^\alpha \int_\Omega \overline{[R(|\xi|^2 + i0)M_\xi(\cdot)](x)} f(x)\, dx\right| \leq C_{\alpha,\varepsilon_0} \|f\|_{L^2_{s+|\alpha|}(\Omega)} \tag{2.34}$$

for all $|\xi| \geq \varepsilon_0$. In particular, we have

$$\left|\partial_\xi^\alpha (\mathscr{F}f)(\xi)\right| \leq \left|\partial_\xi^\alpha (\mathscr{F}_0(jf))(\xi)\right| + C_{\alpha,\varepsilon_0} \|f\|_{L^2_{s+|\alpha|}(\Omega)} \tag{2.35}$$

for all $|\xi| \geq \varepsilon_0$, where $(\mathscr{F}_0 g)(\xi)$ denotes the Fourier transform of $g(x)$ on $\boldsymbol{R}^n$.

(ii) (*Low frequency estimates*). *Let $s > n+1/2$. Then the following estimates hold for all $0 < |\xi| \leq \varepsilon_0$:*

$$\begin{aligned} &\left|\partial_\xi^\alpha \int_\Omega \overline{[R(|\xi|^2 + i0)M_\xi(\cdot)](x)} f(x)\, dx\right| \\ &\qquad \leq C_{\alpha,\varepsilon_0} \bigl\{1 + |\xi|^{n-2-|\alpha|} \bigl|(\log|\xi|)^{\varepsilon(n)}\bigr|\bigr\} \|f\|_{L^2_s(\Omega)} \end{aligned} \tag{2.36}$$

for any $|\alpha| \leq n-2$, and

$$\left|\partial_\xi^\alpha \int_\Omega \overline{[R(|\xi|^2 + i0)M_\xi(\cdot)](x)} f(x)\, dx\right| \leq C_{\alpha,\varepsilon_0} (1 + |\xi|^{n-2-|\alpha|}) \|f\|_{L^2_s(\Omega)} \tag{2.37}$$

for $|\alpha| = n-1, n$, provided $f \in L^2_s(\Omega)$. In particular, we have

$$\begin{aligned} \left|\partial_\xi^\alpha (\mathscr{F}f)(\xi)\right| &\leq \left|\partial_\xi^\alpha (\mathscr{F}_0(jf))(\xi)\right| \\ &\qquad + C_{\alpha,\varepsilon_0} \bigl\{1 + |\xi|^{n-2-|\alpha|} \bigl|(\log|\xi|)^{\varepsilon(n)}\bigr|\bigr\} \|f\|_{L^2_s(\Omega)} \end{aligned} \tag{2.38}$$

for all $0 < |\xi| < \varepsilon_0$ and $|\alpha| \leq n-2$, and

$$\left|\partial_\xi^\alpha (\mathscr{F}f)(\xi)\right| \leq \left|\partial_\xi^\alpha (\mathscr{F}_0(jf))(\xi)\right| + C_{\alpha,\varepsilon_0} (1 + |\xi|^{n-2-|\alpha|}) \|f\|_{L^2_s(\Omega)} \tag{2.39}$$

for all $0 < |\xi| < \varepsilon_0$ and $|\alpha| = n-1, n$.

PROOF. The derivatives of the first term in $(\mathscr{F}f)(\xi)$ are estimated by $|\partial_\xi^\alpha(\mathscr{F}_0(jf))(\xi)|$. Hence we may concentrate on estimating the perturbative term in $(\mathscr{F}f)(\xi)$.

(i) By using the Schwarz inequality, we have

$$\begin{aligned}&\left|\partial_\xi^\alpha\int_\Omega \overline{[R(|\xi|^2+i0)M_\xi(\cdot)](x)}f(x)\,dx\right|\\&\qquad\le \big\|\partial_\xi^\alpha[R(|\xi|^2+i0)M_\xi(\cdot)]\big\|_{L^2_{-s-|\alpha|}(\Omega)}\|f\|_{L^2_{s+|\alpha|}(\Omega)}\end{aligned}\tag{2.40}$$

for $s>1/2$. Since $M_\xi(x)$ has the compact support in $x\in\Omega$, it follows from (2.31) that

$$\big\|\partial_\xi^\beta M_\xi(\cdot)\big\|_{L^2_{s+|\alpha|}(\Omega)}\le C_\beta(1+|\xi|),\quad (\xi\in\boldsymbol{R}^n)\tag{2.41}$$

for any β, and hence, we get, by using Lemma 2.4,

$$\big\|\partial_\xi^\alpha[R(|\xi|^2+i0)M_\xi(\cdot)]\big\|_{L^2_{-s-|\alpha|}(\Omega)}\le\sum_{|\beta|\le|\alpha|}C_{\alpha,\beta,\varepsilon_0}|\xi|^{-1}\big\|\partial_\xi^\beta M_\xi(\cdot)\big\|_{L^2_{s+|\alpha|}(\Omega)}\le C_{\alpha,\varepsilon_0}$$

for all $|\xi|\ge\varepsilon_0$. This estimate together (2.40) imply the required estimate (2.34).

(ii) As to the low frequency part, instead of the additional weight $|\alpha|$ on f, we need to restrict the exponent s to $s>n+1/2$ when we consider the differentiability of the resolvent. In fact, if we choose $d=n$ as $d<s-1/2$ in the part (iii) of Proposition 2.5, then the remainder term in asymptotic expansion (2.13) of $R(|\xi|^2+i0)$ is n-times differentiable in $0<|\xi|\le\varepsilon_0$, and we can estimate

$$\begin{aligned}&\left|\partial_\xi^\alpha\int_\Omega \overline{[R(|\xi|^2+i0)M_\xi(\cdot)](x)}f(x)\,dx\right|\\&\quad\le\big\|\partial_\xi^\alpha[R(|\xi|^2+i0)M_\xi(\cdot)]\big\|_{L^2_{-s}(\Omega)}\|f\|_{L^2_s(\Omega)}\\&\quad\le\sum_{|\beta|\le|\alpha|}C_{\alpha,\beta}\big\|\partial_\xi^{\alpha-\beta}R(|\xi|^2+i0)\big\|_{\mathscr{B}(L^2_s(\Omega),H^2_{-s}(\Omega))}\big\|\partial_\xi^\beta M_\xi(\cdot)\big\|_{L^2_s(\Omega)}\|f\|_{L^2_s(\Omega)}\\&\quad\le C_\alpha\sum_{|\beta|\le|\alpha|}\big\|\partial_\xi^{\alpha-\beta}R(|\xi|^2+i0)\big\|_{\mathscr{B}(L^2_s(\Omega),H^2_{-s}(\Omega))}\|f\|_{L^2_s(\Omega)}\end{aligned}\tag{2.42}$$

for $|\alpha|\le n$, where we used the estimate (2.41). By using the asymptotic expansion (2.13) in Proposition 2.5 we can write

$$R(|\xi|^2+i0)=|\xi|^{n-2}(\log|\xi|)^{\varepsilon(n)}F+G(|\xi|^2)+o(|\xi|^{n-2})\quad \text{in } \mathscr{B}(L^2_s(\Omega),H^2_{-s}(\Omega))$$

