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Scaling limits for weakly pinned random walks with two large deviation minimizers

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Abstract. The scaling limits for d-dimensional random walks perturbed by an attractive force toward the origin are studied under the critical situation that the rate functional of the corresponding large deviation principle admits two minimizers. Our results extend those obtained by [**2**] from the mean-zero Gaussian to non-Gaussian setting under the absence of the wall.

1. Introduction and main result.

It is a general principle in the study of various kinds of scaling limits that the limit points, at least at the level of law of large numbers, can be characterized by variational problems which minimize the rate functionals of the corresponding large deviation principles. However, if the rate functional admits several minimizers, the large deviation principle is not sufficient to give an appropriate answer. This paper discusses such problem, especially for random walks on $\boldsymbol{R}^d$ perturbed by an attractive force toward the origin $0 \in \boldsymbol{R}^d$, motivated by certain models for interfaces or directed polymers.

The mean-zero Gaussian random walks, perturbed by an attractive force toward a subspace M of $\boldsymbol{R}^d$, are studied in [**2**] under the presence or absence of a wall located at the boundary of the upper half space of $\boldsymbol{R}^d$. The present paper investigates the situation that $M = \{0\}$ and the wall is absent. We extend the class of transition probability densities $p(x)$ of the random walks from mean-zero Gaussian (i.e. $p(x) = e^{-|x|^2/2}/(2\pi)^{d/2}$) to general functions satisfying Assumption 1.1 stated below.

1.1. Weakly pinned random walks.

This subsection introduces temporally inhomogeneous Markov chains called the weakly pinned random walks. The macroscopic time parameter of the Markov chains, observed after scaling, runs over the interval $D = [0, 1]$. The range of (microscopic) time for the Markov chains is $D_N = ND \cap \boldsymbol{Z} \equiv \{0, 1, 2, \ldots, N\}$. The

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state space of the Markov chains is $\boldsymbol{R}^d$. The starting point of the (macroscopically scaled) chains at $t=0$ is always specified, while we will or will not specify the arriving point at $t=1$. More precisely, for given $a, b \in \boldsymbol{R}^d$, the starting point of the Markov chains $\phi = (\phi_i)_{i \in D_N}$ is always $aN \in \boldsymbol{R}^d$ (i.e. $\phi_0 = aN$), while, for the arriving point at $i = N$, we consider two cases: under conditioning ϕ as $\phi_N = bN$ (we call Dirichlet case) or without giving any condition on ϕ_N (we call free case). The distributions of the Markov chains ϕ on $(\boldsymbol{R}^d)^{N+1}$ with the strength $\varepsilon \geq 0$ of the pinning force toward the origin 0, imposing the Dirichlet or free conditions at N, are described by the following two probability measures $\mu_N^{D,\varepsilon}$ and $\mu_N^{F,\varepsilon}$ on $(\boldsymbol{R}^d)^{N+1}$, respectively:

$$\mu_N^{D,\varepsilon}(d\phi) = \frac{\mathfrak{p}_N(\phi)}{Z_N^{a,b,\varepsilon}} \delta_{aN}(d\phi_0) \prod_{i=1}^{N-1} \big(\varepsilon\, \delta_0(d\phi_i) + d\phi_i\big) \delta_{bN}(d\phi_N), \tag{1.1}$$

$$\mu_N^{F,\varepsilon}(d\phi) = \frac{\mathfrak{p}_N(\phi)}{Z_N^{a,F,\varepsilon}} \delta_{aN}(d\phi_0) \prod_{i=1}^{N} \big(\varepsilon\, \delta_0(d\phi_i) + d\phi_i\big), \tag{1.2}$$

where

$$\mathfrak{p}_N(\phi) = \prod_{i=1}^{N} p(\phi_i - \phi_{i-1}),$$

with a measurable function $p : \boldsymbol{R}^d \to [0,\infty)$ satisfying $\int_{\boldsymbol{R}^d} p(x)dx = 1$, $d\phi_i$ denotes the Lebesgue measure on $\boldsymbol{R}^d$, and $Z_N^{a,b,\varepsilon}$ and $Z_N^{a,F,\varepsilon}$ are the normalizing constants, respectively. Note that, if $\varepsilon = 0$, ϕ under $\mu_N^{F,0}$ is the random walk with the transition probability $p(y-x)dy, x, y \in \boldsymbol{R}^d$ and its conditioning as $\phi_N = bN$ under $\mu_N^{D,0}$. We always assume the following conditions on the transition probability density p:

ASSUMPTION 1.1.
(1) The function p satisfies $\sup_{x \in \boldsymbol{R}^d} e^{\lambda \cdot x} p(x) < \infty$ for all $\lambda \in \boldsymbol{R}^d$, where $\lambda \cdot x = \sum_{\alpha=1}^d \lambda^\alpha x^\alpha$ denotes the inner product of $\lambda = (\lambda^\alpha)_{\alpha=1}^d$ and $x = (x^\alpha)_{\alpha=1}^d$ in $\boldsymbol{R}^d$. In particular, the Cramér's condition is satisfied:

$$\Lambda(\lambda) \equiv \log \int_{\boldsymbol{R}^d} e^{\lambda \cdot x} p(x) dx < \infty. \tag{1.3}$$

(2) The Legendre transform of Λ defined by

$$\Lambda^*(v) = \sup_{\lambda \in \boldsymbol{R}^d} \{\lambda \cdot v - \Lambda(\lambda)\}, \quad v \in \boldsymbol{R}^d, \tag{1.4}$$

is finite for all $v \in \boldsymbol{R}^d$, and satisfies $\Lambda^* \in C^3(\boldsymbol{R}^d)$.

When $d = 1$, the Markov chain ϕ may be interpreted as the heights of interfaces located in a plane, so that the system is called $(1+1)$-dimensional interface model with δ-pinning at 0, see [**8**]. For general $d \geq 1$, ϕ can be interpreted as the $(1+d)$-dimensional directed polymers, see [**11**].

We will assume that $a, b \neq 0$, since the case $a = 0$ or $b = 0$ is similar or even simpler.

1.2. Scaling limits and large deviation rate functionals.

Let $h^N = \{h^N(t); t \in D\}$ be the macroscopic path of the Markov chain determined from the microscopic one ϕ under a proper scaling, namely, it is defined through a polygonal approximation of $\big(h^N(i/N) = \phi_i/N\big)_{i \in D_N}$ so that

$$h^N(t) = \frac{[Nt] - Nt + 1}{N} \phi_{[Nt]} + \frac{Nt - [Nt]}{N} \phi_{[Nt]+1}, \quad t \in D.$$

Then, the sample path large deviation principle holds for h^N under $\mu_N^{D,\varepsilon}$ and $\mu_N^{F,\varepsilon}$, respectively, on the space $\mathscr{C} = C(D, \boldsymbol{R}^d)$ equipped with the uniform topology as $N \to \infty$, see Theorem 5.1 below (or [**4**], [**13**] for $\mu_N^{F,0}$ when $\varepsilon = 0$). The speeds are N and the unnormalized rate functionals are given by Σ^D and Σ^F, respectively, both of which are of the form:

$$\Sigma(h) = \int_D \Lambda^*(\dot{h}(t))dt - \xi^\varepsilon |\{t \in D; h(t) = 0\}|, \tag{1.5}$$

for $h \in \mathscr{AC}_{a,b} = \{h \in \mathscr{AC}; h(0) = a, h(1) = b\}$ in the Dirichlet case respectively $h \in \mathscr{AC}_{a,F} = \{h \in \mathscr{AC}; h(0) = a\}$ in the free case with certain non-negative constants $\xi^\varepsilon = \xi^{D,\varepsilon}$ or $\xi^{F,\varepsilon}$, where Λ^* is the Legendre transform of Λ defined by (1.4), $|\cdot|$ stands for the Lebesgue measure on D and $\mathscr{AC} = \mathscr{AC}(D, \boldsymbol{R}^d)$ is the family of all absolutely continuous functions $h \in \mathscr{C}$. We define $\Sigma(h) = +\infty$ for h's outside of these spaces. The Cramér's condition (1.3) implies that $\Lambda \in C^\infty(\boldsymbol{R}^d)$ (even real analytic) and strictly convex, and Λ^* is also strictly convex on $\boldsymbol{R}^d$ by Assumption 1.1-(2), see Theorem VII.5.5 of [**6**].

Non-negative constants $\xi^{D,\varepsilon}$ and $\xi^{F,\varepsilon}$ are determined by the thermodynamic limits:

$$\xi^{D,\varepsilon} = \lim_{N \to \infty} \frac{1}{N} \log \frac{Z_N^{0,0,\varepsilon}}{Z_N^{0,0}}, \tag{1.6}$$

$$\xi^{F,\varepsilon} = \lim_{N\to\infty} \frac{1}{N} \log \frac{Z_N^{0,F,\varepsilon}}{Z_N^{0,F}}, \tag{1.7}$$

respectively, where the partition functions are given by taking $a = b = 0$ in the Dirichlet case and $a = 0$ in the free case, and the denominators $Z_N^{0,0}$ and $Z_N^{0,F}$ are defined without pinning effect and equal to their corresponding numerators with $\varepsilon = 0$. See (3.6) below for the explicit formula of $Z_N^{0,0,\varepsilon}$ and (3.11) for $Z_N^{0,F,\varepsilon}$. The constants ξ^ε in (1.5) are defined by $\xi^\varepsilon = \xi^{D,\varepsilon}$ for the functional $\Sigma = \Sigma^D$ and $\xi^\varepsilon = \xi^{F,\varepsilon}$ for Σ^F, respectively.

Explicit formulae determining the free energies $\xi^{D,\varepsilon}$ and $\xi^{F,\varepsilon}$ are found in (3.9) and (3.13) below, respectively. Furthermore, we have the following result which extends Theorem 1.1 of [**2**] to our setting. We denote the mean drift of p by $m = \int_{\boldsymbol{R}^d} xp(x)dx \in \boldsymbol{R}^d$.

THEOREM 1.1.

(1) *The limits $\xi^{D,\varepsilon}$ in* (1.6) *and $\xi^{F,\varepsilon}$ in* (1.7) *exist for every $\varepsilon \geq 0$.*
(2) *There exist two critical values $0 \leq \varepsilon_c^D \leq \varepsilon_c^F$ determined by* (3.8) *and* (3.12) *below, respectively, such that $\xi^{D,\varepsilon} > 0$ if and only if $\varepsilon > \varepsilon_c^D$ (therefore $\xi^{D,\varepsilon} = 0$ if and only if $0 \leq \varepsilon \leq \varepsilon_c^D$) and $\xi^{F,\varepsilon} > 0$ if and only if $\varepsilon > \varepsilon_c^F$ (therefore $\xi^{F,\varepsilon} = 0$ if and only if $0 \leq \varepsilon \leq \varepsilon_c^F$).*
(3) *If $d \geq 3$, $\varepsilon_c^D > 0$, while if $d = 1$ and 2, $\varepsilon_c^D = 0$.*
(4) *We have $\varepsilon_c^D = \varepsilon_c^F$ if and only if $m = 0$, and in this case $\xi^{D,\varepsilon} = \xi^{F,\varepsilon}$ holds for all $\varepsilon \geq 0$. If $m \neq 0$, we have $\varepsilon_c^D < \varepsilon_c^F$ and $\xi^{F,\varepsilon} < \xi^{D,\varepsilon}$ holds for every $\varepsilon > \varepsilon_c^D$.*

The last assertion of Theorem 1.1 can be interpreted as follows. In such a case that the original unperturbed random walk has non-zero drift $m \neq 0$, if the strength ε of the pinning belongs to the range $\varepsilon \in (\varepsilon_c^D, \varepsilon_c^F)$, the weakly pinned random walk is transient (or delocalized) in the free case while it is recurrent (or localized) in the Dirichlet case. This happens because the Dirichlet condition has an effect to make the drift of the Markov chain vanish.

The large deviation principle (Theorem 5.1) immediately implies the concentration properties for $\mu_N = \mu_N^{D,\varepsilon}$ and $\mu_N^{F,\varepsilon}$: for every $\delta > 0$ there exists $c > 0$ such that

$$\mu_N(\mathrm{dist}_\infty(h^N, \mathscr{H}) > \delta) \leq e^{-cN} \tag{1.8}$$

for large enough N, where $\mathscr{H} = \{h^*; \text{minimizers of } \Sigma\}$ with $\Sigma = \Sigma^D$ and Σ^F, respectively, and dist_∞ denotes the distance on $\mathscr{C}$ under the uniform norm $\|\cdot\|_\infty$.

Let us now study the minimizers or their candidates of the rate functionals Σ. Define two functions $\bar{h}_{a,b}$ and $\hat{h}_{a,b;\theta_1,\theta_2}$ on D for $\theta_1, \theta_2 > 0$ such that $\theta_1 + \theta_2 < 1$ by

$$\bar{h}_{a,b}(t) = (1-t)a + tb, \quad t \in D,$$

and

$$\hat{h}_{a,b;\theta_1,\theta_2}(t) = \begin{cases} \frac{(\theta_1-t)a}{\theta_1}, & t \in [0,\theta_1), \\ 0, & t \in [\theta_1, 1-\theta_2], \\ \frac{(t+\theta_2-1)b}{\theta_2}, & t \in (1-\theta_2, 1], \end{cases}$$

respectively. For each $v \in \boldsymbol{R}^d \setminus \{0\}$ and $c \geq -\Lambda^*(0)$, let $s = s(v,c) \geq 0$ be the unique solution of the equation

$$sv \cdot \nabla\Lambda^*(sv) - \Lambda^*(sv)\big(\equiv \Lambda\big(\nabla\Lambda^*(sv)\big)\big) = c, \tag{1.9}$$

where $v \cdot \nabla\Lambda^* = \sum_{\alpha=1}^d v^\alpha \partial\Lambda^*/\partial v^\alpha$. We define $t_1^D, t_2^D > 0$ by $t_1^D = 1/s(-a, \xi^{D,\varepsilon} - \Lambda^*(0)), t_2^D = 1/s(b, \xi^{D,\varepsilon} - \Lambda^*(0))$ and $t_1^F > 0$ by $t_1^F = 1/s(-a, \xi^{F,\varepsilon} - \Lambda^*(0))$, respectively; if such s does not exist, we set $t_1^D = \infty$ etc. Denote the sets of the minimizers of Σ^D and Σ^F by $\mathscr{M}^D$ and $\mathscr{M}^F$, respectively.

