J. Math. Soc. Japan
Vol. 62, No. 4 (2010) pp. 1137–1166
doi: 10.2969/jmsj/06241137

New examples of complete hypersurfaces with constant positive scalar curvature in the Euclidean space

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(Received May 25, 2006)
(Revised Aug. 26, 2009)

Abstract. By using the method of equivariant differential geometry, we construct a new family of noncompact complete hypersurfaces with constant positive scalar curvature in the Euclidean spaces. To do so we make a detailed analysis of the nonlinear ODE of the constant scalar curvature equation.

1. Introduction.

Let M be a complete 2-dimensional surface in the Euclidean space E^3 with constant Gaussian curvature. Then it is well known that M is a sphere, a plane or a cylinder. Generalizations of this result have been attempted by many authors. The typical theorem is due to Thomas [**16**], saying that an Einstein hypersurface of the Euclidean space is locally a flat hypersurface, or a part of the sphere.

If we relax the curvature condition to constant scalar curvature, there are two well know results. The result by Chern [**3**] says that there are no complete graph in E^{n+1} with constant positive scalar curvature. Cheng and Yau [**2**] proved that if M^n is a complete hypersurface in E^{n+1} with constant scalar curvature and nonnegative sectional curvature, then M is isometric to a sphere, a flat manifold or a generalized cylinder $S^p \times \boldsymbol{R}^{n-p}$.

In the famous problem section of the book [**17**], Yau asked whether compact hypersurfaces in $\boldsymbol{R}^{N+1}$ which have constant scalar curvature are isometric to the sphere or not. In 1988, Ros [**13**] solved this problem affirmatively under the additional hypothesis that the hypersurfaces are embedded. So there remains the case when the hypersurfaces are immersed. For the problem, there are several partial answers ([**1**], [**10**]).

Now we consider noncompact complete hypersurfaces. Recently many new results have been proved on constant mean curvature hypersurfaces. Compared

2000 *Mathematics Subject Classification.* Primary 53C40; Secondary 53C42.
Key Words and Phrases. scalar curvature, equivariant differential geometry.
This research was partially supported by Grant-in-Aid for Scientific Research (No. 19540062), Japan Society for the Promotion of Science.

with constant mean curvature hypersurfaces there are less results on constant scalar curvature hypersurfaces. One reason, we think, is the complexity of the scalar curvature equation, but the other reason is the lack of examples. The known examples are flat hypersurfaces, spheres, generalized cylinders $S^p \times \boldsymbol{R}^{n-p}$, the rotational hypersurfaces constructed by Leite [**8**], the complete hypersurface with constant negative scalar curvature constructed by us in [**9**] and the complete hypersurfaces with 0 scalar curvature constructed by Palmas in [**11**] and by Sato in [**15**].

The purpose of this paper is to construct a new family of noncompact complete hypersurfaces in E^{n+1} with constant positive scalar curvature by the method of equivariant differential geometry.

Generalized rotational hypersurfaces were first used by Hsiang et al. [**7**] to construct new examples of compact hypersurfaces with constant mean curvature in the Euclidean space. To an isometric transformation group (G, E^n) with codimension two principal orbit type, which is classified by Hsiang and Lawson in [**6**], we can construct G-invariant hypersurfaces, which are called generalized rotational hypersurfaces. The equation that the scalar curvature is constant is reduced to an ordinary differential equation. In [**10**] we proved the formula of calculating the scalar curvature of generalized rotational hypersurfaces from the volume function of orbits.

In this paper we shall study generalized rotational hypersurfaces of $O(p+1) \times O(q+1)$-type. We will show that the ODE system (2.1) has global solutions. The key idea is to compare the solution of (2.1) with that of (3.1) which has the first integrals and approximates (2.1) asymptotically. This sort of comparison technique has been used in [**4**] and [**5**].

The author would like to thank the referee for his careful reading and many valuable suggestions.

2. Preliminaries.

We consider the standard action of $O(p+1) \times O(q+1)$ on $\boldsymbol{R}^{p+1} \times \boldsymbol{R}^{q+1}$, where we always assume that p and q are greater than one. It is easy to see that the orbit space $\boldsymbol{R}^{p+1} \times \boldsymbol{R}^{q+1}/O(p+1) \times O(q+1)$ can be parametrized by the first quadrant $\boldsymbol{R}_+ \times \boldsymbol{R}_+$. Let $\gamma = \gamma(s) = (x(s), y(s))$ be a curve in the first quadrant $\boldsymbol{R}_+ \times \boldsymbol{R}_+$ parameterized by its arc-length s. Let M_γ be an $O(p+1) \times O(q+1)$-invariant hypersurface in $\boldsymbol{R}^{p+q+2}$ generated by γ. M_γ is diffeomorphic to $S^p \times S^q \times \boldsymbol{R}$. It is easy to see that the principal curvatures of M_γ are $x'y'' - y'x''$, y'/x, $-x'/y$ with multiplicities $1, p$ and q, respectively. Therefore the scalar curvature S of M_γ is

$$S = 2(x'y'' - y'x'')\left(p\frac{y'}{x} - q\frac{x'}{y}\right) + \left(p\frac{y'}{x} - q\frac{x'}{y}\right)^2 - p\frac{y'^2}{x^2} - q\frac{x'^2}{y^2}.$$

Assume that S is constant. Then the above equation is equivalent to the following ODE system.

$$
\begin{aligned}
&\frac{dx}{ds} = \cos\alpha, \\
&\frac{dy}{ds} = \sin\alpha, \\
&\frac{d\alpha}{ds} = \frac{p(p-1)\left(\dfrac{\sin\alpha}{x}\right)^2 - 2pq\dfrac{\sin\alpha}{x}\dfrac{\cos\alpha}{y} + q(q-1)\left(\dfrac{\cos\alpha}{y}\right)^2 - S}{2\left(q\dfrac{\cos\alpha}{y} - p\dfrac{\sin\alpha}{x}\right)},
\end{aligned}
\tag{2.1}
$$

where α is the angle between the tangent vector (x', y') and the x-axis.

Our main theorems of this paper are the following:

THEOREM 2.1. *Suppose that* $2 \le p \le q+1$ *and* $S > 0$. *Let* $0 < x_0 \le \sqrt{p(p-1)/S}$ *and* $0 < y_0 < \sqrt{q(q-1)/S}$. *Then the* ODE *system* (2.1) *has a global solution* $\gamma(s) = (x(s),\ y(s)) \in \boldsymbol{R}_+ \times \boldsymbol{R}_+$ *on* $(-\infty, \infty)$ *for the initial conditions* $x(0) = x_0$, $y(0) = y_0$ *and* $\alpha(0) = 0$ *such that* $y = \sqrt{q(q-1)/S}$ *is the asymptotic line of* γ *as* $s \to \infty$ *and* $x = \sqrt{p(p-1)/S}$ *is the asymptotic line of* γ *as* $s \to -\infty$, *respectively.*

THEOREM 2.2. *Suppose that* $p > q+1 \ge 3$ *and* $S > 0$. *Let* $0 < x_0 \le \sqrt{(p-1)(q-1)/S}$ *and* $0 < y_0 < \sqrt{q(q-1)/S}$. *Then the* ODE *system* (2.1) *has a global solution* $\gamma(s) = (x(s), y(s)) \in \boldsymbol{R}_+ \times \boldsymbol{R}_+$ *on* $(-\infty, \infty)$ *for the initial conditions* $x(0) = x_0$, $y(0) = y_0$ *and* $\alpha(0) = 0$ *such that* $y = \sqrt{q(q-1)/S}$ *is the asymptotic line of* γ *as* $s \to \infty$ *and* $x = \sqrt{p(p-1)/S}$ *is the asymptotic line of* γ *as* $s \to -\infty$, *respectively.*

REMARK. The solution with $y_0 = \sqrt{q(q-1)/S}$ corresponds to the cylinder $\boldsymbol{R}^{p+1} \times S^q(y_0)$.

3. Existence of the solution on $[0, \infty)$.

In this section, we show the following.

THEOREM 3.1. *Suppose that* p *and* q *are greater than one. Let* $x_0 > 0$ *and* $0 < y_0 < \sqrt{q(q-1)/S}$. *Then* (2.1) *has a global solution* $\gamma(s) = (x(s), y(s)) \in \boldsymbol{R}_+ \times \boldsymbol{R}_+$ *on* $[0, \infty)$ *for the initial conditions* $x(0) = x_0$, $y(0) = y_0$ *and* $\alpha(0) = 0$.

We prove this theorem by preparing several lemmas.

Since it holds that

$$\frac{d\alpha}{ds}(0) = \frac{q(q-1)/y_0^2 - S}{2q/y_0} > 0,$$

we have $0 < \alpha(s) < \pi/2$ and $d\alpha/ds > 0$ near $s = 0$.

Lemma 3.2. *Set* $\epsilon = -1/y_0 + \sqrt{1/{y_0}^2 + S}$. *If* $\alpha(s)$ *satisfies the inequality* $0 < \alpha(s) < \pi/2$ *on* $0 < s < s_0$ *for some* s_0, *then there holds*

$$q\frac{\cos\alpha(s)}{y(s)} - p\frac{\sin\alpha(s)}{x(s)} > \epsilon, \quad s \in [0, s_0).$$

Proof. Put $h(s) = q\cos\alpha(s)/y(s) - p\sin\alpha(s)/x(s)$, $X(s) = \sin\alpha(s)/x(s)$ and $Y(s) = \cos\alpha(s)/y(s)$. We see that $h(0) > \epsilon$. Suppose to the contrary that there exists some $s_1 \in (0, s_0)$ such that $h(s) > \epsilon$ for all $0 \le s < s_1$ and $h(s_1) = \epsilon$. Then we have $dh/ds(s_1) \le 0$. Since $h(s_1) = qY(s_1) - pX(s_1) = \epsilon$ and $X(s_1) > 0$, we obtain

$$\begin{aligned}\frac{d\alpha}{ds}(s_1) &= \frac{p(p-1)X(s_1)^2 - 2pqX(s_1)Y(s_1) + q(q-1)Y(s_1)^2 - S}{2(qY(s_1) - pX(s_1))}\\ &= \frac{-(p + p^2/q)X(s_1)^2 - 2p\epsilon/qX(s_1) + (q-1)/q\epsilon^2 - S}{2\epsilon}\\ &< \frac{\epsilon}{2} - \frac{S}{2\epsilon} < 0.\end{aligned}$$

Therefore from the assumption we get

$$\frac{d\alpha}{ds}(s_1) + Y(s_1) < \frac{\epsilon}{2} - \frac{S}{2\epsilon} + \frac{1}{y_0} = 0.$$

Hence we have

$$\begin{aligned}\frac{dX}{ds}(s_1) &= \frac{x(s_1)\cos\alpha(s_1)d\alpha/ds(s_1) - \cos\alpha(s_1)\sin\alpha(s_1)}{x^2} < 0,\\ \frac{dY}{ds}(s_1) &= -\frac{\sin\alpha(s_1)}{y(s_1)}\left(\frac{d\alpha}{ds}(s_1) + Y(s_1)\right) > 0,\end{aligned}$$

which implies

$$\frac{dh}{ds}(s_1) = q\frac{dY}{ds}(s_1) - p\frac{dX}{ds}(s_1) > 0.$$

But this contradicts the inequality $dh/ds(s_1) \leq 0$ we mentioned earlier. □

From Lemma 3.2, if $\alpha(s)$ satisfies the inequality $0 \leq \alpha(s) < \pi/2$ on $[0, s_0)$, then it can be extended to $s = s_0$. Thus (2.1) doesn't have any singularity when $0 < \alpha(s) < \pi/2$.

