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Weak dimension and right distributivity of skew generalized power series rings

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Abstract. Let R be a ring, S a strictly ordered monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. The skew generalized power series ring $R[[S,\omega]]$ is a common generalization of skew polynomial rings, skew power series rings, skew Laurent polynomial rings, skew group rings, and Mal'cev-Neumann Laurent series rings. In the case where S is positively ordered we give sufficient and necessary conditions for the skew generalized power series ring $R[[S,\omega]]$ to have weak dimension less than or equal to one. In particular, for such an S we show that the ring $R[[S,\omega]]$ is right duo of weak dimension at most one precisely when the lattice of right ideals of the ring $R[[S,\omega]]$ is distributive and $\omega(s)$ is injective for every $s \in S$.

Introduction.

In this paper we study relationships between the weak dimension of a skew generalized power series ring (the definition of this construction will be stated in a moment) and properties of the lattice of right ideals of the ring. Recall that a ring R has *weak dimension less than or equal to one* exactly when all right ideals of R are flat, and R is *right distributive* if the lattice of right ideals of R is distributive, i.e. $(I+J)\cap K = (I\cap K)+(J\cap K)$ for any right ideals I, J, K of R. Recall also that a ring R is *right Bezout* if all its finitely generated right ideals are principal, and R is *right $\aleph_0$-injective* if any homomorphism from a countably generated right ideal of R into R extends to a right R-module endomorphism of R. A ring R is said to be *von Neumann regular* if $a \in aRa$ for any $a \in R$, i.e. if R has weak dimension equal to zero.

The motivation for the paper comes from well known results of Jensen [**5**] and Brewer, Rutter and Watkins [**2**] which imply that for any commutative ring R the following conditions are equivalent:

(1) *The power series ring $R[[x]]$ has weak dimension less than or equal to one.*

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(2) $R[[x]]$ *is distributive.*
(3) $R[[x]]$ *is Bezout.*
(4) R *is* $\aleph_0$*-injective von Neumann regular.*

In the noncommutative setting some results on relations between the conditions (1)–(4) were obtained by Herbera ([**4**]) and Karamzadeh and Koochakpoor ([**6**]), and in a more general context of skew power series rings by Tuganbaev (see [**16**, Section 6.5]) and the authors ([**13**]), and they are collected in Theorem 1.1 in Section 1.

The main aim of this paper is to extend Theorem 1.1 to skew generalized power series rings. This is done in Theorem 2.1 in Section 2, where we prove that a skew generalized power series ring $R[[S,\omega]]$ with a positively ordered monoid S of exponents is right duo of weak dimension at most one if and only if the lattice of right ideals of the ring $R[[S,\omega]]$ is distributive and $\omega(s)$ is injective for every $s \in S$ if and only if the ring $R[[S,\omega]]$ is reduced right Bezout if and only if R, S and ω satisfy some conditions explicitly stated in the theorem (see (10) in Theorem 2.1).

The skew generalized power series ring construction, introduced in [**12**], embraces a wide range of classical ring-theoretic extensions, including skew polynomial rings, skew power series rings, skew Laurent polynomial rings, skew group rings, Mal'cev-Neumann Laurent series rings, the "untwisted" versions of all of these, and the "untwisted" rings of generalized power series (see [**14**] for the definition of the last class of rings). In order to recall the skew generalized power series ring construction, we need some definitions.

Throughout this paper, rings are associative, but not necessarily commutative, and they contain an identity element 1 ($1 \neq 0$). If R is a ring, then we write $\mathrm{End}(R)$ to denote the monoid of ring endomorphisms of R (with composition of endomorphisms as the operation).

Let S be a monoid (i.e. a semigroup with identity which is not necessarily commutative), with an operation written multiplicatively, and let $\leq$ be an order relation on the set S. We say that $(S,\leq)$ is an *ordered monoid* if for any $s,t,v \in S$, $s \leq t$ implies $sv \leq tv$ and $vs \leq vt$. Moreover, if for any $s,t,v \in S$, $s < t$ implies $sv < tv$ and $vs < vt$, then $(S,\leq)$ is said to be a *strictly ordered monoid.*

Given a ring R, a strictly ordered monoid $(S,\leq)$ and a monoid homomorphism $\omega\colon S \to \mathrm{End}(R)$, consider the set A of all maps $f : S \to R$ whose support $\mathrm{supp}(f) = \{s \in S \mid f(s) \neq 0\}$ is artinian (i.e., it does not contain any infinite strictly decreasing chains of elements) and narrow (i.e., it does not contain infinite subsets of pairwise order-incomparable elements). If $f, g \in A$ and $s \in S$, it turns out that the set

$$\mathrm{X}_s(f,g) = \{(x,y) \in \mathrm{supp}(f) \times \mathrm{supp}(g) : s = xy\}$$

is finite. Thus one can define the product fg: $S \to R$ of $f, g \in A$ as follows:

$$(fg)(s) = \sum_{(x,y)\in \mathrm{X}_s(f,g)} f(x)\,\omega_x(g(y)) \quad \text{for any} \quad s \in S$$

(by convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above, A becomes a ring, called the *ring of skew generalized power series with coefficients in R and exponents in S*, and denoted by $R[[S,\omega]]$.

We will use the symbol 1 to denote the identity elements of the monoid S, the ring R and the ring $R[[S,\omega]]$. To each $r \in R$ and $s \in S$, we associate elements $\boldsymbol{c}_r, \boldsymbol{e}_s \in R[[S,\omega]]$ defined by

$$\boldsymbol{c}_r(x) = \begin{cases} r & \text{if } x = 1 \\ 0 & \text{if } x \in S \setminus \{1\}, \end{cases} \qquad \boldsymbol{e}_s(x) = \begin{cases} 1 & \text{if } x = s \\ 0 & \text{if } x \in S \setminus \{s\}. \end{cases}$$

It is clear that $r \mapsto \boldsymbol{c}_r$ is a ring embedding of R into $R[[S,\omega]]$ and $s \mapsto \boldsymbol{e}_s$ is a monoid embedding of S into the multiplicative monoid of the ring $R[[S,\omega]]$. Furthermore, we have $\boldsymbol{e}_s\boldsymbol{c}_r = \boldsymbol{c}_{\omega_s(r)}\boldsymbol{e}_s$ for any $r \in R$ and $s \in S$.

In this paper we focus on skew generalized power series rings $R[[S,\omega]]$ with S positively ordered (i.e. $s \geq 1$ for every $s \in S$). The main result of the paper is Theorem 2.1, presented in Section 2, in which we characterize when $R[[S,\omega]]$ is right duo of weak dimension at most one. Since the right distributivity condition is deeply involved in the characterization, to prove Theorem 2.1 we shall first study properties of right distributive skew generalized power series rings in Section 1.

If R is a ring or a monoid, then $U(R)$ stands for the group of units of R. For a ring R, the Jacobson radical of R is denoted by $J(R)$.