If $\xi^{D,\varepsilon} = 0$ or $\xi^{F,\varepsilon} = 0$, the minimizer of Σ^D or Σ^F is unique and given by $\bar{h}^D := \bar{h}_{a,b}$ or $\bar{h}^F := \bar{h}_{a,a+m}$ for each functional. We therefore consider under the condition $\varepsilon > \varepsilon_c^D$ or $\varepsilon > \varepsilon_c^F$.

Lemma 1.2.

(1) *The solution $s = s(v,c)$ of the equation* (1.9) *is unique.*

(2) (Dirichlet case) *If $t_1^D + t_2^D < 1$, $\mathscr{M}^D$ is contained in $\{\bar{h}^D, \hat{h}^D\}$, where $\hat{h}^D := \hat{h}_{a,b;t_1^D,t_2^D}$ (i.e., $\theta_1 = t_1^D, \theta_2 = t_2^D$). If $t_1^D + t_2^D \geq 1$, then $\mathscr{M}^D = \{\bar{h}^D\}$.*

(3) (Free case)

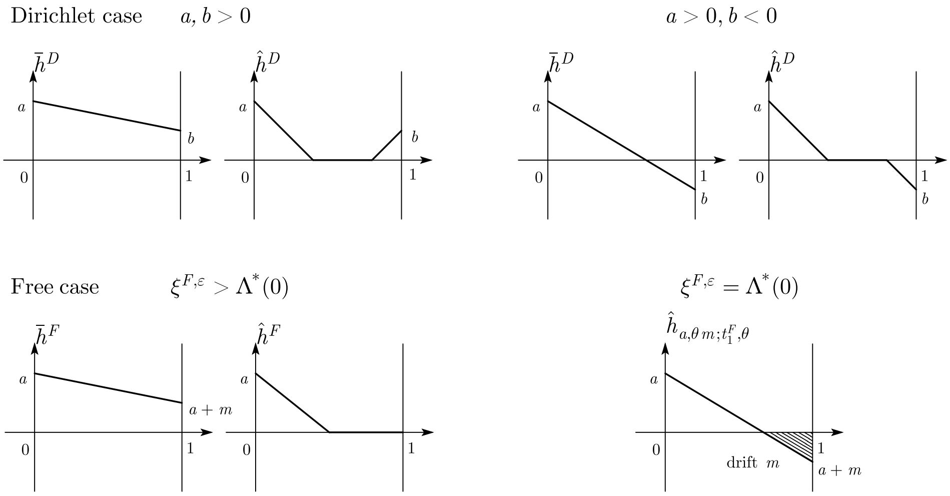
(i) *If $\xi^{F,\varepsilon} > \Lambda^*(0)$ and $t_1^F < 1$, then $\mathscr{M}^F$ is contained in $\{\bar{h}^F, \hat{h}^F\}$, where $\hat{h}^F := \hat{h}_{a,0;t_1^F,0}$ (i.e., $b = 0, \theta_1 = t_1^F, \theta_2 = 0$).*

(ii) *If $\xi^{F,\varepsilon} = \Lambda^*(0)$, $t_1^F < 1$ and $a = -t_1^F m$, then $\mathscr{M}^F$ coincides with the set $\big\{\hat{h}_{a,\theta m;t_1^F,\theta}; \theta \in [0, 1-t_1^F]\big\}$; note that $\bar{h}^F = \hat{h}_{a,(1-t_1^F)m;t_1^F,1-t_1^F}$ in this case.*

(iii) *In all other cases (i.e., if "$\xi^{F,\varepsilon} \geq \Lambda^*(0)$ and $t_1^F \geq 1$" or "$\xi^{F,\varepsilon} = \Lambda^*(0), t_1^F < 1$ and $a \neq -t_1^F m$" or "$0 < \xi^{F,\varepsilon} < \Lambda^*(0)$"), then $\mathscr{M}^F = \{\bar{h}^F\}$.*

The graphs of the functions $\bar{h}^D, \hat{h}^D, \bar{h}^F, \hat{h}^F$ and $\hat{h}_{a,\theta m;t_1^F,\theta}$ in $d = 1$ are shown below.

In the free case, under the condition $\xi^{F,\varepsilon} = \Lambda^*(0)$, the minimizers $\hat{h}_{a,\theta m;t_1^F,\theta}$ starting at a are floated by the drift m without any cost and, once they hit 0, the



price $\Lambda^*(0)$ to stay there balances with the gain $\xi^{F,\varepsilon}$ staying there so that they can leave 0 at any time.

1.3. Main result.

Our concern is in the critical case where $\bar{h}$ and $\hat{h}$ are simultaneously the minimizers of Σ^D, and similar situations for Σ^F. We will exclude the special case appeared in Lemma 1.2-(3)-(ii), for which the set of the minimizers of Σ^F is continuously parameterized by θ. Otherwise, h^N converges to the unique minimizer of Σ as $N \to \infty$ in probability, recall (1.8). We therefore assume the following conditions in each situation:

$$(C)_D \qquad \varepsilon > \varepsilon_c^D,\ t_1^D + t_2^D < 1 \quad \text{and} \quad \Sigma^D(\bar{h}^D) = \Sigma^D(\hat{h}^D),$$
$$(C)_F \qquad \varepsilon > \varepsilon_c^F,\ \xi^{F,\varepsilon} > \Lambda^*(0),\ t_1^F < 1 \quad \text{and} \quad \Sigma^F(\bar{h}^F) = \Sigma^F(\hat{h}^F).$$

We are now in a position to formulate our main result. We say that the limit under μ_N is h^* if $\lim_{N\to\infty} \mu_N(\|h^N - h^*\|_\infty \le \delta) = 1$ holds for every $\delta > 0$. We say that two functions $\bar{h}$ and $\hat{h}$ coexist in the limit under μ_N with probabilities $\bar{\lambda}$ and $\hat{\lambda}$ if $\bar{\lambda}, \hat{\lambda} > 0$, $\bar{\lambda} + \hat{\lambda} = 1$ and $\lim_{N\to\infty} \mu_N(\|h^N - \bar{h}\|_\infty \le \delta) = \bar{\lambda}$, $\lim_{N\to\infty} \mu_N(\|h^N - \hat{h}\|_\infty \le \delta) = \hat{\lambda}$ hold for every $0 < \delta < |a| \wedge |b|$.

THEOREM 1.3.

(1) (Dirichlet case) *Under the condition* $(C)_D$, *the limit under* $\mu_N^{D,\varepsilon}$ *is* $\hat{h}^D$ *if* $d = 1$ *and* $\bar{h}^D$ *if* $d \ge 3$. *If* $d = 2$, $\bar{h}^D$ *and* $\hat{h}^D$ *coexist in the limit under* $\mu_N^{D,\varepsilon}$ *with probabilities* $\bar{\lambda}^{D,\varepsilon}$ *and* $\hat{\lambda}^{D,\varepsilon}$, *respectively, given by* (4.15).

(2) (Free case) *Under the condition* $(C)_F$, *if* $d = 1$, $\bar{h}^F$ *and* $\hat{h}^F$ *coexist in the limit*

under $\mu_N^{F,\varepsilon}$ *with probabilities* $\bar\lambda^{F,\varepsilon}$ *and* $\hat\lambda^{F,\varepsilon}$, *respectively, given by* (4.22). *If* $d \ge 2$, *the limit under* $\mu_N^{F,\varepsilon}$ *is* $\bar h^F$.

Section 2 proves Lemma 1.2. The proof of Theorem 1.3 will be given in Section 4. In particular, this will imply the central limit theorem for the times when the Markov chains first or last touch the origin 0, see Remark 4.1. The conditions $(C)_D$ and $(C)_F$ guarantee that the leading exponential decay rates of the probabilities of the neighborhoods of the two different minimizers balance with each other. This enforces us to study their precise asymptotics, which are discussed in Section 3. The proof of Theorem 1.1 is also given in Section 3. Section 5 is for the sample path large deviation principles. Mogul'skii's result [**13**] for the free case without pinning is extended to the Dirichlet case. In Section 6, we study the critical exponents for the free energies ξ^ε by establishing their asymptotic behavior in ε close to their critical values.

2. Proof of Lemma 1.2.

For each $v \in \boldsymbol{R}^d \setminus \{0\}$, set $f(s) = sv\cdot\nabla\Lambda^*(sv) - \Lambda^*(sv)$ for $s \ge 0$. Then, we see that $f'(s) = s\sum_{\alpha,\beta=1}^d v^\alpha v^\beta \partial^2\Lambda^*/\partial v^\alpha \partial v^\beta(sv) > 0$ for $s > 0$ and $f(0) = -\Lambda^*(0)$, and this proves the assertion (1).

To show (2) and (3), we first notice the following: For $0 \le s_1 < s_2 \le 1$ and $h \in \mathscr{A}\mathscr{C}([s_1, s_2])$ such that $h(s_1) = a$ and $h(s_2) = b$, Jensen's inequality implies that

$$\frac{1}{s_2 - s_1}\int_{s_1}^{s_2} \Lambda^*(\dot h(t))\,dt \ge \Lambda^*\biggl(\frac{1}{s_2 - s_1}\int_{s_1}^{s_2} \dot h(t)\,dt\biggr) = \Lambda^*\biggl(\frac{b-a}{s_2 - s_1}\biggr),$$

in which the equality holds only when $\dot h(t) = (b-a)/(s_2 - s_1)$ because of the strict convexity of Λ^* on $\boldsymbol{R}^d$. Thus the minimizer h of the functional $\int_{s_1}^{s_2} \Lambda^*(\dot h(t))dt$ is linearly interpolating between a and b: $h(t) = (b-a)(t-s_1)/(s_2-s_1) + a$, $t \in [s_1, s_2]$. This means that the graph of any minimizer of Σ must be a line as long as it does not touch 0, therefore, the minimizers of Σ are in the class of functions $\{\bar h_{a,b}, \hat h_{a,b;\theta_1,\theta_2}; \theta_1, \theta_2 > 0, \theta_1 + \theta_2 < 1\}$.

To find the minimizers of Σ in the class of $\{\hat h_{a,b;\theta_1,\theta_2}\}$, we set

$$\begin{aligned} F_{a,b}(\theta_1,\theta_2) &:= \Sigma\bigl(\hat h_{a,b;\theta_1,\theta_2}\bigr) \\ &= \theta_1\Lambda^*\biggl(-\frac{a}{\theta_1}\biggr) + \theta_2\Lambda^*\biggl(\frac{b}{\theta_2}\biggr) + (1-\theta_1-\theta_2)\bigl(\Lambda^*(0) - \xi^\varepsilon\bigr). \end{aligned} \tag{2.1}$$

Then, we have that

$$\frac{\partial F_{a,b}}{\partial \theta_1}(\theta_1,\theta_2) = \Lambda^*\Big(-\frac{a}{\theta_1}\Big) + \Big(\frac{a}{\theta_1}\Big)\cdot\nabla\Lambda^*\Big(-\frac{a}{\theta_1}\Big) - \big(\Lambda^*(0)-\xi^\varepsilon\big),$$

$$\frac{\partial F_{a,b}}{\partial \theta_2}(\theta_1,\theta_2) = \Lambda^*\Big(\frac{b}{\theta_2}\Big) - \Big(\frac{b}{\theta_2}\Big)\cdot\nabla\Lambda^*\Big(\frac{b}{\theta_2}\Big) - \big(\Lambda^*(0)-\xi^\varepsilon\big).$$

If the minimizer of Σ^D is in the class of $\{\hat h_{a,b;\theta_1,\theta_2}\}$, then it satisfies $\partial F_{a,b}/\partial\theta_1 = \partial F_{a,b}/\partial\theta_2 = 0$, which is equivalent to $\theta_1 = t_1^D$ and $\theta_2 = t_2^D$; note that $\hat h_{a,b;\theta_1,\theta_2}$ can not be a minimizer if $\theta_1+\theta_2 = 1$ from the reason mentioned above. This proves the assertion (2).

Let us consider the minimizer of Σ^F in the class of $\{\hat h_{a,b;\theta_1,\theta_2}\}$. Now, $b \in \boldsymbol{R}^d$ also moves as a parameter. The function $F_{a,b}(\theta_1,\theta_2)$, as a function of b, is minimized at $b/\theta_2 = m$ (recall $\Lambda^*(m)=0$), and it becomes $F_a(\theta_1,\theta_2) \equiv F_{a,\theta_2 m}(\theta_1,\theta_2) = \theta_1\Lambda^*(-a/\theta_1) + (1-\theta_1-\theta_2)\big(\Lambda^*(0)-\xi^\varepsilon\big)$. The function $F_a(\theta_1,\theta_2)$, as a function of θ_2, is minimized at $\theta_2 = 0$ if $\xi^\varepsilon > \Lambda^*(0)$ (which proves the assertion (3)-(i)), at $\theta_2 = 1-\theta_1$ if $\xi^\varepsilon < \Lambda^*(0)$ and at all $\theta_2 \in [0,1-\theta_1]$ if $\xi^\varepsilon = \Lambda^*(0)$. In case $\xi^\varepsilon < \Lambda^*(0)$, $\theta_2 = 1-\theta_1$ means that $\hat h_{a,b;\theta_1,\theta_2}$ actually touch 0 only at $t=\theta_1$ (note that we are concerned with the case $m \neq 0$, since $m=0$ implies $\Lambda^*(0)=0$ so that $\xi^\varepsilon < \Lambda^*(0)$ can not happen), and therefore the minimizer of Σ^F must be $\bar h^F$. In case $\xi^\varepsilon = \Lambda^*(0)$, for all $\theta_2 \in [0,1-\theta_1]$, we have $F_a(\theta_1,\theta_2) = \theta_1\Lambda^*(-a/\theta_1)$ which is minimized at $\theta_1 = t_1^F$, so that the candidates of the minimizers are of the form $\hat h_{a,\theta_2 m; t_1^F,\theta_2}$, $\theta_2 \in [0,1-t_1^F]$. Comparing its energy with that of the another candidate $\bar h^F$: $\Sigma^F(\bar h^F) = 0$, it must hold $F_a(t_1^F,\theta_2) = 0$, which is satisfied only when $-a/t_1^F = m$. This proves the assertion (3)-(ii). The proof of Lemma 1.2 is thus concluded.