Next we are going to analyze (2.1) by comparing it with the following ODE system.

$$\begin{aligned} \frac{dx}{ds} &= \cos\alpha, \\ \frac{dy}{ds} &= \sin\alpha, \\ \frac{d\alpha}{ds} &= \frac{q(q-1)\left(\dfrac{\cos\alpha}{y}\right)^2 - S}{2q\dfrac{\cos\alpha}{y}}. \end{aligned} \tag{3.1}$$

THEOREM 3.3. *For any $x_0 > 0$ and y_0 satisfying $0 < y_0 < \sqrt{q(q-1)/S}$, (3.1) has a global solution with the initial conditions $x(0) = x_0$, $y(0) = y_0$ and $\alpha(0) = 0$ (see Figure 1), such that the curve $(x(s), y(s))$ is symmetric with respect to the line parallel to the y-axis at the points where y takes critical values.*

PROOF. We rewrite (3.1) as

$$\begin{aligned} \frac{dx}{dy} &= \cot\alpha, \\ \frac{d\alpha}{dy} &= \frac{q(q-1)\left(\dfrac{\cos\alpha}{y}\right)^2 - S}{2q\sin\alpha\dfrac{\cos\alpha}{y}}, \\ x(y_0) &= x_0,\ \alpha(y_0) = 0. \end{aligned} \tag{3.2}$$

The second equation of (3.2) is explicitly integrable. In fact, putting $Y = \cos\alpha(y)/y$, we have

$$\frac{dY}{dy} = \frac{d}{dy}\left(\frac{\cos\alpha}{y}\right) = \frac{-y\sin\alpha\dfrac{d\alpha}{dy} - \cos\alpha}{y^2} = -\frac{q(q+1)Y^2 - S}{2qyY}.$$

By integrating this equation we obtain

$$\left\{q(q+1)\left(\frac{\cos\alpha}{y}\right)^2 - S\right\}y^{q+1} = \left\{q(q+1)\frac{1}{(y_0)^2} - S\right\}(y_0)^{q+1} > 0. \tag{3.3}$$

We set $f(y) = \{q(q+1)/y^2 - S\}y^{q+1}$. f takes its maximum value at $y_\infty := \sqrt{q(q-1)/S}$. See Figure 5 for the graph of $f(y)$. There is the unique point $\overline{y}_1 > y_0$ which satisfies $f(\overline{y}_1) = f(y_0)$. It is easy to see that $\overline{y}_1 > y_\infty$. The solution curve $(x(s), y(s))$ of (3.1) oscillates between the lines $y = y_0$ and $y = \overline{y}_1$.

Next we are going to show the symmetric property of the solution. First, we consider the solution of (3.1) with the initial conditions $x(0) = 0$, $y(0) = y_0$ and $\alpha(0) = 0$. Set $\tilde{x}(s) := -x(-s)$, $\tilde{y}(s) := y(-s)$, $\tilde{\alpha}(s) := -\alpha(-s)$. Then it is easy to see that $\tilde{x}, \tilde{y}$ and $\tilde{\alpha}$ satisfy (3.1) with the same initial conditions. Therefore the solution curve $(x(s), y(s))$ is symmetric with respect to the y-axis. In general, the solution is symmetric with respect to the line parallel to the y-axis at the points where y takes critical value y_0 or $\overline{y}_1$. □

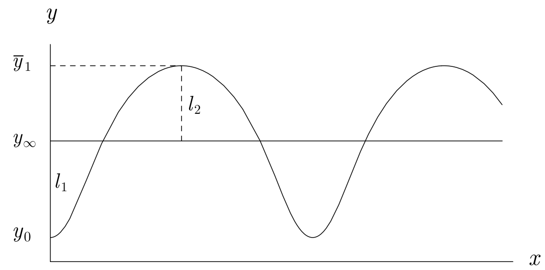

Figure 1.

For later use we prove the following two lemmas (see Figure 1).

Lemma 3.4. *Let $l_1 = y_\infty - y_0$ and $l_2 = \overline{y}_1 - y_\infty$. Then we have $l_1 > l_2$.*

Proof. We set $g(t) = f(y_\infty + t)$ where $f(y)$ is defined earlier in the proof of Theorem 3.3. We also set $h(t) = g(-t) - g(t)$. If we can show that $h(t) > 0$ for all t satisfying $0 < t < y_\infty$, then we can conclude $l_1 > l_2$ (see Figure 5). Since we have

$$g'(t) = -(q+1)(y_\infty + t)^{q-2}(2Sy_\infty t + St^2),$$

we get

$$h'(t) = (q+1)(y_\infty + t)^{q-2} S t^2 + (q+1)(y_\infty - t)^{q-2} + 2(q+1)S y_\infty \cdot t\big\{(y_\infty + t)^{q-2} - (y_\infty - t)^{q-2}\big\} > 0$$

when $0 < t < y_\infty$. Since $h(0) = 0$, we get $h(t) > 0$ for all $0 < t < y_\infty$. □

Let $\overline{x}, \overline{y}$ and $\overline{\alpha}$ be the solutions of (3.1) with the initial conditions $\overline{x}(0) = x_0$, $\overline{y}(0) = y_0$ and $\overline{\alpha}(0) = 0$. Let s_0 be the first $s > 0$ such that $\overline{\alpha}(s) = 0$, that is, $\overline{y}(s_0) = \overline{y}_1$. Since $\overline{\alpha}(s)$ satisfies $0 < \overline{\alpha}(s) < \pi/2$ on $s \in (0, s_0)$, we have $d\overline{y}/ds > 0$ on the same interval. Let us take the inverse $s = s(\overline{y})$ on $\overline{y} \in [y_0, \overline{y}_1]$, and abbreviate the function $\overline{\alpha}(s(\overline{y}))$ as $\overline{\alpha}(\overline{y})$, $\overline{y} \in [y_0, y_1]$ for simplicity. Under these notations, we prove

Lemma 3.5. *Let us denote the maximum of $\overline{\alpha}(y)$, $y \in [y_0, \overline{y}_1]$, by $\overline{\alpha}_0$. Then we have*

$$\cos \overline{\alpha}_0 = \frac{y_0}{y_\infty} \left\{ \frac{(q+1)y_\infty^2}{2y_0^2} - \frac{q-1}{2} \right\}^{1/(q+1)}.$$

Proof. For some $y_* \in [y_0, y_1]$, we have $\overline{\alpha}_0 = \overline{\alpha}(y_*)$. Substituting $d\overline{\alpha}/ds(\overline{\alpha}_0) = 0$ into (3.1), we have $\cos \overline{\alpha}_0 / y_* = 1/y_\infty$. Combining this with (3.3) we get the conclusion. □

Let x, y, and α be the solutions of (2.1) with the initial conditions $x(0) = x_0$, $y(0) = y_0$ and $\alpha(0) = 0$. When $\alpha(s)$ satisfies $0 < \alpha(s) < \pi/2$ on $(0, s_1)$ for some s_1, we have $dy/ds > 0$ on the same interval. Hence, we have the inverse function $s = s(y)$ of $y = y(s)$ on $[0, s_1)$. We abbreviate the function $\alpha(s(y))$ as $\alpha(y)$ for simplicity.

Lemma 3.6. *Under the notations above, we have $\alpha(y) < \overline{\alpha}(y)$ for all $y (> y_0)$ satisfying $0 < \alpha(y) < \pi/2$.*

Proof. First we show that this inequality holds near $y = y_0$.
We can rewrite (2.1) as

$$\frac{dx}{dy} = \cot \alpha,$$

$$\frac{d\alpha}{dy} = \frac{p(p-1)\left(\dfrac{\sin\alpha}{x}\right)^2 - 2pq\dfrac{\sin\alpha}{x}\dfrac{\cos\alpha}{y} + q(q-1)\left(\dfrac{\cos\alpha}{y}\right)^2 - S}{2\sin\alpha\left(q\dfrac{\cos\alpha}{y} - p\dfrac{\sin\alpha}{x}\right)}. \tag{3.4}$$

Since $d\alpha/dy(y_0+0)=+\infty$, we rewrite (3.4) and (3.2) as follows:

$$\begin{aligned}
\frac{dx}{d\alpha} &= \frac{2\cos\alpha\cdot\left(q\dfrac{\cos\alpha}{y}-p\dfrac{\sin\alpha}{x}\right)}{p(p-1)\left(\dfrac{\sin\alpha}{x}\right)^2-2pq\dfrac{\sin\alpha}{x}\dfrac{\cos\alpha}{y}+q(q-1)\left(\dfrac{\cos\alpha}{y}\right)^2-S},\\
\frac{dy}{d\alpha} &= \frac{2\sin\alpha\cdot\left(q\dfrac{\cos\alpha}{y}-p\dfrac{\sin\alpha}{x}\right)}{p(p-1)\left(\dfrac{\sin\alpha}{x}\right)^2-2pq\dfrac{\sin\alpha}{x}\dfrac{\cos\alpha}{y}+q(q-1)\left(\dfrac{\cos\alpha}{y}\right)^2-S}.
\end{aligned}\tag{3.5}$$

$$\begin{aligned}
\frac{d\overline{x}}{d\alpha} &= \frac{2\cos\alpha\cdot q\dfrac{\cos\alpha}{\overline{y}}}{q(q-1)\left(\dfrac{\cos\alpha}{\overline{y}}\right)^2-S},\\
\frac{d\overline{y}}{d\alpha} &= \frac{2\sin\alpha\cdot q\dfrac{\cos\alpha}{\overline{y}}}{q(q-1)\left(\dfrac{\cos\alpha}{\overline{y}}\right)^2-S}.
\end{aligned}\tag{3.6}$$

By a straightforward calculation we have

$$\begin{aligned}
&\frac{dy}{d\alpha}(0)=\frac{d\overline{y}}{d\alpha}(0)=0,\\
&\frac{d^2y}{d\alpha^2}(0)=\frac{d^2\overline{y}}{d\alpha^2}(0)=\frac{2q/y_0}{q(q-1)/y_0^2-S},\\
&\frac{d^3y}{d\alpha^3}(0)=\frac{4}{\{q(q-1)/y_0^2-S\}^2}\,\frac{p}{x_0}\left\{\frac{q(q+1)}{y_0^2}+S\right\}>0,\\
&\frac{d^3\overline{y}}{d\alpha^3}(0)=0.
\end{aligned}$$

Therefore there exists an $\epsilon>0$ such that

$$\overline{y}(\alpha)<y(\alpha) \text{ for all } \alpha \text{ satisfying } 0<\alpha<\epsilon.$$

So there exists an $\epsilon'>0$ such that

$$\alpha(y)<\overline{\alpha}(y) \text{ for all } y \text{ satisfying } y_0<y<y_0+\epsilon',$$

hence Lemma 3.6 holds on the interval $(y_0, y_0 + \epsilon')$.

