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The link of a finitely determined map germ from $\boldsymbol{R}^2$ to $\boldsymbol{R}^2$

By J. A. Moya-Pérez and J. J. Nuño-Ballesteros

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Abstract. Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ be a finitely determined map germ. The link of f is obtained by taking a small enough representative $f : U \subset \boldsymbol{R}^2 \to \boldsymbol{R}^2$ and the intersection of its image with a small enough sphere S^1_ϵ centered at the origin in $\boldsymbol{R}^2$. We will describe the topology of f in terms of the Gauss word associated to its link.

1. Introduction.

The stable singularities of maps from the plane to the plane were studied for the first time by H. Whitney in his famous paper [**13**]. He showed that for a generic smooth map $f : U \subset \boldsymbol{R}^2 \to \boldsymbol{R}^2$, the germ of f at any point $p \in U$ is either regular, of fold type or of cusp type. This means that f is $\mathscr{A}$-equivalent at p to either $(x, y) \mapsto (x, y)$, $(x, y) \mapsto (x, y^2)$ or $(x, y) \mapsto (x, y^3 + xy)$, respectively. Moreover, if we consider also multigerms, then we have to add one more stable singularity, namely the transverse double fold. In Figure 1 we find a typical image which represents a stable map from the plane to the plane.

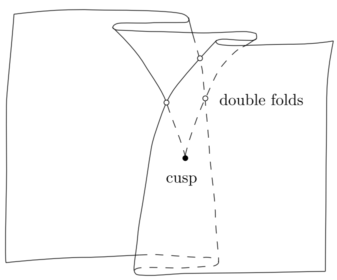

Figure 1.

When $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ is not stable but it is finitely determined, then the origin is an isolated instability by the Mather-Gaffney criterion [**12**]. In particular,

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there is a small enough representative $f : U \subset \boldsymbol{R}^2 \to \boldsymbol{R}^2$ such that f has only simple folds in $U \setminus \{0\}$. The topological structure of f is determined by the so-called link of f, which is obtained by taking the intersection of the image of f with a small enough sphere centered at the origin S^1_ϵ. We use a theorem due to Fukuda [**2**], which ensures that the link of f is a stable map from S^1 to S^1 and that f is topologically equivalent to the cone of its link.

The aim of this paper is to study the topological classification of finitely determined map germs, $f : (\boldsymbol{R}^2, 0) \longrightarrow (\boldsymbol{R}^2, 0)$, by looking at the topological type of the link. In particular, the main motivation of the paper is the following open problem: *Given a stable map $\gamma : S^1 \to S^1$, does there exist a finitely determined map germ $f : (\boldsymbol{R}^2, 0) \longrightarrow (\boldsymbol{R}^2, 0)$ which is topologically equivalent to the cone of γ?*

The main tool will be an adapted version of Gauss words. We show that two finitely determined map germs are topologically equivalent if and only if they have equivalent Gauss words (Corollary 4.4). We also will take special attention to the case that f has corank 1. In this case, f can be written as $f(x, y) = (x, g_x(y))$ and gives a stabilization of $g_0 : (\boldsymbol{R}, 0) \to (\boldsymbol{R}, 0)$. The topology of f is now determined by the two stabilizations g_x^+, with $x > 0$ and g_x^- with $x < 0$. We obtain the topological classification up to multiplicity 5, provided that f is weighted homogeneous (Theorem 5.12).

The topological classification of finitely determined map germs in the complex analytic case $f : (\boldsymbol{C}^2, 0) \to (\boldsymbol{C}^2, 0)$ has been done by Gaffney and Mond in [**5**], [**4**]. Also Nishimura studied in [**9**] the topological $\mathscr{K}$-equivalence of finite map germs $f : (\boldsymbol{R}^n, 0) \longrightarrow (\boldsymbol{R}^n, 0)$ and he obtained that the absolute value of the degree is a complete topological invariant. Finally, we should notice that the techniques used in this paper has been already used by the second author and Marar in [**6**], where they consider Gauss words in the topological classification of finitely determined map germs from $\boldsymbol{R}^2$ to $\boldsymbol{R}^3$.

All map germs considered are real analytic except otherwise stated. We adopt the notation and basic definitions that are usual in singularity theory (e.g., $\mathscr{A}$-equivalence, stability, finite determinacy, etc.), as the reader can find in Wall's survey paper [**12**].

2. Stablility and finite determinacy.

In this section we state the basic definitions and the results that we will need in the remainder of the paper, including the Whitney's characterization of stable maps from the plane to the plane and the Mather-Gaffney finite determinacy criterion.

Two smooth map germs $f, g : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ are $\mathscr{A}$-equivalent if there exist diffeomorphism germs $\phi, \psi : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ such that $g = \psi \circ f \circ \phi^{-1}$. If ϕ, ψ are homeomorphisms instead of diffeomorphisms, then we say that f, g are topologically equivalent.

We say that $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ is k-determined if for any map germ g with the same k-jet, we have that g is $\mathscr{A}$-equivalent to f. We say that f is finitely determined if it is k-determined for some k.

Let $f : U \to \boldsymbol{R}^2$ be a smooth map, where $U \subset \boldsymbol{R}^2$ is an open subset. We denote by $S(f) = \{p \in U : Jf_p = 0\}$ the singular set of f, where Jf is the jacobian determinant. It is a consequence of the Whitney's work [**13**] that f is *stable* if and only if the following two conditions hold:

(1) 0 is a regular value of Jf, so that $S(f)$ is a smooth curve in U.
(2) The restriction $f|_{S(f)} : S(f) \to \boldsymbol{R}^2$ is an immersion with only double transverse points, except at isolated points, where it has simple cusps.

We also find in [**13**] the classification and the normal forms of stable singularities. Assume that f is stable and let $p \in S(f)$.

(1) We say that p is a *fold*, if it is a regular point of $f|_{S(f)}$. The map germ of f at p is $\mathscr{A}$-equivalent to $(x, y) \mapsto (x, y^2)$. The subset of folds is denoted by $S_{1,0}(f)$.
(2) We say that p is a *cusp*, if it is a simple cusp of $f|_{S(f)}$. The map germ of f at p is $\mathscr{A}$-equivalent to $(x, y) \mapsto (x, xy + y^3)$. The subset of cusps is denoted by $S_{1,1}(f)$.
(3) We say that p is a *double fold*, if it is a double transverse point of $f|_{S(f)}$. If $q \in S(f)$ such that $f(p) = f(q)$, with $p \neq q$, then the bi-germ of f at $\{p, q\}$ is $\mathscr{A}$-equivalent to

$$\begin{cases} (x, y) \mapsto (x, y^2), \\ (u, v) \to (v^2, u). \end{cases}$$

Both the stability criterion and the classification of the singular stable points are also true if we consider a holomorphic map $f : U \subset \boldsymbol{C}^2 \to \boldsymbol{C}^2$. Then we consider now a holomorphic map germ $f : (\boldsymbol{C}^2, 0) \to (\boldsymbol{C}^2, 0)$ and we recall the Mather-Gaffney finite determinacy criterion [**12**]. Roughly speaking, f is finitely determined if and only if it has an isolated instability at the origin. To simplify the notation, we state the Mather-Gaffney theorem only in the case of map germs from $(\boldsymbol{C}^2, 0)$ to $(\boldsymbol{C}^2, 0)$, although it is true in a more general form for map germs from $(\boldsymbol{C}^n, 0)$ to $(\boldsymbol{C}^p, 0)$.

THEOREM 2.1. *Let $f : (\boldsymbol{C}^2, 0) \to (\boldsymbol{C}^2, 0)$ be a holomorphic map germ. Then*

f is finitely determined if and only if there is a representative $f : U \subset \boldsymbol{C}^2 \to \boldsymbol{C}^2$ such that

(1) $f^{-1}(0) = \{0\}$,
(2) *the restriction $f|_{U\setminus\{0\}}$ is stable.*

From the condition (2), both the cusps and the double folds are isolated points in $U \setminus \{0\}$. By the curve selection lemma [**7**], we deduce that they are also isolated in U. Thus, we can shrink the neighbourhood U if necessary and get a representative such that $f|_{U\setminus\{0\}}$ is stable with only simple folds.