for $0<|\xi|\le\varepsilon_0$, where $G(|\xi|^2)$ is the polynomial of degree $2[n/2-1]$ in ξ, and the remainder term is n-times differentiable in $0<|\xi|\le\varepsilon_0$. Then we have, for $|\alpha|\le n-2$,

$$\begin{aligned}
&\sum_{|\beta|\le|\alpha|}\big\|\partial_\xi^{\alpha-\beta}R(|\xi|^2+i0)\big\|_{\mathscr{B}(L^2_s(\Omega),H^2_{-s}(\Omega))}\\
&\quad\le\sum_{|\beta|\le|\alpha|}C_{\alpha,\beta}\big\{|\xi|^{n-2-|\alpha|+|\beta|}\big|(\log|\xi|)^{\varepsilon(n)}\big|+\big|\partial_\xi^{\alpha-\beta}G(|\xi|^2)\big|+|\xi|^{n-2-|\alpha|+|\beta|}\big\}\\
&\quad\le C_{\alpha,\varepsilon_0}\big\{1+|\xi|^{n-2-|\alpha|}\big|(\log|\xi|)^{\varepsilon(n)}\big|\big\},
\end{aligned}\tag{2.43}$$

where we used the estimate $\sum_{|\beta|\le|\alpha|}|\partial_\xi^{\alpha-\beta}G(|\xi|^2)|\le C_{\alpha,\varepsilon_0}$ for $|\xi|\le\varepsilon_0$. For $|\alpha|\ge n-1$, the logarithmic factor is negligible. Thus (2.36)–(2.37) follow from (2.42)–(2.43). □

3. Proof of Theorems 1.1 and 1.4.

First we shall prove Theorem 1.1 along an idea of D'Ancona and Spagnolo [**7**]. Let us consider the *linear* problem:

$$\partial_t^2u-c(t)^2\Delta u=0,\quad x\in\Omega,\tag{3.1}$$

for $t\neq0$, with the initial condition

$$u(0,x)=f_0(x),\quad \partial_tu(0,x)=f_1(x),\tag{3.2}$$

and the boundary condition

$$u(t,x)=0,\quad (t,x)\in\boldsymbol{R}\times\partial\Omega.\tag{3.3}$$

Here $c(t)$ satisfies a suitable condition introduced later. We define a new function

$$\tilde c(t)^2=1+\int_\Omega|\nabla u|^2\,dx.\tag{3.4}$$

This defines a map

$$\Theta: c\mapsto\tilde c.$$

If we can find a fixed point of Θ in a suitable space, the solution $u(t,x)$ to (3.1)–(3.3) will be a solution to the original problem (1.1)–(1.3).

Now let us introduce a set $\mathscr{K}$ as follows:

A set $\mathscr{K}$: *Given $\Lambda > 1$, $K > 0$ and $k > 1$, the function $c(t) \in \mathrm{Lip}_{\mathrm{loc}}(\boldsymbol{R})$ belongs to $\mathscr{K} = \mathscr{K}(k, \Lambda, K)$ if the following two conditions are satisfied*:

$$1 \le c(t) \le \Lambda,$$
$$|c'(t)| \le K(1+|t|)^{-k}.$$

The following proposition is crucial in the argument.

Proposition 3.1. *Let $c(t) \in \mathscr{K}$. Then there exist two constants $B > 0$ and $M > 0$ such that if K satisfies $K < B$, then*

$$1 \le \tilde{c}(t) \le 1 + \|\nabla f_0\|_{L^2(\Omega)} + \frac{M}{k-1} |\!|\!|(f_0, f_1)|\!|\!|_{Y_k(\Omega)}, \tag{3.5}$$

$$|\tilde{c}'(t)| \le M(1+|t|)^{-k} |\!|\!|(f_0, f_1)|\!|\!|_{Y_k(\Omega)}. \tag{3.6}$$

Proof. The proof is essentially based on the methods of [**7**, Theorem 1.1] and [**35**, Thoerem 4]. For the solution $u(t,x)$ to (3.1)–(3.3), we define two functions

$$v_\pm(t) = \frac{e^{\pm i\vartheta(t)|D|}}{\sqrt{c(t)}} (\partial_t u \mp i c(t)|D| u),$$

where we put

$$\vartheta(t) = \int_0^t c(s)\, ds.$$

We need two functionals

$$I(r,t) = \left(|D| e^{2ir|D|} v_-(t), v_+(t)\right)_{L^2(\Omega)},$$
$$J(r,t) = \left(|D| e^{2ir|D|} v_+(t), v_+(t)\right)_{L^2(\Omega)} + \left(|D| e^{2ir|D|} v_-(t), v_-(t)\right)_{L^2(\Omega)}$$

for $r, t \in \boldsymbol{R}$. Then it can be checked that

$$2\tilde{c}(t)\tilde{c}'(t) = \mathrm{Im} I(\vartheta(t), t). \tag{3.7}$$

Hence it suffices for our purpose to derive the decay estimate for $I(\vartheta(t),t)$ when $t \geq 0$.

Defining

$$|f|_k = \sup_{r\in\boldsymbol{R}}(1+|r|)^k|f(r)|$$

for every function f on $\boldsymbol{R}$, we shall prove the following estimate:

$$\sup_{t\geq 0}|I(\cdot,t)|_k \leq 2M|(f_0,f_1)|_{Y_k(\Omega)} \tag{3.8}$$

for a suitable constant M depending only on k and Λ.

To begin with, we prove that there exists a constant C_1 such that

$$|I(\cdot,0)|_k + |J(\cdot,0)|_k \leq C_1|(f_0,f_1)|_{Y_k(\Omega)}. \tag{3.9}$$

It follows from the definition of $|(f_0,f_1)|_{Y_k(\Omega)}$ that

$$\begin{aligned}
&\left|\big(|D|e^{2ir|D|}v_-(0),v_+(0)\big)_{L^2(\Omega)}\right| \\
&\quad\leq \left|-c(0)\big(e^{2ir|D|}|D|^{\frac{3}{2}}f_0,|D|^{\frac{3}{2}}f_0\big)_{L^2(\Omega)}\right| + \left|c(0)^{-1}\big(e^{2ir|D|}|D|^{\frac{1}{2}}f_1,|D|^{\frac{1}{2}}f_1\big)_{L^2(\Omega)}\right| \\
&\qquad+\left|i\big(e^{2ir|D|}|D|^{\frac{3}{2}}f_0,|D|^{\frac{1}{2}}f_1\big)_{L^2(\Omega)}+i\big(e^{2ir|D|}|D|^{\frac{1}{2}}f_1,|D|^{\frac{3}{2}}f_0\big)_{L^2(\Omega)}\right| \\
&\quad\leq C(1+|2r|)^{-k}|(f_0,f_1)|_{Y_k(\Omega)},
\end{aligned}$$

which implies that

$$|I(\cdot,0)|_k \leq C|(f_0,f_1)|_{Y_k(\Omega)}.$$

In a similar way, we have the same type estimate for $J(\cdot,0)$. Hence we obtain (3.9).