Now suppose that there exists some $y_1 > y_0$ such that $\alpha(y_1) = \overline{\alpha}(y_1) < \pi/2$ and $\alpha(y) < \overline{\alpha}(y)$ for all y with $y_0 < y < y_1$. We set

$$F(X, Y) := \frac{p(p-1)X^2 - 2pqXY + q(q-1)Y^2 - S}{2(qY - pX)}.$$

When $X, Y > 0$, we have

$$\frac{\partial F}{\partial X} = -\frac{p}{2(qY - pX)^2}\{p(p-1)X^2 - 2(p-1)qXY + q(q+1)Y^2 + S\} < 0.$$

Therefore $F(X, Y)$ is monotone decreasing in X. Hence when $0 < \alpha < \pi/2$ and $x, y > 0$, we have

$$F\left(\frac{\sin\alpha}{x}, \frac{\cos\alpha}{y}\right) < F\left(0, \frac{\cos\alpha}{y}\right).$$

Put $\alpha_1 = \alpha(y_1) = \overline{\alpha}(y_1)$. By using the inequality above we have

$$\begin{aligned}
\frac{d\overline{\alpha}}{dy}(y_1) &= \frac{q(q-1)\left(\dfrac{\cos\alpha_1}{y_1}\right)^2 - S}{2q\sin\alpha_1\dfrac{\cos\alpha_1}{y_1}} = \frac{1}{\sin\alpha_1}F\left(0, \frac{\cos\alpha_1}{y_1}\right) \\
&> \frac{1}{\sin\alpha_1}F\left(\frac{\sin\alpha_1}{x_1}, \frac{\cos\alpha_1}{y_1}\right) \\
&= \frac{p(p-1)\left(\dfrac{\sin\alpha_1}{x_1}\right)^2 - 2pq\dfrac{\sin\alpha_1}{x_1}\dfrac{\cos\alpha_1}{y_1} + q(q-1)\left(\dfrac{\cos\alpha_1}{y_1}\right)^2 - S}{2\sin\alpha_1\left(q\dfrac{\cos\alpha_1}{y_1} - p\dfrac{\sin\alpha_1}{x_1}\right)} \\
&= \frac{d\alpha}{dy}(y_1).
\end{aligned}$$

Therefore $\overline{\alpha}(y) > \alpha(y)$ for y near y_1 and $y > y_1$. Combining this inequality with the above mentioned assumption we obtain $\overline{\alpha}(y) \geq \alpha(y)$ near y_1 and $\overline{\alpha}(y_1) = \alpha(y_1)$, so we get $d\overline{\alpha}/dy(y_1) = d\alpha/dy(y_1)$. But this contradicts with the above inequality. □

LEMMA 3.7. *Denote the point y satisfying $\overline{\alpha}(y) = 0$ $(y > y_0)$ by $\overline{y}_1$. Then there exists a point y_1 satisfying $y_0 < y_1 < \overline{y}_1$ such that $\alpha(y_1) = 0$ (see Figure 2).*

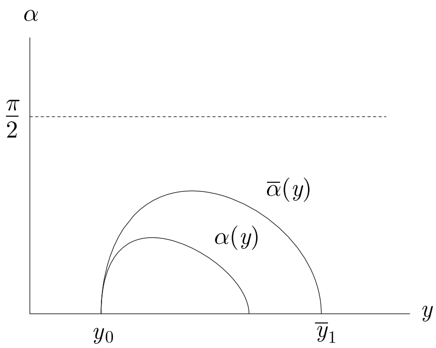

Figure 2.

Proof. The graph of $\overline{\alpha} = \overline{\alpha}(y)$ is in Figure 2. Suppose that there does not exist such a y_1. From Lemma 3.6 we have $0 < \alpha(y) < \overline{\alpha}(y)$ for all $y > y_0$ satisfying $0 < \alpha(y) < \pi/2$. Therefore $y(s)$ is a monotone increasing function and $\alpha(s)$ satisfies $0 < \alpha(s) < \overline{\alpha}_0 < \pi/2$. By using Lemma 3.2, we see that there is no singularity of our solution. Therefore we must have a global solution $(x(s), y(s))$ on $[0,\infty)$ and there exists $y_0 < \hat{y}_1 < \overline{y}_1$ such that $y(s) < \hat{y}_1$ for all $s > 0$, $\lim_{s\to\infty} \alpha = 0$, $\lim_{s\to\infty} y = \hat{y}_1$ and $\lim_{s\to\infty} x = \infty$. By (2.1), we easily conclude that $\hat{y}_1 = y_\infty$. To find a contradiction, we prove the following Claims 1, 2 and 3.

Claim 1. *We can choose sufficiently large s_1 so that $\alpha'(s) < 0$ hold for all $s > s_1$.*

Indeed, since we have $0 < \alpha(s) < \overline{\alpha}_0 < \pi/2$ and $\lim_{s\to\infty} \alpha = 0$, we can choose s_1 so that $\alpha'(s_1) < 0$. Since $x(s)$ and $y(s)$ are monotone increasing functions with $\lim_{s\to\infty} x = \infty$ and $\lim_{s\to\infty} y = \hat{y}_1$, we can also assume that, for all $s > s_1$,

$$\left(-\frac{p(p-1)}{x^2} + \frac{pq}{y^2}\right)(s) > 0, \quad \left(\frac{pq}{x^2} - \frac{q(q-1)}{y^2}\right)(s) < 0,$$

and

$$\left\{\frac{\sin\overline{\alpha}_0}{x}\left(-\frac{p(p-1)}{x^2} + \frac{pq}{y^2}\right) + \frac{\cos\overline{\alpha}_0}{y}\left(\frac{pq}{x^2} - \frac{q(q-1)}{y^2}\right)\right\}(s) < 0. \tag{3.7}$$

If there is no $s_2 > s_1$ such that $\alpha'(s_2) = 0$, then the claim is true. Suppose to the contrary that there is an $s_2 > s_1$ such that $\alpha'(s_2) = 0$ and $\alpha'(s) < 0$ for all $s_1 \le s < s_2$. It is easy to see that $\alpha''(s_2) \ge 0$. On the other hand, since $X' = (\sin\alpha/x)' = -\sin\alpha\cos\alpha/x^2$, $Y' = (\cos\alpha/y)' = -\sin\alpha\cos\alpha/y^2$ at $s = s_2$, we easily obtain

$$\begin{aligned}\alpha''(s_2) &= \frac{\{p(p-1)X^2 - 2pqXY + q(q-1)Y^2 - S\}'}{2(qY-pX)}(s_2)\\ &< \frac{\sin\alpha\cos\alpha}{qY-pX}\left\{\frac{\sin\alpha}{x}\left(-\frac{p(p-1)}{x^2}+\frac{pq}{y^2}\right)+\frac{\cos\alpha}{y}\left(\frac{pq}{x^2}-\frac{q(q-1)}{y^2}\right)\right\}(s_2)\\ &< \frac{\sin\alpha\cos\alpha}{qY-pX}\left\{\frac{\sin\overline{\alpha}_0}{x}\left(-\frac{p(p-1)}{x^2}+\frac{pq}{y^2}\right)+\frac{\cos\overline{\alpha}_0}{y}\left(\frac{pq}{x^2}-\frac{q(q-1)}{y^2}\right)\right\}(s_2)\\ &< 0, \end{aligned}\tag{3.8}$$

where at the last inequality we used (3.7). This contradicts the above inequality $\alpha''(s_2) \geq 0$, proving Claim 1.

Since $\alpha(s)$ is decreasing and $x(s)$ is increasing, we get $\sin\alpha(s)/x(s) < \sin\alpha(s_1)/x(s_1)$ for all $s > s_1$. Set $\alpha(s_1) = \alpha_1$, $x(s_1) = x_1$ and $y(s_1) = y_1$. It follows from $F_X < 0$ that

$$F\left(\frac{\sin\alpha(s)}{x(s)}, \frac{\cos\alpha(s)}{y(s)}\right) > F\left(\frac{\sin\alpha_1}{x_1}, \frac{\cos\alpha(s)}{y(s)}\right). \tag{3.9}$$

We consider the following differential equations.

$$\begin{aligned}\frac{dx}{ds} &= \cos\alpha,\\ \frac{dy}{ds} &= \sin\alpha,\\ \frac{d\alpha}{ds} &= \frac{p(p-1)\left(\dfrac{\sin\alpha_1}{x_1}\right)^2 - 2pq\dfrac{\sin\alpha_1}{x_1}\dfrac{\cos\alpha}{y} + q(q-1)\left(\dfrac{\cos\alpha}{y}\right)^2 - S}{2\left(q\dfrac{\cos\alpha}{y} - p\dfrac{\sin\alpha_1}{x_1}\right)}.\end{aligned}\tag{3.10}$$

Let $\hat{x}, \hat{y}$ and $\hat{\alpha}$ be the solutions of (3.10) for the initial conditions $x(s_1) = x_1$, $y(s_1) = y_1$ and $\alpha(s_1) = \alpha_1$. It is easy to get the first integral of (3.10). In fact, putting $Y = \cos\hat{\alpha}(\hat{y})/\hat{y}$ and $X_1 = \sin\alpha_1/x_1$, we have

$$\frac{dY}{d\hat{y}} = -\frac{p(p-1)X_1^2 - 2p(q+1)X_1Y + q(q+1)Y^2 - S}{2\hat{y}(qY - pX_1)}.$$

By integrating this equation we obtain

$$\begin{aligned}&\left\{q(q+1)\left(\frac{\cos\hat\alpha}{\hat y}\right)^2-2p(q+1)\frac{\sin\alpha_1}{x_1}\frac{\cos\hat\alpha}{\hat y}+p(p-1)\left(\frac{\sin\alpha_1}{x_1}\right)^2-S\right\}\hat y^{q+1}\\&\quad=\left\{q(q+1)\left(\frac{\cos\alpha_1}{y_1}\right)^2-2p(q+1)\frac{\sin\alpha_1}{x_1}\frac{\cos\alpha_1}{y_1}+p(p-1)\left(\frac{\sin\alpha_1}{x_1}\right)^2-S\right\}y_1^{q+1}.\end{aligned}$$

From this we easily see that the solutions $\hat x, \hat y$ and $\hat\alpha$ exist on $(-\infty,\infty)$ and the curve $(\hat x(s),\hat y(s))$ has the $\hat x$-translational invariance like the solutions of (3.1).