1. Preparatory results.

Recall that an ordered monoid $(S, \leq)$ is *positively ordered* if $s \geq 1$ for any $s \in S$. An obvious example of such a monoid is $S_0 = \boldsymbol{N} \cup \{0\}$ under addition, with its natural linear order. It is clear that if σ is an endomorphism of a ring R, then the map $\omega : S_0 \to \mathrm{End}(R)$ given by $\omega(n) = \sigma^n$ for any $n \in S_0$, is a monoid homomorphism, and the ring $R[[S_0,\omega]]$ is isomorphic to the skew power series ring $R[[x;\sigma]]$ whose elements are power series in x, with coefficients in R written on the left, and with multiplication defined by $xa = \sigma(a)x$ for any $a \in R$. Hence, skew power series rings can be considered as a special case of skew generalized power series rings with positively ordered exponents.

The aim of this paper is to identify right duo rings of weak dimension at most one in the class of rings of skew generalized power series with positively ordered exponents. In particular, we extend to this class the following characterization of skew power series rings of weak dimension at most one, given in Theorem 1.1 below.

Recall that a ring R is *right duo* if every right ideal of R is a two-sided ideal. A ring R is said to be *right quasi-duo* (resp. *semicommutative*) if every maximal right ideal of R (resp. the right annihilator of every element of R) is a two-sided ideal of R. A ring R is *abelian* if all idempotents of R are central, and R is *reduced* if it contains no nonzero nilpotent element, i.e. $a^2 = 0$ implies $a = 0$ for any $a \in R$. A ring R is *strongly regular* if for any $a \in R$ there exists $b \in R$ such that $a = a^2b$, or equivalently R is von Neumann regular and abelian. It is well known that for strongly regular rings $\aleph_0$-injectivity is left-right symmetric (see [**16**, 4.88]).

We say that an endomorphism σ of a ring R is *idempotent-stabilizing* if $\sigma(e) = e$ for every idempotent e of R.

Theorem 1.1. *Let σ be an endomorphism of a ring R. Then the following conditions are equivalent*:

(1) $R[[x;\sigma]]$ *has weak dimension less than or equal to one and is right duo.*
(2) $R[[x;\sigma]]$ *has weak dimension less than or equal to one, R is abelian and σ is bijective and idempotent-stabilizing.*
(3) $R[[x;\sigma]]$ *is right duo right distributive.*
(4) $R[[x;\sigma]]$ *is reduced right distributive.*
(5) $R[[x;\sigma]]$ *is right distributive and σ is injective.*
(6) $R[[x;\sigma]]$ *is right duo right Bezout.*
(7) $R[[x;\sigma]]$ *is reduced right Bezout.*
(8) $R[[x;\sigma]]$ *is right quasi-duo right Bezout and σ is injective.*
(9) $R[[x;\sigma]]$ *is semicommutative right Bezout and σ is injective.*
(10) *R is $\aleph_0$-injective strongly regular, and σ is bijective and idempotent-stabilizing.*

Proof. See [**13**, Theorem 1.6]. □

Theorem 1.1 concerns three classes of rings: the rings of weak dimension less than or equal to one, the right distributive rings, and the right Bezout rings. When extending this theorem to skew generalized power series rings, it will be convenient to explore some well known connections between these classes of rings. We collect them in the following:

Proposition 1.2. *Let R be a ring. Then*

(i) ([**16**, 4.21(2)]) *If R is right Bezout and reduced, then R has weak dimension less than or equal to one.*

(ii) ([**16**, 5.16(1)]) *If the factor ring $R/J(R)$ is strongly regular, then R is right distributive if and only if R is right Bezout.*

(iii) ([**16**, 2.35]) *If R is right Bezout and right quasi-duo, then R is right distributive.*

In this section we study relations between the weak dimension, the right distributivity, and the right Bezout condition of skew generalized power series rings $R[[S,\omega]]$ with positively ordered exponents. The following result will allow us to apply in these studies the important connection given in Proposition 1.2(ii).

Lemma 1.3. *Let R be a ring, $(S,\leq)$ a positively strictly ordered monoid, $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism, and let $A = R[[S,\omega]]$. Then*

$$J(A) = \{f \in A : f(1) \in J(R)\}$$

and $A/J(A) \cong R/J(R)$.

Proof. Since S is positively ordered, $(fg)(1) = f(1)g(1)$ for any $f, g \in A$, and it follows that the map $\varphi : A \to R/J(R)$, $\varphi(f) = f(1) + J(R)$, is a ring epimorphism with $\ker\varphi = \{f \in A : f(1) \in J(R)\}$. Hence to complete the proof, it suffices to show that $\ker\varphi = J(A)$. If $f \in \ker\varphi$, then $f(1) \in J(R)$ and thus for any $g \in A$ we have $(1 - gf)(1) = 1 - g(1)f(1) \in 1 + J(R) \subseteq U(R)$. Hence [**12**, Proposition 2.2] implies that $1 - gf \in U(A)$, and thus $f \in J(A)$ by [**7**, Lemma 4.1]. Therefore, $\ker\varphi \subseteq J(A)$, and since $A/\ker\varphi \cong R/J(R)$ is Jacobson semisimple, from [**7**, Proposition 4.6] we deduce that $\ker\varphi = J(A)$. □

We will often use the following property of flat right ideals of a ring.

Lemma 1.4 (see [**16**, 4.23]). *Let a, b, c, d be elements of a ring R such that $ab = cd$ and $aR + cR$ is a flat right ideal of R. Then there exist $f, g, h, k \in R$ such that $af = cg$, $(1 - f)b = hd$, $ah = ck$ and $(1 - k)d = gb$.*

Recall that an element s of a monoid S is said to be *left cancellative* if for any $u, v \in S$, $su = sv$ implies $u = v$. A monoid S is called a *left cancellative monoid* if all elements of S are left cancellative. *Right cancellative monoids* are defined similarly, and monoids that are left and right cancellative are called *cancellative monoids*. The following lemma implies, in particular, that for any positively strictly ordered monoid $(S,\leq)$, if the ring $R[[S,\omega]]$ has weak dimension less than or equal to one, then S is left cancellative.