Let us prove (3.8). By the definition of $v_\pm(t)$ we have

$$v_\pm'(t) = -\frac{c'(t)}{2c(t)}e^{\pm 2i\vartheta(t)|D|}v_\mp(t). \tag{3.10}$$

Differentiating $I(r,t)$ and $J(r,t)$ with respect to t and plugging (3.10) into the resulting ones, we get

$$\partial_t I(r,t) = -\frac{c'(t)}{2c(t)} J(r-\vartheta(t),t),$$
$$\partial_t J(r,t) = -\frac{c'(t)}{c(t)} \big(I(r+\vartheta(t),t) + \overline{I(-r+\vartheta(t),t)} \big).$$

Write these equations into integral equation:

$$\begin{aligned} I(r,t) &= I(r,0) - \frac{1}{2}\int_0^t \frac{c'(s)}{c(s)} J(r-\vartheta(s),0)\,ds \\ &\quad + \frac{1}{2}\int_0^t \frac{c'(s)}{c(s)} \int_0^s \frac{c'(\sigma)}{c(\sigma)} \\ &\quad \times \big(I(r-\vartheta(s)+\vartheta(\sigma),\sigma) + \overline{I(-r+\vartheta(s)+\vartheta(\sigma),\sigma)} \big)\,d\sigma ds. \end{aligned} \tag{3.11}$$

Defining $[g]_k = \sup_{t\ge 0} |g(\cdot,t)|_k$, we see from (3.11) that

$$(1+|r|)^k |I(r,t)| \le |I(\cdot,0)|_k + I_1 + I_2, \tag{3.12}$$

where

$$I_1 = \frac{K}{2} |J(\cdot,0)|_k (1+|r|)^k \int_0^t (1+s)^{-k} (1+|r-\vartheta(s)|)^{-k}\,ds,$$
$$\begin{aligned} I_2 = \frac{K^2}{2} [I]_k (1+|r|)^k &\int_0^t (1+s)^{-k} \int_0^s (1+\sigma)^{-k} \\ &\times \big\{ (1+|r-\vartheta(s)+\vartheta(\sigma)|)^{-k} + (1+|-r+\vartheta(s)+\vartheta(\sigma)|)^{-k} \big\}\,d\sigma ds. \end{aligned}$$

Notice that

$$\vartheta'(s) = c(s) \ge 1, \quad \vartheta(s) = \int_0^s c \le \Lambda s, \quad \frac{1+\vartheta(s)}{1+s} \le \Lambda.$$

Then changing variable $\rho = \vartheta(s)$ in I_1 and using (3.9), we have

$$\begin{aligned} I_1 &\le \frac{C_1 K \Lambda^k |(f_0,f_1)|_{Y_k(\Omega)}}{2} (1+|r|)^k \int_0^t (1+\vartheta(s))^{-k} (1+|r-\vartheta(s)|)^{-k}\,ds \\ &\le \frac{C_1 K \Lambda^k |(f_0,f_1)|_{Y_k(\Omega)}}{2} (1+|r|)^k \int_0^\infty (1+\rho)^{-k} (1+|r-\rho|)^{-k} \frac{d\rho}{\vartheta'(s)} \end{aligned}$$

$$\leq \frac{2^{k-1}C_1 K\Lambda^k}{k-1} |(f_0, f_1)|_{Y_k(\Omega)}, \tag{3.13}$$

where we have used the following well-known inequality: If θ_1 and θ_2 are real numbers with $\max(\theta_1, \theta_2) > 1$, then

$$\int_0^\infty (1+|t-s|)^{-\theta_1}(1+s)^{-\theta_2}\, ds \leq c(\theta_1,\theta_2)(1+|t|)^{-\min(\theta_1,\theta_2)}$$

holds with $c(\theta_1, \theta_2) = 2^{\min(\theta_1,\theta_2)}/(\max(\theta_1,\theta_2)-1)$. In a similar way, we can treat the term I_2, and we have

$$\begin{aligned} I_2 \leq{}& \frac{K^2\Lambda^{2k}}{2}[I]_k(1+|r|)^k \int_0^t (1+\vartheta(s))^{-k}\int_0^s (1+\vartheta(\sigma))^{-k} \\ &\times \left\{(1+|r-\vartheta(s)+\vartheta(\sigma)|)^{-k} + (1+|\vartheta(s)+\vartheta(\sigma)-r|)^{-k}\right\} d\sigma ds \\ \leq{}& \frac{2^{2k}K^2\Lambda^{2k}}{(k-1)^2}[I]_k. \end{aligned} \tag{3.14}$$

Applying the estimate (3.9) to the first term in the right-hand side of (3.12) and combining this with (3.13)–(3.14), we arrive at

$$[I]_k \leq C_1|(f_0,f_1)|_{Y_k(\Omega)} + \frac{2^{k-1}C_1K\Lambda^k}{k-1}|(f_0,f_1)|_{Y_k(\Omega)} + \frac{2^{2k}K^2\Lambda^{2k}}{(k-1)^2}[I]_k.$$

If K satisfies

$$K < \frac{1}{\sqrt{2}}\frac{k-1}{2^k\Lambda^k} \equiv B,$$

then

$$[I]_k \leq 2\left(C_1 + \frac{2^{k-1}C_1B\Lambda^k}{k-1}\right)|(f_0,f_1)|_{Y_k(\Omega)} \equiv 2M|(f_0,f_1)|_{Y_k(\Omega)},$$

which proves (3.8).

We now apply the estimate (3.8) to obtain the decay estimate of $\tilde{c}'(t)$. It follows from (3.7) that

$$|\tilde{c}'(t)| \leq \frac{1}{2}|I(\vartheta(t),t)| \leq M|(f_0,f_1)|_{Y_k(\Omega)}(1+t)^{-k}, \tag{3.15}$$

which proves the estimate (3.6).

It remains to prove the estimate (3.5). The first inequality is obvious, if we recall the definition (3.4) of $\tilde{c}(t)$. As to the second inequality, integrating (3.15) and using

$$\tilde{c}(t) \leq \tilde{c}(0) + \int_0^\infty |\tilde{c}'(\tau)|\, d\tau,$$

we get

$$\tilde{c}(t) \leq \tilde{c}(0) + \frac{M}{k-1} |(f_0, f_1)|_{Y_k(\Omega)}.$$

Since $\tilde{c}^2(0) = 1 + \|\nabla f_0\|_{L^2}^2$, we obtain (3.5). The proof of Proposition 3.1 is now complete. □

Proof of Theorem 1.1. We employ the Schauder-Tychonoff fixed point theorem. Let $c(t) \in \mathscr{K}$, and we fix the data $(f_0, f_1) \in Y_k(\Omega)$. Then it follows from Proposition 3.1 that the map

$$\Theta : c(t) \mapsto \tilde{c}(t)$$

maps $\mathscr{K}$ into itself provided that the quantity $\|\nabla f_0\|_{L^2(\Omega)}^2 + |(f_0, f_1)|_{Y_k(\Omega)}$ is sufficiently small. Now $\mathscr{K}$ may be regarded as the convex subset of the Fréchet space $L^\infty_{\mathrm{loc}}(\boldsymbol{R})$, and we endow $\mathscr{K}$ with the induced topology.

