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# Any 7-colorable knot can be colored by four colors

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**Abstract.** Any 7-colorable 1-knot has a 7-colored diagram such that exactly four colors of seven are assigned to the arcs of the diagram.

## 1. Introduction.

A *knot* is a circle smoothly embedded in $\boldsymbol{R}^3$. Any knot diagram $D$ is regarded as a disjoint union of arcs obtained from the projected planar curves by cutting the lower paths at crossings.

For an odd prime $p$, we consider an assignment of an element of $\boldsymbol{Z}_p$ to each arc of $D$. It is called a *$p$-coloring* [**2**], [**3**], [**4**] if $a + c = 2b$ holds in $\boldsymbol{Z}_p$ near each crossing, where the lower arcs are colored by $a$ and $c$ and the upper arc is colored by $b$. In this paper, we represent the elements of $\boldsymbol{Z}_p$ by $\{0, 1, \dots, p-1\}$. A $p$-coloring is *trivial* if all arcs of $D$ have the same color, and otherwise *non-trivial*. A knot which has a diagram with a non-trivial $p$-coloring is called a *$p$-colorable knot*.

For non-trivial $p$-colorings of knots, diagrams do not necessarily have all colors $0, 1, \dots, p-1$ of $\boldsymbol{Z}_p$. In [**5**], [**7**], the minimal number of colors assigned to the arcs among all non-trivially $p$-colored diagrams of a $p$-colorable knot $K$ is studied. We denote it by $C_p(K)$. A lower bound was given in [**7**], [**10**]:

Lemma 1.1 ([**7**], [**10**]). *If $p > 3$, then any non-trivial $p$-coloring for $D$ uses at least four colors of $0, 1, \dots, p-1$; namely, $C_p(K) \geq 4$ for any $p$-colorable knot $K$.*

In [**7**], Kauffman and Lopes studied $r$-colorings defined for any integer $r > 1$ which is not necessarily prime and odd. The above lemma is a consequence of Corollary 3.1 and Proposition 3.5 in [**7**].

For 5-colorable knots, Satoh gave the following theorem:

Theorem 1.2 ([**10**]). *Any 5-colorable knot has a 5-colored diagram with exactly four colors.*

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By Lemma 1.1, Theorem 1.2 shows that $C_5(K) = 4$ for any 5-colorable knot $K$.

For 7-colorable knots, it is known that there exist infinitely many knots each of which has a non-trivially 7-colored diagram with exactly four colors, see [**8**]. For all 7-colorable knots, Satoh gave the following question: *Does it hold $C_7(K) = 4$ for any 7-colorable knot $K$?* Our first theorem gives the answer:

Theorem 1.3. *Any 7-colorable knot has a 7-colored diagram with exactly four colors.*

By Lemma 1.1, Theorem 1.3 implies $C_7(K) = 4$ for any 7-colorable knot $K$.

Remark 1.4. Kauffman and Lopes [**7**] gave the following conjecture about $p$-colorable $(2, n)$-torus knots $T(2, n)$: For any positive integers $p$ and $n$ such that the least common prime divisor of $p$ and $n$ is equal to $2k + 1$ for some integer $k > 2$, $C_p(T(2, n)) = k + 2$. It says that $C_7(T(2, 7)) = 5$. By Theorem 1.3, we have $C_7(T(2, 7)) = 4$. This is a counterexample to the conjecture.

A 7-coloring is also defined for diagrams of 2-dimensional knots (2-spheres in $\boldsymbol{R}^4$). For the family of ribbon 2-knots, we have the following theorem:

Theorem 1.5. *Any 7-colorable ribbon 2-knot has a 7-colored diagram with exactly four colors.*

Section 2 is devoted to examine about 7-colorable 1-knots and we give the proof of Theorem 1.3. In Section 3, we examine 7-colorable ribbon 2-knots and prove Theorem 1.5.