Remark 2.1. The condition (1.9) is known as the Young's relation, which prescribes the free boundary points t_1^D, t_2^D and t_1^F.

3. Precise asymptotics for the partition functions.

This section establishes the precise asymptotic behavior of the ratios of partition functions associated with the Markov chains in $\boldsymbol{R}^d$ with pinning at 0 and starting at 0 (and reaching 0 in the Dirichlet case), which were mentioned in Section 1.2 to determine $\xi^{D,\varepsilon}$ and $\xi^{F,\varepsilon}$. In particular, these will imply the statements in Theorem 1.1. A similar result is obtained by [**2**].

We introduce several notation; see Section 5.5 of [**8**] when $d=1$. For $\lambda \in \boldsymbol{R}^d$, we define the Cramér transform p_λ of p by

$$p_\lambda(x) = e^{\lambda\cdot x - \Lambda(\lambda)}p(x), \quad x \in \boldsymbol{R}^d.$$

Note that, under Assumption 1.1, the function Λ is in $C^\infty(\boldsymbol{R}^d)$ and strictly convex, since its Hesse matrix $(\partial^2\Lambda(\lambda)/\partial\lambda^\alpha\partial\lambda^\beta)_{1\le\alpha,\beta\le d}$ is equal to the covariance matrix $\mathscr{Q}(\lambda) = (q^{\alpha\beta}(\lambda))_{1\le\alpha,\beta\le d}$ of p_λ, which is strictly positive definite. Here, $q^{\alpha\beta}(\lambda) = \int_{\boldsymbol{R}^d}(x^\alpha - v^\alpha(\lambda))(x^\beta - v^\beta(\lambda))p_\lambda(x)dx$ and $v^\alpha(\lambda) = \int_{\boldsymbol{R}^d} x^\alpha p_\lambda(x)dx$; in particular, $m = v(0)$. Two functions $v = v(\lambda) : \boldsymbol{R}^d \to \boldsymbol{R}^d$ and $\lambda = \lambda(v) : \boldsymbol{R}^d \to \boldsymbol{R}^d$ are defined by

$$\begin{aligned} &v = v(\lambda) := \nabla\Lambda(\lambda)\Big(= \int_{\boldsymbol{R}^d} xp_\lambda(x)dx\Big), \quad \lambda \in \boldsymbol{R}^d, \\ &\lambda = \lambda(v) := \nabla\Lambda^*(v), \quad v \in \boldsymbol{R}^d. \end{aligned} \tag{3.1}$$

Note that $\lambda = \lambda(v)$ is the inverse function of $v = v(\lambda)$: $\lambda = \lambda(v) \Leftrightarrow v = v(\lambda)$ under Assumption 1.1 and the supremum in the right hand side of (1.4) for $\Lambda^*(v)$ is attained at $\lambda = \lambda(v)$:

$$\Lambda^*(v) = \lambda(v)\cdot v - \Lambda(\lambda(v)), \tag{3.2}$$

cf. Theorem VII.5.5 of [**6**] and Lemma 2.2.31 (b) of [**4**]. See also Exercise 2.2.24 of [**4**] for $\Lambda^* \in C^\infty(\boldsymbol{R})$ when $d = 1$.

3.1. Dirichlet case.

For $0 \le j < k \le N$, we denote by $\mu_{j,k}^{a,b}$ the probability measure on $(\boldsymbol{R}^d)^{\{j,\dots,k\}} = \big\{\phi = (\phi_i)_{j\le i\le k}; \phi_i \in \boldsymbol{R}^d\big\}$ without pinning under the Dirichlet conditions $\phi_j = aN$ and $\phi_k = bN$:

$$\mu_{j,k}^{a,b}(d\phi) = \frac{\mathfrak{p}_{j,k}(\phi)}{Z_{j,k}^{a,b}}\delta_{aN}(d\phi_j)\prod_{i=j+1}^{k-1} d\phi_i\,\delta_{bN}(d\phi_k), \tag{3.3}$$

where $\mathfrak{p}_{j,k}(\phi) = \prod_{i=j+1}^k p(\phi_i - \phi_{i-1})$ and $Z_{j,k}^{a,b} = Z_{k-j}^{a,b}\,\big(= Z_{k-j}^{a,b,N}\big)$ is the normalizing constant. Then, we have the following lemma. A similar result for random walks on $\boldsymbol{Z}^d$ can be found in Proposition B.2 of [**3**]. Recall that the matrices $\mathscr{Q}(\lambda)$ are strictly positive definite for all $\lambda \in \boldsymbol{R}^d$ from the definition.

LEMMA 3.1. *As $n \to \infty$ by keeping $n/N \to r \in (0,1]$, we have*

$$Z_n^{a,b} \sim \frac{1}{(2\pi n)^{d/2}\sqrt{\det Q\big(\frac{b-a}{r}\big)}}\exp\Big\{ - n\Lambda^*\Big(\frac{(b-a)N}{n}\Big)\Big\},$$

where $\sim$ means that the ratio of both sides tends to 1 and $Q(v) := \mathscr{Q}(\lambda(v))$ is the *covariance matrix of $p_{\lambda(v)}$ for $v \in \boldsymbol{R}^d$; recall that p_λ is the Cramér transform of p and the function $\lambda(v)$ is defined by* (3.1). *In particular, we have*

$$Z_n^{0,0} \sim \frac{1}{(2\pi n)^{d/2}\sqrt{\det Q}} e^{-n\Lambda^*(0)}, \tag{3.4}$$

as $n \to \infty$, where $Q := Q(0)$ is the covariance matrix of $p_{\lambda(0)}$.

PROOF. From its definition, the normalizing constant $Z_n^{a,b}$ can be rewritten as $Z_n^{a,b} = p^{n*}((b-a)N)$ in terms of the n-fold convolution p^{n*} of p. However, by a simple computation recalling (3.2), we can rewrite $p^{n*}(x)$ as

$$p^{n*}(x) = e^{-n\Lambda^*(x/n)} \big(p_{\lambda(x/n)}\big)^{n*}(x). \tag{3.5}$$

Define the probability densities $\tilde{p}_v$ and $q_{n,v}$ for $v \in \boldsymbol{R}^d$ by $\tilde{p}_v(x) = p_{\lambda(v)}(x+v)$ and $q_{n,v}(x) = n^{d/2}\big(\tilde{p}_v\big)^{n*}(\sqrt{n}x)$, respectively. Note that the mean of $\tilde{p}_v$ is 0 and its covariance matrix is $Q(v)$ (i.e., same as that of $p_{\lambda(v)}$) and $q_{n,v}$ is the distribution density of $n^{-1/2}\sum_{i=1}^n X_i^{(v)}$, where $\{X_i^{(v)}\}_{i=1}^n$ is an i.i.d. sequence with distribution densities $\tilde{p}_v$. Since Assumption 1.1-(1) implies $\sup_{|v|\le K}\sup_{x\in\boldsymbol{R}^d}\tilde{p}_v(x) < \infty$ for every $K>0$, the local limit theorem, which holds uniformly in v and formulated in Lemma 3.2 below applied for $p^{(v)} = \tilde{p}_v$, proves

$$\lim_{n\to\infty}\sup_{|v|\le K}\left| q_{n,v}(0) - \frac{1}{(2\pi)^{d/2}\sqrt{\det Q(v)}}\right| = 0.$$

This shows

$$\sup_{|v|\le K}\left|\big(p_{\lambda(v)}\big)^{n*}(nv) - \frac{1}{(2\pi n)^{d/2}\sqrt{\det Q(v)}}\right| = o\big(n^{-d/2}\big),$$

as $n \to \infty$, since $(\tilde{p}_v)^{n*}(x) = (p_{\lambda(v)})^{n*}(x+nv)$. Therefore, in particular by taking $v = (b-a)N/n$ which runs over a certain bounded set of $\boldsymbol{R}^d$ as long as $n/N \to r > 0$, the identity (3.5) with $x = (b-a)N$ shows that

$$\begin{aligned} &p^{n*}((b-a)N) \\ &\quad = \left(\frac{1}{(2\pi n)^{d/2}\sqrt{\det Q\big(\frac{b-a}{r}\big)}} + o\big(n^{-d/2}\big)\right)\exp\left\{-n\Lambda^*\left(\frac{(b-a)N}{n}\right)\right\}. \end{aligned}$$

Thus, the proof of the lemma is concluded. $\square$

We need to extend Theorem 19.1 of [**1**] in the following form, in which the random variables depend on an extra parameter v running over a certain set Θ and the local limit theorem is established uniformly in v. The proof is essentially the same so that it is omitted.

LEMMA 3.2. *For each $v \in \Theta$, let an $\boldsymbol{R}^d$-valued i.i.d. sequence $\{X_n^{(v)}\}_{n=1}^\infty$ be given. We assume that $E[X_n^{(v)}] = 0$, $\mathrm{Cov}(X_n^{(v)}) = V^{(v)}$, which is a symmetric positive definite matrix, and the distribution of $X_n^{(v)}$ has a density $p^{(v)}(x)$. Then, if $\sup_{v\in\Theta}\sup_{x\in\boldsymbol{R}^d} p^{(v)}(x) < \infty$ and if $c_1 I \le V^{(v)} \le c_2 I$ hold for all $v \in \Theta$ with some constants $0 < c_1 < c_2 < \infty$ and the $d \times d$ identity matrix I, the distribution of $n^{-1/2}\sum_{i=1}^n X_i^{(v)}$ has a density $q_n^{(v)}(x)$ and it holds that*

$$\lim_{n\to\infty} \sup_{v\in\Theta} \sup_{x\in\boldsymbol{R}^d} \left| q_n^{(v)}(x) - \phi_{0,V^{(v)}}(x) \right| = 0,$$

where $\phi_{0,V}(x)$ stands for the density of the Gaussian distribution on $\boldsymbol{R}^d$ with mean 0 and covariance V.

The partition function $Z_N^{0,0,\varepsilon}$ is determined by

$$Z_N^{0,0,\varepsilon} = \int_{(\boldsymbol{R}^d)^{N+1}} \mathfrak{p}_N(\phi)\delta_0(d\phi_0) \prod_{i=1}^{N-1} \big(\varepsilon\delta_0(d\phi_i) + d\phi_i\big)\delta_0(d\phi_N), \tag{3.6}$$

and $Z_N^{0,0} = Z_N^{0,0,0}$, i.e. $\varepsilon = 0$. Let us define the function $g : [0,\infty) \to [0,\infty]$ by

$$g(x) = \sum_{n=1}^\infty x^n Z_n^{0,0}. \tag{3.7}$$

Note that g is increasing, $g(0) = 0$, $g(x) < \infty$ if and only if $x \in [0, e^{\Lambda^*(0)}]$ when $d \ge 3$ and $g(x) < \infty$ if and only if $x \in [0, e^{\Lambda^*(0)})$ when $d = 1, 2$ by (3.4). Set

$$\varepsilon_c^D = \frac{1}{g(e^{\Lambda^*(0)})}. \tag{3.8}$$

In particular, $\varepsilon_c^D > 0$ if $d \ge 3$ and $\varepsilon_c^D = 0$ if $d = 1, 2$. For each $\varepsilon > \varepsilon_c^D$, we determine $x = x^\varepsilon \in (0, e^{\Lambda^*(0)})$ as the unique solution of $g(x) = 1/\varepsilon$ and introduce two positive constants:

$$\xi^{D,\varepsilon} = \Lambda^*(0) - \log x^\varepsilon \quad \text{and} \quad C^{D,\varepsilon} = \frac{(2\pi)^{d/2}\sqrt{\det Q}}{\varepsilon^2 x^\varepsilon g'(x^\varepsilon)}. \tag{3.9}$$

Proposition 3.3. *For each* $\varepsilon > \varepsilon_c^D$, *we have the precise asymptotics as* $N \to \infty$ *for the ratio of two partition functions*:

$$\frac{Z_N^{0,0,\varepsilon}}{Z_N^{0,0}} \sim C^{D,\varepsilon} N^{d/2} e^{N\xi^{D,\varepsilon}}.$$

Proof. We first note the renewal equation for $Z_N^{0,0,\varepsilon}$, $N \geq 2$ with $Z_1^{0,0,\varepsilon} = Z_1^{0,0} = 1$:

$$Z_N^{0,0,\varepsilon} = Z_N^{0,0} + \varepsilon \sum_{i=1}^{N-1} Z_i^{0,0} Z_{N-i}^{0,0,\varepsilon}, \tag{3.10}$$

see Lemma 2.1 in [**2**]. Then, in a very similar manner to the proof of Proposition 2.2 in [**2**] (remind that the partition functions in [**2**] in the Gaussian case have an extra factor $(2\pi)^{dn/2}$ because p is unnormalized there), taking $u_0 = a_0 = b_0 = 0$ and $u_n = (x^\varepsilon)^n Z_n^{0,0,\varepsilon}$, $a_n = \varepsilon (x^\varepsilon)^n Z_n^{0,0}$, $b_n = (x^\varepsilon)^n Z_n^{0,0}$ for $n \geq 1$ in the present setting and noting that $\sum_{n=0}^\infty a_n = 1$, the renewal theory applied for the equation for $\{u_n\}$ obtained from (3.10) shows that

$$\lim_{n\to\infty} (x^\varepsilon)^n Z_n^{0,0,\varepsilon} = \frac{\sum_{n=0}^\infty b_n}{\sum_{n=0}^\infty n a_n} = \frac{1}{\varepsilon^2 x^\varepsilon g'(x^\varepsilon)}.$$

The conclusion is shown by combining this with (3.4). □

The free energy $\xi^{D,\varepsilon}$ defined by (1.6) is, if exists, non-negative and non-decreasing in ε, since $Z_n^{0,0,\varepsilon}$ is increasing in ε. Therefore, since (3.9) implies $\lim_{\varepsilon \downarrow \varepsilon_c^D} \xi^{D,\varepsilon} = 0$, we see that $\xi^{D,\varepsilon} = 0$ for $0 \leq \varepsilon \leq \varepsilon_c^D$.