Claim 2. *Set $\hat y_2=\max_{\boldsymbol{R}}\hat y(s)$. If we choose s_1 sufficiently large, then $\hat y_2>y_\infty$ and $\hat\alpha(y)>0$ for all y satisfying $y_1<y<\hat y_2$. In particular, $\hat\alpha(y_\infty)>0$.*

We prove Claim 2 as follows: Since $\hat\alpha=0$ at the point where $\hat y=\hat y_2$, we obtain

$$\begin{aligned}&\left\{q(q+1)\left(\frac{1}{\hat y_2}\right)^2-2p(q+1)\frac{\sin\alpha_1}{x_1}\frac{1}{\hat y_2}+p(p-1)\left(\frac{\sin\alpha_1}{x_1}\right)^2-S\right\}\hat y_2^{q+1}\\&\quad=\left\{q(q+1)\left(\frac{\cos\alpha_1}{y_1}\right)^2-2p(q+1)\frac{\sin\alpha_1}{x_1}\frac{\cos\alpha_1}{y_1}+p(p-1)\left(\frac{\sin\alpha_1}{x_1}\right)^2-S\right\}y_1^{q+1}.\end{aligned}$$

$\hat y_2$ depends only on p,q,α_1,x_1,y_1 and S. If we let $\alpha_1\to0$, $x_1\to\infty$ and $y_1\to y_\infty$, then $\hat y_2\to\sqrt{q(q+1)/S}>y_\infty$ from the above equation. Thus if we choose s_1 sufficiently large, then we have $\hat y_2>y_\infty$. It is easy to see that $\hat\alpha(y)>0$ for all $y_1<y<\hat y_2$, proving Claim 2.

We have the following comparison inequality similar to Lemma 3.6.

Claim 3. *If we choose s_1 sufficiently large, then $\alpha(y)>\hat\alpha(y)$ holds for all y satisfying $y_1<y<y_\infty$.*

The proof of Claim 3 is almost the same as that of Lemma 3.6. Indeed first we show this inequality for all s near s_1. We compute

$$\frac{dy}{d\alpha}(\alpha_1)=\frac{d\hat y}{d\alpha}(\alpha_1)=\frac{\sin\alpha_1}{F\left(\dfrac{\sin\alpha_1}{x_1},\dfrac{\cos\alpha_1}{y_1}\right)},\tag{3.11}$$

and

$$\begin{aligned}
\frac{d^2y}{d\alpha^2}(\alpha_1) &= \frac{\cos\alpha_1}{F\left(\dfrac{\sin\alpha_1}{x_1}, \dfrac{\cos\alpha_1}{y_1}\right)} \\
&\quad - \frac{\sin\alpha_1}{F\left(\dfrac{\sin\alpha_1}{x_1}, \dfrac{\cos\alpha_1}{y_1}\right)^2} F_X\left(\frac{\sin\alpha_1}{x_1}, \frac{\cos\alpha_1}{y_1}\right) \frac{x_1\cos\alpha_1 - \sin\alpha_1\cos\alpha_1}{x_1^2} \\
&\quad + \frac{\sin\alpha_1}{F\left(\dfrac{\sin\alpha_1}{x_1}, \dfrac{\cos\alpha_1}{y_1}\right)^2} F_Y\left(\frac{\sin\alpha_1}{x_1}, \frac{\cos\alpha_1}{y_1}\right) \frac{y_1\sin\alpha_1 + \sin\alpha_1\cos\alpha_1}{y_1^2} \\
&> \frac{\cos\alpha_1}{F\left(\dfrac{\sin\alpha_1}{x_1}, \dfrac{\cos\alpha_1}{y_1}\right)} \\
&\quad + \frac{\sin\alpha_1}{F\left(\dfrac{\sin\alpha_1}{x_1}, \dfrac{\cos\alpha_1}{y_1}\right)^2} F_Y\left(\frac{\sin\alpha_1}{x_1}, \frac{\cos\alpha_1}{y_1}\right) \frac{y_1\sin\alpha_1 + \sin\alpha_1\cos\alpha_1}{y_1^2} \\
&= \frac{d^2\hat{y}}{d\alpha^2}(\alpha_1),
\end{aligned}$$

where we used the fact that x_1 is large and $\alpha_1 > 0$ is small if s_1 is sufficiently large.

Therefore there exists an $\epsilon > 0$ such that

$$\hat{y}(\alpha) < y(\alpha) \quad \text{for all } \alpha \text{ satisfying } \alpha_1 - \epsilon < \alpha < \alpha_1.$$

Hence there exists an $\epsilon' > 0$ such that

$$\alpha(y) > \hat{\alpha}(y) \quad \text{for all } y \text{ satisfying } y_1 < y < y_1 + \epsilon'.$$

Now we are going to show the inequality $\alpha(y) > \hat{\alpha}(y)$ for all y satisfying $y_1 < y < y_\infty$. Suppose to the contrary that there exists $y_2 > y_1$ such that there holds $\alpha(y) > \hat{\alpha}(y)$ for all y satisfying $y_1 < y < y_2$ and $\alpha(y_2) = \hat{\alpha}(y_2)$. Set $x_2 = x(y_2)$ and $\alpha_2 = \alpha(y_2)$. Then we get from (3.9)

$$\begin{aligned}
\frac{d\alpha}{dy}(y_2) &= \frac{1}{\sin\alpha_2} F\left(\frac{\sin\alpha_2}{x_2}, \frac{\cos\alpha_2}{y_2}\right) > \frac{1}{\sin\alpha_2} F\left(\frac{\sin\alpha_1}{x_1}, \frac{\cos\alpha_2}{y_2}\right) \\
&= \frac{d\hat{\alpha}}{dy}(y_2).
\end{aligned}$$

Thus there exists an ϵ such that

$$\alpha(y) > \hat{\alpha}(y) \quad \text{for all } y \text{ satisfying } y_2 < y < y_2 + \epsilon.$$

Combining this inequality and the assumptions $\alpha(y) > \hat{\alpha}(y)$, $y \in (y_1, y_2)$, and $\alpha(y_2) = \hat{\alpha}(y_2)$, we obtain $d\hat{\alpha}/dy(y_2) = d\alpha/dy(y_2)$ which contradicts the above inequality. Therefore Claim 3 is proved.

From Claim 2, Claim 3 and $\lim_{y \to y_\infty - 0} \alpha(y) = 0$, we get a contradiction, which proves Lemma 3.7. □

From Lemma 3.7 we see that the solutions $x(s), y(s)$ and $\alpha(s)$ exist on the interval $[0, s_1]$ such that $0 < \alpha(s) < \pi/2$, $dx/ds > 0$ and $dy/ds > 0$ for $s \in (0, s_1)$ and $\alpha(s_1) = 0$. We have $y(s_1) = y_1$ and set $x(s_1) = x_1$.

Lemma 3.8. $y_1 > y_\infty = \sqrt{q(q-1)/S}$.

Proof. First, suppose that $y_1 < y_\infty$. Then

$$\frac{d\alpha}{ds}(s_1) = \frac{y_1}{2q}\left\{\frac{q(q-1)}{y_1^2} - S\right\} > 0.$$

Therefore we have $\alpha(s) > 0$ for all $s_1 < s < s_1 < s + \epsilon$ for some ϵ. On the other hand, we have $\alpha(s) > 0$ for all $s_1 - \epsilon < s < s_1$ and $\alpha(s_1) = 0$. So we get $d\alpha/ds(s_1) = 0$, which contradicts the above inequality. Next, suppose that $y_1 = y_\infty$. Then the solution of (2.1) with the initial conditions $x(s_1) = x_1$, $y(s_1) = y_1$ and $\alpha(s_1) = 0$ is $y(s) \equiv y_1$. This is a contradiction. □

Therefore we have the following Figure (see Figure 3).

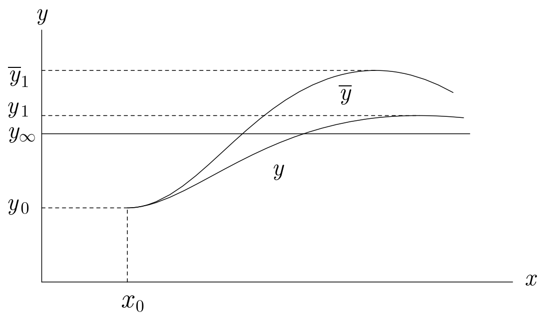

Figure 3.

Since

$$\frac{d\alpha}{ds}(s_1) = \frac{y_1}{2q}\left\{\frac{q(q-1)}{y_1^2} - S\right\} < 0,$$

we have $\alpha(s) < 0$ for all $s_1 < s < s_1 + \epsilon$ for some $\epsilon > 0$. We are going to show that there exists an s_2 satisfying $s_1 < s_2 < \infty$ such that $0 < \alpha(s) < \pi/2$, $dx/ds > 0$ and $dy/ds < 0$ for $s \in (s_1, s_2)$ and $\alpha(s_2) = 0$. To do so, we denote the solutions of (3.1) with the initial conditions $x(s_1) = x_1$, $y(s_1) = y_1$ and $\alpha(s_1) = 0$ as $\tilde{x}(s), \tilde{y}(s)$ and $\tilde{\alpha}(s)$. Similar to the proof of Theorem 3.3, we can show that the solutions $\tilde{x}, \tilde{y}$ and $\tilde{\alpha}$ exist on $(-\infty, \infty)$ and the curve $(\tilde{x}(s), \tilde{y}(s))$ has the $\tilde{x}$-translational invariance. Set $\tilde{y}_2 = \min_{\boldsymbol{R}} \tilde{y}_2(s)$ and denote the first $s > s_1$ satisfying $\tilde{y}(s) = \tilde{y}_2$ as $\tilde{s}_2$. It is easy to see that $y_0 < \tilde{y}_2 < y_\infty$. We have $-\pi/2 < \tilde{\alpha}(s) < 0$, $d\tilde{x}/ds > 0$ and $d\tilde{y}/ds < 0$ for $s \in (s_1, \tilde{s}_2)$ and $\alpha(\tilde{s}_2) = 0$. We set $\tilde{x}(\tilde{s}_2) = \tilde{x}_2$. Let us take the inverse $s = s(\tilde{y})$ on $\tilde{y} \in [\tilde{y}_2, y_1]$, and abbreviate the function $\tilde{\alpha}(s(\tilde{y}))$ as $\tilde{\alpha}(\tilde{y})$, $\tilde{y} \in [\tilde{y}_2, y_1]$ for simplicity.

When $\alpha(s)$ satisfies $-\pi/2 < \alpha(s) < 0$ on (s_1, s_2) for some s_2, we have $dy/ds < 0$ on the same interval. Hence, we have the inverse function $s = s(y)$ of $y = y(s)$ on $[s_1, s_2)$. We abbreviate the function $\alpha(s(y))$ as $\alpha(y)$ for simplicity.