Coming back to the real case, we consider now an analytic map germ $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$. If we denote by $\hat{f} : (\boldsymbol{C}^2, 0) \to (\boldsymbol{C}^2, 0)$ the complexification of f, it is well known that f is finitely determined if and only if $\hat{f}$ is finitely determined. Then, we have the following immediate consequence of the Mather-Gaffney finite determinacy criterion.

Corollary 2.2. *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ be a finitely determined map germ. Then there is a representative $f : U \subset \boldsymbol{R}^2 \to \boldsymbol{R}^2$ such that*

(1) $f^{-1}(0) = \{0\}$,
(2) *the restriction $f|_{U\setminus\{0\}}$ is stable with only simple folds.*

Given a map germ $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$, we denote $\Delta(f) = f(S(f))$ and we define $X(f)$ as the closure of $f^{-1}(\Delta(f)) \setminus S(f)$. If f is finitely determined, then the three set germs $S(f)$, $\Delta(f)$ and $X(f)$ are plane curves with an isolated singularity at the origin and they will play an important role in the topological classification of f.

3. The link of a germ.

In this section we recall an important result due to Fukuda, which tell us that any finitely determined map germ, $f : (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^p, 0)$, with $n \le p$, has a conic structure over its link. The link is obtained by intersecting the image of a representative of f with a small enough sphere centered in the origin of $\boldsymbol{R}^p$. In order to simplify the notation, we only state the result in our case $n = p = 2$.

We denote by $J^r(\boldsymbol{R}^2, \boldsymbol{R}^2)$ the r-jet space and if $s \ge r$ we have the natural projection $\pi^s_r : J^s(\boldsymbol{R}^2, \boldsymbol{R}^2) \to J^r(\boldsymbol{R}^2, \boldsymbol{R}^2)$.

Theorem 3.1. *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ be a finitely determined map germ. Then, up to $\mathscr{A}$-equivalence, there is a representative $f : U \subset \boldsymbol{R}^2 \to \boldsymbol{R}^2$ and $\epsilon_0 > 0$, such that, for any ϵ with $0 < \epsilon \le \epsilon_0$ we have:*

(1) $\widetilde{S}^1_\epsilon = f^{-1}(S^1_\epsilon)$ *is diffeomorphic to S^1.*

(2) *The map $f|_{\widetilde{S}^1_\epsilon} : \widetilde{S}^1_\epsilon \to S^1_\epsilon$ is stable, in other words, it is a Morse function all of whose critical values are distinct.*
(3) *f is topologically equivalent to the cone of $f|_{\widetilde{S}^1_\epsilon}$.*

Proof. Assume that f is r-determined for some r and let $W = \{j^r f(0)\}$, where $j^r f(0)$ denotes the r-jet of f. By Fukuda theorem [**2**], there is an s, and a closed semi-algebraic subset Σ_W of $(\pi^s_r)^{-1}(W)$ having codimension ≥ 1 such that for any C^∞ mapping $g : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ with $j^s g(0)$ belonging to $(\pi^s_r)^{-1}(W) \setminus \Sigma_W$, there exists $\epsilon_0 > 0$ such that (1), (2) and (3) hold, for any ϵ with $0 < \epsilon \leq \epsilon_0$. Since $(\pi^s_r)^{-1}(W) \setminus \Sigma_W \neq \emptyset$, we can take a map $g : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ with $j^s g(0) \in (\pi^s_r)^{-1}(W) \setminus \Sigma_W$. This implies that $j^r g(0) = j^r f(0)$ and g is $\mathscr{A}$-equivalent to f. $\square$

Definition 3.2. Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ be a finitely determined map germ. We say that the stable map $f|_{\widetilde{S}^1_\epsilon} : \widetilde{S}^1_\epsilon \to S^1_\epsilon$ is the *link* of f, where f is a representative such that (1), (2) and (3) of Theorem 3.1 hold for any ϵ with $0 < \epsilon \leq \epsilon_0$. This link is well defined, up to $\mathscr{A}$-equivalence.

Since any finitely determined map germ is topologically equivalent to the cone of its link, we have the following immediate consequence.

Corollary 3.3. *Two finitely determined map germs $f, g : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ are topologically equivalent if and only if their associated links are topologically equivalent.*

4. The Gauss word.

We recall that a Gauss word is a word which contains each letter exactly twice, one with exponent $+1$ and another one with exponent -1. They were introduced originally by Gauss to describe the topology of closed curves in the plane $\boldsymbol{R}^2$ or in the sphere S^2. Here, we use the same terminology of Gauss word to represent a different type of word, adapted to our particular case of stable maps $S^1 \to S^1$.

Let S^1 be the unit 1-sphere with the anti-clockwise orientation and let us choose a base point $z_0 \in S^1$. Any point $x \in S^1$ can be written in a unique way as $x = z_0 e^{i\alpha}$, with $\alpha \in [0, 2\pi)$. Given $x = z_0 e^{i\alpha}$ and $y = z_0 e^{i\beta}$, with $\alpha, \beta \in [0, 2\pi)$, we denote $x \leq y$ if $\alpha \leq \beta$. If S^1 is considered with the clockwise orientation, then we write $x = z_0 e^{-i\alpha}$, with $\alpha \in [0, 2\pi)$ and the order relation is defined in an analogous way.

Definition 4.1. Let $\gamma : S^1 \to S^1$ be a stable map, that is, such that all its singularities are of Morse type and its critical values are distinct. We fix orientations in each S^1 and we also choose base points $z_0 \in S^1$ in the source and

$a_0 \in S^1$ in the target.

Suppose that γ has r critical values labeled by r letters $a_1, \ldots, a_r \in S^1$ and let us denote their inverse images by $z_1, \ldots, z_k \in S^1$. We assume they are ordered such that $a_0 \leq a_1 < \cdots < a_r$ and $z_0 \leq z_1 < \cdots < z_k$ and following the orientation of each S^1.

We define a map $\sigma : \{1, \ldots, k\} \to \{a_1, \ldots, a_r, \overline{a}_1, \ldots, \overline{a}_r\}$ in the following way: given $i \in \{1, \ldots, k\}$, then $\gamma(z_i) = a_j$ for some $j \in \{1, \ldots, r\}$; we define $\sigma(i) = a_j$, if z_i is a regular point and $\sigma(i) = \overline{a}_j$, if z_i is a singular point. We call *Gauss word* to the sequence $\sigma(1) \ldots \sigma(k)$.

Example 4.2.

(1) Let $\gamma : S^1 \to S^1$ be the link of the fold $f(x, y) = (x, y^2)$. There are only 2 critical values and 2 inverse images, one for each critical value. The Gauss word is $\overline{ab}$ (Figure 2).

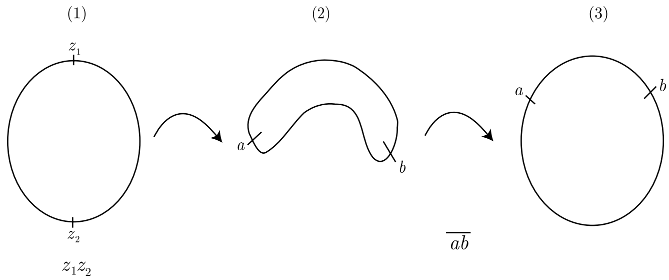


Figure 2.

(2) Let $\gamma : S^1 \to S^1$ be the link of the cusp $f(x, y) = (x, xy + y^3)$. There are 2 critical values and 4 inverse images, two for each critical value. The Gauss word in this case is $a\overline{ba}b$ (Figure 3).