Compactness of $\mathscr{K}$. Since $\mathscr{K}$ is uniformly bounded and equi-continuous on every compact t-interval, one can deduce from the Ascoli-Arzelà theorem that $\mathscr{K}$ is relatively compact in $L^\infty_{\mathrm{loc}}(\boldsymbol{R})$, and it is sequentially compact. This means that every sequence $\{c_j(t)\}_{j=1}^\infty$ in $\mathscr{K}$ has a subsequence, denoted by the same, converging to some $c(\cdot) \in \mathrm{Lip}_{\mathrm{loc}}(\boldsymbol{R})$:

$$c_j(t) \underset{(j\to\infty)}{\to} c(t) \quad \text{in } L^\infty_{\mathrm{loc}}(\boldsymbol{R}), \quad \|c(\cdot)\|_{L^\infty(\boldsymbol{R})} \leq \Lambda,$$

where we used the equation

$$c_j(t) - c_j(t') = \int_{t'}^t c_j'(\tau)\, d\tau, \tag{3.16}$$

and the assumption that

$$|c_j'(t)| \leq K(1+|t|)^{-k} \in L^1(\boldsymbol{R}). \tag{3.17}$$

Moreover, the derivative $c'(t)$ exists almost everywhere on $\boldsymbol{R}$. Now, for the derivative $c'(t)$, if we prove that

$$|c'(t)| \le K(1+|t|)^{-k} \quad \text{a.e. } t \in \boldsymbol{R}, \tag{3.18}$$

then $c(t) \in \mathscr{K}$, which proves the compactness of $\mathscr{K}$. We prove (3.18). Let $t_0 \in \boldsymbol{R}$ be an arbitrarily point where $c(t)$ is differentiable. Since we have, by (3.16)–(3.17),

$$\left|\frac{1}{2h}\{c_j(t_0+h)-c_j(t_0-h)\}\right| = \left|\frac{1}{2h}\int_{t_0-h}^{t_0+h} c_j'(t)\,dt\right| \le \frac{1}{2h}\int_{t_0-h}^{t_0+h} K(1+|t|)^{-k}\,dt,$$

for $h>0$, we can take the limit in this equation with respect to j, so that

$$\left|\frac{1}{2h}\{c(t_0+h)-c(t_0-h)\}\right| \le \frac{1}{2h}\int_{t_0-h}^{t_0+h} K(1+|t|)^{-k}\,dt.$$

Then, letting $h \to +0$, we conclude that

$$|c'(t_0)| \le K(1+|t_0|)^{-k}.$$

Since t_0 is arbitrary, we get (3.18).

Continuity of Θ *on* $\mathscr{K}$. We may consider the case $t>0$, since the case $t<0$ can be treated in the same way. Let us take a sequence $\{c_m(t)\}$ in $\mathscr{K}$ such that

$$c_m(t) \to c(t) \in \mathscr{K} \quad \text{in } L^\infty_{\text{loc}}(0,\infty) \quad (m\to\infty),$$

and let $u_m(t,x)$ and $u(t,x)$ be corresponding solutions to $c_m(t)$ and $c(t)$, respectively, with fixed data $(f_0,f_1) \in Y_k(\Omega)$. Then we prove that the images $\tilde{c}_m(t) := \Theta(c_m(t))$ and $\tilde{c}(t) := \Theta(c(t))$ satisfy

$$\tilde{c}_m(t) \to \tilde{c}(t) \quad \text{in } L^\infty_{\text{loc}}(0,\infty) \quad (m\to\infty). \tag{3.19}$$

The functions $v_m := u_m - u$, $m = 1,2,\ldots$, solve the following initial-boundary value problem:

$$\begin{cases} \partial_t^2 v_m - c(t)^2\Delta v_m = \{c_m(t)^2 - c(t)^2\}\Delta u_m, & (t,x) \in \boldsymbol{R}\times\Omega, \\ v_m(0,x) = 0, \quad \partial_t v_m(0,x) = 0, & x \in \Omega, \\ v_m(t,x) = 0, & (t,x) \in \boldsymbol{R}\times\partial\Omega. \end{cases}$$

Differentiate the energy $E(v_m(t))$ for v_m with respect to t, where

$$E(v_m(t)) = \|v_m'(t)\|^2_{L^2(\Omega)} + c(t)^2\|\nabla v_m(t)\|^2_{L^2(\Omega)}, \quad (' = \partial_t).$$

Then we get

$$\begin{aligned} E'(v_m(t)) &= -2\{c_m(t)^2 - c(t)^2\}\mathrm{Re}(\Delta u_m(t), v_m'(t))_{L^2(\Omega)} \\ &\quad + 2c(t)c'(t)\|\nabla v_m(t)\|^2_{L^2(\Omega)} \\ &\le 2\big|c_m(t)^2 - c(t)^2\big|\|u_m(t)\|_{H^{3/2}(\Omega)}\|v_m'(t)\|_{H^{1/2}(\Omega)} + 2\frac{c'(t)}{c(t)}E(v_m(t)). \end{aligned} \tag{3.20}$$

Since $\|u_m(t)\|_{H^{3/2}(\Omega)}$ and $\|v_m'(t)\|_{H^{1/2}(\Omega)}$ are bounded by the quantity $\|f_0\|_{H^{3/2}(\Omega)} + \|f_1\|_{H^{1/2}(\Omega)}$, we integrate (3.20) and apply Gronwall's lemma to obtain, for all $t \ge 0$,

$$E(v_m(t)) \le C\biggl(\int_0^t \big|c_m(\tau)^2 - c(\tau)^2\big|\, d\tau\biggr)\bigl(\|f_0\|_{H^{3/2}(\Omega)} + \|f_1\|_{H^{1/2}(\Omega)}\bigr)^2 e^{2\int_0^\infty \frac{|c'(\tau)|}{c(\tau)}\, d\tau},$$

which implies that

$$\left.\begin{aligned} \nabla u_m(t) &\to \nabla u(t) \\ u_m'(t) &\to u'(t) \end{aligned}\right\} \quad \text{in } L^\infty_{\mathrm{loc}}(0,\infty; L^2(\Omega)) \text{ as } m \to \infty.$$

Hence we get (3.19), which proves the continuity of Θ. □

Completion of the proof of Theorem 1.1. By using the Schauder–Tychonoff fixed point theorem, we can show that Θ has a fixed point in $\mathscr{K}$, and hence, we conclude that the solution $u(t,x)$ to (3.1)–(3.3) is the solution to (1.1)–(1.3). The uniqueness of solutions is obvious. This proves Theorem 1.1. □

Proof of Theorem 1.4. We need a decay estimate of some oscillatory integrals.

Lemma 3.2. *Let $n \ge 3$. Assume that $\boldsymbol{R}^n \setminus \Omega$ is star-shaped with respect to the origin. Let $f_1 \in H^{\gamma_1+1/2}_{s(k),0}(\Omega)$ and $f_2 \in H^{\gamma_2+1/2}_{s(k),0}(\Omega)$ for some $s(k) > \max(n + 1/2, k + n/2)$, $k \in (1, n]$, and for some $\gamma_1, \gamma_2 > n/2$. Consider the oscillatory integral of the form*

$$F(\tau)=\int_{\boldsymbol{R}^n} e^{i\tau|\xi|}(\mathscr{F}f_1)(\xi)(\mathscr{F}f_2)(\xi)|\xi|\,d\xi, \quad (\tau\in\boldsymbol{R}).$$

Then

$$|F(\tau)|\le C(1+|\tau|)^{-k}\|f_1\|_{H^{\gamma_1+1/2}_{s(k)}(\Omega)}\|f_2\|_{H^{\gamma_2+1/2}_{s(k)}(\Omega)}.$$

The proof of Lemma 3.2 is rather long and will be postponed in the last part of this section.