Lemma 1.5. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism such that the ring $R[[S,\omega]]$ has weak dimension* *less than or equal to one. Then S is cancellative if and only if for any $s, t, w \in S$, $stw = sw$ implies $t = 1$.*

PROOF. The "only if" part is obvious. To prove the "if" part, assume that

$$stw = sw \text{ implies } t = 1 \text{ for all } s, t, w \in S, \tag{1.1}$$

and consider any $s, u, v \in S$ with $su = sv$. Then in the ring $A = R[[S, \omega]]$ we have $\boldsymbol{e}_s \boldsymbol{e}_u = \boldsymbol{e}_s \boldsymbol{e}_v$, and thus by Lemma 1.4 there exist $f, g, h, k \in A$ such that

$$\boldsymbol{e}_s f = \boldsymbol{e}_s g, \quad (1-f)\boldsymbol{e}_u = h\boldsymbol{e}_v, \quad \boldsymbol{e}_s h = \boldsymbol{e}_s k \text{ and } (1-k)\boldsymbol{e}_v = g\boldsymbol{e}_u. \tag{1.2}$$

Suppose that $u \notin Sv$. Then (1.1) and (1.2) imply that

$$0 = (h\boldsymbol{e}_v)(u) = ((1-f)\boldsymbol{e}_u)(u) = 1 - f(1)$$

and

$$0 = ((1-k)\boldsymbol{e}_v)(u) = (g\boldsymbol{e}_u)(u) = g(1).$$

Thus from the first part of (1.2) we obtain

$$1 = \omega_s(1) = \omega_s(f(1)) = (\boldsymbol{e}_s f)(s) = (\boldsymbol{e}_s g)(s) = \omega_s(g(1)) = \omega_s(0) = 0,$$

and this contradiction shows that $u = tv$ for some $t \in S$. Hence $stv = su = sv$, and (1.1) implies that $t = 1$. Thus $u = v$, which proves that S is left cancellative. Similarly one can show that S is right cancellative. □

We now turn to right distributive skew generalized power series rings. We will often use the following characterization of right distributive rings (see [**15**, Theorem 1.6]).

PROPOSITION 1.6. *A ring R is right distributive if and only if for any $a, b \in R$ there exist $x, y \in R$ such that $x + y = 1$, $ax \in bR$ and $by \in aR$.*

Since every idempotent of a right distributive ring is central (see [**15**, Corollary 2 of Proposition 1.1]), the following proposition implies that for any strictly ordered monoid $(S, \leq)$, if the ring $R[[S, \omega]]$ is right distributive, then ω_s is idempotent-stabilizing for any $s \in S$.

PROPOSITION 1.7. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid, and*

$\omega : S \to \mathrm{End}(R)$ *a monoid homomorphism such that the ring* $R[[S,\omega]]$ *is abelian. Then for any* $s \in S$*,* ω_s *is idempotent-stabilizing.*

PROOF. Set $A = R[[S,\omega]]$ and consider any $s \in S$ and $e = e^2 \in R$. Then $\boldsymbol{c}_e = \boldsymbol{c}_e^2$ in A, and since A is abelian, we obtain $\boldsymbol{c}_e\boldsymbol{e}_s = \boldsymbol{e}_s\boldsymbol{c}_e = \boldsymbol{c}_{\omega_s(e)}\boldsymbol{e}_s$. Hence $\omega_s(e) = e$. □

COROLLARY 1.8. *Let* R *be a ring,* $(S,\leq)$ *a strictly ordered monoid, and* $\omega : S \to \mathrm{End}(R)$ *a monoid homomorphism such that the ring* $R[[S,\omega]]$ *is right distributive. Then for any* $s \in S$*,* ω_s *is idempotent-stabilizing.*

The following lemma implies, in particular, that for any positively strictly ordered monoid $(S,\leq)$, if the ring $R[[S,\omega]]$ is right distributive, then S is left cancellative.

LEMMA 1.9. *Let* R *be a ring,* $(S,\leq)$ *a strictly ordered monoid, and* $\omega : S \to \mathrm{End}(R)$ *a monoid homomorphism such that the ring* $R[[S,\omega]]$ *is right distributive. Then* S *is left cancellative if and only if for any* $s,t \in S$*,* $st = s$ *implies* $t = 1$*.*

PROOF. The "only if" part is clear. For the "if" part, assume that

$$st = s \ \text{ implies } \ t = 1 \ \text{ for any } \ s,t \in S, \tag{1.3}$$

and consider any $s,u,v \in S$ with $su = sv$. Then in the ring $A = R[[S,\omega]]$ we have $\boldsymbol{e}_s\boldsymbol{e}_u = \boldsymbol{e}_s\boldsymbol{e}_v$, and thus by Proposition 1.6 there exist $f,g,h,k \in A$ such that

$$f + g = 1, \quad \boldsymbol{e}_u f = \boldsymbol{e}_v h \ \text{ and } \ \boldsymbol{e}_v g = \boldsymbol{e}_u k. \tag{1.4}$$

Suppose that $u \notin vS$ and $v \notin uS$. Then (1.3) and (1.4) imply that

$$\omega_u(f(1)) = (\boldsymbol{e}_u f)(u) = (\boldsymbol{e}_v h)(u) = 0,$$

and analogously one shows that $\omega_v(g(1)) = 0$. Since $su = sv$, it follows that $\omega_{su}(f(1)) = \omega_{su}(g(1)) = 0$. Now from the first part of (1.4) we obtain

$$1 = \omega_{su}(1) = \omega_{su}((f+g)(1)) = \omega_{su}(f(1)) + \omega_{su}(g(1)) = 0 + 0 = 0,$$

and this contradiction shows that $u \in vS$ or $v \in uS$. In the first case $u = vt$ for some $t \in S$. Hence $sv = su = svt$, and (1.3) implies that $t = 1$, which leads to $u = v$, as desired. The case when $v \in uS$ follows similarly. □

Let S be a monoid. Recall that S is a *right chain monoid* if the right ideals of S are totally ordered by set inclusion ([**3**]), i.e. for any $s, t \in S$ we have $sS \subseteq tS$ or $tS \subseteq sS$. Recall also that S is said to be *right duo* if all right ideals of S are two-sided ideals, i.e. $St \subseteq tS$ for any $t \in S$.

Lemma 1.10. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism such that ω_s is injective for any $s \in S$. Then S is a right chain monoid in each of the following cases*:

(i) *If $R[[S,\omega]]$ is right distributive and S is left cancellative.*
(ii) *If $R[[S,\omega]]$ has weak dimension less than or equal to one, and S is cancellative and right duo.*

Proof. (i) Assume that S is not a right chain monoid. Then there exist $s, t \in S$ with $sS \not\subseteq tS$ and $tS \not\subseteq sS$. Since the ring $R[[S,\omega]]$ is right distributive, by Proposition 1.6 for some $f, g, h \in R[[S,\omega]]$ we have

$$\boldsymbol{e}_s f = \boldsymbol{e}_t g \quad \text{and} \quad \boldsymbol{e}_t(1-f) = \boldsymbol{e}_s h. \tag{1.5}$$

Since $s \notin tS$, the left cancellativity of S and the first part of (1.5) imply that

$$0 = (\boldsymbol{e}_t g)(s) = (\boldsymbol{e}_s f)(s) = \omega_s(f(1)),$$

and since ω_s is injective, $f(1) = 0$ follows. Hence, using that $t \notin sS$, from the second part of (1.5) we obtain

$$0 = (\boldsymbol{e}_s h)(t) = (\boldsymbol{e}_t(1-f))(t) = \omega_t((1-f)(1)) = \omega_t(1-f(1)) = \omega_t(1) = 1,$$

a contradiction.