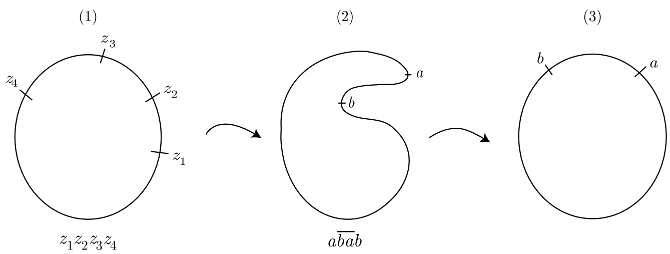


Figure 3.

It is obvious that the Gauss word is not uniquely determined, since it depends on the chosen orientations and base points in each S^1. Different choices will produce the following changes in the Gauss word:

(1) a cyclic permutation in the letters $a_1, \dots, a_r$;
(2) a cyclic permutation in the sequence $\sigma(1)\dots\sigma(k)$;
(3) a reversion in the set of the letters $a_1, \dots, a_r$;
(4) a reversion in the sequence $\sigma(1)\dots\sigma(k)$.

We say that two Gauss words are equivalent if they are related through these four operations. Under this equivalence, the Gauss word is now well defined.

In order to simplify the notation, given a stable map $\gamma : S^1 \to S^1$, we denote by $w(\gamma)$ the associated Gauss word and by $\simeq$ the equivalence relation between Gauss words. We also denote by $\deg(\gamma)$ the topological degree. Then, we can state the main result of this section.

THEOREM 4.3. *Let $\gamma, \delta : S^1 \to S^1$ be two stable maps. Then γ, δ are topologically equivalent if and only if*

$$\begin{cases} w(\gamma) \simeq w(\delta), & \text{if } \gamma, \delta \text{ are singular,} \\ |\deg(\gamma)| = |\deg(\delta)|, & \text{if } \gamma, \delta \text{ are regular.} \end{cases}$$

PROOF. We choose orientations in the source and the target of $\gamma : S^1 \to S^1$ and we also choose base points $z_0 \in S^1$ and $a_0 \in S^1$. We denote by $a_1, \dots, a_r$ the critical values of γ and by $z_1, \dots, z_k$ their inverse images. Assume they are ordered such that $a_0 \le a_1 < \dots < a_r$ and $z_0 \le z_1 < \dots < z_k$ and following the orientation of each S^1. Let $\sigma(1)\dots\sigma(k)$ be the Gauss word of γ.

Suppose that $\delta : S^1 \to S^1$ is topologically equivalent to γ. Then, there are homeomorphisms $\phi, \psi : S^1 \to S^1$ such that $\delta = \psi \circ \gamma \circ \phi^{-1}$. We choose the orientations in the source and the target induced by the orientations of γ and the homeomorphisms ϕ, ψ. We denote $z_i' = \phi(z_i)$ with $i = 0, \dots, k$ and $a_j' = \psi(a_j)$ with $j = 0, \dots, r$. We take z_0' and a_0' as base points in the source and the target respectively. Then, $a_1', \dots, a_r'$ are the critical points of δ and $z_1', \dots, z_k'$ are their inverse images and all of them are well ordered with respect to the chosen base points and orientations. If we label the critical values also with the letters $a_1, \dots, a_r$, then δ has the same Gauss word $\sigma(1)\dots\sigma(k)$.

If γ, δ are topologically equivalent, then we always have the equality $|\deg(\gamma)| = |\deg(\delta)|$. In fact, since any homeomorphism has degree ± 1, we obtain

$$\deg(\delta) = \deg(\psi \circ \gamma \circ \phi^{-1}) = \deg(\psi)\deg(\gamma)\deg(\phi^{-1}) = \pm \deg(\gamma).$$

We show now the converse. We divide the proof into several cases.

Case 1: γ, δ are singular and $w(\gamma) = w(\delta)$. We adopt the following notation:

(1) $a_1, \dots, a_r$ are the critical values of γ and $z_1, \dots, z_k$ are their inverse images,
(2) $a'_1, \dots, a'_r$ are the critical values of δ and $z'_1, \dots, z'_k$ are their inverse images.

We assume that all the points are well ordered with respect to the chosen base point and orientation in each corresponding S^1. The fact that $w(\gamma) = w(\delta)$ implies that $\gamma(z_i) = a_j$ if and only if $\delta(z'_i) = a'_j$.

We define the circle intervals

$$J_j = [a_j, a_{j+1}], \quad I_i = [z_i, z_{i+1}], \quad K_j = [a'_j, a'_{j+1}], \quad H_i = [z'_i, z'_{i+1}],$$

with $j = 1, \dots, r$ and $i = 1, \dots, k$ (we set $a_{r+1} = a_1$, $a'_{r+1} = a'_1$, $z_{k+1} = z_1$ and $z'_{k+1} = z'_1$).

For each $j = 1, \dots, r$ we choose a homeomorphism $\psi_j : J_j \to K_j$ such that $\psi_j(a_j) = a'_j$ and we construct the homeomorphism $\psi : S^1 \to S^1$ by taking $\psi|_{J_j} = \psi_j$.

For each $i = 1, \dots, k$, suppose that $\gamma(z_i) = a_j$ and $\delta(z'_i) = a'_j$. Then the restrictions $\gamma_i = \gamma|_{I_i} : I_i \to J_j$ and $\delta_i = \delta|_{H_i} : H_i \to K_j$ are also homeomorphisms. We define the homeomorphism $\phi_i : I_i \to H_i$ by $\phi_i = \delta_i^{-1} \circ \psi_j \circ \gamma_i$ (see Figure 4). Finally, we construct the homeomorphism $\phi : S^1 \to S^1$ by taking $\phi|_{I_i} = \phi_i$. This homeomorphism verifies that $\delta = \psi \circ \gamma \circ \phi^{-1}$ and hence, γ, δ are topologically equivalent.

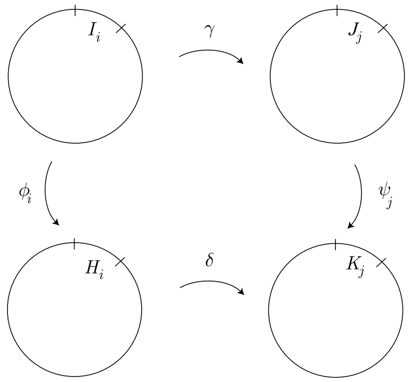


Figure 4.

Case 2: γ, δ are singular and $w(\gamma) \simeq w(\delta)$.

In this case, we can define a new map $\tilde{\delta}$ which is topologically equivalent to δ and such that $w(\gamma) = w(\tilde{\delta})$. Then, the result follows from case 1.

In fact, given $\theta \in [0, 2\pi)$, we denote by $T_\theta : S^1 \to S^1$ the rotation with angle θ, that is, $T_\theta(z) = e^{i\theta} z$. We also denote the inversion $I : S^1 \to S^1$, where $I(z) = z^{-1}$.

(1) If $w(\gamma), w(\delta)$ are related through a cyclic permutation in the letters $a_1, \dots, a_r$, then there is θ such that $w(\gamma) = w(T_\theta \circ \delta)$.
(2) If $w(\gamma), w(\delta)$ are related through a cyclic permutation in the sequence $\sigma(1) \dots \sigma(k)$, then there is θ such that $w(\gamma) = w(\delta \circ T_\theta)$.
(3) If $w(\gamma), w(\delta)$ are related through a reversion in the set of the letters $a_1, \dots, a_r$, then $w(\gamma) = w(I \circ \delta)$.
(4) If $w(\gamma), w(\delta)$ are related through a reversion in the sequence $\sigma(1) \dots \sigma(k)$, then $w(\gamma) = w(\delta \circ I)$.

Case 3: γ, δ are regular and $\deg(\gamma) = \deg(\delta)$.