3.2. Free case.

We next consider the case with the free condition at $t = 1$ (or microscopically at $i = N$). The partition function $Z_N^{0,F,\varepsilon}$ is determined by

$$Z_N^{0,F,\varepsilon} = \int_{(\boldsymbol{R}^d)^{N+1}} \mathfrak{p}_N(\phi)\delta_0(d\phi_0) \prod_{i=1}^N \big(\varepsilon\delta_0(d\phi_i) + d\phi_i\big), \tag{3.11}$$

and we have $Z_N^{0,F} (= Z_N^{0,F,0}) = 1$. Recall the function g defined by (3.7) and set

$$\varepsilon_c^F = \frac{1}{g(1)}. \tag{3.12}$$

We see that $\varepsilon_c^F \geq \varepsilon_c^D$ from $\Lambda^*(0) \geq 0$ and $\varepsilon_c^F = \varepsilon_c^D$ is equivalent to $\Lambda^*(0) = 0$, namely, $m \equiv \int_{\boldsymbol{R}^d} xp(x)dx = 0$. For each $\varepsilon > \varepsilon_c^F$, we define two positive constants:

$$\xi^{F,\varepsilon} = -\log x^\varepsilon \quad \text{and} \quad C^{F,\varepsilon} = \frac{1}{\varepsilon x^\varepsilon (1-x^\varepsilon) g'(x^\varepsilon)}. \tag{3.13}$$

PROPOSITION 3.4. *For each $\varepsilon > \varepsilon_c^F$, we have the precise asymptotics as $N \to \infty$:*

$$\frac{Z_N^{0,F,\varepsilon}}{Z_N^{0,F}} \sim C^{F,\varepsilon} e^{N\xi^{F,\varepsilon}}.$$

PROOF. We first note the renewal equation for $Z_N^{0,F,\varepsilon}$, $N \geq 1$ with $Z_0^{0,F,\varepsilon} = 1$:

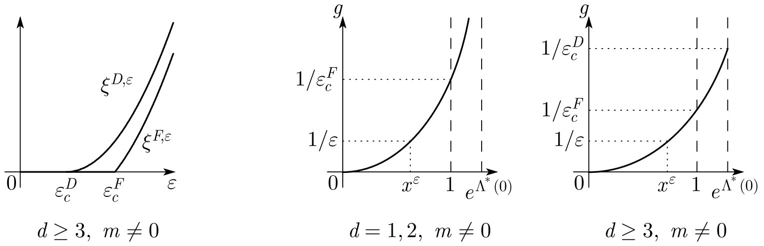
$$Z_N^{0,F,\varepsilon} = Z_N^{0,F} + \varepsilon \sum_{i=1}^N Z_i^{0,0} Z_{N-i}^{0,F,\varepsilon},$$

see Lemma 2.4 in [**2**]. Then, in a similar manner to Proposition 2.5 in [**2**], taking $u_0 = b_0 = 1, a_0 = 0$ and $u_n = (x^\varepsilon)^n Z_n^{0,F,\varepsilon}$, $a_n = \varepsilon (x^\varepsilon)^n Z_n^{0,0}$, $b_n = (x^\varepsilon)^n Z_n^{0,F} = (x^\varepsilon)^n$ for $n \geq 1$ in the present setting, an application of the renewal theory shows that

$$\lim_{n\to\infty} (x^\varepsilon)^n Z_n^{0,F,\varepsilon} = \frac{1}{\varepsilon x^\varepsilon (1-x^\varepsilon) g'(x^\varepsilon)}.$$

Note that the limit is finite only if $\sum_{n=0}^\infty b_n = \sum_{n=0}^\infty (x^\varepsilon)^n < \infty$, that is $x^\varepsilon < 1$, i.e., $\varepsilon > \varepsilon_c^F$. The conclusion is now shown by recalling $Z_N^{0,F} = 1$. $\square$

Since (3.13) implies $\lim_{\varepsilon \downarrow \varepsilon_c^F} \xi^{F,\varepsilon} = 0$, we see that $\xi^{F,\varepsilon} = 0$ for $0 \leq \varepsilon \leq \varepsilon_c^F$ for the free energy $\xi^{F,\varepsilon}$ defined by (1.7).



4. Proof of Theorem 1.3.

In this section, we give the proof of Theorem 1.3 under the conditions $(C)_D$ and $(C)_F$. Recall the definition (3.3) of the probability measure $\mu_{j,k}^{a,b}$ on $(\boldsymbol{R}^d)^{\{j,\dots,k\}}$ for $0 \le j < k \le N$. The corresponding measure with pinning is denoted by $\mu_{j,k}^{a,b,\varepsilon}$.

4.1. Proof of Theorem 1.3-(1).

Under the measure $\mu_{j,k}^{a,b}$, the macroscopic path determined from $(\phi_i)_{j\le i\le k}$ concentrates on the straight line $g_{[j/N,k/N]}^{a,b}(t)$ between $(j/N,a)$ and $(k/N,b)$, in particular, $g_{[0,1]}^{a,b} = \bar{h}^D$. More precisely, by the large deviation principle (cf. Proposition 5.2 below), we have the following lemma.

LEMMA 4.1. *For any $\delta' > 0$, there exists $c(\delta') > 0$ and $N_0(\delta') \in \boldsymbol{N}$ such that for any $a, b \in \boldsymbol{R}^d$, $0 \le j < k \le N$:*

$$\mu_{j,k}^{a,b}\left(\max_{i:j\le i\le k}\left|\frac{\phi_i}{N} - g_{[j/N,k/N]}^{a,b}\left(\frac{i}{N}\right)\right| \ge \delta'\right) \le e^{-c(\delta')N}$$

for $N \ge N_0(\delta')$.

We write

$$\gamma_{j,k}^{a,b}(\delta) := \mu_{j,k}^{a,b}\big(\big\|h_{[j/N,k/N]}^N - \hat{h}_{[j/N,k/N]}\big\|_\infty \le \delta\big),$$

where $\hat{h} = \hat{h}^D$ in this subsection, and $f_{[u,v]}$ is the restriction of a function $f : [0,1] \to \boldsymbol{R}^d$ to the subinterval $[u,v]$ of $[0,1]$. The probability $\gamma_{j,k}^{a,b,\varepsilon}(\delta)$ is similarly defined with pinning, i.e., under $\mu_{j,k}^{a,b,\varepsilon}$. We sometimes write $U_\delta(\hat{h}_{[u,v]})$ for the δ-neighborhood with respect to $\|\cdot\|_\infty$ in the space of functions on $[u,v]$ of $\hat{h}_{[u,v]}$; when the subscript $[u,v]$ is dropped, it is considered on $[0,1]$. We similarly write $U_\delta(\bar{h})$ for $\bar{h} = \bar{h}^D$.

To complete the proof of Theorem 1.3-(1), it suffices to evaluate the limit

$$\lim_{N\to\infty}\frac{\mu_N^{D,\varepsilon}\big(h^N \in U_\delta(\hat{h})\big)}{\mu_N^{D,\varepsilon}\big(h^N \in U_\delta(\bar{h})\big)}$$

for arbitrarily small $\delta > 0$; recall the concentration property (1.8) or (4.14) below.

Let i_ℓ and i_r be the times when the Markov chains ϕ first respectively last touch 0, namely, $i_\ell = \min\{i \in D_N; \phi_i = 0\}$ and $i_r = \max\{i \in D_N; \phi_i = 0\}$, where we define $\min\emptyset = N$ (in the Dirichlet case), $= N+1$ (in the free case discussed later) and $\max\emptyset = 0$. An expansion of the product measure $\prod_{i=1}^{N-1}(\varepsilon\delta_0(d\phi_i) + d\phi_i)$

in (1.1) by specifying i_ℓ and i_r gives rise to

$$\begin{aligned} R_N^D &:= \frac{Z_N^{a,b,\varepsilon}}{Z_N^{a,b}} \mu_N^{D,\varepsilon}\big(h^N \in U_\delta(\hat{h})\big) \\ &= \gamma_{0,N}^{a,b}(\delta) + \sum_{j=1}^{N-1} \varepsilon \Xi_{N,j,j}^{\varepsilon} \gamma_{0,j}^{a,0}(\delta)\gamma_{j,N}^{0,b}(\delta) \\ &\quad + \sum_{0<j<k<N} \varepsilon^2 \Xi_{N,j,k}^{\varepsilon} \gamma_{0,j}^{a,0}(\delta)\gamma_{j,k}^{0,0,\varepsilon}(\delta)\gamma_{k,N}^{0,b}(\delta) \\ &=: I_N^1 + I_N^2 + I_N^3, \end{aligned} \tag{4.1}$$

where

$$\Xi_{N,j,k}^{\varepsilon} = \frac{Z_j^{a,0} Z_{k-j}^{0,0,\varepsilon} Z_{N-k}^{0,b}}{Z_N^{a,b}} \tag{4.2}$$

for $0 < j \le k < N$. In fact, I_N^1 covers all paths without touching 0: $i_\ell = N, i_r = 0$ and I_N^2 is for those touching 0 once: $0 < i_\ell = i_r (= j) < N$, while I_N^3 is for those touching 0 at least twice: $0 < i_\ell(= j) < i_r(= k) < N$. We set $Z_0^{0,0,\varepsilon} = 1$ to define $\Xi_{N,j,j}^{\varepsilon}$. If δ is chosen small enough, we have from Lemma 4.1

$$I_N^1 + I_N^2 \le e^{-cN} \tag{4.3}$$

for N sufficiently large, with $c > 0$.

By Lemma 3.1, the ratio of the partition functions in (4.2) has the asymptotics for $j < k$ as $N \to \infty$:

$$\Xi_{N,j,k}^{\varepsilon} \sim \alpha_{N,j,k} e^{-N\tilde{f}(s_1,s_2)} \frac{Z_{k-j}^{0,0,\varepsilon}}{Z_{k-j}^{0,0}}, \tag{4.4}$$

where $s_1 = j/N$, $s_2 = (N-k)/N$,

$$\tilde{f}(s_1, s_2) := s_1 \Lambda^*\Big(-\frac{a}{s_1}\Big) + s_2 \Lambda^*\Big(\frac{b}{s_2}\Big) + (1 - s_1 - s_2)\Lambda^*(0) - \Lambda^*(b-a), \tag{4.5}$$

and

$$\alpha_{N,j,k} = \frac{1}{(2\pi)^d}\left[\frac{N}{j(k-j)(N-k)}\right]^{d/2}\left[\frac{\det Q(b-a)}{\det Q\big(-\frac{a}{s_1}\big)\ \det Q\ \det Q\big(\frac{b}{s_2}\big)}\right]^{1/2}.$$

In the part I_N^3, we decompose the summation in j and k into the part over

$$A := \big\{(j,k); |j - Nt_1| \le N^{3/5},\ |k - N(1-t_2)| \le N^{3/5}\big\}, \tag{4.6}$$

and over its complement, where $t_1 = t_1^D$ and $t_2 = t_2^D$ are determined by the Young's relation (1.9). We always assume that N is large enough so that $Nt_1 + N^{3/5} < N(1-t_2) - N^{3/5}$. Using Proposition 3.3, we get

$$\begin{aligned}\sum_{(j,k)\notin A} & \Xi^{\varepsilon}_{N,j,k}\gamma^{a,0}_{0,j}(\delta)\gamma^{0,0,\varepsilon}_{j,k}(\delta)\gamma^{0,b}_{k,N}(\delta) \\ &\le \sum_{(j,k)\notin A}\Xi^{\varepsilon}_{N,j,k} \le C\sum_{(j,k)\notin A}\alpha_{N,j,k}(k-j)^{d/2}e^{-Nf(s_1,s_2)},\end{aligned}$$

for some $C>0$, where

$$f(s_1,s_2) = \tilde f(s_1,s_2) - \xi^{D,\varepsilon}(1-s_1-s_2). \tag{4.7}$$

However, since the third condition in $(C)_D$ is equivalent to $f(t_1,t_2)=0$ and the Young's relation (1.9) implies $\partial f/\partial s_1(t_1,t_2) = \partial f/\partial s_2(t_1,t_2) = 0$, the Taylor's theorem gives the expansion of $f(s_1,s_2)$:

$$\begin{aligned} f(s_1,s_2) = {} & \frac{1}{2t_1^3}(a\cdot\nabla)^2\Lambda^*\Big(-\frac{a}{t_1}\Big)(s_1-t_1)^2 + \frac{1}{2t_2^3}(b\cdot\nabla)^2\Lambda^*\Big(\frac{b}{t_2}\Big)(s_2-t_2)^2 \\ & + O(|s_1-t_1|^3 + |s_2-t_2|^3), \end{aligned} \tag{4.8}$$

for s_1 and s_2 close to t_1 and t_2, respectively; we use the condition $\Lambda^* \in C^3(\boldsymbol{R}^d)$ required in Assumption 1.1-(2). Therefore, since $f(s_1,s_2)>0$ except $(s_1,s_2) = (t_1,t_2)$, we have

$$Nf(s_1,s_2) \ge CN^{1/5},$$

on the complement A^c with some $C>0$ and thus

$$\sum_{(j,k)\notin A}\Xi^{\varepsilon}_{N,j,k} \le e^{-cN^{1/5}} \tag{4.9}$$

for some $c > 0$, and large enough N.