Lemma 3.9. *Under these notations above, we have $\tilde{\alpha}(y) < \alpha(y) < 0$ for all $y < y_1$ satisfying $-\pi/2 < \alpha(y) < 0$.*

Proof. The proof is similar to that of Lemma 3.6. First we show that this inequality holds near $y = y_1$. We rewrite (3.1) as

$$\begin{aligned} &\frac{d\tilde{x}}{d\tilde{y}} = \cot\tilde{\alpha}, \\ &\frac{d\tilde{\alpha}}{d\tilde{y}} = \frac{q(q-1)\left(\dfrac{\cos\tilde{\alpha}}{\tilde{y}}\right)^2 - S}{2q\sin\tilde{\alpha}\dfrac{\cos\tilde{\alpha}}{\tilde{y}}}, \\ &\tilde{x}(y_1) = x_1, \quad \tilde{\alpha}(y_1) = 0. \end{aligned} \tag{3.12}$$

Since $d\tilde{\alpha}/d\tilde{y}(y_1 - 0) = -\infty$, we rewrite this equation as follows.

$$\frac{d\tilde{x}}{d\alpha} = \frac{2\cos\alpha \cdot q\dfrac{\cos\alpha}{\tilde{y}}}{q(q-1)\left(\dfrac{\cos\alpha}{\tilde{y}}\right)^2 - S},$$

$$\frac{d\tilde{y}}{d\alpha} = \frac{2\sin\alpha \cdot q\dfrac{\cos\alpha}{\tilde{y}}}{q(q-1)\left(\dfrac{\cos\alpha}{\tilde{y}}\right)^2 - S}, \tag{3.13}$$

$$\tilde{x}(0) = x_1, \quad \tilde{y}(0) = y_1,$$

where we have changed the variable $\tilde{\alpha}$ to α.

Similar to the calculation in the proof of Lemma 3.6, we obtain

$$\begin{aligned}
&\frac{dy}{d\alpha}(0) = \frac{d\tilde{y}}{d\alpha}(0) = 0,\\
&\frac{d^2y}{d\alpha^2}(0) = \frac{d^2\tilde{y}}{d\alpha^2}(0) = \frac{2q/y_1}{q(q-1)/y_1^2 - S},\\
&\frac{d^3y}{d\alpha^3}(0) = 4\left\{-\frac{p/x_1}{q(q-1)/y_1^2 - S} - \frac{q/y_1}{\{q(q-1)/y_1^2 - S\}^2}\left(-\frac{2pq}{x_1y_1}\right)\right\} > 0,\\
&\frac{d^3\tilde{y}}{d\alpha^3}(0) = 0.
\end{aligned}$$

Therefore there exists an $\epsilon > 0$ such that

$$\tilde{y}(\alpha) > y(\alpha) \quad \text{for all } \alpha \text{ satisfying } -\epsilon < \alpha < 0.$$

Thus there exists an $\epsilon' > 0$ such that

$$\tilde{\alpha}(y) < \alpha(y) \quad \text{for all } y \text{ satisfying } y_1 - \epsilon' < y < y_1.$$

Now suppose that there exists some $y_2 < y_1$ such that $-\pi/2 < \tilde{\alpha}(y_2) = \alpha(y_2) < 0$ and $\tilde{\alpha}(y) < \alpha(y)$ for all $y_2 < y < y_1$. Set $\alpha_2 = \alpha(y_2)$ $(= \tilde{\alpha}(y_2))$ and $x_2 = x(y_2)$. Since $-\pi/2 < \alpha_2 < 0$ and $F(X, Y)$ is monotone decreasing in X, we have $F(0, \cos\alpha_2/y_2) < F(\sin\alpha_2/x_2, \cos\alpha_2/y_2)$. Then

$$\begin{aligned}
\frac{d\tilde{\alpha}}{dy}(y_2) &= \frac{F\left(0, \dfrac{\cos\alpha_2}{y_2}\right)}{\sin\alpha_2} > \frac{F\left(\dfrac{\sin\alpha_2}{x_2}, \dfrac{\cos\alpha_2}{y_2}\right)}{\sin\alpha_2}\\
&= \frac{d\alpha}{dy}(y_2).
\end{aligned}$$

Therefore $\tilde{\alpha}(y) < \alpha(y)$ for y near y_2 and $y < y_2$. On the other hand we have $\tilde{\alpha}(y) < \alpha(y)$ for all $y_2 < y < y_2 + \epsilon$ and $\tilde{\alpha}(y_2) = \alpha(y_2)$. Therefore we get

$d\tilde{\alpha}/dy(y_2) = d\alpha/dy(y_2)$, which contradicts the above inequality. □

LEMMA 3.10. *There exists a point y_2 satisfying $\tilde{y}_2 < y_2 < y_1$ such that $\alpha(y_2) = 0$.*

PROOF. When $-\pi/2 < \alpha(s) < 0$ we have $q \cos\alpha(s)/y(s) - p \sin\alpha(s)/x(s) > 0$. Lemma 3.9 yields $y(s) > \tilde{y}_2 > 0$. Therefore the numerator of the expression of $d\alpha/ds$ in (2.1) is bounded. Thus we don't have any singularity of the solutions $x(s), y(s)$ and $\alpha(s)$ while $-\pi/2 < \alpha(s) < 0$. Suppose that there does not exist $y < y_1$ with $\alpha(y) = 0$. Then we have global solutions on $[s_1, \infty)$ and there exists $y_2 < y_1$ such that $\lim_{s\to\infty} \alpha = 0$, $\lim_{s\to\infty} y = y_2$ and $\lim_{s\to\infty} x = \infty$. It is easy to see that $y_2 = y_\infty$. From the similar argument as in the proof of Lemma 3.7, we get a contradiction. □

LEMMA 3.11. $y_2 < y_\infty$.

PROOF. We omit the proof, because all arguments follow almost verbatim the corresponding arguments of Lemma 3.8. □

Now we can prove the main theorem, Theorem 3.1, of this section. In fact, applying Lemmas 3.7, 3.8, 3.10 and 3.11 repeatedly, we easily get global solutions $x(s), y(s)$ and $\alpha(s)$ on $[0, \infty)$.

Next we analyze the shape of the solution curve $(x(s), y(s))$ obtained in Theorem 3.1.

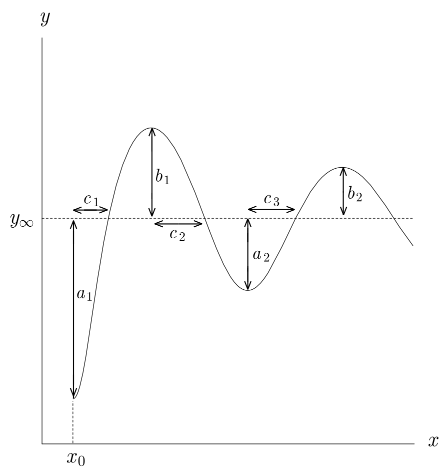

Figure 4.

THEOREM 3.12. *The solutions in Theorem* 3.1 *satisfy the followings* (*see Figure* 4 *for the definitions of* a_k, b_k *and* c_k).

1) $a_k > b_k$ *for all* $k \in \boldsymbol{N}$.
2) $a_1 > a_2 > \cdots$ *and* $b_1 > b_2 > \cdots$.
3) *There exists a constant* $M(y_0, q, S) > 0$ *such that* $c_k \geq M(y_0, q, S)$ *for all* $k \in \boldsymbol{N}$.

PROOF.

1) follows from Lemma 3.4.

2) From Lemmas 3.7 and 3.10 we have $y_\infty < y_1 < \overline{y}_1$ and $y_0 < \tilde{y}_2 < y_2 < y_\infty$. Set $y_2 = y(s_2)$, $x_2 = x(s_2)$. Let $\overline{x}, \overline{y}$ and $\overline{\alpha}$ be the solutions of (3.1) for the initial conditions $\overline{x}(0) = x_2$, $\overline{y}(0) = y_2$ and $\overline{\alpha}(0) = 0$. Set $\overline{y}_3 = \max_{\boldsymbol{R}} \overline{y}$. It is easy to see that $\overline{y}_3 < \overline{y}_1$. From Lemma 3.7 there exists an $s_3 > s_2$ such that $\alpha(s_3) = 0$ and $y_3 < \overline{y}_3$, where $y_3 = y(s_3)$. Let $\tilde{x}, \tilde{y}$ and $\tilde{\alpha}$ be the solutions of (3.1) for the initial conditions $\tilde{x}(0) = x_3$, $\tilde{y}(0) = y_3$ and $\tilde{\alpha}(0) = 0$. Set $\tilde{y}_4 = \min_{\boldsymbol{R}} \tilde{y}$. It is easy to see that $\tilde{y}_2 < \tilde{y}_4$. From Lemma 3.10 there exists an $s_4 > s_3$ such that $\alpha(s_4) = 0$ and $y_4 > \tilde{y}_4$, where $y_4 = y(s_4)$. We can repeat this process inductively and this proves 2).

3) For the odd natural number k we have $c_k = \int_{y_{k-1}}^{y_\infty} \cot \alpha(y) dy$. We are going to estimate c_1. From Lemmas 3.5 and 3.6 we have $0 \leq \alpha(y) < \overline{\alpha}(y) \leq \overline{\alpha}_0$, which implies

$$c_1 \geq \int_{y_0}^{y_\infty} \cot \overline{\alpha}(y) dy \geq (y_\infty - y_0) \cot \overline{\alpha}_0.$$

We set $a(t) = \{(q+1)y_\infty^2/2t^2 - (q-1)/2\}^{1/(q+1)}$. From Lemma 3.5 we obtain $\cos \overline{\alpha}_0 = y_0/y_\infty a(y_0)$ and it is easy to see that

$$\lim_{y_0 \to y_\infty} a(y_0) = 1,$$

$$\lim_{y_0 \to y_\infty} a'(y_0) = -\frac{1}{y_\infty},$$

$$\lim_{y_0 \to y_\infty} a''(y_0) = \frac{3-q}{y_\infty^2}.$$

Thus by using l'Hospital's rule and $\lim_{y_0 \to y_\infty} \cos \overline{\alpha}_0 = 1$ we get

$$\begin{aligned} &\lim_{y_0 \to y_\infty} (y_\infty - y_0)^2 \cot^2 \overline{\alpha}_0 \\ &= \lim_{y_0 \to y_\infty} \frac{(y_\infty - y_0)^2}{1 - y_0^2/y_\infty^2 \cdot a(y_0)^2} \end{aligned}$$

$$= -y_\infty^2 \lim_{y_0 \to y_\infty} \frac{y_0 - y_\infty}{y_0 \cdot a(y_0)^2 + y_0^2 \cdot a(y_0) \cdot a'(y_0)}$$

$$= -y_\infty^2 \lim_{y_0 \to y_\infty} \big\{ a(y_0)^2 + 4y_0 \cdot a(y_0) \cdot a'(y_0) + y_0^2 \cdot (a'(y_0))^2 + y_0^2 \cdot a(y_0) \cdot a''(y_0) \big\}^{-1}$$

$$= \frac{q}{S}.$$

So we get $\lim_{y_0 \to y_\infty} c_1 \geq \sqrt{q/S}$. We can repeat the same argument for other c_k with odd k. Thus we can conclude that there exists a constant $M(y_0, q, S) > 0$ such that $c_k \geq M(y_0, q, S)$ holds for all odd integer $k \in \boldsymbol{N}$. Similar argument can be applied to c_k for even integer k, so we omit the proof. □

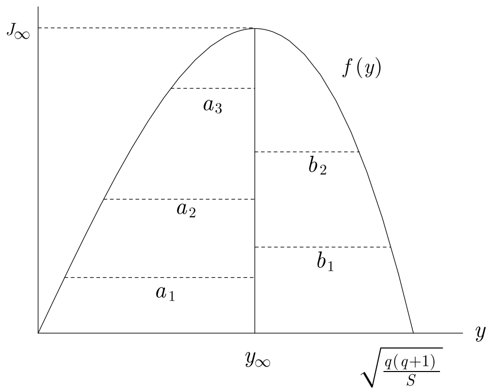

Figure 5.