We choose a point $a_0 \in S^1$ and let us denote $\gamma^{-1}(a_0) = \{z_1, \dots, z_k\}$ and $\delta^{-1}(a_0) = \{z'_1, \dots, z'_k\}$. We assume that the points are well ordered in each corresponding S^1. We consider the intervals $I_i = [z_i, z_{i+1}]$ and $H_i = [z'_i, z'_{i+1}]$ and the restrictions $\gamma_i = \gamma|_{I_i} : I_i \to S^1$ and $\delta_i = \delta|_{H_i} : H_i \to S^1$, with $i = 1, \dots, k$.

For each $i = 1, \dots, k$, we define the homeomorphism $\phi_i : I_i \to H_i$ by taking $\phi_i = \delta_i^{-1} \circ \gamma_i$ on the interior of I_i, $\phi_i(z_i) = z'_i$ and $\phi_i(z_{i+1}) = z'_{i+1}$. We construct the homeomorphism $\phi : S^1 \to S^1$ by taking $\phi|_{I_i} = \phi_i$. This homeomorphism verifies that $\delta = \gamma \circ \phi^{-1}$ and γ, δ are topologically equivalent.

Case 4: γ, δ are regular and $\deg(\gamma) = -\deg(\delta)$.

We have that $\deg(\gamma) = \deg(\delta \circ I)$ and hence, this is a consequence of Case 3. $\square$

Given a finitely determined map germ $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$, we denote by $w(f)$ the Gauss word of its link and by $\deg(f)$ the local topological degree. Then we have the following immediate consequence of Corollary 3.3 and Theorem 4.3

COROLLARY 4.4. *Let $f, g : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ be two finitely determined map germs. Then f, g are topologically equivalent if and only if*

$$\begin{cases} w(f) \simeq w(g), & \text{if } f, g \text{ are singular outside the origin,} \\ |\deg(f)| = |\deg(g)|, & \text{if } f, g \text{ are regular outside the origin,} \end{cases}$$

REMARK 4.5. If f is regular outside the origin and $|deg(f)| = r$, then f is topologically equivalent to the germ $z \to z^r$, with $z = x + iy$.

REMARK 4.6. If $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ is a finitely determined map germ, then we can compute Gauss word of the link of f just by looking at the relative position of the branches of the three curves $S(f)$, $\Delta(f)$ and $X(f)$. This construction is useful sometimes because we do not need to compute explicitly the link of f. We take a small enough representative $f : U \subset \boldsymbol{R}^2 \to \boldsymbol{R}^2$ such that

(1) $f^{-1}(0) = \{0\}$,
(2) the restriction $f|_{U \setminus \{0\}}$ is stable with only simple folds.

The three curves $S(f)$, $\Delta(f)$ and $X(f)$ are plane curves which are smooth outside the origin. By shrinking the neighbourhood U if necessary, we can assume that the three curves are simply connected.

The discriminant $\Delta(f)$ has a tree structure with one vertex at the origin and r adjacent edges labeled by r letters $a_1, \dots, a_r$. Analogously, $S(f) \cup X(f)$ has also a tree structure with one vertex at the origin and k adjacent edges labeled by $Z_1, \dots, Z_k$. We assume that the edges are well ordered $a_1 < \cdots < a_r$ and $Z_1 < \cdots < Z_k$ with respect to some chosen base points and orientations in the source and the target. We define the map $\sigma : \{1, \dots, k\} \to \{a_1, \dots, a_r, \overline{a}_1, \dots, \overline{a}_r\}$ in the following way: given $i \in \{1, \dots, k\}$, then $\gamma(Z_i) = a_j$ for some $j \in \{1, \dots, r\}$; we define $\sigma(i) = a_j$, if $Z_i \subset X(f)$ and $\sigma(i) = \overline{a}_j$, if $Z_i \subset S(f)$. Then, $\sigma(1) \dots \sigma(k)$ is equal to the Gauss word of the link of f.

5. Topological classification of corank 1 map germs.

In this section we study the topological classification of finitely determined map germs of corank 1. The main tool will be the Gauss word, which is a complete topological invariant, as we have seen in Section 4.

First of all, we should remark that if $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ has corank ≤ 1, then after taking smooth changes of coordinates in the source and the target, we can write f in the form $f(x, y) = (x, g_x(y))$, in other words, f can be seen as a 1-parameter unfolding of the germ $g_0 : (\boldsymbol{R}, 0) \to (\boldsymbol{R}, 0)$.

The first consequence is that the topological degree of f can be only $-1, 0$ or 1. In fact, if $g_0(y) = a_n y^n + a_{n+1} y^{n+1} + \dots$ with $a_n \neq 0$, then

$$\deg(f) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd and } a_n > 0, \\ -1, & \text{if } n \text{ is odd and } a_n < 0. \end{cases}$$

Another consequence is that the *multiplicity* of f is equal to n. In general, the multiplicity of $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ is defined as

$$m(f) = \dim_{\boldsymbol{R}} \frac{\boldsymbol{R}\{x, y\}}{\langle f_1, f_2 \rangle},$$

where f_1, f_2 denote the components of f and $\boldsymbol{R}\{x,y\}$ is the local algebra of germs of analytic functions $(\boldsymbol{R}^2,0) \to \boldsymbol{R}$.

Next, we state a result due to J. H. Rieger ([**11**]) which gives a classification of corank 1 map germs according to its 2-jet. We denote by $\Sigma^1 J^2(2,2)$ the space of 2-jets of corank 1 map germs from $(\boldsymbol{R}^2,0)$ to $(\boldsymbol{R}^2,0)$ and $\mathscr{A}^2$ denotes the space of 2-jets of diffeomorphisms in the source and target.

LEMMA 5.1. *There exist three orbits in $\Sigma^1 J^2(2,2)$ under the action of $\mathscr{A}^2$, which are*

$$(x,y^2), \quad (x,xy), \quad (x,0).$$

It is well known that the fold $f(x,y)=(x,y^2)$ is 2-determined. Thus, if a map germ has 2-jet equivalent to (x,y^2), then it is in fact $\mathscr{A}$-equivalent to the fold. Hence, we do not need to consider this case.

5.1. Classification of germs with 2-jet of type (x,xy).

Now, we center our attention in germs with 2-jet $\mathscr{A}$-equivalent to (x,xy). We will prove that a map germ of this type is topologically equivalent to the fold, (x,y^2), or the simple cusp, $(x,xy+y^3)$. First of all, we state an important result, due to J. Damon [**1**].

THEOREM 5.2. *Let $f_0:(\boldsymbol{R}^n,0)\to(\boldsymbol{R}^p,0)$ be a weighted homogeneous finitely determined map germ. Then, any polynomial unfolding of f_0 with positive weighted degrees is topologically trivial.*

In our case, we will show that any finitely determined map germ $f:(\boldsymbol{R}^2,0)\to(\boldsymbol{R}^2,0)$ with 2-jet of type (x,xy) is semi-weighted homogeneous, that is, we can write $f=f_0+h$ where f_0 is weighted homogeneous and finitely determined and h has only terms of higher weighted degree. By Damon's result, this implies that f is topologically equivalent to the initial part f_0 and we can complete the topological classification.

THEOREM 5.3. *Let $f:(\boldsymbol{R}^2,0)\to(\boldsymbol{R}^2,0)$ be a finitely determined map germ with 2-jet of type (x,xy) and with multiplicity n. Then, f is topologically equivalent either to the fold (x,y^2) if n is even, or to the cusp $(x,xy+y^3)$ if n is odd.*

PROOF. We can assume, without loss of generality that f is polynomial and that it is written in the form

$$f(x,y)=(x,xy+a_n y^n+\cdots),$$

with $a_n \neq 0$ and where n is the multiplicity of f. We have that $f_0(x, y) = (x, xy + a_n y^n)$ is weighted homogeneous of weights $(n-1, 1)$ and weighted degrees $(n-1, n)$ and any other monomial appearing in $h = f - f_0$ has weighted degree $> n$.