(ii) Let $s, t \in S$, and assume that $sS \not\subseteq tS$. Since S is right duo, $st = tp$ for some $p \in S$. Hence $\boldsymbol{e}_s\boldsymbol{e}_t = \boldsymbol{e}_t\boldsymbol{e}_p$, and since $R[[S,\omega]]$ has weak dimension less than or equal to one, by Lemma 1.4 there exist $f, g, h \in R[[S,\omega]]$ such that

$$\boldsymbol{e}_s f = \boldsymbol{e}_t g \quad \text{and} \quad (1-f)\boldsymbol{e}_t = h\boldsymbol{e}_p. \tag{1.6}$$

Since $s \notin tS$ and S is cancellative, it follows from (1.6) that

$$0 = (\boldsymbol{e}_t g)(s) = (\boldsymbol{e}_s f)(s) = \omega_s(f(1)),$$

and since ω_s is injective, we obtain $f(1) = 0$. Hence the second part of (1.6) implies that

$$(h\boldsymbol{e}_p)(t) = [(1-f)\boldsymbol{e}_t](t) = (1-f)(1) = 1 - f(1) = 1, \tag{1.7}$$

and thus $t = xp$ for some $x \in S$. Therefore $tp = st = sxp$, and since S is cancellative, $t = sx$ follows. Hence $tS \subseteq sS$, and thus S is a right chain monoid. □

In the following lemma we characterize totally ordered right chain monoids (cf. [**10**, Lemma 4]).

Lemma 1.11. *Let $(S, \leq)$ be an ordered right chain monoid. Then the order $\leq$ is total if and only if for any $s \in S$ we have $s \leq 1$ or $s \geq 1$.*

Proof. Assume that for any $s \in S$ we have $s \leq 1$ or $s \geq 1$, and let $x, y \in S$. Since S is a right chain monoid, we may assume that $x = ys$ for some $s \in S$. If $s \leq 1$ (resp. $s \geq 1$), then $x \leq y$ (resp. $x \geq y$). Hence the order $\leq$ is total. The opposite implication is obvious. □

By Theorem 1.1, in the case where σ is injective, the right distributivity of the skew power series ring $R[[x;\sigma]]$ forces the strong regularity of the coefficient ring R. As we will see in the proof of Lemma 1.13, this property is a consequence of the following result (cf. [**16**, 4.58]).

Lemma 1.12. *For any ring R and an endomorphism σ of R, the following conditions are equivalent*:

(1) *R is strongly regular and σ is idempotent-stabilizing.*
(2) *σ is injective and for any $a \in R$ there exists $b \in R$ such that $\sigma(a) = \sigma(a)ab$.*

Proof.
(1) ⇒ (2). Since R is strongly regular, each element of R is a product of a unit and an idempotent, and thus (1) implies that σ is injective. Moreover, for any $a \in R$ there exists $b \in R$ such that $a = a^2b$ and $ab = (ab)^2$, and it follows from (1) that $\sigma(a) = \sigma(a)ab$.

(2) ⇒ (1). We first show that R is a reduced ring. For, let $a \in R$ and $a^2 = 0$. Then $\sigma(\sigma(a)a)\sigma(a)a = \sigma(\sigma(a)a^2)a = 0$, and since by (2) we have $\sigma(\sigma(a)a) \in \sigma(\sigma(a)a)\sigma(a)aR$, it follows that $\sigma(\sigma(a)a) = 0$. Hence $\sigma(a)a = 0$ by the injectivity of σ, and we deduce from (2) that $\sigma(a) = 0$, which implies that $a = 0$, as desired.

Next we show that the ring R is strongly regular. By (2), for any $a \in R$ there exists $b \in R$ such that $\sigma(a) = \sigma(a)ab$. Hence $\sigma(a)c = 0$ for $c = 1 - ab$, and since R is reduced, it follows from (2) that $\sigma(ac) \in \sigma(ac)acR \subseteq \sigma(a)RcR = \{0\}$. Thus

$ac = 0$ by the injectivity of σ, which shows that $a = a^2b$.

Finally we prove that σ is idempotent-stabilizing. Let $e = e^2 \in R$. By (2) there exist $c, d \in R$ such that $\sigma(e) = \sigma(e)ec$ and $\sigma(1-e) = \sigma(1-e)(1-e)d$. Thus $1 = \sigma(e)+\sigma(1-e) = \sigma(e)ec+\sigma(1-e)(1-e)d$, and since R is reduced, e is central and $e = \sigma(e)ec = \sigma(e)$ follows. $\square$

If $(S, \leq)$ is a nontrivial positively ordered monoid, then clearly S is not a group, i.e. $S \setminus U(S) \neq \emptyset$. Therefore, when skew generalized power series rings with positively ordered exponents are considered, as it is in this paper, then the following lemma gives some necessary conditions for such a power series ring to be right distributive.

LEMMA 1.13. *Let R be a ring, $(S, \leq)$ a strictly ordered left cancellative monoid, and let $\omega : S \to \mathrm{End}(R)$ be a monoid homomorphism such that the ring $R[[S, \omega]]$ is right distributive. Then*

(i) *For any $s \in S \setminus U(S)$ and $a \in R$ there exists $b \in R$ such that $\omega_s(a) = \omega_s(a)ab$.*
(ii) *If ω_{s_0} is injective for some $s_0 \in S \setminus U(S)$, then*
  (a) *R is strongly regular.*
  (b) *ω_s is bijective for any $s \in S$.*
  (c) *If the order $\leq$ is total, then the ring $R[[S, \omega]]$ is reduced.*

PROOF.
(i) Let $s \notin U(S)$ and $a \in R$. By Proposition 1.6 there exist $f, g, h, k \in R[[S, \omega]]$ such that $f + g = 1$, $\boldsymbol{c}_a f = \boldsymbol{e}_s h$ and $\boldsymbol{e}_s g = \boldsymbol{c}_a k$. Note that $st \neq 1$ for any $t \in S$ (otherwise $sts = s$; thus $ts = 1$ by the left cancellativity of S, and we obtain $s \in U(S)$, a contradiction). Hence

$$af(1) = (\boldsymbol{c}_a f)(1) = (\boldsymbol{e}_s h)(1) = 0.$$

Since S is left cancellative, we obtain also that

$$\omega_s(g(1)) = (\boldsymbol{e}_s g)(s) = (\boldsymbol{c}_a k)(s) = ak(s).$$

Thus

$$\omega_s(a) = \omega_s(a)\omega_s(1) = \omega_s(a)[\omega_s(f(1)) + \omega_s(g(1))] = \omega_s(a)\omega_s(g(1)) = \omega_s(a)ak(s).$$

(ii) (a) This follows from (i) and Lemma 1.12.