## 2. 7-colorable 1-knots.

Let $K$ be a 7-colorable knot which has a 7-colored diagram $D$ with exactly four colors. Then the colors given to the diagram $D$ can be assumed to be 0, 1, 2 and 4, see [**8**]. What we prove is that all non-trivial 7-colored diagrams can be transformed to diagrams colored by the four elements 0, 1, 2 and 4.

For a crossing of a 7-colored diagram, we say that the color of the crossing is $\{a|b|c\}$ (or $\{c|b|a\}$) or the crossing is of $\{a|b|c\}$-type (or of $\{c|b|a\}$-type) if the lower arcs are colored by $a$ and $c$ and the upper arc is colored by $b$.

For the proof of Theorem 1.3, we prepare the following lemmas:

Lemma 2.1. *Any 7-colorable knot has a non-trivially 7-colored diagram $D$ with no arc colored by 6.*

Proof. Let $D$ be a non-trivially 7-colored diagram. We need three steps for the proof. The first step is to prove that any 7-colorable knot has a non-trivially

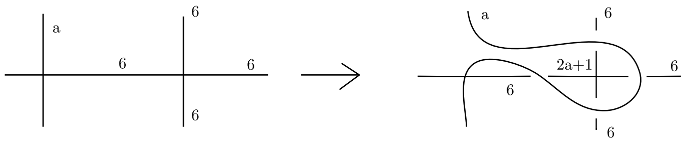


Figure 1. Deformation near a crossing of $\{6|6|6\}$-type.

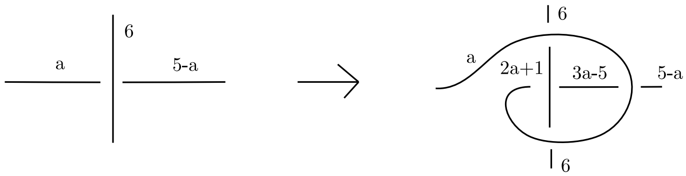


Figure 2. Deformation near a crossing of $\{a|6|5-a\}$-type.

7-colored diagram $D$ with no crossing whose color is $\{6|6|6\}$. The second step is to prove that any 7-colorable knot has a non-trivially 7-colored diagram $D$ with no crossing whose color is $\{*|6|*\}$, where $*$ represents any element of $\boldsymbol{Z}_p$. The final step is to complete the proof.

Assume that $D$ has a crossing of $\{6|6|6\}$-type. It is easy to see that there exists an adjacent pair of crossings $P$ and $Q$ such that $P$ is of $\{6|6|6\}$-type and $Q$ is of $\{6|a|2a+1\}$-type or of $\{a|6|5-a\}$-type for some $a \neq 6$. We deform the arc with the color $a$ such that the arc detours around $P$ passing over the arcs as in Figure 1, where we do not indicate over-under information at $P$ and $Q$. Then the color of $P$ changes into $\{2a+1|2a+1|2a+1\}$. And the new crossings are of $\{2a+1|a|6\}$-type. Notice that $2a+1 \neq 6$ for any $a \neq 6$. We repeat the deformation above if the obtained diagram still has a crossing of $\{6|6|6\}$-type.

We may assume that $D$ has no crossing whose color is $\{6|6|6\}$ by the above deformation. Assume that $D$ has a crossing of $\{a|6|5-a\}$-type for some $a \neq 6$. Then we deform the arc with the color $a$ such that the arc detours around the crossing as in Figure 2. Then the color of the original crossing changes into $\{a|2a+1|3a+2\}$, and the new crossings are of $\{6|a|2a+1\}$-type and of $\{3a+2|a|5-a\}$-type. Notice that $2a+1 \neq 6$ and $3a+2 \neq 6$ for any $a \neq 6$. We repeat the deformation above if the obtained diagram still has a crossing of $\{a|6|5-a\}$-type for some $a \neq 6$.