If we define the unfolding $F(t, x, y) = (t, f_0(x, y) + th(x, y))$, then F is a polynomial unfolding of f_0 with positive weighted degrees in the sense of [**1**]. We will show that f_0 is finitely determined and by Damon's result, F is topologically trivial. In particular, we deduce that f is topologically equivalent to f_0.

Let $\hat{f}_0$ be the complexification of f_0. The jacobian determinant of $\hat{f}_0$ is $x + na_n y^{n-1}$ and thus, the singular curve $S(\hat{f}_0)$ is smooth. The restriction of $\hat{f}_0$ to $S(\hat{f}_0)$ is the map $y \mapsto (-na_n y^{n-1}, -(n-1)a_n y^n)$, which is an injective immersion outside the origin. By the stability criterion, $\hat{f}_0$ is stable outside the origin. Therefore, $\hat{f}_0$ (and hence f_0) is finitely determined by the Mather-Gaffney criterion (see Theorem 2.1).

To finish the proof, it only remains to show that f_0 is topologically equivalent to the fold if the multiplicity n is even, or to the cusp if n is odd. Since the singular curve $S(f_0)$ is smooth, the discriminant $\Delta(f_0)$ has only one branch. Thus, the link of f_0 has only 2 critical values.

If n is even, then $\deg(f_0) = 0$. The only stable map $\gamma : S^1 \to S^1$ with 2 critical values and degree 0 is that with Gauss word $\overline{ab}$. Hence, f_0 is topologically equivalent to the fold (x, y^2). Analogously, if n is odd, then $\deg(f_0) = \pm 1$. Again the only stable map $\gamma : S^1 \to S^1$ with 2 critical values and degree ± 1 is that with Gauss word $a\overline{b}\overline{a}b$. In this case, f_0 is topologically equivalent to the cusp $(x, xy + y^3)$. □

5.2. Classification of germs with 2-jet of type $(x, 0)$.

The germs with 2-jet of type $(x, 0)$ is the biggest class inside the corank 1 map germs and we cannot expect to obtain a complete classification. We will restrict ourselves to the weighted homogenous case and also to map germs with multiplicity ≤ 5, although the techniques can be also used to classify more degenerate singularities.

We assume that f is written in the form $f(x, y) = (x, g_x(y))$, and we look at f as a 1-parameter unfolding of the germ $g_0 : (\boldsymbol{R}, 0) \to (\boldsymbol{R}, 0)$. If f has multiplicity n, then g_0 has type A_{n-1} (i.e., it is $\mathscr{A}$-equivalent to y^n) and the $\mathscr{A}_e$-versal unfolding of the A_{n-1} singularity is

$$G(a_1, \ldots, a_{n-2}, y) = y^n + a_{n-2} y^{n-2} + \cdots + a_1 y.$$

As a consequence, after taking smooth changes of coordinates in the source and the target, we can assume that f is written in the form:

$$f(x,y) = \bigl(x, y^n + a_{n-2}(x)y^{n-2} + \cdots + a_1(x)y\bigr),$$

for some germs $a_i : (\boldsymbol{R},0) \to (\boldsymbol{R},0)$, $i = 1,\ldots,n-2$. The germ $a = (a_1,\ldots,a_{n-2}) : (\boldsymbol{R},0) \to (\boldsymbol{R}^{n-2},0)$ defines a curve in the space of parameters of the versal unfolding. This will allow us to control the functions g_x by looking at the versal deformation.

Another important point in the classification is that if $f(x,y) = (x, g_x(y))$ is finitely determined, then f is a stabilization of g_0. This means that there is a representative $f : U = (-\epsilon,\epsilon)\times V \to \boldsymbol{R}^2$ such that for any x, with $0 < |x| < \epsilon$, $g_x : V \to \boldsymbol{R}$ is stable (that is, g_x is a Morse function with distinct critical values).

Proposition 5.4. *Let $f : (\boldsymbol{R}^2,0) \to (\boldsymbol{R}^2,0)$ be a finitely determined map germ given by $f(x,y) = (x, g_x(y))$. Then, f is a stabilization of g_0.*

Proof. By Corollary 2.2, if f is finitely determined, we can choose a representative $f : U \subset \boldsymbol{R}^2 \to \boldsymbol{R}^2$ such that

(1) $f^{-1}(0) = \{0\}$,
(2) the restriction $f|_{U\setminus\{0\}}$ is stable with only simple folds.

We take V a neighbourhood of 0 in $\boldsymbol{R}$, and then take ϵ sufficiently small, so that we can assume that $U = (-\epsilon,\epsilon)\times V$. Let us take x, with $0 < |x| < \epsilon$. If g_x has a degenerate singularity at $y \in V$, then $g_x'(y) = g_x''(y) = 0$ and f should have a cusp at $(x,y) \in U \setminus \{0\}$. Analogously, if g_x is singular at two distinct point $y_1, y_2 \in V$ with $g_x(y_1) = g_x(y_2)$, then $g_x'(y_1) = g_x'(y_2) = 0$ and f should have a double fold at $(x,y_1),(x,y_2) \in U \setminus \{0\}$. □

Let $f : (\boldsymbol{R}^2,0) \to (\boldsymbol{R}^2,0)$ be a finitely determined map germ given by $f(x,y) = (x, g_x(y))$. We take a representative $f : U = (-\epsilon,\epsilon)\times V \to \boldsymbol{R}^2$ such that $g_x : V \to \boldsymbol{R}$ is stable for any x, with $0 < |x| < \epsilon$. Since g_0 has isolated singularity, by shrinking U if necessary, we can also assume that $g_0^{-1}(0) = \{0\}$ in V and that g_0 is regular in $V \setminus \{0\}$.

Because of the stability, all the functions $g_x : V \to \boldsymbol{R}$ are $\mathscr{A}$-equivalent if $-\epsilon < x < 0$ and we will denote by g_x^- one of these functions. Analogously, all the functions $g_x : V \to \boldsymbol{R}$ are $\mathscr{A}$-equivalent if $0 < x < \epsilon$ and we will denote by g_x^+ one of these functions. We associate a partial Gauss word to each of the functions g_x^-, g_x^+ in a similar way to Definition 4.1.

Definition 5.5. Let $g : V \to \boldsymbol{R}$ be one of the functions g_x^- or g_x^+. Let $a_1,\ldots,a_r \in \boldsymbol{R}$ be the critical values of g and let $y_1,\ldots,y_k \in V$ their inverse images. Assume all of them are ordered such that $a_1 < \cdots < a_r$ and $y_1 < \cdots < y_k$. We define the *partial Gauss word* of g as $\sigma(1)\ldots\sigma(k)$, where $\sigma(i) = a_j$ if $g(y_i) = a_j$

and y_i is regular or $\sigma(i) = \overline{a}_j$ if $g(y_i) = a_j$ and y_i is singular.

Definition 5.6. Assume that g_x^+ and g_x^- have r and s critical values respectively and let $\sigma^+(1)\dots\sigma^+(k)$ and $\sigma^-(1)\dots\sigma^-(\ell)$ their respective partial Gauss words. We denote by ϕ the map $\phi(a_j) = a_{r+s-j+1}$ and $\phi(\overline{a}_j) = \overline{a}_{r+s-j+1}$, for $j = 1,\dots,s$. Then we define the *union* of the partial Gauss words as the Gauss word with $r+s$ critical values defined by

$$\sigma^+(1)\cdots\sigma^+(k)\phi(\sigma^-(\ell))\cdots\phi(\sigma^-(1)).$$

Theorem 5.7. *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ be a finitely determined map germ given by $f(x, y) = (x, g_x(y))$. Then the Gauss word of f is equivalent to the union of the partial Gauss words of g_x^+ and g_x^-.*

Proof. We will compute the Gauss word of f by following the construction of Remark 4.6. Take a small enough representative $f : U = (-\epsilon, \epsilon) \times V \to \boldsymbol{R}^2$ such that

(1) $f^{-1}(0) = \{0\}$,
(2) f is stable with only simple folds in $U \setminus \{0\}$,
(3) the three curves $S(f)$, $\Delta(f)$ and $X(f)$ are simply connected.