(b) Let $s \in S$. Then (a), Corollary 1.8 and Lemma 1.12 imply that ω_s is injective. Thus, to complete the proof it suffices to show that if $a \in R$, then

$a \in \omega_s(R)$. By Proposition 1.6 there exist $f, h, k \in R[[S,\omega]]$ with $\boldsymbol{c}_a\boldsymbol{e}_s f = \boldsymbol{e}_s h$ and $\boldsymbol{e}_s(1-f) = \boldsymbol{c}_a\boldsymbol{e}_s k$. Therefore, applying also the left cancellativity of S, we obtain

$$a\omega_s(f(1)) = (\boldsymbol{c}_a\boldsymbol{e}_s f)(s) = (\boldsymbol{e}_s h)(s) = \omega_s(h(1)) \in \omega_s(R) \tag{1.8}$$

and

$$1 - \omega_s(f(1)) = (\boldsymbol{e}_s(1-f))(s) = (\boldsymbol{c}_a\boldsymbol{e}_s k)(s) = a\omega_s(k(1)). \tag{1.9}$$

By (a) there exist $u_1, u_2 \in U(R)$ and central idempotents $e_1, e_2 \in R$ such that

$$f(1) = u_1 e_1 \quad \text{and} \quad k(1) = u_2 e_2.$$

From (1.8) and Corollary 1.8 we obtain $a\omega_s(u_1)e_1 \in \omega_s(R)$, and thus

$$ae_1 \in \omega_s(R)\omega_s(u_1^{-1}) \subseteq \omega_s(R). \tag{1.10}$$

On the other hand, by multiplying (1.9) by $1 - e_1$, we obtain

$$1 - e_1 = a\omega_s(u_2)e_2(1-e_1). \tag{1.11}$$

Now by multiplying (1.11) by $1 - e_2$ we obtain $(1-e_1)(1-e_2) = 0$, and thus $e_2(1-e_1) = 1 - e_1$. Hence by (1.11) we have $1 - e_1 = a\omega_s(u_2)(1-e_1)$. Therefore

$$a(1-e_1) = (1-e_1)\omega_s\big(u_2^{-1}\big) = \omega_s\big((1-e_1)u_2^{-1}\big) \in \omega_s(R),$$

which together with (1.10) implies that $a \in \omega_s(R)$.

(c) By [**9**, Theorem 3.9], to prove that $R[[S,\omega]]$ is reduced, it suffices to show that for any $s \in S$ and $a \in R$, $a\omega_s(a) = 0$ implies $a = 0$. But this is obvious, since by (a), a is a product of a unit and a central idempotent e, and by Corollary 1.8 we have $\omega_s(e) = e$. □

In the next lemma we give some necessary conditions for a skew generalized power series ring to be right duo.

Lemma 1.14. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism such that the ring $R[[S,\omega]]$ is right duo. Then*

(i) *The ring R and the monoid S are right duo, and ω_s is idempotent-stabilizing for any $s \in S$.*

(ii) *If $s \in S$ is left cancellative, then ω_s is bijective.*

PROOF.

(i) Since $A = R[[S,\omega]]$ is right duo, for any $a, b \in R$ there exists $f \in A$ such that $\boldsymbol{c}_{ba} = \boldsymbol{c}_b \boldsymbol{c}_a = \boldsymbol{c}_a f$. Hence $ba = \boldsymbol{c}_{ba}(1) = (\boldsymbol{c}_a f)(1) = \boldsymbol{c}_a(1)f(1) = af(1) \in aR$ proves that R is right duo. Similarly, for any $s, t \in S$ there exists $g \in A$ with $\boldsymbol{e}_{st} = \boldsymbol{e}_s \boldsymbol{e}_t = \boldsymbol{e}_t g$. Now $(\boldsymbol{e}_t g)(st) = \boldsymbol{e}_{st}(st) = 1$ implies that $st \in tS$, and thus S is right duo.

Since any right duo ring is abelian, Lemma 1.7 completes the proof of (i).

(ii) Set $A = R[[S,\omega]]$ and assume that $s \in S$ is left cancellative. We first show that ω_s is surjective. Since A is right duo, for any $r \in R$ there exists $h \in A$ such that $\boldsymbol{c}_r \boldsymbol{e}_s = \boldsymbol{e}_s h$. Hence, using also that s is left cancellative, we obtain $r = (\boldsymbol{c}_r \boldsymbol{e}_s)(s) = (\boldsymbol{e}_s h)(s) = \omega_s(h(1)) \in \omega_s(R)$, and thus ω_s is a surjection.

To prove that ω_s is injective, we adapt the proof of [**8**, Theorem 1]. The case where $s \in U(S)$ is obvious. Thus we assume that $s \notin U(S)$. Let $a \in R$ be such that $\omega_s(a) = 0$. Since ω_s is surjective, $a = \omega_s(b)$ for some $b \in R$. Since A is right duo, there exists $k \in A$ such that

$$\boldsymbol{c}_a \boldsymbol{e}_s + \boldsymbol{e}_{s^3} = \boldsymbol{e}_s(\boldsymbol{c}_b + \boldsymbol{c}_a \boldsymbol{e}_s + \boldsymbol{e}_{s^2}) = (\boldsymbol{c}_b + \boldsymbol{c}_a \boldsymbol{e}_s + \boldsymbol{e}_{s^2})k. \tag{1.12}$$

Since s is left cancellative and $s \notin U(S)$, taking values of (1.12) at 1, s and s^3, respectively, we obtain the following equations:

$$\begin{aligned} &0 = bk(1), \qquad a = bk(s) + a\omega_s(k(1)), \\ &1 = bk(s^3) + a\omega_s(k(s^2)) + \omega_{s^2}(k(s)). \end{aligned} \tag{1.13}$$

From the first equation of (1.13) we obtain $0 = \omega_s(bk(1)) = a\omega_s(k(1))$, and thus the second equation of (1.13) implies that $a = bk(s)$, which leads to $0 = a\omega_s(k(s))$. Applying ω_{s^2} to the third equation of (1.13), we obtain $\omega_{s^4}(k(s)) = 1$. Hence, since A is right duo, it follows that $\boldsymbol{e}_{s^3} = \boldsymbol{e}_{s^3}\boldsymbol{c}_{\omega_s(k(s))} \in \boldsymbol{c}_{\omega_s(k(s))}A$, and thus $1 = \omega_s(k(s))d$ for some $d \in R$. Hence $a = a\omega_s(k(s))d = 0 \cdot d = 0$, proving that ω_s is bijective. $\square$

In the case where the coefficient ring R is a finite direct product of rings, the following result will allow us to represent the ring $R[[S,\omega]]$ as a direct product of skew generalized power series rings.