Put $\gamma_1=\gamma_2=s_0-1/2$ in Lemma 3.2. Recall the definition of $Y_k(\Omega)$. If we choose $(\mathscr{F}f_1)(\xi),(\mathscr{F}f_2)(\xi)$ as $|\xi|(\mathscr{F}f_0)(\xi)$ in Lemma 3.2, one has

$$\left|\left(e^{i\tau|\xi|}|\xi|^{\frac{3}{2}}\mathscr{F}f_0,|\xi|^{\frac{3}{2}}\mathscr{F}f_0\right)_{L^2(\boldsymbol{R}^n)}\right|\le C(1+|\tau|)^{-k}\||D|f_0\|^2_{H^{s_0}_{s(k)}(\Omega)}.$$

If we choose $(\mathscr{F}f_1)(\xi),(\mathscr{F}f_2)(\xi)$ as $(\mathscr{F}f_1)(\xi)$ in Lemma 3.2, one has

$$\left|\left(e^{i\tau|\xi|}|\xi|^{\frac{1}{2}}\mathscr{F}f_1,|\xi|^{\frac{1}{2}}\mathscr{F}f_1\right)_{L^2(\boldsymbol{R}^n)}\right|\le C(1+|\tau|)^{-k}\|f_1\|^2_{H^{s_0}_{s(k)}(\Omega)}.$$

If we choose $(\mathscr{F}f_1)(\xi)$ as $|\xi|(\mathscr{F}f_0)(\xi)$, and $(\mathscr{F}f_2)(\xi)$ as $(\mathscr{F}f_1)(\xi)$ in Lemma 3.2, respectively, one has

$$\left|\left(e^{i\tau|\xi|}|\xi|^{\frac{3}{2}}\mathscr{F}f_0,|\xi|^{\frac{1}{2}}\mathscr{F}f_1\right)_{L^2(\boldsymbol{R}^n)}\right|\le C(1+|\tau|)^{-k}\||D|f_0\|_{H^{s_0}_{s(k)}(\Omega)}\|f_1\|_{H^{s_0}_{s(k)}}.$$

These estimates imply Example 1.3:

$$H^{s_0+1}_{s(k),0}(\Omega)\times H^{s_0}_{s(k),0}(\Omega)\subset Y_k(\Omega).$$

Therefore, applying Theorem 1.1, we conclude that the $H^{3/2}$ solution $u(t,x)$ exists globally in $\boldsymbol{R}$. Thus, by using Theorem 2 from [**1**, Arosio and Garavaldi] we can readily check that $u(t,x)$ belongs to $\cap_{j=0,1,2}C^j(\boldsymbol{R};H^{s_0+1-j}(\Omega))$. The proof of Theorem 1.4 is finished. □

Proof of Lemma 3.2. First, we observe that $F(\tau)$ is bounded in $\tau\in\boldsymbol{R}$, provided that $f_1\in H^{1/2}(\Omega)$ and $f_2\in H^{1/2}(\Omega)$. In fact, by using the generalized Parseval identity (2.32) and diagonalization property (2.33), we have

$$|F(\tau)|\le\int_{\boldsymbol{R}^n}|(\mathscr{F}f_1)(\xi)|\,|(\mathscr{F}f_2)(\xi)|\,|\xi|\,d\xi\le\|f_1\|_{H^{\frac{1}{2}}(\Omega)}\|f_2\|_{H^{\frac{1}{2}}(\Omega)}.$$

Hence we have only to prove the case $|\tau| \geq 1$. To prove the decay estimate of $F(\tau)$, we divide it into the high frequency part and low frequency one. Recall the number ε_0 in Proposition 2.5. Inserting the cut-off function $\chi(\xi) \in C^\infty(\boldsymbol{R}^n)$ equal to one for $|\xi| \geq \varepsilon_0$ and 0 for $|\xi| \leq \varepsilon_0/2$, we write

$$\begin{aligned} F(\tau) &= F_1(\tau) + F_2(\tau) \\ &= \int_{\boldsymbol{R}^n} e^{i\tau|\xi|}\chi(\xi)(\mathscr{F}f_1)(\xi)(\mathscr{F}f_2)(\xi)|\xi|\,d\xi \\ &\quad + \int_{\boldsymbol{R}^n} e^{i\tau|\xi|}(1-\chi(\xi))(\mathscr{F}f_1)(\xi)(\mathscr{F}f_2)(\xi)|\xi|\,d\xi. \end{aligned}$$

We shall derive the decay estimate of $F_1(\tau)$ for $|\tau| \geq 1$. On the support of $\chi(\xi)$, we see that $|\nabla_\xi(\tau|\xi|)| = |\tau|$. Since the support of the amplitude function is away from the origin, and since k-fold ξ-derivatives of the amplitude function decay as $|\xi| \to \infty$ for any integer k, we can perform k-fold integration by parts with an operator $P = \frac{\nabla_\xi(\tau|\xi|)}{i|\nabla_\xi(\tau|\xi|)|^2} \cdot \nabla_\xi$; thus we find that

$$F_1(\tau) = \int_{\boldsymbol{R}^n} e^{i\tau|\xi|}(P^*)^k\big[\chi(\xi)|\xi|^{-\gamma_1-\gamma_2}(\mathscr{F}|D|^{\gamma_1+1/2}f_1)(\xi)(\mathscr{F}|D|^{\gamma_1+1/2}f_2)(\xi)\big]\,d\xi,$$

where we used the diagonalization property (2.33). Then, by using (2.35) from Lemma 2.8, we can write

$$\begin{aligned} |F_1(\tau)| &\leq C_k|\tau|^{-k} \sum_{|\nu|\leq|\mu|\leq k} \int_{\boldsymbol{R}^n} \langle\xi\rangle^{-\gamma_1-\gamma_2} \\ &\quad \times \left|\partial_\xi^{\mu-\nu}\big\{\big(\mathscr{F}[|D|^{\gamma_1+\frac{1}{2}}f_1]\big)(\xi)\big\}\right|\left|\partial_\xi^{\nu}\big\{\big(\mathscr{F}[|D|^{\gamma_2+\frac{1}{2}}f_2]\big)(\xi)\big\}\right| d\xi \\ &\leq C_k|\tau|^{-k} \sum_{|\nu|\leq|\mu|\leq k} \int_{\boldsymbol{R}^n} \langle\xi\rangle^{-\gamma_1-\gamma_2} \\ &\quad \times \Big[\left|\partial_\xi^{\mu-\nu}\big\{\big(\mathscr{F}_0[j|D|^{\gamma_1+\frac{1}{2}}f_1]\big)(\xi)\big\}\right| + C\|f_1\|_{H^{\gamma_1+\frac{1}{2}}_{s+|\mu-\nu|}(\Omega)}\Big] \\ &\quad \times \Big[\left|\partial_\xi^{\nu}\big\{\big(\mathscr{F}_0[j|D|^{\gamma_2+\frac{1}{2}}f_2]\big)(\xi)\big\}\right| + C\|f_2\|_{H^{\gamma_2+\frac{1}{2}}_{s+|\nu|}(\Omega)}\Big]\,d\xi \end{aligned}$$