The first two conditions imply that if $0 < |x| < \epsilon$, then $g_x : V \to \boldsymbol{R}$ is stable and that $g_0^{-1}(0) = \{0\}$.

We show that the three curves $S(f)$, $\Delta(f)$ and $X(f)$ are transverse to the vertical lines $\{x\} \times \boldsymbol{R}$ if $0 < |x| < \epsilon$. In fact, $S(f)$ is defined by equation $g_x'(y) = 0$ and the intersection with $\{x\} \times \boldsymbol{R}$ is not transverse if in addition $g_x''(y) = 0$, but this should imply that $(x, y) \in U \setminus \{0\}$ is a cusp of f.

Now, if $\alpha(t) = (x(t), y(t))$ is a local parametrization of $S(f)$ near a point $(x, y) \in S(f)$ with $x \neq 0$, then $f(\alpha(t)) = (x(t), g_{x(t)}(y(t)))$ gives a local parametrization of $\Delta(f)$ near the point $f(x, y) \in \Delta(f)$. Since $x'(t) \neq 0$, $\Delta(f)$ is also transverse to $\{x\} \times \boldsymbol{R}$ at $f(x, y)$. A similar argument shows that the same is true for $X(f)$.

The transversality of $S(f)$ with the vertical lines, together with the fact that $S(f)$ is simply connected, imply that $S(f) \cap (\{0\} \times \boldsymbol{R}) = \{0\}$. In particular, g_0 is regular in $V \setminus \{0\}$ and we have a good representative in order to define the partial Gauss words.

We take points x_1, x_2 with $-\epsilon < x_2 < 0 < x_1 < \epsilon$. We assume that:

(1) $\Delta(f) \cap (\{x_1\} \times \boldsymbol{R}) = \{a_1, \dots, a_r\}$,
(2) $f^{-1}(\Delta(f)) \cap (\{x_1\} \times \boldsymbol{R}) = \{y_1, \dots, y_k\}$,
(3) $\Delta(f) \cap (\{x_2\} \times \boldsymbol{R}) = \{a_1', \dots, a_s'\}$,

(4) $f^{-1}(\Delta(f)) \cap (\{x_2\} \times \boldsymbol{R}) = \{y'_1, \ldots, y'_\ell\}$.

We choose the indices such that all the points are well ordered and we denote the corresponding partial Gauss words by $\sigma^+(1) \ldots \sigma^+(k)$ and $\sigma^-(1) \ldots \sigma^-(\ell)$ respectively.

By transversality, the curve $\Delta(f)$ has $r+s$ edges $A_1, \ldots, A_{r+s}$ which we can label so that $a_1 \in A_1, \ldots, a_r \in A_r, a'_s \in A_{r+1}, \ldots, a'_1 \in A_{r+s}$. Moreover, the edges are well ordered following the standard orientation of $\boldsymbol{R}^2$. Analogously, $f^{-1}(\Delta(f))$ has $k+\ell$ edges $Z_1, \ldots, Z_{k+\ell}$ which can be labeled such that $y_1 \in Z_1, \ldots, y_k \in Z_k, y'_\ell \in Z_{k+1}, \ldots, y'_1 \in Z_{k+\ell}$ and they are well ordered. We deduce from Definition 5.6 that the associated Gauss word is exactly the union of the two partial Gauss words (see Figure 5). $\square$

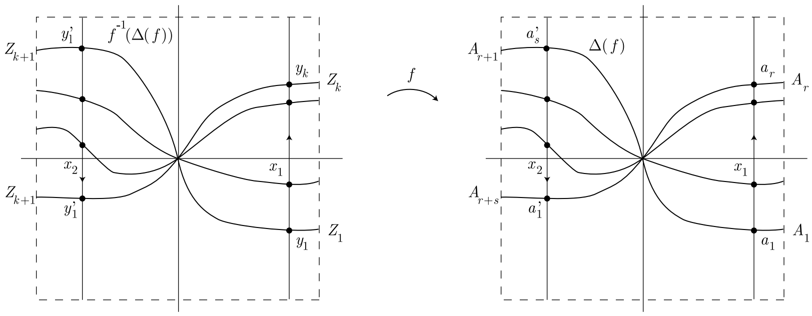


Figure 5.

Corollary 5.8. *Let $f, \tilde{f} : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ be two finitely determined map germs given by $f(x,y) = (x, g_x(y))$ and $\tilde{f}(x,y) = (x, \tilde{g}_x(y))$. If the two partial Gauss words of $f, \tilde{f}$ are equal, then $f, \tilde{f}$ are topologically equivalent.*

Remark 5.9. The condition that the partial Gauss words are equal is necessary condition, because in general, the union of equivalent partial Gauss words does not give equivalent Gauss words.

Remark 5.10. The Gauss word of a stable map $\gamma : S^1 \to S^1$ has always the following property: at two consecutive positions of the Gauss word, we must have two consecutive letters (either overlined or not). In the remaining of the paper, we use this property to simplify the notation of the Gauss words in the following way: each time that we find a group of the form $a_i\overline{a_j}a_i$ in the Gauss word, then we substitute it by just $\overline{a_j}$. For instance, the Gauss word $\overline{a}bc\overline{d}c\overline{b}cb$ can be simplified with this operation (see Figure 6):

$$\bar{a}bc\bar{d}c\overline{bc}b \to \bar{a}b\overline{db}cb \to \overline{adbc}.$$

It is obvious that given a simplified Gauss word, we can recover the complete Gauss word just by adding the missing consecutive letters.

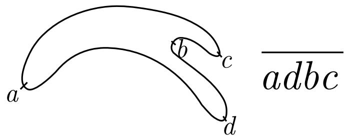


Figure 6.

EXAMPLE 5.11. Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ be a finitely determined map with 2-jet of type $(x, 0)$ and multiplicity 3. We assume that f is written in its prenormal form

$$f(x, y) = (x, y^3 + u(x)y),$$

where $u(x) = u_k x^k + \cdots$ and $u_k \neq 0$. If $x \neq 0$, we have two possibilities for the stabilization $g_x(y) = y^3 + u(x)y$. If $u(x) > 0$, then g_x is regular and the partial Gauss word is $\emptyset$. Otherwise, if $u(x) < 0$, then g_x has 2 critical values and the partial Gauss word is $a\overline{ba}b$ (see Figure 7).

Figure 7.

By taking the union of the partial Gauss words we get 3 possibilities for f:

(1) If k is odd, then g_x^+ and g_x^- are the 2 stabilizations of y^3. Hence, the link of f has 2 critical values and the Gauss word is $a\overline{ba}b$.
(2) If k is even and $u_k > 0$, then both g_x^+ and g_x^- are regular. Hence, the link of f is regular and the Gauss word is $\emptyset$.
(3) If k is even and $u_k < 0$, then both g_x^+ and g_x^- are singular. Hence, the link of f has 4 critical values and the Gauss word is $a\overline{ba}bc\overline{dc}d$.

The pictures and the normal forms for these three topological classes can be found in the first three entries of degree 1 in Table 1.

Analogously, if f has multiplicity 4, a similar analysis can be done. We have three stabilizations of y^4 with partial Gauss words: (a) $\overline{a}$, (b) $c\overline{acb}c$ and (c) $c\overline{bcac}$ (see Figure 8).