For $(j,k) \in A$, the expansion (4.8) shows

$$f(s_1, s_2) = c_1(s_1 - t_1)^2 + c_2(s_2 - t_2)^2 + O(N^{-6/5}),$$

where

$$c_1 = \frac{1}{2t_1^3}(a \cdot \nabla)^2 \Lambda^* \Big(-\frac{a}{t_1}\Big), \quad c_2 = \frac{1}{2t_2^3}(b \cdot \nabla)^2 \Lambda^* \Big(\frac{b}{t_2}\Big). \tag{4.10}$$

Furthermore, the straight lines $g^{a,0}_{[0,s_1]}$ and $g^{0,b}_{[1-s_2,1]}$ are within distance $\delta/2$ to the restrictions of $\hat{h}_{[0,s_1]}$ and $\hat{h}_{[1-s_2,1]}$, respectively, if N is large enough, and therefore, using Lemma 4.1 and Theorem 5.1 below (in fact, Proposition 5.7 is sufficient), we get

$$\begin{aligned} \sum_{(j,k)\in A} \Xi^{\varepsilon}_{N,j,k}(1 - e^{-cN}) &\le \sum_{(j,k)\in A} \Xi^{\varepsilon}_{N,j,k} \gamma^{a,0}_{0,j}(\delta)\gamma^{0,0,\varepsilon}_{j,k}(\delta)\gamma^{0,b}_{k,N}(\delta) \\ &\le \sum_{(j,k)\in A} \Xi^{\varepsilon}_{N,j,k}, \end{aligned} \tag{4.11}$$

for some $c > 0$. It therefore suffices to estimate $\sum_{(j,k)\in A} \Xi^{\varepsilon}_{N,j,k}$. By using (4.4), Proposition 3.3 and substituting $j - [Nt_1]$ and $k - [N(1-t_2)]$ into j and k, we have by a Riemann sum approximation

$$\begin{aligned} \varepsilon^2 \sum_{(j,k)\in A} \Xi^{\varepsilon}_{N,j,k} &\sim C_1 N^{-d/2} \sum_{|j| \le N^{3/5}} e^{-c_1(j/\sqrt{N})^2} \sum_{|k| \le N^{3/5}} e^{-c_2(k/\sqrt{N})^2} \\ &\sim C_1 N^{1-d/2} \int_{-\infty}^{\infty} e^{-c_1 x^2}\, dx \int_{-\infty}^{\infty} e^{-c_2 x^2}\, dx \\ &= \frac{C_1 \pi}{\sqrt{c_1 c_2}} N^{1-d/2}, \end{aligned} \tag{4.12}$$

as $N \to \infty$, with

$$C_1 = \frac{\varepsilon^2 C^{D,\varepsilon}}{(4\pi^2 t_1 t_2)^{d/2}} \left[\frac{\det Q(b-a)}{\det Q\big(-\frac{a}{t_1}\big) \, \det Q \, \det Q\big(\frac{b}{t_2}\big)} \right]^{1/2},$$

where $C^{D,\varepsilon}$ is the constant given in (3.9).

Summarizing, we get from (4.1), (4.3), (4.9) and (4.12), for sufficiently large N

$$\begin{aligned} R_N^D &= \frac{C_1\pi}{\sqrt{c_1c_2}} N^{1-d/2}(1-O(e^{-cN})) + O\big(e^{-cN^{1/5}}\big) + O(e^{-cN}) \\ &\sim \frac{C_1\pi}{\sqrt{c_1c_2}} N^{1-d/2}. \end{aligned} \tag{4.13}$$

On the other hand, the definition (1.1) of $\mu_N^{D,\varepsilon}$ implies for every $0<\delta<|a|\wedge|b|$ that

$$\frac{Z_N^{a,b,\varepsilon}}{Z_N^{a,b}} \mu_N^{D,\varepsilon}\big(h^N \in U_\delta(\bar h)\big) = \mu_N^{D,0}\big(h^N \in U_\delta(\bar h)\big) \sim 1,$$

where $\bar h = \bar h^D$. Comparing with (4.13), we have the conclusion of Theorem 1.3-(1) by recalling that (1.8) implies

$$\lim_{N\to\infty} \big\{\mu_N^{D,\varepsilon}\big(h^N \in U_\delta(\hat h)\big) + \mu_N^{D,\varepsilon}\big(h^N \in U_\delta(\bar h)\big)\big\} = 1. \tag{4.14}$$

In particular, if $d=2$, the coexistence of $\bar h$ and $\hat h$ occurs in the limit with probabilities

$$\big(\bar\lambda^{D,\varepsilon}, \hat\lambda^{D,\varepsilon}\big) := \left(\frac{1}{1+C_2}, \frac{C_2}{1+C_2}\right), \tag{4.15}$$

where $C_2 = C_1\pi/\sqrt{c_1c_2} > 0$.

4.2. Proof of Theorem 1.3-(2).

Let $\mu_N^{a,F} (= \mu_N^{F,0})$ be the measure defined on $(\boldsymbol{R}^d)^{D_N}$ without pinning and having the normalizing constant $Z_N^{a,F} (= Z_N^{a,F,0})$:

$$\mu_N^{a,F}(d\phi) = \frac{\mathfrak{p}_N(\phi)}{Z_N^{a,F}} \delta_{aN}(d\phi_0) \prod_{i=1}^N d\phi_i. \tag{4.16}$$

For $0 \le j < k \le N$, one can define the measure $\mu_{j,k}^{0,F,\varepsilon}$ on $(\boldsymbol{R}^d)^{\{j,\dots,k\}}$ with pinning, the condition $\phi_j = 0$ at j, and the free condition (no specific condition) at k, having the normalizing constant $Z_{k-j}^{0,F,\varepsilon}$. The expansion of the product measure $\prod_{i=1}^N (\varepsilon\delta_0(d\phi_i) + d\phi_i)$ in (1.2) by specifying $0 < i_\ell \le N+1$ leads to

$$
\begin{aligned}
R_N^F &:= \frac{Z_N^{a,F,\varepsilon}}{Z_N^{a,F}} \mu_N^{F,\varepsilon}\big(h^N \in U_\delta(\hat{h})\big) \\
&= \mu_N^{a,F}\big(h^N \in U_\delta(\hat{h})\big) \\
&\quad + \sum_{j=1}^{N} \varepsilon \Xi_{N,j}^{F,\varepsilon} \mu_{0,j}^{a,0}\big(h_{[0,j/N]}^N \in U_\delta(\hat{h}_{[0,j/N]})\big) \mu_{j,N}^{0,F,\varepsilon}\big(h_{[j/N,1]}^N \in U_\delta(\hat{h}_{[j/N,1]})\big) \\
&=: I_N^{1,F} + I_N^{2,F},
\end{aligned}
\tag{4.17}
$$

where $\hat{h} = \hat{h}^F$ in this subsection and

$$
\Xi_{N,j}^{F,\varepsilon} = \frac{Z_j^{a,0} Z_{N-j}^{0,F,\varepsilon}}{Z_N^{a,F}}
$$

for $1 \le j \le N$. Noting that $Z_n^{a,F} = Z_n^{0,F} = 1$ and recalling Lemma 3.1 for $Z_j^{a,0}$, we see that

$$
\Xi_{N,j}^{F,\varepsilon} \sim (2\pi j)^{-d/2} \bigg(\det Q\Big(-\frac{a}{s_1}\Big)\bigg)^{-1/2} e^{-N\tilde{f}(s_1)} \cdot \frac{Z_{N-j}^{0,F,\varepsilon}}{Z_{N-j}^{0,F}},
$$

where $s_1 = j/N$ and $\tilde{f}(s_1) = s_1 \Lambda^*(-a/s_1)$.

We put here

$$
A := \{ j; |j - Nt_1| \le N^{3/5} \},
$$

where $t_1 = t_1^F$, and arrive in the same way as in Section 4.1, using the large deviation estimate for $\mu_{0,j}^{a,0}$ and $\mu_{j,N}^{0,F,\varepsilon}$ (cf. Theorem 5.1 below), to

$$
R_N^F = \varepsilon \sum_{j\in A} \Xi_{N,j}^{F,\varepsilon}\big(1 - O(e^{-cN})\big) + O\big(e^{-cN^{1/5}}\big) + O(e^{-cN}),
\tag{4.18}
$$

for some $c > 0$. Furthermore, we get by Proposition 3.4,

$$
\varepsilon \sum_{j\in A} \Xi_{N,j}^{F,\varepsilon} \sim \varepsilon C^{F,\varepsilon} (2\pi)^{-d/2} \bigg(\det Q\Big(-\frac{a}{t_1}\Big)\bigg)^{-1/2} \sum_{j\in A} (Ns_1)^{-d/2} e^{-Nf^F(s_1)},
$$

where $C^{F,\varepsilon}$ is the constant given in (3.13) and $f^F(s) = \tilde{f}(s) - \xi^{F,\varepsilon}(1-s)$. By the final condition in $(C)_F$, the Young's relation (1.9) and the Taylor's theorem, we have the expansion of f^F:

$$f^F(s_1) = \frac{1}{2t_1^3}(a \cdot \nabla)^2 \Lambda^* \Big(-\frac{a}{t_1} \Big)(s_1 - t_1)^2 + O(|s_1 - t_1|^3), \tag{4.19}$$

for s_1 close to t_1. This finally proves, recalling (4.18), that

$$\begin{aligned} R_N^F &\sim C_3 N^{-d/2} \sum_{|j| \le N^{3/5}} e^{-c_3 (j/\sqrt{N})^2} \\ &\sim C_3 N^{(1-d)/2} \int_{-\infty}^{\infty} e^{-c_3 x^2}\, dx = C_3 \sqrt{\frac{\pi}{c_3}} N^{(1-d)/2}, \end{aligned} \tag{4.20}$$

as $N \to \infty$, with

$$C_3 = \frac{\varepsilon C^{F,\varepsilon}}{(2\pi t_1)^{d/2} \sqrt{\det Q\big(-\frac{a}{t_1} \big)}} \quad \text{and} \quad c_3 = \frac{1}{2t_1^3}(a \cdot \nabla)^2 \Lambda^* \Big(-\frac{a}{t_1} \Big). \tag{4.21}$$

On the other hand, for every $0 < \delta < |a|$, we have that

$$\frac{Z_N^{a,F,\varepsilon}}{Z_N^{a,F}} \mu_N^{F,\varepsilon}\big(h^N \in U_\delta(\bar h)\big) = \mu_N^{F,0}\big(h^N \in U_\delta(\bar h)\big) \sim 1,$$

where $\bar h = \bar h^F$. Comparing this with (4.20), and recalling (1.8), the conclusion of Theorem 1.3-(2) is proved. In particular, if $d = 1$, the coexistence of $\bar h$ and $\hat h$ occurs in the limit with probabilities

$$\big(\bar\lambda^{F,\varepsilon}, \hat\lambda^{F,\varepsilon}\big) := \Big(\frac{1}{1 + C_4}, \frac{C_4}{1 + C_4} \Big), \tag{4.22}$$

where $C_4 = C_3 \sqrt{\pi / c_3} > 0$.

REMARK 4.1. Consider the times i_ℓ and i_r when the Markov chains first respectively last touch 0 under the scaling: $X = (i_\ell - t_1 N)/\sqrt{N}$ and $Y = (i_r - (1 - t_2)N)/\sqrt{N}$. Then, the following central limit theorem can be shown in a similar manner to [**2**] based on the computations leading to (4.12), (4.13), (4.20) and others: Under $\mu_N^{D,\varepsilon}$, conditioned on the event $\{i_\ell \le N - 1\}$ if $d \ge 2$, the pair of random variables (X, Y) weakly converges to (U_1, U_2) as $N \to \infty$, where $U_1 = N(0, 1/2c_1)$ and $U_2 = N(0, 1/2c_2)$ are mutually independent centered Gaussian random variables, and c_1 and c_2 are given by (4.10), while under $\mu_N^{F,\varepsilon}$ conditioned on the event $\{i_\ell \le N\}$, X weakly converges to $U = N(0, 1/2c_3)$ as $N \to \infty$, where c_3 is given by (4.21).

5. Large deviation principle.

The goal of this section is to show the sample path large deviation principle (LDP). Here we do not require the conditions $(C)_D$ nor $(C)_F$.

5.1. Formulation of results.

THEOREM 5.1. *The LDP holds for $h^N = \{h^N(t); t \in D\}$ distributed under $\mu_N = \mu_N^{D,\varepsilon}$ and $\mu_N^{F,\varepsilon}$ on the space $\mathscr{C}$ as $N \to \infty$ with the speed N and the good rate functionals $I = I^D$ and I^F of the form*:

$$I(h) = \begin{cases} \Sigma(h) - \inf_{\mathscr{AC}} \Sigma, & \text{if } h \in \mathscr{AC}, \\ +\infty, & \text{otherwise}, \end{cases} \tag{5.1}$$

with $\Sigma = \Sigma^D$ and Σ^F given by (1.5), *where $\mathscr{AC} = \mathscr{AC}_{a,b}$ and $\mathscr{AC}_{a,F}$, respectively, and $\inf_{\mathscr{AC}} \Sigma$ is taken over the space $\mathscr{AC}$. Namely, for every open set $\mathfrak{O}$ and closed set $\mathfrak{C}$ of $\mathscr{C}$ equipped with the uniform topology, we have that*

$$\begin{aligned} \liminf_{N\to\infty} \frac{1}{N} \log \mu_N(h^N \in \mathfrak{O}) &\ge - \inf_{h\in\mathfrak{O}} I(h), \\ \limsup_{N\to\infty} \frac{1}{N} \log \mu_N(h^N \in \mathfrak{C}) &\le - \inf_{h\in\mathfrak{C}} I(h), \end{aligned} \tag{5.2}$$

in each of two situations.

5.2. The LDP without pinning.

We will show the LDP for $\{h^N\}_N$ distributed under $\mu_N^{a,b} = \mu_N^{D,0}$. The LDP for $\mu_N^{a,F}$, i.e., the case with the free condition at the right end point, was established by Mogul'skii [**13**]; see also Section 5.1 of [**4**].