Following the idea of [**4**] we are going to show that $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 0$. First we need two lemmas.

LEMMA 3.13. *We set* $J(s) = q(q+1)y(s)^{q-1}\cos^2\alpha(s) - Sy(s)^{q+1}$.

(1) *J is a monotone increasing function of s which satisfies* $x(s) \geq q(p-1)/Sy_0$.
(2) *Set* $J_\infty = 2q\{q(q-1)/S\}^{(q-1)/2}$. *Then we have* $J(s) \leq J_\infty$.
(3) $\lim_{s \to \infty} J(s) = J_\infty$ *holds if and only if* $\lim_{s \to \infty} y(s) = y_\infty$ *and* $\lim_{s \to \infty} \alpha(s) = 0$.

PROOF.
(1) Straightforward calculation shows

$$\frac{dJ}{ds} = \frac{p(q+1)y^{q-2}\sin^2\alpha}{x\left(q\dfrac{\cos\alpha}{y} - p\dfrac{\sin\alpha}{x}\right)}\left\{q(q+1)\cos^2\alpha + Sy^2 - \frac{q(p-1)y\cos\alpha\sin\alpha}{x}\right\}$$
$$\geq \frac{p(q+1)y^{q-1}\sin^2\alpha}{x\left(q\dfrac{\cos\alpha}{y} - p\dfrac{\sin\alpha}{x}\right)}\left\{Sy_0 - \frac{q(p-1)}{x}\right\} \geq 0.$$

(2) Since $J \leq q(q+1)y^{q-1} - Sy^{q+1}$ and the right hand side of this inequality attains its maximum at $y = y_\infty$, (2) follows.
(3) We omit the proof, since it is easily shown from (1) and (2). □

LEMMA 3.14. *Let $x_0 > 0$, $s_0 > 0$ and y_0, y_1 be the constants satisfying $y_0 < y_1 < y_\infty$. We consider the solutions of* (2.1) *for the initial conditions $x(s_0) = x_0$, $y(s_0) = y_0$, and $\alpha(s_0) = 0$. We choose $s_1 > s_0$ such that $y(s_1) = y_1$. If x_0 is sufficiently large, then there exists a constant k_0 depending only on y_0, y_1, p, q, S such that $x(s_1) - x(s_0) \leq k_0$.*

PROOF. We denote the solutions of the following equations with the initial conditions $x(s_0) = x_0$, $y = y_0$, $\alpha = 0$ as $\overline{x}(s), \overline{y}(s)$ and $\overline{\alpha}(s)$.

$$\begin{aligned}
\frac{dx}{ds} &= \cos\alpha, \\
\frac{dy}{ds} &= \sin\alpha, \\
\frac{d\alpha}{ds} &= \frac{p(p-1)\epsilon^2 - 2pq\epsilon\dfrac{\cos\alpha}{y} + q(q-1)\left(\dfrac{\cos\alpha}{y}\right)^2 - S}{2\left(q\dfrac{\cos\alpha}{y} - p\epsilon\right)}.
\end{aligned} \tag{3.14}$$

The equations have the following first integral.

$$\left\{q(q+1)\left(\frac{\cos\overline{\alpha}}{\overline{y}}\right)^2 - 2p(q+1)\epsilon\left(\frac{\cos\overline{\alpha}}{\overline{y}}\right) + p(p-1)\epsilon^2 - S\right\}\overline{y}^{q+1} = c,$$

where c is a constant. We choose ϵ such that

$$\epsilon < \frac{pq/y_0 - \sqrt{(pq/y_0)^2 - p(p-1)\{q(q-1)/y_0^2 - S\}}}{p(p-1)}.$$

Then $d\overline{\alpha}/ds(s_0) > 0$. The curve $\overline{\gamma}(s) = (\overline{x}(s), \overline{y}(s))$ can be given by a periodic function $\overline{y} = f(\overline{x}), f(\overline{x}+T_0) = f(\overline{x})$ for some $T_0 > 0$. When $\epsilon \to 0$, $\overline{\gamma}$ converges to the solution of (3.2). Hence it is easy to see that we can choose ϵ sufficiently small so that the maximum value of $\overline{y}$ is greater than y_1. Let $x_0 \geq 1/\epsilon$, then we can compare $\alpha(y)$ with $\overline{\alpha}(y)$. Since $dy/d\alpha(0) = d\overline{y}/d\alpha(0) = 0$ and $d^2y/d\alpha^2(0) < d^2\overline{y}/d\alpha^2(0)$, similar argument as in the proof of Lemma 3.6 shows that $\alpha(y) > \overline{\alpha}(y)$ for $y > y_0$. Hence $x(y) < \overline{x}(y)$ (see Figure 6). Therefore $x(s_1) - x(s_0) = x(y_1) - x_0 < \overline{x}(y_1) - x_0$. We put $k_0 = \overline{x}(y_1) - x_0$. It is easy to see that k_0 depends only on y_0, y_1, p, q and S. □

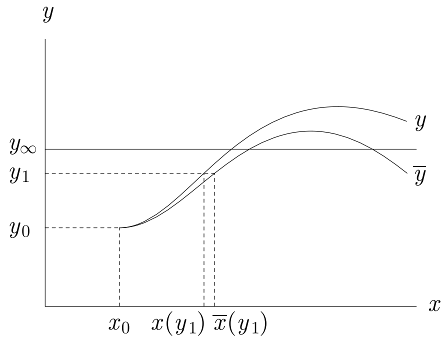

Figure 6.

THEOREM 3.15. *Under the same notations used in Theorem* 3.12, *we have*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0,$$

that is, the line $y = y_\infty$ *is the asymptotic line of our solution curve* $\gamma(s) = (x(s), y(s))$ *as* $s \to \infty$.

PROOF. It is easy to see from Figure 5 that $\lim_{n\to\infty} a_n = 0$ implies $\lim_{n\to\infty} b_n = 0$, or vise versa. Suppose that we don't have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. Then, putting $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$, we have $a > 0$ and $b > 0$. From Theorem 3.12, there holds, $y(s) \leq y_\infty - a$ or $y(s) \geq y_\infty + b$ when $\alpha(s) = 0$ (see Figure 7). Also there exist constants $\overline{a}$ and $\overline{b}$ satisfying $a < \overline{a} < y_\infty$ and $b < \overline{b}$ such that $y_\infty - \overline{a} < y(s) < y_\infty + \overline{b}$ holds for all $s > 0$. By Lemma 3.13 J is an increasing function of s with an upper bound J_∞. Hence there exists $\lim_{s\to\infty} J$. From Lemma 3.13 we have $\lim_{s\to\infty} J = c < J_\infty$. We denote the positive solutions of the equation $f(y) = q(q+1)y^{q-1} - Sy^{q+1} = (J_\infty + c)/2$ as $y_\infty - \delta$ and $y_\infty + \overline{\delta}$. From the definition we have $y_\infty - a < y_\infty - \delta < y_\infty + \overline{\delta} < y_\infty + b$. If $y(s)$ satisfies the inequality $y_\infty - \delta \leq y(s) \leq y_\infty + \overline{\delta}$, then it follows from $J < c$

that

$$\begin{aligned} q(q+1)(y_\infty+\overline{b})^{q-1}\sin^2\alpha(s) &\geq q(q+1)y(s)^{q-1}\sin^2\alpha(s) \\ &> q(q+1)y(s)^{q-1} - Sy(s)^{q+1} - c \\ &= f(y(s)) - c > \frac{J_\infty - c}{2} > 0. \end{aligned}$$

That is, there exists a positive constant k_0 depending only on $q, \overline{b}, c, S$ such that

$$|\sin\alpha(s)| \geq k_0, \quad y(s) \in [y_\infty - \delta, y_\infty + \overline{\delta}]. \tag{3.15}$$

Hereafter $k_0, k_1, \ldots$ denote positive constants depending only on $p, q, a, b, c, \overline{a}, \overline{b}$ and S. Let s_0 be a point satisfying $\alpha(s_0) = 0$ with y taking the local minimum. We set s_1, s_2 as the first point satisfying $s > s_0$ and $y(s_1) = y_\infty - \delta$, $y(s_2) = y_\infty + \overline{\delta}$, respectively. We set s_3 as the first point satisfying $s > s_0$ with y taking the local maximum, and set s_4 as the first point satisfying $s > s_0$ with $y(s_4)$ taking the local minimum (see Figure 7).

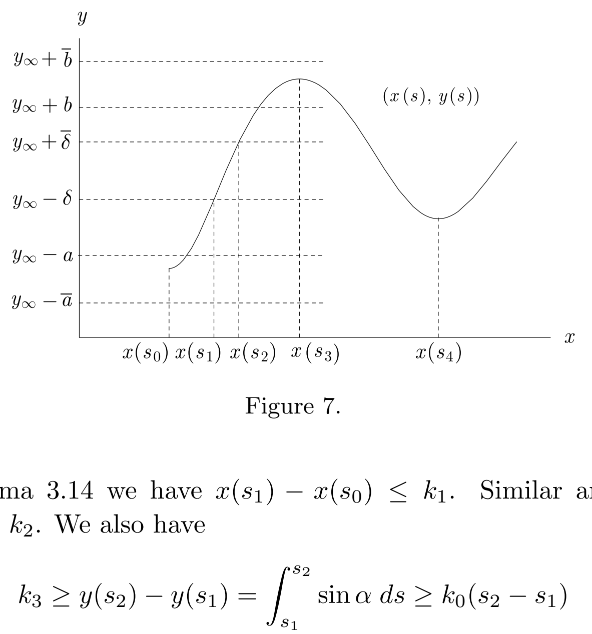

Figure 7.