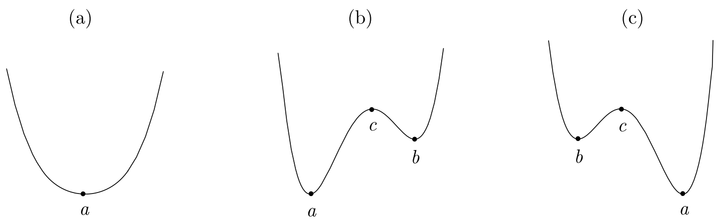


Figure 8.

The possible Gauss words for f are obtained by taking all the possible combinations between these 3 stabilizations. We see that (a)+(b) is equivalent to (a)+(c) and that (b)+(b) is also equivalent to (c)+(c). Then, there are only 4 non-equivalent possibilities, namely (a)+(a), (a)+(b), (b)+(b) and (b)+(c). The corresponding Gauss words are respectively:

(1) $\overline{ab}$,
(2) $\overline{acbd}$,
(3) $c\overline{bca}cd\overline{edf}d$,
(4) $c\overline{bca}cd\overline{fde}d$.

The pictures and the normal forms for these four topological classes can be found in the first four entries in Table 1.

A similar analysis can be done for higher multiplicity, just by looking at the stabilizations of the germ y^n and taking the possible unions which are not equivalent. For y^5, there are seven stabilizations of y^5 with partial Gauss words: (a) $\emptyset$, (b) $a\overline{ba}b$, (c) $ab\overline{dbca}cd$, (d) $a\overline{ba}bc\overline{dc}d$, (e) $ab\overline{cadb}cd$, (f) $ab\overline{dacb}cd$ and (g) $ab\overline{cbda}cd$ (see Figures 7 and 9).

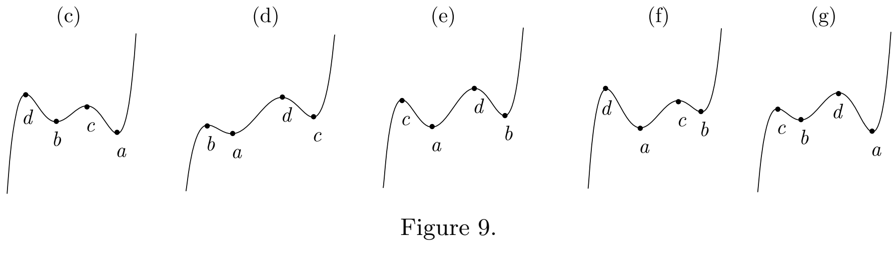


Figure 9.

We finish this section with the classification of weighted homogeneous map germs of multiplicity ≤ 5.

Theorem 5.12. *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ be a weighted homogeneous finitely determined map germ given by $f(x, y) = (x, g_x(y))$, with 2-jet of type $(x, 0)$ and multiplicity ≤ 5. Then, f is topologically equivalent to one of the germs of Tables 1 and 2, depending on the topological configuration of its associated link and its topological degree.*

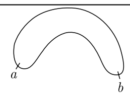
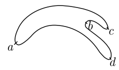
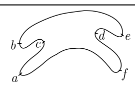
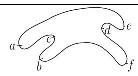
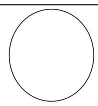
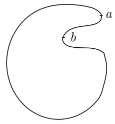
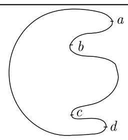
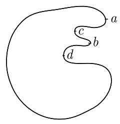
Degree	Germ	Associated link
0	$(x, y^4 + x^2y^2)$	 $\overline{ab}$
	$(x, y^4 - xy^2 - x^2y)$	 $\overline{adbc}$
	$\left(x, y^4 - x^4y^2 + \frac{1}{4}x^6y\right)$	 $c\overline{bca}cdf\overline{ded}$
	$\left(x, y^4 - x^2y^2 - \frac{1}{4}x^3y\right)$	 $c\overline{acb}cdf\overline{ded}$
1	$(x, y^3 + x^2y)$	 $\emptyset$
	$(x, y^3 + x^3y)$	 $a\overline{ba}b$
	$(x, y^3 - x^2y)$	 $a\overline{ba}bc\overline{dc}d$
	$\left(x, y^5 + 2xy^3 + \frac{1}{2}x^2y\right)$	 $ab\overline{dbca}cd$

Table 1.

Proof. The cases of multiplicity 3 and 4 have been analyzed in Example 5.11. Hence, we assume that the multiplicity is 5. We distinguish 2 cases:

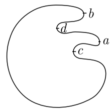
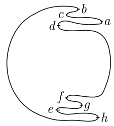
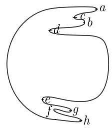
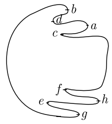
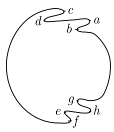
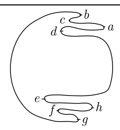
Degree	Germ	Associated link
1	$(x, y^5 + 3xy^3 + 2x^2y)$	 $ab\overline{cadb}cd$
	$\left(x, y^5 - 3x^2y^3 + \frac{5}{4}x^3y^2 + x^4y\right)$	 $ab\overline{dacb}cdef\overline{hegf}gh$
	$(x, y^5 - 5x^2y^3 + x^4y)$	 $ab\overline{dbca}cdef\overline{hfge}gh$
	$\left(x, y^5 - \frac{3}{2}x^2y^3 + \frac{1}{2}x^4y\right)$	 $ab\overline{cadb}cde\overline{fgeh}fgh$
	$(x, y^5 - 3x^2y^3 + 3x^4y)$	 $a\overline{ba}bc\overline{dc}de\overline{fe}fg\overline{hg}h$
	$\left(x, y^5 - \frac{7}{2}x^4y^3 + 2x^6y^2 + x^8y\right)$	 $ab\overline{dacb}cdef\overline{gfhe}gh$

Table 2.

Case A: f is homogeneous. Since $g_x(y)$ is homogeneous of degree 5, we have the symmetry $g_{-x}(-y) = -g_x(y)$. Let us denote by $\sigma(1)\dots\sigma(k)$ the partial Gauss word of g_x^+ with r letters. Then the partial Gauss word of g_x^- is $\tau(\sigma(k))\dots\tau(\sigma(1))$, where $\tau(a_i) = a_{r-i+1}$. By Theorem 5.7, the Gauss word of f is $\sigma(1)\dots\sigma(k)\phi(\sigma(1))\dots\phi(\sigma(k))$, where $\phi(a_i) = a_{r+i}$.

Consider the seven stabilizations of y^5 (a),...,(g) given in Figures 7 and 9. The stabilization (f) is symmetric to (g), but each one of the remaining cases is its own symmetric. Thus, we obtain six possible combinations, namely, (a)+(a), (b)+(b), (c)+(c), (d)+(d), (e)+(e) and (f)+(g). The corresponding Gauss words are respectively:

(1) $\emptyset$,

(2) $a\overline{ba}bc\overline{dc}d$,

(3) $ab\overline{dbca}cdef\overline{hfge}gh$,
(4) $a\overline{ba}bc\overline{dc}de\overline{fe}fg\overline{hg}h$,
(5) $ab\overline{cadb}cdef\overline{gehf}gh$,
(6) $ab\overline{dacb}cdef\overline{hegf}gh$.

Case B: f is weighted homogeneous. We suppose now that $g_x(y)$ is a weighted homogeneous polynomial of weights w_1 and w_2, with $w_1 \neq w_2$. We can write this polynomial in the form

$$g_x(y) = x^r y^s \big(a_0(x^{w_2})^d + a_1(x^{w_2})^{d-1}y^{w_1} + \cdots + a_d(y^{w_1})^d\big).$$

Since f is finitely determined, we must have necessarily $r = 0$ and $s \leq 2$. We have two possible cases:

(1) $f(x,y) = (x, y(a_0x^{dw_2} + a_1x^{(d-1)w_2}y^{w_1} + \cdots + a_dy^{dw_1}))$,
(2) $f(x,y) = (x, y^2(a_0x^{dw_2} + a_1x^{(d-1)w_2}y^{w_1} + \cdots + a_{d-1}x^{w_2}y^{(d-1)w_1} + a_dy^{dw_1}))$,

where either $dw_1 + 1 = 5$ or $dw_1 + 2 = 5$. In addition, we can also take into account the following restrictions:

(a) As a consequence of the prenormal form of map germs of corank 1, if $w_1 = 1$, we can take $a_{d-1} = 0$.
(b) The functions $\phi_1(x) = x^{w_2}$ if w_2 is odd and $\phi_2(x^2) = x^{w_2}$ if w_2 is even are homeomorphisms. Thus, we only consider $w_2 = 1, 2$.