5.2.1. Results.

Let $\mathscr{C}_{a,b}$ be the family of all $h \in \mathscr{C}$ such that $h(0) = a$ and $h(1) = b$. We set

$$\Sigma_0(h) = \begin{cases} \int_D \Lambda^*(\dot h(t))\, dt, & \text{if } \ h \in \mathscr{AC}_{a,b}, \\ +\infty, & \text{if } \ h \in \mathscr{C}_{a,b} \setminus \mathscr{AC}_{a,b}, \end{cases}$$

and

$$I_0(h) = \Sigma_0(h) - \inf_{\mathscr{AC}_{a,b}} \Sigma_0.$$

PROPOSITION 5.2. *The family of macroscopic paths $\{h^N\}_N$ distributed under $\mu_N^{a,b} = \mu_N^{D,0}$ satisfies the LDP on the space $\mathscr{C}_{a,b}$ with speed N and the good rate functional $I_0(h)$, namely, for every open set $\mathfrak{O}$ and closed set $\mathfrak{C}$ of $\mathscr{C}_{a,b}$, we have the lower and upper bounds* (5.2) *for $\mu_N^{a,b}$ and I_0 in place of μ_N and I, respectively.*

REMARK 5.1. Deuschel, Giacomin and Ioffe [**5**] proved the LDP for $\mu_N^{a,b}$ in the L^2-topology, even for the Markov fields rather than the Markov chains discussed in this paper, under the log-concavity condition on p. Such condition was needed to characterize all (infinite-volume) Gibbs measures for the corresponding gradient fields, which are simply the superpositions of certain product measures in our setting. Therefore, their method would work also in our setting. To improve the topology, one may show the exponential tightness which is actually easy; see Corollary 4.2.6 of [**4**].

We will follow the method used by Guo, Papanicolaou and Varadhan [**12**] to show the equivalence of ensemble for a sequence of canonical (conditional) probability measures, with an external field depending on t. This will be applied to show the law of large numbers (LLN) for the perturbed measure. Then, we will use the Cramér's trick to prove Proposition 5.2.

5.2.2. LLN for a perturbed measure.

For a step function λ on D, we introduce the perturbed measure $\mu_{N,\lambda}^{a,b}$ by

$$\mu_{N,\lambda}^{a,b}(d\phi) = \frac{\mathfrak{p}_N(\phi)}{Z_{N,\lambda}^{a,b}} \prod_{i=1}^{N} e^{\lambda(\frac{i}{N})\cdot(\phi_i - \phi_{i-1})} \prod_{i=1}^{N-1} d\phi_i,$$

under the boundary conditions $\phi_0 = aN$, $\phi_N = bN$.

Let $h \in \mathscr{C}_{a,b}$ be a polygon with corners at $t = k/m$, $0 \le k \le m$, $m \in \boldsymbol{N}$. We assume that N is divisible by m for simplicity. We define the step function λ_h by $\lambda_h(t) = \lambda(\dot{h}(t)), t \in D$.

PROPOSITION 5.3. *For the polygon h, we have that*

$$\lim_{N\to\infty} \mu_{N,\lambda_h}^{a,b}\big(\|h^N - h\|_\infty \ge \delta\big) = 0$$

for every $\delta > 0$.

Proof.

Step 1: The exponential tightness of the distributions on the space $\mathscr{C}_{a,b}$ of $\{h^N\}$ under $\mu_{N,\lambda}^{a,b}$ will be shown later, see Lemma 5.6 below. Then, the conclusion follows by showing the convergence of $\langle h^N, J\rangle$ to $\langle h, J\rangle$ in probability as $N \to \infty$ for every test function $J \in C^\infty(D, \boldsymbol{R}^d)$. To this end, it suffices to show that $\langle \dot{h}^N, J\rangle$ converges to $\langle \dot{h}, J\rangle$ in probability for every test function J.

Step 2: Note that

$$\langle \dot{h}^N, J\rangle = \frac{1}{N}\sum_{i=1}^N \eta_i \cdot \tilde{J}_i,$$

where $\eta_i = \phi_i - \phi_{i-1}, 1 \le i \le N$ and $\tilde{J}_i = N\int_{(i-1)/N}^{i/N} J(t)\,dt$.

We define the probability measure $\nu_{N,\lambda}$ on $(\boldsymbol{R}^d)^N$ by

$$\nu_{N,\lambda}(d\eta) = \frac{1}{\tilde{Z}_{N,\lambda}}\prod_{i=1}^N p(\eta_i) e^{\lambda(\frac{i}{N})\cdot\eta_i} d\eta_i, \quad \eta = (\eta_i)_{i=1}^N \in (\boldsymbol{R}^d)^N.$$

The conditional probability measure of $\nu_{N,\lambda}$ on the hyperplane $\{\eta | \frac{1}{N}\sum_{i=1}^N \eta_i = b-a\}$ is denoted by $\nu_{N,\lambda}^{b-a}$:

$$\nu_{N,\lambda}^{b-a}(\cdot) := \nu_{N,\lambda}\biggl(\,\cdot\,\biggm|\frac{1}{N}\sum_{i=1}^N \eta_i = b-a\biggr).$$

Let $f_{N,\lambda}(x)$ be the probability density of $\frac{1}{N}\sum_{i=1}^N \eta_i$ under the distribution $\nu_{N,\lambda}$, i.e.,

$$f_{N,\lambda}(x)dx = \nu_{N,\lambda}\biggl(\frac{1}{N}\sum_{i=1}^N \eta_i \in dx\biggr), \quad x \in \boldsymbol{R}^d.$$

The following lemma is an extension of Theorem 3.4 of [**12**] to the case with non-constant external field λ:

Lemma 5.4. *We have that*

$$\lim_{N\to\infty}\frac{1}{N}\log f_{N,\lambda}(y) = -\min_{\substack{x_1,\dots,x_m\in\boldsymbol{R}^d\ s.t.\\ \frac{1}{m}(x_1+\cdots+x_m)=y}} \frac{1}{m}\sum_{\ell=1}^m\{\Lambda^*(x_\ell) - \lambda_\ell\cdot x_\ell + \Lambda(\lambda_\ell)\},$$

uniformly in y on every compact subset of $\boldsymbol{R}^d$, where λ_ℓ is the value of the step function $\lambda(t)$ on the ℓth interval $D_\ell = ((\ell-1)/m, \ell/m]$, $1 \le \ell \le m$.

PROOF. Let X_ℓ be the average of η over the domain $\tilde{D}_\ell := ND_\ell \cap \boldsymbol{Z} \equiv ((\ell-1)N/m, \ell N/m] \cap \boldsymbol{Z}$:

$$X_\ell := \frac{m}{N} \sum_{i \in \tilde{D}_\ell} \eta_i,$$

and let $f^{(\ell)}_{N/m,\lambda}(x_\ell), x_\ell \in \boldsymbol{R}^d$ be the probability density of X_ℓ under $\nu_{N,\lambda}$. Then, noting the independence of $\{X_1, \dots, X_m\}$ under $\nu_{N,\lambda}$, we see that $f_{N,\lambda}(x)dx$ is nothing but the distribution of $\frac{1}{m}(x_1 + \cdots + x_m)$ under the product probability measure

$$\prod_{\ell=1}^m f^{(\ell)}_{N/m,\lambda}(x_\ell)\, dx_\ell.$$

This implies that

$$\begin{aligned} f_{N,\lambda}(y) = m \int_{(\boldsymbol{R}^d)^{m-1}} & f^{(m)}_{N/m,\lambda}\big(my - (x_1 + \cdots + x_{m-1})\big) \\ & \cdot \prod_{\ell=1}^{m-1} f^{(\ell)}_{N/m,\lambda}(x_\ell)\, dx_\ell, \quad y \in \boldsymbol{R}^d. \end{aligned} \tag{5.3}$$

In fact, taking any test function $\varphi \in C_0^\infty(\boldsymbol{R}^d)$, one can rewrite the integral $\int_{\boldsymbol{R}^d} \varphi(y) f_{N,\lambda}(y)\, dy$ by change of variables and obtains (5.3). However, from Theorem 3.4 in [**12**] applied for $f^{(\ell)}_{N/m,\lambda}$ (we take $-\log p(x) - \lambda_\ell \cdot x + \Lambda(\lambda_\ell)$ as the potential $\phi(x)$ in [**12**]), we see that

$$\lim_{N\to\infty} \frac{m}{N} \log f^{(\ell)}_{N/m,\lambda}(x) = -(\Lambda^*)^{(\ell)}(x),$$

uniformly in x on every compact subset of $\boldsymbol{R}^d$, where

$$\begin{aligned} (\Lambda^*)^{(\ell)}(v) &= \sup_{\lambda \in \boldsymbol{R}^d} \{\lambda \cdot v - \Lambda(\lambda + \lambda_\ell) + \Lambda(\lambda_\ell)\} \\ &= \Lambda^*(v) - \lambda_\ell \cdot v + \Lambda(\lambda_\ell). \end{aligned} \tag{5.4}$$

Now, the combination of (5.3) and (5.4) proves the conclusion. □

We now return to the proof of Proposition 5.3. Our goal is to show that $\frac{1}{N}\sum_{i=1}^N \eta_i \cdot \tilde{J}_i$ converges to $\langle \dot{h}, J\rangle$ in probability under $\nu_{N,\lambda}^{b-a}$ with $\lambda = \lambda_h$. To show this, we estimate by the exponential Chebyshev's inequality

$$\begin{aligned}
&\frac{1}{N}\log \nu_{N,\lambda}^{b-a}\left(\left|\frac{1}{N}\sum_{i=1}^N \eta_i \cdot \tilde{J}_i - \langle \dot{h}, J\rangle\right| > \delta\right)\\
&\le \frac{1}{N}\log\left[\int e^{N\theta\{\frac{1}{N}\sum_{i=1}^N \eta_i\cdot \tilde{J}_i - \langle \dot{h}, J\rangle - \delta\}}\, d\nu_{N,\lambda}^{b-a}\right.\\
&\qquad\qquad \left. + \int e^{-N\theta\{\frac{1}{N}\sum_{i=1}^N \eta_i\cdot \tilde{J}_i - \langle \dot{h}, J\rangle + \delta\}}\, d\nu_{N,\lambda}^{b-a}\right] \qquad (5.5)
\end{aligned}$$

for every $\theta > 0$. For the first integral on the right hand side, we have that

$$\int e^{\theta\sum_{i=1}^N \eta_i\cdot\tilde{J}_i}\, d\nu_{N,\lambda}^{b-a} = \left.\frac{\int 1_{\{\frac{1}{N}\sum_{i=1}^N \eta_i \in dx\}} e^{\theta\sum_{i=1}^N \eta_i\cdot\tilde{J}_i}\, d\nu_{N,\lambda}}{\nu_{N,\lambda}\big(\frac{1}{N}\sum_{i=1}^N \eta_i \in dx\big)}\right|_{x=b-a}.$$

The denominator is equal to $f_{N,\lambda}(x)dx$, while the numerator is equal to

$$\frac{\tilde{Z}_{N,\lambda}^\theta}{\tilde{Z}_{N,\lambda}} f_{N,\lambda}^\theta(x)dx.$$

Here $f_{N,\lambda}^\theta$ is the probability density of $\frac{1}{N}\sum_{i=1}^N \eta_i$ under the distribution

$$\nu_{N,\lambda}^\theta(d\eta) = \frac{1}{\tilde{Z}_{N,\lambda}^\theta}\prod_{i=1}^N p(\eta_i) e^{\lambda(\frac{i}{N})\cdot\eta_i} e^{\theta\eta_i\cdot\tilde{J}_i} d\eta_i.$$

If J is a step function on D, which takes constant-value J_ℓ on each subinterval $D_\ell, 1 \le \ell \le m$, we can apply Lemma 5.4 also for $f_{N,\lambda}^\theta$ by taking $\lambda_\ell + \theta J_\ell$ in place of λ_ℓ and have that

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\log f_{N,\lambda}^\theta(y)\\
&\quad = -\min_{\substack{x_1,\dots,x_m\in \boldsymbol{R}^d \text{ s.t.}\\ \frac{1}{m}(x_1+\cdots+x_m)=y}} \frac{1}{m}\sum_{\ell=1}^m \big\{\Lambda^*(x_\ell) - (\lambda_\ell + \theta J_\ell)\cdot x_\ell + \Lambda(\lambda_\ell+\theta J_\ell)\big\},
\end{aligned}$$

uniformly in y on every compact subset of $\boldsymbol{R}^d$. On the other hand, we have

$$\frac{1}{N}\log\frac{\tilde Z^{\theta}_{N,\lambda}}{\tilde Z_{N,\lambda}} = \frac{1}{N}\log E^{\nu_{N,\lambda}}\big[e^{\theta\sum_{i=1}^N \eta_i\cdot\tilde J_i}\big]$$
$$= \frac{1}{N}\log\left[\frac{\prod_{\ell=1}^m (e^{\Lambda(\lambda_\ell+\theta J_\ell)})^{N/m}}{\prod_{\ell=1}^m (e^{\Lambda(\lambda_\ell)})^{N/m}}\right] = \frac{1}{m}\sum_{\ell=1}^m \log\frac{e^{\Lambda(\lambda_\ell+\theta J_\ell)}}{e^{\Lambda(\lambda_\ell)}}.$$

These computations are summarized into

$$\lim_{N\to\infty}\frac{1}{N}\log\int e^{\theta\sum_{i=1}^N \eta_i\cdot\tilde J_i - N\theta\langle\dot h, J\rangle - N\theta\delta}\,d\nu^{b-a}_{N,\lambda}$$
$$= -\min_{\substack{x_1,\dots,x_m\in\boldsymbol{R}^d \text{ s.t.}\\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m}\sum_{\ell=1}^m \big\{\Lambda^*(x_\ell)-(\lambda_\ell+\theta J_\ell)\cdot x_\ell\big\}$$
$$+ \min_{\substack{x_1,\dots,x_m\in\boldsymbol{R}^d \text{ s.t.}\\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m}\sum_{\ell=1}^m \big\{\Lambda^*(x_\ell)-\lambda_\ell\cdot x_\ell\big\} - \theta\langle\dot h, J\rangle - \theta\delta. \tag{5.6}$$

We prepare the following lemma to prove that the right hand side of (5.6) is negative if $\theta > 0$ is sufficiently small.

Lemma 5.5. *For a step function λ satisfying $\int_0^1 v(\lambda(t))\,dt = b-a$, the minimizer of the variational problem*

$$\min_{\substack{x_1,\dots,x_m\in\boldsymbol{R}^d \textit{ s.t.}\\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m}\sum_{\ell=1}^m \big\{\Lambda^*(x_\ell)-\lambda_\ell\cdot x_\ell\big\}$$

is given by $\bar x = \{\bar x_\ell = v(\lambda_\ell)\}_{\ell=1}^m$.