PROPOSITION 1.15. *Let $R_1, R_2, \ldots, R_n$ be rings and let $R = \prod_{i=1}^n R_i$. For any $i \in \{1, 2, \ldots, n\}$ let $\tau_i : R_i \to R$ and $\pi_i : R \to R_i$ be the natural injection and the natural projection, respectively. Let $(S, \leq)$ be a strictly ordered monoid* *and* $\omega : S \to \mathrm{End}(R)$ *a monoid homomorphism such that* $\omega_s \circ \tau_i(R_i) \subseteq \tau_i(R_i)$ *for any* $s \in S$ *and* $i \in \{1, 2, \dots, n\}$. *Then for every* $i \in \{1, 2, \dots, n\}$ *the map* $\omega_i : S \to \mathrm{End}(R_i)$ *defined by*

$$\omega_{i,s} = \pi_i \circ \omega_s \circ \tau_i \quad \textit{for any } s \in S$$

is a monoid homomorphism and the ring $R[[S, \omega]]$ *is isomorphic to the ring* $\prod_{i=1}^{n} R_i[[S, \omega_i]]$.

Proof. Since by assumption for any $i \in \{1, 2, \dots, n\}$ and $s \in S$ we have $\omega_s \circ \tau_i(R_i) \subseteq \tau_i(R_i)$, it easily follows that $\omega_{i,s}(1) = 1$. Now to complete the proof, it suffices to repeat arguments of the proof of [**12**, Proposition 2.1]. □

In the proof of Theorem 2.1 we will need the following characterization of finite products of division rings (see [**10**, Corollary 13]). Recall that a ring R is said to be *orthogonally finite* if R has no infinite set of orthogonal idempotents.

Lemma 1.16. *A ring* R *is orthogonally finite strongly regular if and only if* R *is a finite direct product of division rings.*

To prove Theorem 2.1, we will also need the following generalization of [**10**, Lemma 11].

Lemma 1.17. *Let* $(S, \leq)$ *be a right chain positively strictly ordered monoid, and let* $t \in S$. *Then*

(i) *For any* $s \in S$ *there exists a unique element* $s^{(t)} \in S$ *such that* $st = ts^{(t)}$.
(ii) *Let* R *be a ring, and* $\omega : S \to \mathrm{End}(R)$ *a monoid homomorphism such that* ω_t *is bijective. For any* $f \in R[[S, \omega]]$ *define* $f^{(t)} : S \to R$ *by* $f^{(t)}(x) = \omega_t^{-1}(f(s))$ *if* $x = s^{(t)}$ *for some* $s \in S$, *and* $f^{(t)}(x) = 0$ *otherwise. Then* $f^{(t)} \in R[[S, \omega]]$ *and* $f\boldsymbol{e}_t = \boldsymbol{e}_t f^{(t)}$.

Proof. We claim that for any $s \in S$, $st \in tS$. Otherwise, since S is a right chain monoid, $t = stv$ for some $v \in S \setminus \{1\}$. Since $(S, \leq)$ is positively ordered, $s \geq 1$ and $v > 1$, and we obtain $t = stv > t$, a contradiction that proves our claim. Hence there exists $s^{(t)} \in S$ such that $st = ts^{(t)}$. Since by Lemma 1.11 the order $\leq$ is total, such an element $s^{(t)}$ is unique, and the proof of (i) is complete. Furthermore, for any $s_1, s_2 \in S$ we have $s_1 \leq s_2 \Leftrightarrow s_1^{(t)} \leq s_2^{(t)}$. Thus for any $f \in R[[S, \omega]]$ the map $f^{(t)} : S \to R$ is well-defined and $f^{(t)} \in R[[S, \omega]]$. The rest of (ii) is easy to verify. □

2. Main result.

We are now in a position to prove the main result of this paper, which extends Theorem 1.1 to skew generalized power series rings with positively ordered exponents. Recall that a monoid S is *cyclic* if for some $s \in S$ we have $S = \{s^n : n \in \boldsymbol{N} \cup \{0\}\}$. A ring R is a *right chain ring* if its right ideals are totally ordered by inclusion ([**1**]).

Theorem 2.1. *Let R be a ring, $(S, \leq)$ a nontrivial positively strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. Then the following conditions are equivalent*:

(1) *$R[[S,\omega]]$ has weak dimension less than or equal to one and is right duo.*
(2) *$R[[S,\omega]]$ has weak dimension less than or equal to one, R is abelian, S is a right chain monoid, and ω_s is bijective and idempotent-stabilizing for any $s \in S$.*
(3) *$R[[S,\omega]]$ is right duo right distributive.*
(4) *$R[[S,\omega]]$ is reduced right distributive.*
(5) *$R[[S,\omega]]$ is right distributive and ω_s is injective for any $s \in S$.*
(6) *$R[[S,\omega]]$ is right duo right Bezout.*
(7) *$R[[S,\omega]]$ is reduced right Bezout.*
(8) *$R[[S,\omega]]$ is right quasi-duo right Bezout and ω_s is injective for any $s \in S$.*
(9) *$R[[S,\omega]]$ is semicommutative right Bezout and ω_s is injective for any $s \in S$.*
(10) *ω_s is bijective and idempotent-stabilizing for any $s \in S$, and either*
(a) *S is cyclic and R is $\aleph_0$-injective strongly regular*
or
(b) *S is not cyclic, S is a right chain monoid and R is a finite direct product of division rings.*

Proof. Set $A = R[[S,\omega]]$.

(1) $\Rightarrow$ (2). Since S is positively strictly ordered, S is cancellative by Lemma 1.5. Thus Lemmas 1.14 and 1.10(ii) imply that ω_s is bijective and idempotent-stabilizing for any $s \in S$, and that S is a right chain monoid. Moreover, R is right duo by Lemma 1.14(i), and thus R is abelian.

(2) $\Rightarrow$ (10). If S is cyclic, say generated by s, then $A \cong R[[x;\sigma]]$, where $\sigma = \omega_s$, and in this case this implication follows from Theorem 1.1. Therefore, we assume that S is not cyclic, and we show that if (2) holds, then condition (b) of (10) is satisfied.

To prove (b), we will apply Lemma 1.16. We first show that the ring R is strongly regular. Let $a \in R$ and choose any $s \in S \setminus \{1\}$. Since $\boldsymbol{c}_a \boldsymbol{e}_s = \boldsymbol{e}_s \boldsymbol{c}_{\omega_s^{-1}(a)}$, Lemma 1.4 implies that $\boldsymbol{c}_a f = \boldsymbol{e}_s g$ and $(1-f)\boldsymbol{e}_s = h\boldsymbol{c}_{\omega_s^{-1}(a)}$ for some $f, g, h \in A$. Hence

$$af(1) = (\boldsymbol{c}_a f)(1) = (\boldsymbol{e}_s g)(1) = \boldsymbol{e}_s(1)g(1) = 0,$$

and since S is cancellative by Lemma 1.5, we obtain also that

$$1 - f(1) = [(1 - f)\boldsymbol{e}_s](s) = (h\boldsymbol{c}_{\omega_s^{-1}(a)})(s) = h(s)a.$$

Thus $a = a(1 - f(1)) = ah(s)a$, which proves that R is von Neumann regular. Since R is abelian, it follows that R is strongly regular.