Depending on the possible values of d and w_1 and with the above restrictions, we find only 4 possibilities for f:

$$f(x,y) = \begin{cases} (x, y^5 + ax^4y^3 + bx^6y^2 + cx^8y), \\ (x, y^5 + axy^3 + bx^2y), \\ (x, y^5 + ax^2y^2), \\ (x, y^5 + axy^2). \end{cases}$$

In the first and third cases, we have the symmetry $g_{-x}(y) = g_x(y)$. Thus, we will have one of the following combinations (a)+(a), (b)+(b), (c)+(c), (d)+(d), (e)+(e), (f)+(f) or (g)+(g). Since (f)+(f) is equivalent to (g)+(g), we only need to add one more topological type to our list, namely that with Gauss word $ab\overline{dacb}cdef\overline{gfhe}gh$.

In the second case, f is finitely determined if and only if $(20b - 9a^2)(4b - a^2)(10b + 81^2)b \neq 0$. This gives a partition of the (a, b) plane into 8 connected components. By taking a point in each connected component we find all the possible topological types of f. We find two more new types corresponding to the combinations (a)+(c) and (a)+(e), with Gauss word $a\overline{b}\overline{d}\overline{b}\overline{c}acd$ and $abc\overline{a}\overline{d}\overline{b}cd$ respectively. Finally, in the fourth case, f is finitely determined if and only if $a \neq 0$ and we get two connected components, but do not get any new topological type in this case.

We remark that all the types that appear in Tables 1 and 2 can be realized by considering the normal forms listed there. We have used the software `singR2R2` developed by A. Montesinos [**8**] in order to check that each normal form gives the desired topological type. □

6. The number of cusps and double folds of germs of corank 1.

In this last section, we give some results related to the number of cusp and double fold points that appear near the origin in a stable perturbation of a finitely determined map germ of corank 1.

Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ be a finitely determined map germ. We denote by $\hat{f} : (\boldsymbol{C}^2, 0) \to (\boldsymbol{C}^2, 0)$ the complexification of f and we consider $\hat{F} = (t, \hat{f}_t)$ a stabilization of $\hat{f}$ (i.e., $\hat{f}_0 = \hat{f}$ and if $t \neq 0$, then $\hat{f}_t$ is stable in a small enough neighbourhood U). Then, we denote

$$c(f) = \text{ the number of cusps of } \hat{f}_t \text{ in } U,$$

$$d(f) = \text{ the number of double folds of } \hat{f}_t \text{ in } U.$$

These two numbers $c(f), d(f)$ were introduced for the first time by Rieger [**11**] for the case of corank 1 and independently $c(f)$ was studied by Fukuda-Ishikawa in [**3**] for general case. Both numbers $c(f), d(f)$ are invariants of f which do not depend on the stabilization $\hat{F}$. Moreover, they can be computed algebraically in terms of the dimensions of some local algebras associated to f (see [**4**], [**5**]). In fact, if f has corank 1 and f has the form $f(x, y) = (x, g_x(y))$, then we have

$$c(f) = \dim_{\boldsymbol{R}} \frac{\boldsymbol{R}\{x, y\}}{\langle g_x', g_x'' \rangle},$$

where g_x', g_x'' denote the first and second partial derivatives of g_x with respect to y and $\boldsymbol{R}\{x, y\}$ is the local algebra of germs of analytic functions $(\boldsymbol{R}^2, 0) \to \boldsymbol{R}$.

If F is a real stabilization of f, then the numbers $c(f_t)$ and $d(f_t)$ of cusps and double folds of f_t respectively, depend on the stabilization F. However, $c(f_t)$ is congruent modulo 2 to the invariant $c(f)$ (see [**3**]).

Theorem 6.1. *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ be a finitely determined map germ of corank 1 given by $f(x, y) = (x, g_x(y))$. Then,*

$$\deg(g'_x, g''_x) = \frac{n(g_x^-) - n(g_x^+)}{2} \equiv c(f) \mod 2,$$

where $n(g_x^+)$ and $n(g_x^-)$ denote the number of critical values of g_x^+ and g_x^- respectively.

Proof. The equality is a direct consequence of Theorem C (2) in [**10**] applied to $g'_x : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}, 0)$. In fact, $n(g_x^+)$ is the number of branches of $(g'_x)^{-1}(0)$ which lie in the half region $x > 0$ and $n(g_x^-)$ is the number of branches of $(g'_x)^{-1}(0)$ which lie in the half region $x < 0$. Moreover, since f is finitely determined we have that

$$c(f) = \dim_{\boldsymbol{R}} \frac{\boldsymbol{R}\{x, y\}}{\langle g'_x, g''_x \rangle} < \infty.$$

Then, the result of Nishimura, Fukuda and Aoki implies that

$$n(g_x^-) - n(g_x^+) = 2 \deg(g'_x, g''_x).$$

Finally, the congruence follows from the relation between the multiplicity and the local degree of $(g'_x, g''_x) : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$. □

Remark 6.2. Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ be a finitely determined map germ of corank 1 and multiplicity $m(f)$. Then:

(1) $c(f) = 0$ if and only if $m(f) \leq 2$,
(2) $d(f) = 0$ if and only if $m(f) \leq 3$.

We can take a prenormal form

$$f(x, y) = (x, y^k + a_1(x)y^{k-2} + \cdots + a_{k-2}(x)y),$$

for some functions $a_i : (\boldsymbol{R}, 0) \to (\boldsymbol{R}, 0)$, where $k = m(f)$. We use a result of Gaffney and Mond [**4**]. We set

$$B_C = \{u \in \boldsymbol{C}^{k-1} : g_u \text{ has a degenerate critical point}\},$$

$$B_D = \{u \in \boldsymbol{C}^{k-1} : g_u \text{ has two critical points having the same critical value}\},$$

with $g_u(x) = y^k + u_1 y^{k-2} + \cdots + u_{k-2} y$. We denote by b_C and b_D the reduced equations of B_C and B_D respectively. Then,

$$c(f) = \nu(b_C \circ a), \quad d(f) = \nu(b_D \circ a),$$

where $a = (a_1, \dots, a_{k-2}) : (\boldsymbol{C}, 0) \to (\boldsymbol{C}^{k-2}, 0)$ and $\nu(h)$ denotes the order of a function h.

As a consequence, $c(f) \geq 1$ if and only if the polynomial b_C is not constant, in other words, if the set B_C is not empty. But this means that $c(f) \geq 1$ if and only if there is a map germ f_0 such that $m(f_0) = k$ and $c(f_0) \geq 1$. Analogously, $d(f) \geq 1$ if and only if there is a map germ f_0 such that $m(f_0) = k$ and $d(f_0) \geq 1$.

If we consider $f_0(x, y) = (x, y^k + xy)$, it is easy to compute

$$c(f_0) = k - 2, \quad d(f_0) = \frac{(k-2)(k-3)}{2},$$

which implies the desired result.

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J. A. Moya-Pérez
Departament de Geometria i Topologia
Universitat de València
Campus de Burjassot
46100 Burjassot, Spain
E-mail: Juan.Moya@uv.es

J. J. Nuño-Ballesteros
Departament de Geometria i Topologia
Universitat de València
Campus de Burjassot
46100 Burjassot, Spain
E-mail: Juan.Nuno@uv.es