From Lemma 3.14 we have $x(s_1) - x(s_0) \leq k_1$. Similar argument shows $x(s_3) - x(s_2) \leq k_2$. We also have

$$\begin{aligned} k_3 \geq y(s_2) - y(s_1) &= \int_{s_1}^{s_2} \sin\alpha\, ds \geq k_0(s_2 - s_1) \\ &\geq k_0 \int_{s_1}^{s_2} \cos\alpha\, ds = k_0(x(s_2) - x(s_1)). \end{aligned}$$

Combining these inequalities we obtain $x(s_3) - x(s_0) \leq k_4$. Similar argument shows $x(s_4) - x(s_3) \leq k_5$. Therefore we obtain $x(s_4) - x(s_0) \leq k_6$. Set $T = k_6$. Now we have

$$J(s_4) - J(s_0) \geq \int_{s_1}^{s_2} \frac{dJ}{ds} ds \geq \int_{s_1}^{s_2} k_7 \frac{\sin^2 \alpha}{x} ds \geq \frac{k_8}{x(s_0) + T}.$$

Let $s_8 < s_{12} < \cdots < s_{4m} < \cdots$ be the points satisfying $\alpha(s_{4m}) = 0$ with y taking the local minimum. Then by the similar argument as above we have

$$J(s_{4(m+1)}) - J(s_{4m}) \geq \frac{k_8}{x(s_0) + (m+1)T}, \quad (m \in \boldsymbol{N}).$$

From this inequality we get

$$\lim_{s \to \infty} (J(s) - J(s_0)) \geq k_8 \sum_{m=0}^{\infty} \frac{1}{x(s_0) + (m+1)T} = \infty.$$

This contradicts the assumption $J \leq c$. □

Remark. It seems that $a_1 > b_1 > a_2 > b_2 > \cdots$, but at the moment we cannot prove this inequality.

4. Existence of the solution on $(-\infty, 0]$.

In this section we prove the following theorems.

Theorem 4.1. *Suppose that $2 \leq p \leq q + 1$ and $S > 0$. Let $0 < x_0 \leq \sqrt{p(p-1)/S}$ and $0 < y_0 < \sqrt{q(q-1)/S}$. Then the* ODE (2.1) *has a global solution $\gamma(s) = (x(s), y(s))$ on $(-\infty, 0]$ for the initial conditions $x(0) = x_0$, $y(0) = y_0$ and $\alpha(0) = 0$.*

Theorem 4.2. *Suppose that $p > q + 1 \geq 3$ and $S > 0$. Let $0 < x_0 \leq \sqrt{(p-1)(q-1)/S}$ and $0 < y_0 < \sqrt{q(q-1)/S}$. Then the* ODE (2.1) *has a global solution $\gamma(s) = (x(s), y(s))$ on $(-\infty, 0]$ for the initial conditions $x(0) = x_0$, $y(0) = y_0$ and $\alpha(0) = 0$.*

For the convenience of treatment we interchange the role of x and y, and we prove the following theorems which are equivalent to Theorem 4.1 and Theorem 4.2.

Theorem 4.3. *Suppose that $p \geq q - 1 \geq 1$ and $S > 0$. Let $0 < x_0 < \sqrt{p(p-1)/S}$ and $0 < y_0 \leq \sqrt{q(q-1)/S}$. Then the* ODE (2.1) *has a global* *solution* $\gamma(s) = (x(s), y(s))$ *on* $[0,\infty)$ *for the initial conditions* $x(0) = x_0$, $y(0) = y_0$ *and* $\alpha(0) = -\pi/2$.

THEOREM 4.4. *Suppose that* $2 \le p < q-1$ *and* $S > 0$. *Let* $0 < x_0 < \sqrt{p(p-1)/S}$ *and* $0 < y_0 \le \sqrt{(p-1)(q-1)/S}$. *Then the* ODE (2.1) *has a global solution* $\gamma(s) = (x(s), y(s))$ *on* $[0,\infty)$ *for the initial conditions* $x(0) = x_0$, $y(0) = y_0$ *and* $\alpha(0) = -\pi/2$.

We consider the solutions of (2.1) with the initial conditions $x(0) = x_0 < \sqrt{p(p-1)/S}$, $y(0) = y_0 > 0$ and $\alpha(0) = -\pi/2$. Set $\gamma(s) = (x(s), y(s))$. Then

$$\alpha'(0) = \frac{x_0}{2p}\left\{\frac{p(p-1)}{x_0^2} - S\right\} > 0.$$

LEMMA 4.5. *We put* $x_\infty = \sqrt{p(p-1)/S}$ *and* $y_\infty = \sqrt{q(q-1)/S}$. *Set* $\Omega_1 = \{(x,y) \mid 0 < x \le x_\infty,\ 0 < y \le y_\infty\}$. *Then when* $-\pi/2 < \alpha(s) < 0$ *and* $\gamma(s) \in \Omega_1$, *we have* $\alpha'(s) > 0$.

PROOF. Since $0 < x < x_\infty$, $0 < y < y_\infty$ and $-\pi/2 < \alpha(s) < 0$, we have

$$\begin{aligned}\frac{d\alpha}{ds} &> \frac{p(p-1)\left(\dfrac{\sin\alpha}{x}\right)^2 + q(q-1)\left(\dfrac{\cos\alpha}{y}\right)^2 - S}{2\left(q\dfrac{\cos\alpha}{y} - p\dfrac{\sin\alpha}{x}\right)} \\ &\ge \frac{S(\cos^2\alpha + \sin^2\alpha - 1)}{2\left(q\dfrac{\cos\alpha}{y} - p\dfrac{\sin\alpha}{x}\right)} = 0.\end{aligned}$$

□

Set $D = \{(x,y) \mid 0 < x,\ 0 < y \le \sqrt{(q-1)/p}\,x\}$ (see Figure 8).

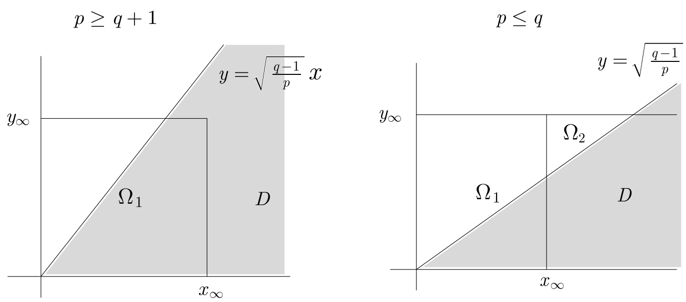

Figure 8.

LEMMA 4.6. *If there exist $s_0 > 0$ and $\epsilon > 0$ such that $-\pi/2 < \alpha(s) < 0$, $\alpha'(s) > 0$ and $\gamma(s) \in D$ for all $s \in (s_0 - \epsilon, s_0)$, then $\alpha'(s_0) > 0$.*

PROOF. The argument is similar to the proof of Claim 1 in Lemma 3.7. Suppose, to the contrary, that $\alpha'(s_0) = 0$. When $\gamma(s_0) \in D$, we have $p/x(s_0)^2 - (q-1)/y(s_0)^2 \le 0$ and $-(p-1)/x(s_0)^2 + q/y(s_0)^2 > 0$. Then we have from (3.8)

$$\begin{aligned}\frac{d^2\alpha}{ds^2}(s_0) &= \frac{\sin\alpha\cos\alpha}{qY - pX}\left\{\frac{p\sin\alpha}{x}\left(-\frac{p-1}{x^2} + \frac{q}{y^2}\right) + \frac{q\cos\alpha}{y}\left(\frac{p}{x^2} - \frac{q-1}{y^2}\right)\right\}(s_0)\\ &> 0. \end{aligned} \tag{4.1}$$

Therefore we have $\alpha'(s) > 0$ for all s satisfying $s_0 < s < s_0 + \epsilon'$ for some ϵ'. By combining with the assumptions $\alpha'(s) > 0$ on $(s_0 - \epsilon, s_0)$ and $\alpha'(s_0) = 0$, we get $d^2\alpha/ds^2(s_0) = 0$. But this contradicts the above inequality. □

LEMMA 4.7. *Suppose that $p \ge q + 1 \ge 3$. Let $0 < x_0 < \sqrt{p(p-1)/S}$ and $0 < y_0 \le \sqrt{q(q-1)/S}$. Let $\gamma(s)$ be the solution of (2.1) for the initial conditions $x(0) = x_0$, $y(0) = y_0$ and $\alpha(0) = -\pi/2$. Then there exists an $s_1 > 0$ such that $\alpha(s_1) = 0$, $x(s_1) > x_0$ and $0 < y(s_1) < y_0$.*

PROOF. Since $p \ge q + 1$, when there holds $-\pi/2 < \alpha(s) < 0$, $0 < s < s_0$, for some s_0, the solution $\gamma(s)$ is contained in $\Omega_1 \cup D$. From Lemmas 4.5 and 4.6, we have $\alpha'(s) > 0$. Suppose, to the contrary, that there is no s_1 with $\alpha(s_1) = 0$. Then by using Lemma 3.9 and Lemma 4.6 we have the following three possible cases.

Case 1: The solution exists on $[0, \infty)$ and there exists $0 < y'_\infty < y_0$ such that $\alpha(s) \nearrow 0$, $y(s) \searrow y'_\infty$ and $x(s) \to \infty$ as $s \to \infty$.

In this case we have $X(s) = \sin\alpha(s)/x(s) \to 0$ and $Y(s) = \cos\alpha(s)/y(s) \to 1/y'_\infty > 1/y_\infty$ as $s \to \infty$, which implies

$$\frac{d\alpha}{ds} \to \left\{\frac{q(q-1)}{(y'_\infty)^2} - S\right\}\frac{y'_\infty}{2q} > 0 \text{ as } s \to \infty.$$

So there exists some $s > 0$ such that $\alpha(s) = 0$ getting a contradiction.

Case 2: The solution exists on $[0, \infty)$ and $\alpha(s) \nearrow 0$, $y(s) \searrow 0$ and $x(s) \to \infty$ as $s \to \infty$.

In this case we have $X(s) \to 0$ and $Y(s) \to \infty$ as $s \to \infty$. Then we get $d\alpha/ds \nearrow \infty$, which contradicts $\alpha(s) \nearrow 0$.

Case 3: There exists s_1, α_1 and x_1 such that $0 < s_1 < \infty$, $-\pi/2 < \alpha_1 \leq 0$, $0 < x_1 < \infty$, and $\alpha(s) \nearrow \alpha_1$, $y(s) \searrow 0$, $x(s) \nearrow x_1$ as $s \to s_1 - 0$.