for any $k \in \boldsymbol{N}$ and $s > 1/2$. Here, by using the properties of the Fourier transform on $\boldsymbol{R}^n$ and the Schwarz inequality, we have

$$\int_{\boldsymbol{R}^n} \langle\xi\rangle^{-\gamma_1-\gamma_2} \left|\partial_\xi^{\mu-\nu}\left\{\left(\mathscr{F}_0[j|D|^{\gamma_1+\frac{1}{2}}f_1]\right)(\xi)\right\}\right| d\xi \leq C\|f_1\|_{H^{\gamma_1+\frac{1}{2}}_{|\mu-\nu|}(\Omega)},$$

$$\int_{\boldsymbol{R}^n} \langle\xi\rangle^{-\gamma_1-\gamma_2} \left|\partial_\xi^{\nu}\left\{\left(\mathscr{F}_0[j|D|^{\gamma_2+\frac{1}{2}}f_2]\right)(\xi)\right\}\right| d\xi \leq C\|f_2\|_{H^{\gamma_2+\frac{1}{2}}_{|\nu|}(\Omega)},$$

and hence, we get

$$\begin{aligned}
|F_1(\tau)| \leq C_k|\tau|^{-k}\sum_{a\leq k}\Big(&\|f_1\|_{H^{\gamma_1+\frac{1}{2}}_{k-a}(\Omega)}\|f_2\|_{H^{\gamma_2+\frac{1}{2}}_{a}(\Omega)} + \|f_1\|_{H^{\gamma_1+\frac{1}{2}}_{k-a}(\Omega)}\|f_2\|_{H^{\gamma_2+\frac{1}{2}}_{s+a}(\Omega)} \\
&+ \|f_1\|_{H^{\gamma_1+\frac{1}{2}}_{s+k-a}(\Omega)}\|f_2\|_{H^{\gamma_2+\frac{1}{2}}_{a}(\Omega)} + \|f_1\|_{H^{\gamma_1+\frac{1}{2}}_{s+k-a}(\Omega)}\|f_2\|_{H^{\gamma_2+\frac{1}{2}}_{s+a}(\Omega)}\Big) \\
\leq C_k|\tau|^{-k}\sum_{a\leq k}&\|f_1\|_{H^{\gamma_1+\frac{1}{2}}_{s+k-a}(\Omega)}\|f_2\|_{H^{\gamma_2+\frac{1}{2}}_{s+a}(\Omega)} \qquad (3.21)
\end{aligned}$$

for any $k \in \boldsymbol{N}$ and $s > 1/2$.

We now turn to the estimate of $F_2(\tau)$. For brevity, we denote the symbol in the integral $F_2(\tau)$ by

$$A(\xi) = (1-\chi(\xi))(\mathscr{F}f_1)(\xi)(\mathscr{F}f_2)(\xi)|\xi|.$$

Making change of variable $\xi = \lambda\omega$ ($\lambda = |\xi|, \omega \in \boldsymbol{S}^{n-1}$), we have

$$F_2(\tau) = \int_{\boldsymbol{S}^{n-1}}\int_0^\infty e^{i\lambda\tau}A(\lambda\omega)\lambda^{n-1}\,d\lambda d\omega.$$

We shall prove the following:

$$|F_2(\tau)| \leq C_k|\tau|^{-k}\|f_1\|_{L^2_{s(k)}(\Omega)}\|f_2\|_{L^2_{s(k)}(\Omega)} \tag{3.22}$$

for $s(k) > \max(n+1/2, k+n/2)$ and $k = 1,\dots,n$. Writing

$$\begin{aligned}
\left|\partial_\lambda^{k-1}(A(\lambda\omega)\lambda^{n-1})\right| &= \left|\partial_\lambda^{k-1}\{\lambda^n(1-\chi(\lambda\omega))(\mathscr{F}f_1)(\lambda\omega)(\mathscr{F}f_2)(\lambda\omega)\}\right| \\
&\leq C_{n,k}\sum_{a=0}^{k-1}\lambda^{n-k+a+1}\sum_{a_1+a_2\leq a}\left|\partial_\lambda^{a_1}(\mathscr{F}f_1)(\lambda\omega)\right|\left|\partial_\lambda^{a_2}(\mathscr{F}f_2)(\lambda\omega)\right|
\end{aligned}$$

and applying (2.38)–(2.39) in Lemma 2.8 to the last factor, we get

$$
\begin{aligned}
&\left|\partial_\lambda^{k-1}(A(\lambda\omega)\lambda^{n-1})\right| \\
&\quad \le C_{n,k}\sum_{a=0}^{k-1}\lambda^{n-k+a+1}\sum_{a_1+a_2\le a} \\
&\qquad \times \left[\left|\partial_\lambda^{a_1}(\mathscr{F}_0(jf_1))(\lambda\omega)\right| + \{1+\lambda^{n-2-a_1}|(\log\lambda)^{\varepsilon(n)}|\}\|f_1\|_{L^2_s(\Omega)}\right] \\
&\qquad \times \left[\left|\partial_\lambda^{a_2}(\mathscr{F}_0(jf_2))(\lambda\omega)\right| + \{1+\lambda^{n-2-a_2}|(\log\lambda)^{\varepsilon(n)}|\}\|f_2\|_{L^2_s(\Omega)}\right]
\end{aligned}
$$

for $s > n + 1/2$, which implies that

$$
\partial_\lambda^{k-1}(A(\lambda\omega)\lambda^{n-1}) \to 0 \quad \text{as } \lambda \to 0
$$

provided $k = 1, \dots, n$. Here the logarithmic terms are negligible for $|a_1| = n-1$ or $|a_2| = n-1$. Then integrating by parts, we get

$$
\begin{aligned}
&\left|\int_{\boldsymbol{S}^{n-1}}\int_0^\infty e^{i\lambda\tau}A(\lambda\omega)\lambda^{n-1}\,d\lambda d\omega\right| \\
&\qquad \le C_k|\tau|^{-k}\int_{\boldsymbol{S}^{n-1}}\int_0^\infty \left|\partial_\lambda^k(A(\lambda\omega)\lambda^{n-1})\right| d\lambda d\omega
\end{aligned} \tag{3.23}
$$

for $k = 0, 1, \dots, n$. Writing the perturbative term in the representation $(\mathscr{F}f)(\xi)$ as $V(f)(\xi)$, we have

$$
(\mathscr{F}f)(\xi) = (\mathscr{F}_0[jf])(\xi) + V(f)(\xi),
$$

and hence,

$$
\begin{aligned}
A(\lambda\omega) &= (1-\chi(\lambda\omega))\big\{\lambda(\mathscr{F}_0[jf_1])(\lambda\omega)(\mathscr{F}_0[jf_2])(\lambda\omega) + \lambda V(f_1)(\xi)(\mathscr{F}_0[jf_2])(\lambda\omega) \\
&\qquad + \lambda(\mathscr{F}_0[jf_1])(\lambda\omega)V(f_2)(\lambda\omega) + \lambda V(f_1)(\lambda\omega)V(f_2)(\lambda\omega)\big\} \\
&= A_1(\lambda\omega) + A_2(\lambda\omega) + A_3(\lambda\omega) + A_4(\lambda\omega).
\end{aligned}
$$