Now we show that R is a finite direct product of division rings. By Lemma 1.16, we need only prove that R is orthogonally finite. Suppose, for a contradiction, that there exists an infinite sequence $e_1, e_2, e_3, \ldots$ of nonzero orthogonal idempotents of R. By [**10**, Lemma 7], in S there exist an element t and a sequence $(s_n)_{n\in\boldsymbol{N}}$ such that

$$s_1 < s_2 < s_3 < \cdots < t.$$

Define $p \in A$ by $p(s_i) = e_i$ for all $i \in \boldsymbol{N}$, and $p(x) = 0$ for $x \in S \setminus \{s_1, s_2, s_3, \ldots\}$, and let $p^{(t)} \in A$ be defined as in Lemma 1.17(ii). Then by Lemma 1.17 we have $p\boldsymbol{e}_t = \boldsymbol{e}_t p^{(t)}$, and by Lemma 1.4 there exist $f, g, h \in A$ with

$$pf = \boldsymbol{e}_t g \quad \text{and} \quad \boldsymbol{e}_t = f\boldsymbol{e}_t + hp^{(t)}. \tag{2.1}$$

We claim that

$$e_j f(1) = 0 \quad \textit{for any} \quad j \in \boldsymbol{N}. \tag{2.2}$$

To see this, note that since S is positively ordered and $s_j < t$, we have $s_j \notin tS$, and thus from the first part of (2.1) we obtain

$$\begin{aligned} 0 = (\boldsymbol{e}_t g)(s_j) = (pf)(s_j) &= p(s_j)\omega_{s_j}(f(1)) + \sum_{\substack{(x,y)\in \mathrm{X}_{s_j}(p,f) \\ x \neq s_j}} p(x)\omega_x(f(y)) \\ &= e_j\omega_{s_j}(f(1)) + e_{k_1}\omega_{s_{k_1}}(f(y_1)) + e_{k_2}\omega_{s_{k_2}}(f(y_2)) + \cdots + e_{k_m}\omega_{s_{k_m}}(f(y_m)) \end{aligned}$$

for some $m \in \boldsymbol{N}$, $y_1, y_2, \ldots, y_m \in S$, and $k_1, k_2, \ldots, k_m \in \boldsymbol{N} \setminus \{j\}$. Multiplying the above equation by e_j from the left, we obtain

$$0 = e_j\omega_{s_j}(f(1)) = \omega_{s_j}(e_j)\omega_{s_j}(f(1)) = \omega_{s_j}(e_j f(1)).$$

Since ω_{s_j} is injective, $e_j f(1) = 0$ follows, completing the proof of (2.2).

On the other hand, applying the definition of $p^{(t)}$ and the second part of (2.1), we obtain

$$\begin{aligned} 1 = \boldsymbol{e}_t(t) &= (f\boldsymbol{e}_t)(t) + (hp^{(t)})(t) \\ &= f(1) + \sum_{(x,y)\in \mathrm{X}_t(h,p^{(t)})} h(x)\omega_x(p^{(t)}(y)) \\ &= f(1) + h(x_1)e_{i_1} + h(x_2)e_{i_2} + \cdots + h(x_n)e_{i_n} \end{aligned}$$

for some $n, i_1, \ldots, i_n \in \boldsymbol{N} \cup \{0\}$ and $x_1, x_2, \ldots, x_n \in S$. Take any $j \in \boldsymbol{N} \setminus \{i_1, i_2, \ldots, i_n\}$. Since $e_{i_d} e_j = 0$ for all $d \in \{1, 2, \ldots, n\}$, from the above equation it follows that $e_j = f(1)e_j$. But $e_j f(1) = 0$ by (2.2), and we obtain $e_j = e_j f(1) e_j = 0$, a contradiction.

(10) $\Rightarrow$ (3). If S is cyclic, then this implication follows from Theorem 1.1. Assume that S is not cyclic. Then $R = D_1 \times \cdots \times D_n$ for some division rings $D_1, \ldots, D_n$. Furthermore, if $s \in S$, then $\omega_s(e) = e$ for any idempotent $e \in R$, and thus $\omega_s \circ \tau_i(D_i) \subseteq \tau_i(D_i)$ for any $1 \leq i \leq n$, where $\tau_i : D_i \to R$ is the natural injection. Hence by Proposition 1.15 we have $A \cong D_1[[S, \omega_1]] \times \cdots \times D_n[[S, \omega_n]]$, where for any $i \in \{1, \ldots, n\}$, $\omega_i : S \to \mathrm{End}(D_i)$ is a monoid homomorphism such that $\omega_{i,s} = \omega_i(s)$ is bijective for any $s \in S$. Since S is a right chain monoid, the order $\leq$ is total by Lemma 1.11, and thus [**11**, Theorem 4.7] implies that for any $i \in \{1, \ldots, n\}$, $D_i[[S, \omega_i]]$ is a right chain ring and any nonzero principal right ideal of this ring is generated by $\boldsymbol{e}_t$ for some $t \in S$. Thus by Lemma 1.17(ii), $D_i[[S, \omega_i]]$ is a right duo ring for any $i \in \{1, \ldots, n\}$. Therefore, being a finite direct product of right chain right duo rings, A is a right distributive right duo ring.

(3) $\Rightarrow$ (6). Since S is left cancellative by Lemma 1.9, it follows from Lemmas 1.14(ii) and 1.13(ii) that R is strongly regular. Hence Lemma 1.3 implies that $A/J(A)$ is strongly regular, and thus R is right Bezout by Proposition 1.2(ii).

(6) $\Rightarrow$ (8). Proposition 1.2(iii) implies that A is right distributive, and thus S is left cancellative by Lemma 1.9. Now (8) follows from Lemma 1.14(ii).

(8) $\Rightarrow$ (7). Proposition 1.2(iii) implies that A is right distributive. From Lemmas 1.9, 1.10(i) and 1.11 it follows that the order $\leq$ is total, and thus by Lemma 1.13(ii)(c), A is reduced.

(7) $\Rightarrow$ (9). We show first that for any $s \in S$, ω_s is injective. For this, assume that $a \in R$ and $\omega_s(a) = 0$. Then in A we have $(\boldsymbol{c}_a \boldsymbol{e}_s)^2 = \boldsymbol{c}_{a\omega_s(a)} \boldsymbol{e}_{s^2} = 0$, and since A is reduced, $\boldsymbol{c}_a \boldsymbol{e}_s = 0$ follows. Hence $a = (\boldsymbol{c}_a \boldsymbol{e}_s)(s) = 0$, which proves that ω_s is injective. Since every reduced ring is semicommutative, so is A.