In this case, it is easy to see that $x(s), y(s)$ and $\alpha(s)$ are continuous on $[0, s_1]$. We know $Y(s) \nearrow \infty$ as $s \to s_1 - 0$. Therefore $d\alpha/ds \to \infty$ as $s \to s_1 - 0$. So we can rewrite $\alpha = \alpha(s)$ as $s = s(\alpha)$ near $s = s_1$. Then (2.1) becomes (3.5). Consider the solutions of (3.5) with the initial conditions $x(\alpha_1) = x_1$ and $y(\alpha_1) = y_1$. Then the functions $x(\alpha), y(\alpha)$ defined by $x(\alpha) = x_1$, $y(\alpha) = 0$ become the solutions of (3.5). This contradicts the uniqueness of solutions. □

Now we consider the case $p \leq q$. We set

$$\Omega_2 = \left\{ (x,y) \,\middle|\, \sqrt{\frac{p(p-1)}{S}} < x < \sqrt{\frac{pq}{S}}, \sqrt{\frac{q-1}{p}}\, x < y < \sqrt{\frac{q(q-1)}{S}} \right\}.$$

LEMMA 4.8. *Let $p = q$ or $p = q-1$ with $p, q \geq 2$. Let $0 < x_0 < \sqrt{p(p-1)/S}$ and $0 < y_0 \leq \sqrt{q(q-1)/S}$. Let $\gamma(s)$ be the solution of (2.1) for the initial conditions $x(0) = x_0$, $y(0) = y_0$ and $\alpha(0) = -\pi/2$. If there exist $s_0 > 0$ and $\epsilon > 0$ such that $-\pi/2 < \alpha(s) < 0$, $\alpha'(s) > 0$ and $\gamma(s) \in \Omega_2$ for all $s_0 - \epsilon < s < s_0$, then $\alpha'(s_0) > 0$.*

PROOF. Suppose, to the contrary, that $\alpha'(s_0) = 0$. From (4.1) we have

$$\frac{d^2\alpha}{ds^2}(s_0) = \frac{\sin\alpha\cos\alpha}{qY - pX}\left\{\frac{p\sin\alpha}{x}\left(-\frac{p-1}{x^2} + \frac{q}{y^2}\right) + \frac{q\cos\alpha}{y}\left(\frac{p}{x^2} - \frac{q-1}{y^2}\right)\right\}. \tag{4.2}$$

We are going to show that the right hand side of (4.2) is positive. If this is proved, then we get a contradiction from the similar argument as in the proof of Lemma 4.6.

Set $\hat{x} = 1/x$ and $\hat{y} = 1/y$. Then (4.2) is changed to

$$\begin{aligned}\frac{d^2\alpha}{ds^2}(s_0) = {} & \frac{\sin\alpha\cos\alpha}{q\cos\alpha\cdot\hat{y} - p\sin\alpha\cdot\hat{x}}\hat{x}^3\cos\alpha\cdot p\{-(p-1) + qt^2\} \\ & \cdot\left\{\tan\alpha + \frac{q}{p}\cdot\frac{pt - (q-1)t^3}{-(p-1) + qt^2}\right\}(s_0),\end{aligned} \tag{4.3}$$

where $t = \hat{y}/\hat{x}$. The domain Ω_2 is transformed to

$$\hat{\Omega}_2 = \left\{ (\hat{x},\hat{y}) \,\middle|\, \sqrt{\frac{S}{pq}} < \hat{x} < \sqrt{\frac{S}{p(p-1)}}, \sqrt{\frac{S}{q(q-1)}} < \hat{y} < \sqrt{\frac{p}{q-1}}\hat{x} \right\}.$$

The condition $\alpha'(s_0) = 0$ is equivalent to

$$\begin{aligned} &p(p-1)\sin^2\alpha(s_0)\hat{x}(s_0)^2 - 2pq\sin\alpha(s_0)\cos\alpha(s_0)\hat{x}(s_0)\hat{y}(s_0) \\ &\quad + q(q-1)\cos^2\alpha(s_0)\hat{y}(s_0)^2 - S = 0. \end{aligned}$$

Put

$$\begin{aligned} h(u) = {}& p(p-1)\sin^2\alpha(s_0)u^2 - 2pq\sin\alpha(s_0)\cos\alpha(s_0)\hat{y}(s_0)u \\ &+ q(q-1)\cos^2\alpha(s_0)\hat{y}(s_0)^2 - S. \end{aligned}$$

From $-\pi/2 < \alpha(s_0) < 0$ and $\sqrt{S/q(q-1)} < \hat{y}(s_0)$ we have

$$h\left(\sqrt{\frac{S}{p(p-1)}}\right) > S\sin^2\alpha(s_0) + S\cos^2\alpha(s_0) - S = 0.$$

Since $\sqrt{(q-1)/p}\,\hat{y}(s_0) < \hat{x}(s_0) < \sqrt{S/p(p-1)}$ and $h(\hat{x}(s_0)) = 0$, we must have $h(\sqrt{(q-1)/p}\,\hat{y}(s_0)) < 0$. This implies

$$\begin{aligned} &\frac{(q-p+1)(q-1)}{2}\cos 2\alpha(s_0) - q\sqrt{p(q-1)}\sin 2\alpha(s_0) \\ &\quad < \frac{S}{\hat{y}(s_0)^2} - \frac{(p+q-1)(q-1)}{2}. \end{aligned}$$

Combining with $\hat{y}(s_0) > \sqrt{S/q(q-1)}$, we obtain

$$\frac{(q-p+1)(q-1)}{2}(\cos 2\alpha(s_0) - 1) - q\sqrt{p(q-1)}\sin 2\alpha(s_0) < 0. \tag{4.4}$$

From $-\pi < 2\alpha(s_0) < 0$ and (4.4), we obtain $-\pi/2 < \alpha(s_0) < \alpha_2$, where $\alpha_2 < 0$ is defined by

$$\tan\alpha_2 = -\frac{2\sqrt{p}q}{(q-p+1)\sqrt{q-1}}.$$

Now we estimate the right hand side of (4.3). From $p \geq q-1$ we have $-(p-1) + qt^2 > 0$.

Let $p = q$. Then from $1 < t < \sqrt{p/(p-1)}$ we get

$$\frac{pt-(p-1)t^3}{-(p-1)+pt^2} < \sqrt{\frac{p}{p-1}}.$$

Thus we have

$$\tan\alpha(s_0) + \left\{\frac{pt-(p-1)t^3}{-(p-1)+pt^2}\right\}(s_0) < -\frac{2\sqrt{p}p}{\sqrt{p-1}} + \sqrt{\frac{p}{p-1}} < 0,$$

which yields $\alpha''(s_0) > 0$.

Now let $p = q-1$. We have $\sqrt{(p-1)/(p+1)} < t < 1$. Then

$$\begin{aligned}\tan\alpha + \frac{p+1}{p}\cdot\frac{pt-pt^3}{-(p-1)+(p+1)t^2} \\ \leq -(p+1)+(p+1)\left\{\frac{1-(p-1)/(p+1)}{-(p-1)+(p+1)(p-1)}\right\} < 0,\end{aligned}$$

which yields $\alpha''(s_0) > 0$. □

Lemma 4.9. *Let $p = q$ or $p = q-1$ with $p, q \geq 2$. Suppose that $S > 0$. Let $0 < x_0 < \sqrt{p(p-1)/S}$ and $0 < y_0 \leq \sqrt{q(q-1)/S}$. Let $x(s), y(s)$ and $\alpha(s)$ be the solutions of* (2.1) *for the initial conditions $x(0) = x_0$, $y(0) = y_0$ and $\alpha(0) = -\pi/2$. Then there exists a positive s_1 such that $\alpha(s_1) = 0$, $x(s_1) > 0$ and $0 < y(s_1) < y_0$.*

Proof. When there holds $-\pi/2 < \alpha(s) < 0$, $s \in (0, s_0)$, for some s_0, then the solution $\gamma(s)$ is contained in $\Omega_1 \cup \Omega_2 \cup D$. From Lemmas 4.5, 4.6 and 4.8 we have $\alpha'(s) > 0$. The rest of the proof is the same as the proof of Lemma 4.7, so we shall omit it. □

Proof of Theorem 4.3. From Lemma 4.7 and Lemma 4.9 there exists $0 < s_1 < \infty$ such that $\alpha(s_1) = 0$, $x(s_1) > x_0$ and $0 < y(s_1) < y_0$. Once we have $\alpha(s_1) = 0$, then we can apply Theorem 3.1 to get a global solution on $[0, \infty)$. □

Proof of Theorem 2.1. Combining Theorem 3.1, Theorem 3.17 and Theorem 4.1 we easily get the conclusion. □

Finally we treat the case $2 \leq p < q-1$.

Lemma 4.10. *Suppose that $2 \leq p < q-1, S > 0$. Let $0 < x_0 < \sqrt{p(p-1)/S}$ and $0 < y_0 \leq \sqrt{(p-1)(q-1)/S}$. Let $x(s), y(s)$ and $\alpha(s)$ be the solutions of* (2.1) *for the initial conditions $x(0) = x_0$, $y(0) = y_0$ and $\alpha(0) = -\pi/2$. Then there exists a positive s_1 such that $\alpha(s_1) = 0$, $x(s_1) > 0$ and $0 < y(s_1) < y_0$.*

PROOF. If we start from (x_0, y_0) at $s = 0$, then when there holds $-\pi/2 < \alpha(s) < 0$, $s \in (0, s_0)$ for some s_0, we have $(x(s), y(s)) \in \Omega_1 \cup D$. Therefore from Lemma 4.5 and Lemma 4.6 we have $\alpha'(s) > 0$. Similar argument as in the proof of Lemma 4.7 shows that there exists $s_1 > 0$ such that $\alpha(s_1) = 0$. □

PROOF OF THEOREM 4.4. Using Lemma 4.10 and Theorem 3.1 we get the conclusion. □

PROOF OF THEOREM 2.2. Combining Theorem 3.1, Theorem 3.17 and Theorem 4.4 we get the conclusion. □

Our solution curve is like Figure 9.

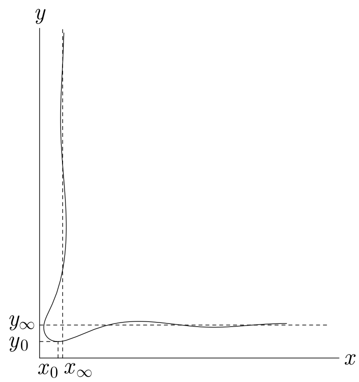

Figure 9.

REMARKS.

1. Since these solution curves are not congruent in general, we have infinitely many non congruent complete hypersurfaces with constant scalar curvature 1 for all $p, q \geq 2$.
2. For the other isometric transformation groups (G, E^n) of cohomogeneity 2 we can give the constant scalar curvature equations [**10**] by using Reilly's formula (see [**12**] and [**14**]). Those equations are more complicated than the one treated in this paper, but it seems that there are many complete hypersurfaces with constant scalar curvature. These problems are treated in the near future.
3. In Theorem 2.2 we need a stronger condition $0 < x_0 \leq \sqrt{(p-1)(q-1)/S}$ than that of Theorem 2.1. But from numerical computations it seems that even if $p > q+1$ there exist global solutions for the same initial conditions as Theorem 2.1. This is an open problem.

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