By Lemma A from [**8**] (cf. Lemma A.1 from [**7**]), we have

$$
\int_{\boldsymbol{S}^{n-1}}\int_0^\infty \left|\partial_\lambda^k(A_1(\lambda\omega)\lambda^{n-1})\right| d\lambda d\omega \le C_k\sum_{a\le k}\|f_1\|_{L^2_{k-a}(\Omega)}\|f_2\|_{L^2_a(\Omega)}, \tag{3.24}
$$

provided $k = 1, \dots, n+1$. As to $A_2(\lambda\omega)$, we have

$$
\begin{aligned}
&\int_{\boldsymbol{S}^{n-1}}\int_0^\infty \left|\partial_\lambda^k(A_2(\lambda\omega)\lambda^{n-1})\right| d\lambda d\omega \\
&\quad\le \sum_{|\alpha|\le k} C_{k,\alpha}\int_{\boldsymbol{S}^{n-1}}\int_{0<\lambda<\varepsilon_0} \left|\partial_\lambda^{k-|\alpha|}(\lambda^n V(f_1)(\lambda\omega))\right|\left\|\partial_\xi^\alpha(\mathscr{F}_0[jf_2])\right\|_{L^\infty(\boldsymbol{R}^n)} d\lambda d\omega.
\end{aligned}
$$

Here, on the support of $A_2(\lambda\omega)$, using (2.36)–(2.37) in Lemma 2.8 we see that

$$
\begin{aligned}
&\sum_{|\alpha|\le k} \left|\partial_\lambda^{k-|\alpha|}(\lambda^n V(f_1)(\lambda\omega))\right|\left\|\partial_\xi^\alpha(\mathscr{F}_0[jf_2])\right\|_{L^\infty(\boldsymbol{R}^n)} \\
&\quad\le C_k\sum_{a\le k}\left\{1+\lambda^{2n-2-k+a}|(\log\lambda)^{\varepsilon(n)}|\right\}\|f_1\|_{L^2_s(\Omega)}\|f_2\|_{L^2_{a+s'}(\Omega)}
\end{aligned}
$$

for $k=1,\dots,n$, provided $s>n+1/2$ and $s'>n/2$. The functions $\lambda^{2n-2-k}|(\log\lambda)^{\varepsilon(n)}|$ are integrable over $(0,\varepsilon_0)$, since $n\ge 3$, and hence, we get, for $k=1,\dots,n$,

$$
\int_{\boldsymbol{S}^{n-1}}\int_0^\infty \left|\partial_\lambda^k(A_2(\lambda\omega)\lambda^{n-1})\right| d\lambda d\omega \le C_k\|f_1\|_{L^2_s(\Omega)}\|f_2\|_{L^2_{k+s'}(\Omega)} \tag{3.25}
$$

for $s>n+1/2$ and $s'>n/2$. We have also the estimate (3.25) for $A_3(\lambda\omega)$ in a similar way. As to $A_4(\lambda\omega)$, using again (2.36)–(2.37) in Lemma 2.8, we see that

$$
\begin{aligned}
\left|\partial_\lambda^k(A_4(\lambda\omega)\lambda^{n-1})\right| &\le C_k\sum_{a\le k}\left|\partial_\lambda^{k-a}\{\lambda^{n/2}V(f_1)(\lambda\omega)\}\right|\left|\partial_\lambda^a\{\lambda^{n/2}V(f_2)(\lambda\omega)\}\right| \\
&\le C_k\sum_{a\le k}\left\{\lambda^{(n-2)+n/2-k+a}|(\log\lambda)^{\varepsilon(n)}|+\lambda^{n/2-k+a}\right\} \\
&\qquad\times\left\{\lambda^{(n-2)+n/2-a}|(\log\lambda)^{\varepsilon(n)}|+\lambda^{n/2-a}\right\}\|f_1\|_{L^2_s(\Omega)}\|f_2\|_{L^2_s(\Omega)} \\
&\le C_k\left\{\lambda^{n-2}|(\log\lambda)^{\varepsilon(n)}|+1\right\}^2\lambda^{n-k}\|f_1\|_{L^2_s(\Omega)}\|f_2\|_{L^2_s(\Omega)}
\end{aligned}
$$

for $s>n+1/2$. Since $n\ge 3$, we get

$$
\left|\int_{\boldsymbol{S}^{n-1}}\int_0^\infty \partial_\lambda^k(A_4(\lambda\omega)\lambda^{n-1})\, d\lambda d\omega\right| \le C_k\|f_1\|_{L^2_s(\Omega)}\|f_2\|_{L^2_s(\Omega)} \tag{3.26}
$$

for $s>n+1/2$ and $k=1,\dots,n$. Thus the required estimate (3.22) follows from (3.23)–(3.26).

Combining the estimates (3.21)–(3.22), we arrive at the estimate

$$|F(\tau)| \le C_k(1+|\tau|)^{-k}\|f_1\|_{H^{\gamma_1+1/2}_{s(k)}(\Omega)}\|f_2\|_{H^{\gamma_2+1/2}_{s(k)}(\Omega)} \tag{3.27}$$

for any $s(k) > \max(n+1/2, k+n/2)$ and $k = 1, \dots, n$. But then (3.27) holds also for any *real* $k \in [1, n]$, if we use a method similar to the one used in [**33**, Lemma 2.1]. Actually $F(\tau)$ can be regarded as the particular form of bilinear operators $T(f_1, f_2)(\tau)$ from $H^{\gamma_1+1/2}_{s(k)}(\Omega) \times H^{\gamma_2+1/2}_{s(k)}(\Omega)$ to the space $Z_k = \{g(t) : (1+|t|)^k|g(t)| \in L^\infty(\Omega)\}$ defined by

$$T(f_1, f_2)(\tau) = \int_{\boldsymbol{R}^n} e^{i\tau|\xi|}(\mathscr{F}f_1)(\xi)(\mathscr{F}f_2)(\xi)|\xi|\, d\xi,$$

where Z_k is the Banach space with the norm $\|g(t)\|_{Z_k} = \text{ess.sup}_{t\in\boldsymbol{R}}(1+|t|)^k|g(t)|$. It is readily seen, by the complex interpolation between k and $k+1$ $(0 < \theta < 1)$ with $k = 1, \dots, n-1$, that $[Z_k, Z_{k+1}]_\theta$ and $\big[H^{\gamma+1/2}_{s(k)}(\Omega), H^{\gamma+1/2}_{s(k+1)}(\Omega)\big]_\theta$ $(\gamma = \gamma_1$ or $\gamma_2)$ coincide with $Z_{k+\theta}$ and $H^{\gamma+1/2}_{s(k+\theta)}(\Omega)$, respectively. Hence the multilinear interpolation theorem (see Theorem 4.4.1 of [**3**]) implies that

$$\|T(f_1, f_2)(\tau)\|_{Z_{k+\theta}} \le C_{k+\theta}\|f_1\|_{H^{\gamma_1+1/2}_{s(k+\theta)}(\Omega)}\|f_2\|_{H^{\gamma_2+1/2}_{s(k+\theta)}(\Omega)}.$$

Then putting $\varkappa = k+\theta$, we have

$$|T(f_1, f_2)(\tau)| \le C_\varkappa(1+|\tau|)^{-\varkappa}\|f_1\|_{H^{\gamma_1+1/2}_{s(\varkappa)}(\Omega)}\|f_2\|_{H^{\gamma_2+1/2}_{s(\varkappa)}(\Omega)}$$

for any $\varkappa \in [1, n]$, which proves Lemma 3.2. □

4. Proof of Theorem 1.5.

The following lemma can be found in [**20**, Theorem 1.1] which is proved in the whole space $\boldsymbol{R}^n$. However, it holds even in the exterior domains.