(9) $\Rightarrow$ (5). By Proposition 1.2(ii) and Lemma 1.3, it suffices to show that R is strongly regular. For this, consider any $a \in R$. Since S is nontrivial, there exists

$s \in S\backslash\{1\}$. Since A is right Bezout, there exist $f, g, h, k \in A$ with $\boldsymbol{c}_a = (\boldsymbol{c}_a f + \boldsymbol{e}_s g)h$ and $\boldsymbol{e}_s = (\boldsymbol{c}_a f + \boldsymbol{e}_s g)k$. Since S is positively ordered, it follows that

$$a = \boldsymbol{c}_a(1) = [(\boldsymbol{c}_a f + \boldsymbol{e}_s g)h](1) = (\boldsymbol{c}_a fh)(1) + (\boldsymbol{e}_s gh)(1) = af(1)h(1),$$

and thus

$$\omega_s(a) = \omega_s(a)\omega_s(f(1))\omega_s(h(1)). \tag{2.3}$$

Moreover

$$0 = \boldsymbol{e}_s(1) = [(\boldsymbol{c}_a f + \boldsymbol{e}_s g)k](1) = (\boldsymbol{c}_a fk)(1) + (\boldsymbol{e}_s gk)(1) = af(1)k(1),$$

and thus

$$0 = \omega_s(a)\omega_s(f(1))\omega_s(k(1)). \tag{2.4}$$

Furthermore,

$$\begin{aligned} 1 = \boldsymbol{e}_s(s) &= [(\boldsymbol{c}_a f + \boldsymbol{e}_s g)k](s) = (\boldsymbol{c}_a fk)(s) + (\boldsymbol{e}_s gk)(s) \\ &= a(fk)(s) + \omega_s(g(1))\omega_s(k(1)). \end{aligned}$$

By (2.4), $\omega_s(k(1))$ belongs to the right annihilator of $\omega_s(a)\omega_s(f(1))$, which by assumption is an ideal of R, and thus using (2.3) we obtain

$$\begin{aligned} \omega_s(a)\omega_s(g(1))\omega_s(k(1)) &= [\omega_s(a)\omega_s(f(1))\omega_s(h(1))]\omega_s(g(1))\omega_s(k(1)) \\ &= \omega_s(a)\omega_s(f(1))[\omega_s(h(1))\omega_s(g(1))]\omega_s(k(1)) = 0. \end{aligned}$$

Hence

$$\omega_s(a) = \omega_s(a)1 = \omega_s(a)[a(fk)(s) + \omega_s(g(1))\omega_s(k(1))] = \omega_s(a)a(fk)(s).$$

Thus by Lemma 1.12 the ring R is strongly regular.

(5) $\Rightarrow$ (4) follows from Lemmas 1.9, 1.10(i), 1.11 and 1.13(ii)(c).

(4) $\Rightarrow$ (10). The same argument as in the proof of (7) $\Rightarrow$ (9) implies that all the ω_s's are injective. Since S is left cancellative by Lemma 1.9, it follows from Lemma 1.13(ii) and Corollary 1.8 that for any $s \in S$, ω_s is bijective and idempotent-stabilizing.

If S is cyclic, then part (a) of (10) follows from Theorem 1.1. Thus we assume that S is not cyclic and prove part (b). By Lemma 1.10(i), S is a right chain monoid. By Lemmas 1.13(ii)(a) and 1.16, to prove the rest of (b) it suffices to show that R is orthogonally finite. For this it suffices to modify slightly the proof of the implication (2) $\Rightarrow$ (10). Suppose, for a contradiction, that there exists an infinite sequence $e_1, e_2, e_3, \dots$ of nonzero orthogonal idempotents of R. Since S is a positively ordered right chain monoid that is not cyclic, by [**10**, Lemma 7] in S there exist an element t and a sequence $(s_n)_{n\in\boldsymbol{N}}$ such that

$$s_1 < s_2 < s_3 < \dots < t.$$

Define $p \in A$ by $p(s_i) = e_i$ for all $i \in \boldsymbol{N}$, and $p(x) = 0$ for $x \in S \setminus \{s_1, s_2, s_3, \dots\}$. Since A is right distributive, by Proposition 1.6 there exist $f, g, h \in A$ with

$$pf = \boldsymbol{e}_t g \quad \text{and} \quad \boldsymbol{e}_t = \boldsymbol{e}_t f + ph. \tag{2.5}$$

From the second part of (2.5) we obtain

$$\begin{aligned} 1 = \boldsymbol{e}_t(t) = (\boldsymbol{e}_t f)(t) + (ph)(t) &= \omega_t(f(1)) + \sum_{(x,y)\in \mathrm{X}_t(p,h)} p(x)\omega_x(h(y)) \\ &= \omega_t(f(1)) + p(x_1)\omega_{x_1}(h(y_1)) + p(x_2)\omega_{x_2}(h(y_2)) + \dots + p(x_n)\omega_{x_n}(h(y_n)) \\ &= \omega_t(f(1)) + e_{i_1}\omega_{x_1}(h(y_1)) + e_{i_2}\omega_{x_2}(h(y_2)) + \dots + e_{i_n}\omega_{x_n}(h(y_n)) \end{aligned}$$

for some $n, i_1, \dots, i_n \in \boldsymbol{N} \cup \{0\}$ and $x_1, x_2, \dots, x_n$, $y_1, y_2, \dots, y_n \in S$. Take any $j \in \boldsymbol{N} \setminus \{i_1, i_2, \dots, i_n\}$. Since $e_j e_{i_d} = 0$ for all $d \in \{1, 2, \dots, n\}$, from the above equation it follows that $e_j = e_j\omega_t(f(1)) = \omega_t(e_j f(1))$. But the first part of (2.5) implies that $e_j f(1) = 0$ (see the proof of (2.2)), and we obtain $e_j = \omega_t(0) = 0$, a contradiction.

(3) $\Rightarrow$ (1). We already know that (3) implies (7), and thus to get (1) it suffices to apply Proposition 1.2(i). $\square$

We close this paper by pointing out that the "positively ordered" assumption is essential in Theorem 2.1, i.e. if $(S, \leq)$ is not assumed to be positively ordered, then the conditions (1)–(10) in Theorem 2.1 need not be equivalent. For instance, if R is a commutative artinian chain ring that is not a domain, and $(S, \leq)$ is a nontrivial totally ordered commutative group, and $\omega : S \to \mathrm{End}(R)$ is the trivial monoid homomorphism, then [**11**, Theorem 4.6] implies that $R[[S, \omega]]$ is a commutative chain ring that is not reduced, and thus any of the conditions (3), (5), (6), (8), (9) is satisfied but none of the conditions (1), (2), (4), (7), (10) holds.

For a more concrete example, one can consider $R = \boldsymbol{Z}/4\boldsymbol{Z}$, the ring of integers modulo 4, and $S = \boldsymbol{Z}$, the additive group of integers with its natural total order $\leq$.

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