Proof. At the minimal point $x = \{x_\ell\}_{\ell=1}^m$, $\nabla\Lambda^*(x_\ell) - \lambda_\ell = c$ should be satisfied with a constant $c \in \boldsymbol{R}^d$ chosen as $\frac{1}{m}\sum_{\ell=1}^m v(\lambda_\ell + c) = b-a$. But this is fulfilled by $c = 0$. □

Lemma 5.5 can be applied for the first variational problem in the right hand side of (5.6) as well. In fact, choosing $c(\theta) \in \boldsymbol{R}^d$ in such a way that $\int_0^1 v(\lambda(t) + \theta J(t) + c(\theta))\,dt = b-a$, we can rewrite the first variational problem into

$$\min_{\substack{x_1,\dots,x_m\in\boldsymbol{R}^d \text{ s.t.}\\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m}\sum_{\ell=1}^m \big\{\Lambda^*(x_\ell)-(\lambda_\ell+\theta J_\ell+c(\theta))\cdot x_\ell\big\} + c(\theta)\cdot(b-a),$$

which is equal to

$$\frac{1}{m}\sum_{\ell=1}^{m}\left\{\Lambda^*(v(\lambda_\ell+\theta J_\ell+c(\theta)))-(\lambda_\ell+\theta J_\ell+c(\theta))\cdot v(\lambda_\ell+\theta J_\ell+c(\theta))\right\}+c(\theta)\cdot(b-a),$$

by Lemma 5.5. We expand this formula in θ. Then, since $c(0)=0$, the main term (the first order term) coincides with the second term in the right hand side of (5.6) by noting Lemma 5.5 again. The second order term (the term of order θ in the expansion) is given by

$$\begin{aligned}\frac{\theta}{m}\sum_{\ell=1}^{m}&\big\{\nabla\Lambda^*(v(\lambda_\ell))\cdot\nabla v(\lambda_\ell)(J_\ell+c'(0))-(J_\ell+c'(0))\cdot v(\lambda_\ell)\\&\qquad-\lambda_\ell\cdot\nabla v(\lambda_\ell)(J_\ell+c'(0))\big\}+\theta c'(0)\cdot(b-a)\\&=-\frac{\theta}{m}\sum_{\ell=1}^{m}J_\ell\cdot v(\lambda_\ell)=-\theta\langle\dot h,J\rangle,\end{aligned}$$

recall that $\nabla\Lambda^*(v(\lambda_\ell))=\lambda_\ell$, $\frac{1}{m}\sum_{\ell=1}^{m}v(\lambda_\ell)=b-a$ and note that $\nabla v(\lambda)$ defines a $d\times d$ matrix. This exactly cancels with the term $-\theta\langle\dot h,J\rangle$ appearing in (5.6) and we have proved that the right hand side of (5.6) is strictly negative if $\theta>0$ is sufficiently small.

We can treat the second integral in the right hand side of (5.5) in a similar manner, and this completes the proof of Proposition 5.3. □

The final task of this subsection is to establish the exponential tightness of the distributions on the space $\mathscr{C}_{a,b}$ of $\{h^N\}$ under $\mu_{N,\lambda}^{a,b}$. In fact, once the next lemma is shown, this follows in a similar manner to the proof of Lemma 5.1.7 in [**4**].

Lemma 5.6. *Let λ be a step function on D as in Lemma 5.4. Then, for every $\delta<1$, we have that*

$$\limsup_{N\to\infty}\frac{1}{N}\log E^{\mu_{N,\lambda}^{a,b}}\left[e^{\delta\sum_{i=1}^{N}\Lambda^*(\phi_i-\phi_{i-1})}\right]<\infty.$$

Proof. For $\delta<1$, let $p^{(\delta)}(x)$ be the probability density defined by

$$p^{(\delta)}(x)=\frac{1}{z^{(\delta)}}p(x)e^{\delta\Lambda^*(x)},$$

where $z^{(\delta)} = \int_{\boldsymbol{R}^d} p(x)e^{\delta\Lambda^*(x)}\,dx < \infty$ if $\delta < 1$ from Lemma 5.1.14 in [**4**]. Then, $p^{(\delta)}$ satisfies the Cramér's condition:

$$\Lambda^{(\delta)}(\lambda) = \log \int_{\boldsymbol{R}^d} e^{\lambda\cdot x} p^{(\delta)}(x)\,dx < \infty. \tag{5.7}$$

Indeed, by applying Lemma 5.1.14 in [**4**] for the Cramér transform $p_{\bar\lambda}$ of p, we see that

$$\int_{\boldsymbol{R}^d} e^{\delta(\Lambda_{\bar\lambda})^*(x)} p_{\bar\lambda}(x)\,dx < \infty, \tag{5.8}$$

for all $\delta < 1$ and $\bar\lambda \in \boldsymbol{R}$, where

$$\Lambda_{\bar\lambda}(\lambda) \equiv \log \int_{\boldsymbol{R}^d} e^{\lambda\cdot x} p_{\bar\lambda}(x)\,dx = \Lambda(\lambda + \bar\lambda) - \Lambda(\bar\lambda)$$

and $(\Lambda_{\bar\lambda})^*$ is its Legendre transform

$$(\Lambda_{\bar\lambda})^*(v) \equiv \sup_{\lambda\in\boldsymbol{R}^d} \{\lambda\cdot v - \Lambda_{\bar\lambda}(\lambda)\} = \Lambda^*(v) - \bar\lambda\cdot v + \Lambda(\bar\lambda).$$

Inserting this into (5.8), we see that

$$\int_{\boldsymbol{R}^d} e^{(1-\delta)\bar\lambda\cdot x} p^{(\delta)}(x)\,dx < \infty,$$

which implies (5.7) by taking $\bar\lambda = \lambda/(1-\delta)$ for each $\lambda \in \boldsymbol{R}$.

Let $\nu_{N,\lambda}^{(\delta)}$ be the probability measure $\nu_{N,\lambda}$ defined by taking $p^{(\delta)}$ in place of p, that is,

$$\nu_{N,\lambda}^{(\delta)}(d\eta) = \frac{1}{\tilde Z_{N,\lambda}^{(\delta)}} \prod_{i=1}^N p^{(\delta)}(\eta_i) e^{\lambda(\frac{i}{N})\cdot\eta_i}\,d\eta_i,$$

with the normalizing constant $\tilde Z_{N,\lambda}^{(\delta)}$ and let $f_{N,\lambda}^{(\delta)}(x)$ be the probability density of $\frac{1}{N}\sum_{i=1}^N \eta_i$ under the distribution $\nu_{N,\lambda}^{(\delta)}$. Then, since $p^{(\delta)}$ satisfies the Cramér's condition, Lemma 5.4 can be applied for $p^{(\delta)}$ and we obtain that

$$\lim_{N\to\infty}\frac{1}{N}\log f_{N,\lambda}^{(\delta)}(y) = -\min_{\substack{x_1,\dots,x_m\in\boldsymbol{R}^d \text{ s.t.}\\ \frac{1}{m}(x_1+\cdots+x_m)=y}}\frac{1}{m}\sum_{\ell=1}^{m}\big\{(\Lambda^{(\delta)})^*(x_\ell)-\lambda_\ell\cdot x_\ell+\Lambda^{(\delta)}(\lambda_\ell)\big\}, \tag{5.9}$$

which is finite for each $y\in\boldsymbol{R}^d$.

We now rewrite the expectation in the statement of the lemma as

$$\begin{aligned} E^{\mu_{N,\lambda}^{a,b}}\big[e^{\delta\sum_{i=1}^N\Lambda^*(\phi_i-\phi_{i-1})}\big] &= \frac{\int 1_{\{\frac{1}{N}\sum_{i=1}^N\eta_i\in dx\}}e^{\delta\sum_{i=1}^N\Lambda^*(\eta_i)}\,d\nu_{N,\lambda}}{\nu_{N,\lambda}\big(\frac{1}{N}\sum_{i=1}^N\eta_i\in dx\big)}\Bigg|_{x=b-a}\\ &= \frac{(z^{(\delta)})^N\tilde{Z}_{N,\lambda}^{(\delta)}}{\tilde{Z}_{N,\lambda}}\times\frac{f_{N,\lambda}^{(\delta)}(b-a)}{f_{N,\lambda}(b-a)}. \end{aligned}$$

However, it is easy to see that

$$\frac{1}{N}\log\tilde{Z}_{N,\lambda}^{(\delta)} = \frac{1}{N}\sum_{i=1}^N\Lambda^{(\delta)}\left(\lambda\left(\frac{i}{N}\right)\right) = \frac{1}{m}\sum_{\ell=1}^m\Lambda^{(\delta)}(\lambda_\ell)<\infty.$$

This holds also for $\frac{1}{N}\log\tilde{Z}_{N,\lambda}$; take $\delta=0$. Thus, (5.9) together with this formula taken $\delta=0$ (for $f_{N,\lambda}(b-a)$) completes the proof of the lemma recalling that $z^{(\delta)}<\infty$. $\square$

5.2.3. Proof of the lower bound in Proposition 5.2.

Let h be the polygon considered in Proposition 5.3 and denote $\lambda=\lambda_h$. Then, for every $\delta>0$, we have

$$\mu_N^{a,b}(\|h^N-h\|_\infty\le\delta) = \frac{Z_{N,\lambda}^{a,b}}{Z_N^{a,b}}E^{\mu_{N,\lambda}^{a,b}}\big[e^{-\sum_{i=1}^N\lambda(\frac{i}{N})\cdot(\phi_i-\phi_{i-1})},\|h^N-h\|_\infty\le\delta\big]. \tag{5.10}$$

Here,

$$\begin{aligned} Z_{N,\lambda}^{a,b} &= \int_{(\boldsymbol{R}^d)^{N-1}}\mathfrak{p}_N(\phi)\prod_{i=1}^N e^{\lambda(\frac{i}{N})\cdot(\phi_i-\phi_{i-1})}\prod_{i=1}^{N-1}d\phi_i\Bigg|_{\phi_0=aN,\phi_N=bN}\\ &= \int\prod_{i=1}^N p(\eta_i)e^{\lambda(\frac{i}{N})\cdot\eta_i}d\eta_i\Bigg|_{\frac{1}{N}\sum_{i=1}^N\eta_i=b-a} \end{aligned}$$

$$= \frac{\int 1_{\{\frac{1}{N}\sum_{i=1}^N \eta_i \in dx\}} \prod_{i=1}^N p(\eta_i) e^{\lambda(\frac{i}{N})\cdot\eta_i} d\eta_i}{dx}\Bigg|_{x=b-a}$$
$$= \tilde{Z}_{N,\lambda} f_{N,\lambda}(b-a),$$

and $Z_N^{a,b} = \tilde{Z}_{N,0} f_{N,0}(b-a)$. Since it holds that

$$\left| \sum_{i=1}^N \lambda\left(\frac{i}{N}\right) \cdot (\phi_i - \phi_{i-1}) - N \int_0^1 \lambda(t) \cdot \dot{h}(t)\, dt \right| \le 2N\delta \|\lambda\|_{L^1(D)}$$

on the event $\{\|h^N - h\|_\infty \le \delta\}$, we have from (5.10) that

$$\begin{aligned} &\liminf_{N\to\infty} \frac{1}{N} \log \mu_N^{a,b}(\|h^N - h\|_\infty \le \delta) \\ &\quad \ge \lim_{N\to\infty} \frac{1}{N} \log \frac{\tilde{Z}_{N,\lambda}}{\tilde{Z}_{N,0}} + \lim_{N\to\infty} \frac{1}{N} \log \frac{f_{N,\lambda}(b-a)}{f_{N,0}(b-a)} \\ &\quad\quad - \int_0^1 \lambda(t) \cdot \dot{h}(t)\, dt - 2\delta \|\lambda\|_{L^1(D)} + \lim_{N\to\infty} \frac{1}{N} \log \mu_{N,\lambda}^{a,b}(\|h^N - h\|_\infty \le \delta). \end{aligned} \tag{5.11}$$

However, by the computations made in the last subsection, the first term in the right hand side of (5.11) is equal to

$$\frac{1}{m} \sum_{\ell=1}^m \Lambda(\lambda_\ell),$$

while the second in (5.11) is equal to

$$\begin{aligned} &- \min_{\substack{x_1,\dots,x_m \in \boldsymbol{R}^d \text{ s.t.} \\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m} \sum_{\ell=1}^m \left\{ \Lambda^*(x_\ell) - \lambda_\ell \cdot x_\ell + \Lambda(\lambda_\ell) \right\} \\ &+ \min_{\substack{x_1,\dots,x_m \in \boldsymbol{R}^d \text{ s.t.} \\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m} \sum_{\ell=1}^m \Lambda^*(x_\ell). \end{aligned}$$

Proposition 5.3 implies that the last term in (5.11) is 0. Thus, we have that

$$
\begin{aligned}
&\liminf_{N\to\infty} \frac{1}{N} \log \mu_N^{a,b}(\|h^N - h\|_\infty \le \delta) \\
&\ge - \min_{\substack{x_1,\dots,x_m \in \boldsymbol{R}^d \text{ s.t.} \\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m} \sum_{\ell=1}^m \{\Lambda^*(x_\ell) - \lambda_\ell \cdot x_\ell\} \\
&\qquad + \min_{\substack{x_1,\dots,x_m \in \boldsymbol{R}^d \text{ s.t.} \\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m} \sum_{\ell=1}^m \Lambda^*(x_\ell) - \int_0^1 \lambda(t)\cdot \dot{h}(t)\, dt - 2\delta \|\lambda\|_{L^1(D)} \\
&\ge - \int_0^1 \Lambda^*(\dot{h}(t))\, dt + \inf \Sigma_0 - 2\delta\|\lambda\|_{L^1(D)} \\
&= -I_0(h) - 2\delta \|\lambda\|_{L^1(D)}.
\end{aligned} \tag{5.12}
$$

Here, the second inequality follows from

$$
\begin{aligned}
&\min_{\substack{x_1,\dots,x_m \in \boldsymbol{R}^d \text{ s.t.} \\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m} \sum_{\ell=1}^m \{\Lambda^*(x_\ell) - \lambda_\ell \cdot x_\ell\} \\
&\qquad = \frac{1}{m} \sum_{\ell=1}^m \left\{\Lambda^*(v(\lambda_\ell)) - \lambda_\ell \cdot v(\lambda_\ell)\right\} = - \int_0^1 \Lambda(\lambda(t))\, dt
\end{aligned}
$$

by Lemma 5.5 and (3.2),

$$
\min_{\substack{x_1,\dots,x_m \in \boldsymbol{R}^d \text{ s.t.} \\ \frac{1}{m}(x_1+\cdots+x_m)=b-a}} \frac{1}{m} \sum_{\ell=1}^m \Lambda^*(x_\ell) = \inf_{\substack{g:\text{ polygons s.t.} \\ g(0)=a, g(1)=b}} \int_0^1 \Lambda^*(\dot{g}(t))\, dt \ge \inf \Sigma_0
$$

and $\lambda(t)\cdot \dot{h}(t) = \Lambda^*(\dot{h}(t)) + \Lambda(\lambda(t))$ by (3.2).