LEMMA 4.1. *Let $n \ge 1$. For the linear problem* (3.1)–(3.3)*, we assume that the function $c(t) \in \mathrm{Lip}_{\mathrm{loc}}(\boldsymbol{R})$ satisfies*

$$\inf_{t\in\boldsymbol{R}} c(t) > 0,$$
$$c(t) = c_{\pm\infty} + o(1) \quad \textit{for some } c_{\pm\infty} > 0 \textit{ as } t \to \pm\infty.$$

If $c(t) - c_{\pm\infty}$ is integrable on $\boldsymbol{R}$, i.e., the function

$$\Psi(t) = \int_0^t (c(\tau) - c_{\pm\infty})\, d\tau$$

has the finite limits as $t \to \pm\infty$, *then for each finite energy solution* $u(t,x)$ *to* (3.1)–(3.3), *there exist finite energy solutions* $u_\pm(t,x)$ *to equations* $\partial_t^2 u_\pm - c_{\pm\infty}^2 \Delta u_\pm = 0$ *in* Ω *with the boundary condition* $u_\pm(t,x) = 0$ *on* $\partial\Omega$, *such that*

$$\begin{aligned}&\|\nabla u_\pm(t,\cdot) - \nabla u(t,\cdot)\|_{L^2(\Omega)} + \|\partial_t u_\pm(t,\cdot) - \partial_t u(t,\cdot)\|_{L^2(\Omega)} \\ &\quad = O\Big(\int_{|t|}^{+\infty} (c(\tau) - c_{\pm\infty})\, d\tau\Big)\end{aligned}$$

as $t \to \pm\infty$.

Let $c(t) \in \mathscr{K}$. Then we have proved in Section 3 that the solution u to (3.1)–(3.3) is also the solution to (1.1)–(1.3) provided that the data are sufficient small. Furthermore, we have

$$c(t) = \sqrt{1 + \|\nabla u(t,\cdot)\|_{L^2(\Omega)}^2}. \tag{4.1}$$

It follows from Proposition 3.1 that there exists constants $c_{\pm\infty} > 0$ such that

$$c(t) - c_{\pm\infty} = O(|t|^{-k+1}) \quad \text{as } t \to \pm\infty.$$

Then the existence of scattering states $u_\pm(t,x)$ and the asymptotic behaviour (1.6) in Theorem 1.5 follow from Lemma 4.1, since $\Psi(t)$ has the finite limits as $t \to \pm\infty$ on account of our assumption $k > 2$.

We prove the equation (1.7). Since $u_\pm(t,x)$ are the free waves with propagation speeds $c_{\pm\infty}$ and have the finite energy, they satisfy the property of an equipartition of energy, so that the potential energy $(c_{\pm\infty}^2/2)\|\nabla u_\pm(t)\|_{L^2(\Omega)}^2$ are asymptotically equal to the kinetic energy $(1/2)\|\partial_t u_\pm(t)\|_{L^2(\Omega)}^2$ (see [**10**]). This means that

$$\begin{aligned}\lim_{t\to-\infty} \frac{c_{\pm\infty}^2}{2}\|\nabla u_\pm(t)\|_{L^2(\Omega)}^2 &= \lim_{t\to\pm\infty} \frac{1}{2}\|\partial_t u_\pm(t)\|_{L^2(\Omega)}^2 \\ &= \frac{1}{4}\big(c_{\pm\infty}^2\|\nabla u_\pm(0)\|_{L^2(\Omega)}^2 + \|\partial_t u_\pm(0)\|_{L^2(\Omega)}^2\big).\end{aligned} \tag{4.2}$$

Putting

$$\lim_{t\to\pm\infty} \|\nabla u_\pm(t)\|_{L^2(\Omega)} = \lambda_{\pm\infty},$$

we have, by using the first equality in (4.2),

$$\lim_{t\to\pm\infty} \|\partial_t u_\pm(t)\|^2_{L^2(\Omega)} = c^2_{\pm\infty}\lambda^2_{\pm\infty}, \tag{4.3}$$

and by (4.1), $c_{\pm\infty}$ and $\lambda_{\pm\infty}$ are related with

$$c^2_{\pm\infty} = 1 + \lambda^2_{\pm\infty}. \tag{4.4}$$

Thus, combining (4.3)–(4.4), we get

$$\lim_{t\to\pm\infty} \|\partial_t u_\pm(t)\|^2_{L^2(\Omega)} = c^2_{\pm\infty}(c^2_{\pm\infty} - 1).$$

Combining this limit and the second equality in (4.2), we conclude that $c_{\pm\infty}$ satisfy (1.7).

Finally, we prove the equality:

$$c_{+\infty} = c_{-\infty}. \tag{4.5}$$

We define the energy for the equation (1.1) to be

$$\|u(t)\|^2_E = \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)} + \frac{1}{4}\|\nabla u(t)\|^4_{L^2(\Omega)} + \frac{1}{2}\|\partial_t u(t)\|^2_{L^2(\Omega)}.$$

Multiplying the equation (1.1) by $\partial_t u$ and integrating it over Ω, we get the following energy identity:

$$\|u(t)\|_E = \|u(0)\|_E. \tag{4.6}$$

Letting $t \to \pm\infty$ in (4.6) and using the asymptotics (1.6), we have

$$\frac{1}{2}\lambda^2_{\pm\infty} + \frac{1}{4}\lambda^4_{\pm\infty} + \frac{1}{2}\big(1 + \lambda^2_{\pm\infty}\big)\lambda^2_{\pm\infty} = \|u(0)\|^2_E. \tag{4.7}$$

Put

$$\phi(\lambda) = \frac{1}{2}\lambda^2 + \frac{1}{4}\lambda^4 + \frac{1}{2}(1+\lambda^2)\lambda^2 \quad \text{for } \lambda > 0.$$

Then it follows from (4.7) that

$$\phi(\lambda_{+\infty}) = \phi(\lambda_{-\infty}). \tag{4.8}$$

By the injectivity of $\phi(\lambda)$ we conclude that $\lambda_{+\infty} = \lambda_{-\infty}$, and hence, $c_{+\infty} = c_{-\infty}$. The proof of Theorem 1.5 is complete. □

ACKNOWLEDGEMENTS. The author would like to express his sincere gratitude to the referee for his/her careful reading of the manuscript and giving the author many valuable comments.

ADDED IN PROOF. After the author submitted this paper, he has been acquainted with a paper of K. Kajitani, (Advances in Phase Space Analysis of PDEs (A. Bore, D. Del Santo, and M. K. V. Murthy, eds.), Progress in Nonlinear Differential Equations and Their Applications, vol. 78, Birkhäuser, Boston, 2009, pp. 141–153.), where a similar result of Theorem 1.1 is proved for a slightly wider class in "the whole space $\boldsymbol{R}^n$". Also for the exterior problem, after some trivial modifications, the global well-posedness can be proved for the class of Kajitani.

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