Now take an arbitrary open set $\mathfrak{O}$ of $\mathscr{C}_{a,b}$. Then, since $\{\|h^N - h\|_\infty \le \delta\} \subset \{h^N \in \mathfrak{O}\}$ for every polygon $h \in \mathfrak{O}$ and every sufficiently small $\delta > 0$, we see from (5.12) that

$$
\liminf_{N\to\infty} \frac{1}{N} \log \mu_N^{a,b}(h^N \in \mathfrak{O}) \ge - \inf_{h \in \mathfrak{O}:\text{polygons}} I_0(h).
$$

However, the (local Lipschitz) continuity of Λ^* implies that

$$
\inf_{h\in\mathfrak{O}} I_0(h) = \inf_{h \in \mathfrak{O}:\text{polygons}} I_0(h)
$$

and this completes the proof of the lower bound in the proposition.

5.2.4. Proof of the upper bound in Proposition 5.2.

For the upper bound, it is enough to show the following estimate for every $g \in \mathscr{A}\mathscr{C}_{a,b}$:

$$\limsup_{N\to\infty} \frac{1}{N} \log \mu_N^{a,b}(\|h^N - g\|_\infty < \delta) \le -I_0(g) + \theta, \tag{5.13}$$

for every $\theta > 0$ with some $\delta > 0$ (depending on θ), see the remark below (5.15). The exponential tightness for $\mu_N^{a,b}$ follows from Lemma 5.6.

For every $g \in \mathscr{A}\mathscr{C}_{a,b}$, since Assumption 1.1-(1) implies $\sup_{x\in\boldsymbol{R}^d} p(x) < \infty$, by Lemma 3.1, we have

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N} \log \mu_N^{a,b}(\|h^N - g\|_\infty < \delta) \\
&\quad\le \limsup_{N\to\infty} \frac{1}{N} \log \mu_{N-1}^{a,F}\big(\|h^N - g\|_{\infty,[0,1-1/N]} < \delta\big) + \Lambda^*(b-a).
\end{aligned}$$

By the relation

$$\begin{aligned}
&\mu_{N-1}^{a,F}\big(\|h^N - g\|_{\infty,[0,1-1/N]} < \delta\big) \\
&\quad= \mu_{N-1}^{a,F}\left(\left\|h^{N-1} - \frac{N}{N-1} g\left(\frac{N-1}{N}\,\cdot\right)\right\|_\infty < \frac{N}{N-1}\delta\right)
\end{aligned}$$

and the continuity of g, we can get

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N} \log \mu_{N-1}^{a,F}\big(\|h^N - g\|_{\infty,[0,1-1/N]} < \delta\big) \\
&\quad\le \limsup_{N\to\infty} \frac{1}{N} \log \mu_{N-1}^{a,F}(\|h^{N-1} - g\|_\infty < 2\delta).
\end{aligned}$$

Finally, by the LD upper bound for $\mu_N^{a,F}$, the relation $\Lambda^*(b-a) = \inf_{\mathscr{A}\mathscr{C}_{a,b}} \Sigma_0$ and the lower semi-continuity of $\Sigma_0(h)$, we have

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N} \log \mu_N^{a,b}(\|h^N - g\|_\infty < \delta) \\
&\quad\le -\inf_{h\in\{\|h-g\|_\infty\le 2\delta\}} \Sigma_0(h) + \inf_{\mathscr{A}\mathscr{C}_{a,b}} \Sigma_0 \le -I_0(g) + \theta,
\end{aligned}$$

for every $\theta > 0$ with some $\delta > 0$ (depending on θ).

5.3. Proof of Theorem 5.1.

For the proof of Theorem 5.1 for $\mu_N^{D,\varepsilon}$, it is enough to show the following two estimates for every $g \in \mathscr{AC}_{a,b}$:

$$\liminf_{N\to\infty} \frac{1}{N} \log \mu_N^{D,\varepsilon}(\|h^N - g\|_\infty < \delta) \geq -I^D(g), \tag{5.14}$$

for every $\delta > 0$, and

$$\limsup_{N\to\infty} \frac{1}{N} \log \mu_N^{D,\varepsilon}(\|h^N - g\|_\infty < \delta) \leq -I^D(g) + \theta, \tag{5.15}$$

for every $\theta > 0$ with some $\delta > 0$ (depending on θ), where I^D is defined by (5.1) with $\Sigma = \Sigma^D$ and $\mathscr{AC} = \mathscr{AC}_{a,b}$. This step of reduction is standard, for instance, see (6.6) and the estimate just above (6.11) in [**10**].

The proof of the lower bound (5.14) is similar to Section 4.3.1 of [**2**]. The only difference is that we should replace $\Sigma_0(a, b; t_1^1, t_2^K)$ in Lemma 4.6 of [**2**] by

$$\Sigma_0\big(a, b; t_1^1, t_2^K\big) = t_1^1 \Lambda^*\bigg(-\frac{a}{t_1^1}\bigg) + t_2^K \Lambda^*\bigg(\frac{b}{t_2^K}\bigg).$$

In fact, from (4.4), Proposition 3.3 and the formula (4.5) for $\tilde{f}(s_1, s_2)$, one can show that

$$\lim_{N\to\infty} \frac{1}{N} \log \frac{Z_N^{a,b}}{Z_N^{a,b,\varepsilon}} = -\Sigma_0(\bar{h}^D) + \inf_{\mathscr{AC}_{a,b}} \Sigma(h). \tag{5.16}$$

The equality (1) in Lemma 4.6 of [**2**] follows from (5.16) and Proposition 3.3 recalling Lemma 3.1. Another inequality (2) in that lemma is a consequence of Propositions 5.2 and 5.7 stated below. All other arguments are exactly the same.

Proposition 5.7. *For every $\delta > 0$, there exist $C, c > 0$ such that*

$$\mu_N^\varepsilon(\|h^N\|_\infty \geq \delta) \leq Ce^{-cN}$$

for $\mu_N^\varepsilon = \mu_N^{0,0,\varepsilon}$ and $\mu_N^{0,F,\varepsilon}$.

This proposition is shown in Proposition 4.3 of [**2**] or Proposition 2.1 of [**9**] for the Gaussian case. The general case can be proved from Proposition 5.2 by tracing the method used in Section 2.2 of [**9**], which is based on a renewal theory.

The proof of the upper bound (5.15) is also similar to Section 4.3.2 of [**2**]. We should replace $\int_{D\setminus I} |\dot{g}(t)|^2\, dt/2$ with $\int_{D\setminus I} \Lambda^*(\dot{g}(t))\, dt$ in the statement of Lemma 4.7 in [**2**] and the estimate on $I_N^j(\delta)$ in its proof with

$$I_N^j(\delta) \le \exp\left\{ N\left(-\int_0^s \Lambda^*(\dot{g}(t))dt + s\Lambda^*\left(-\frac{a}{s}\right) + \theta\right)\right\}.$$

Otherwise, all arguments are the same.

For the proof of Theorem 5.1 for $\mu_N^{F,\varepsilon}$, we may modify some arguments in the proof for $\mu_N^{D,\varepsilon}$ as indicated in Section 4.4 of [**2**].

6. Critical exponents for the free energies.

This section studies the asymptotic behavior of the free energies $\xi^{D,\varepsilon}$ and $\xi^{F,\varepsilon}$ near the critical values ε_c^D and ε_c^F, respectively; recall (3.8), (3.9), (3.12) and (3.13) for the definition of these quantities. The results are summarized in the following proposition.

PROPOSITION 6.1.
(1) (Dirichlet case) *As* $\varepsilon \downarrow \varepsilon_c^D$, *we have that*

$$\xi^{D,\varepsilon} \sim \begin{cases} C_d\big(\varepsilon-\varepsilon_c^D\big)^2, & d=1,3, \\ e^{-2\pi\sqrt{\det Q}/\varepsilon}, & d=2, \\ -\frac{C_4(\varepsilon-\varepsilon_c^D)}{\log(\varepsilon-\varepsilon_c^D)}, & d=4, \\ C_d\big(\varepsilon-\varepsilon_c^D\big), & d\ge 5, \end{cases}$$

where $C_1 = 1/(2\det Q)$, $C_3 = 2\pi^2\det Q/(\varepsilon_c^D)^4$, $C_4 = 4\pi^2\sqrt{\det Q}/(\varepsilon_c^D)^2$ *and* $C_d = 1/\big((\varepsilon_c^D)^2\sum_{n=1}^\infty n e^{n\Lambda^*(0)} Z_n^{0,0}\big)$ *for* $d \ge 5$.

(2) (Free case)

(i) *If* $m=0$, $\xi^{F,\varepsilon}$ *behaves exactly in the same way as* $\xi^{D,\varepsilon}$.

(ii) *If* $m \ne 0$, *as* $\varepsilon \downarrow \varepsilon_c^F$, *we have that*

$$\xi^{F,\varepsilon} \sim C_d^F\big(\varepsilon-\varepsilon_c^F\big),$$

for every $d\ge 1$, *where* $C_d^F = 1/\big((\varepsilon_c^F)^2\sum_{n=1}^\infty nZ_n^{0,0}\big)$.

For the proof of the proposition, we prepare a lemma which establishes the asymptotic behavior of the function:

$$q_d(x) = (2\pi)^{d/2}\sqrt{\det Q}\, g\big(e^{\Lambda^*(0)}x\big), \quad 0 \le x \le 1,$$

as $x \uparrow 1$, where $g(x) \equiv g_d(x)$ is the function defined by (3.7). We only consider the case $1 \le d \le 4$, since the case $d \ge 5$ is easy.

LEMMA 6.2. *As $x \uparrow 1$, we have that*

$$q_d(x) \sim \begin{cases} \sqrt{\pi}(1-x)^{-1/2}, & d=1, \\ -\log(1-x), & d=2, \end{cases}$$

and

$$q_d(1) - q_d(x) \sim \begin{cases} 2\sqrt{\pi}(1-x)^{1/2}, & d=3, \\ -(1-x)\log(1-x), & d=4. \end{cases}$$

PROOF. Let $f_d(x) = \sum_{n=1}^\infty x^n/n^{d/2}$, $0 \le x \le 1$, be the function defined by (A.1) of [**2**], whose asymptotics as $x \uparrow 1$ can be found in Lemma A.3 there. Then, we have that

$$q_d(x) - f_d(x) = \sum_{n=1}^\infty \frac{x^n}{n^{d/2}}\big\{(2\pi n)^{d/2}\sqrt{\det Q}\, e^{n\Lambda^*(0)} Z_n^{0,0} - 1\big\}.$$

However, since (3.4) in Lemma 3.1 shows that the difference in the braces in the right hand side tends to 0 as $n \to \infty$, one can show that, for every $\delta > 0$, there exists $C_\delta > 0$ such that

$$|q_d(x) - f_d(x)| \le \delta f_d(x) + C_\delta.$$

If $d = 1, 2$, since $f_d(x) \to \infty$ as $x \uparrow 1$, this implies that the asymptotics of q_d are the same as f_d. To show the asymptotics of $q_d(1) - q_d(x)$ for $d = 3, 4$, we see that, for every $\delta > 0$, there exists $C_\delta > 0$ such that

$$\big|x q_d'(x) - f_{d-2}(x)\big| \le \delta f_{d-2}(x) + C_\delta.$$

This can be proved similarly as above. Since $f_{d-2}(x) \to \infty$ as $x \uparrow 1$, this shows the asymptotics for $d = 3, 4$, cf. the proof of Lemma A.3. $\square$

Proof of Proposition 6.1. The assertion (1) for $1 \le d \le 4$ follows from Lemma 6.2 in a similar manner to the proof of Proposition A.1 of [**2**] recalling that $q_d(e^{-\xi^{D,\varepsilon}}) = (2\pi)^{d/2}\sqrt{\det Q}/\varepsilon$. The proof of the assertion (1) for $d \ge 5$ is easy from

$$g\big(e^{\Lambda^*(0)}\big) - g\big(e^{-\xi^{D,\varepsilon}+\Lambda^*(0)}\big) = \frac{1}{\varepsilon_c^D} - \frac{1}{\varepsilon}.$$

Indeed, the left hand side is asymptotically equivalent to $\xi^{D,\varepsilon}e^{\Lambda^*(0)}g'(e^{\Lambda^*(0)}-) = \xi^{D,\varepsilon}\sum_{n=1}^\infty ne^{n\Lambda^*(0)}Z_n^{0,0}$, while the right hand side behaves as $(\varepsilon-\varepsilon_c^D)/(\varepsilon_c^D)^2$; note that the series appeared above converges. The proof of the assertion (2) is immediate, since we have $\xi^{F,\varepsilon} = \xi^{D,\varepsilon}$ and $\varepsilon_c^F = \varepsilon_c^D$ if $m = 0$. The proof of the assertion (3) is similar as above by noting that

$$g(1) - g\big(e^{-\xi^{F,\varepsilon}}\big) = \frac{1}{\varepsilon_c^F} - \frac{1}{\varepsilon}. \qquad \square$$

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