We may assume that the upper arc of any crossing of $D$ does not have the color 6 by the above deformation. Hence, each arc with the color 6 connects a pair of crossings directly whose colors are $\{6|a|2a+1\}$ and $\{6|b|2b+1\}$ for some $a, b \neq 6$. When $b = a$ ($b = 2a+1$ or otherwise), we deform the arc colored by $2a+1$ ($2a+1$ or $a$) as shown in the right side of Figure 3 (in the middle side of Figure 3 or in the left side of Figure 3, respectively) so that the arc with the

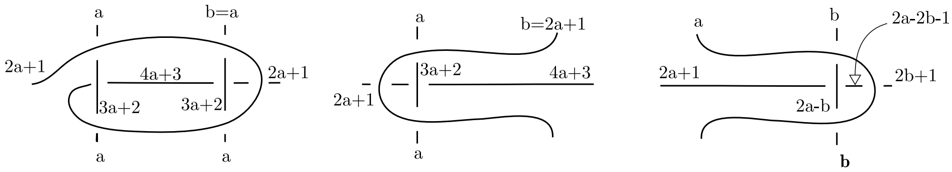


Figure 3. Deformation for eliminating the color 6.

| $a\backslash b$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | $*$ | $*$ | 5 | 4 | 3 | 2 |
| 1 | 2 | $*$ | 0 | $*$ | 5 | 4 |
| 2 | 4 | 3 | $*$ | 1 | 0 | $*$ |
| 3 | $*$ | 5 | 4 | $*$ | 2 | 1 |
| 4 | 1 | 0 | $*$ | 5 | $*$ | 3 |
| 5 | 3 | 2 | 1 | 0 | $*$ | $*$ |

Table 1. Table of $2a-b$.

| $a\backslash b$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | $*$ | $*$ | 2 | 0 | 5 | 3 |
| 1 | 1 | $*$ | 4 | $*$ | 0 | 5 |
| 2 | 3 | 1 | $*$ | 4 | 2 | $*$ |
| 3 | $*$ | 3 | 1 | $*$ | 4 | 2 |
| 4 | 0 | 5 | $*$ | 1 | $*$ | 4 |
| 5 | 2 | 0 | 5 | 3 | $*$ | $*$ |

Table 2. Table of $2a-2b-1$.

color 6 is eliminated. Notice that each new arc which is obtained by the above deformation does not have the color 6, refer to Tables 1 and 2. We repeat the deformation above if the obtained diagram still has an arc whose color is 6. □

LEMMA 2.2. *Any 7-colorable knot has a non-trivially 7-colored diagram $D$ with no arc colored by 5 or 6.*

PROOF. Let $D$ be a non-trivially 7-colored diagram. By Lemma 2.1, we may assume that no arc of $D$ has the color 6. As in the proof of Lemma 2.1, we need three steps for the proof. The first step is to prove that any 7-colorable knot has a non-trivially 7-colored diagram $D$ with no crossing whose color is $\{5|5|5\}$. The second step is to prove that any 7-colorable knot has a non-trivially 7-colored diagram $D$ with no crossing whose color is $\{*|5|*\}$. The final step is to complete the proof. Notice that for these steps, the resulting diagrams have no arc colored by 6.

Assume that $D$ has a crossing of $\{5|5|5\}$-type. There exists an adjacent pair of crossings $P$ and $Q$ such that $P$ is of $\{5|5|5\}$-type and $Q$ is of $\{5|a|2a+2\}$-type or of $\{a|5|3-a\}$-type for some $a \neq 5, 6$. By the assumption, if $Q$ is of $\{5|a|2a+2\}$-type, then the color $2a+2$ does not equal 6, i.e., $a \neq 2$ and if $Q$ is of $\{a|5|3-a\}$-type, then the color $3-a$ does not equal 6, i.e., $a \neq 4$. It implies that $Q$ with the color $\{a|5|3-a\}$ is of $\{0|5|3\}$-type or $\{1|5|2\}$-type. When $Q$ is of $\{5|a|2a+2\}$-type, deform the arc with the color $a$ near $Q$ such that the arc detours

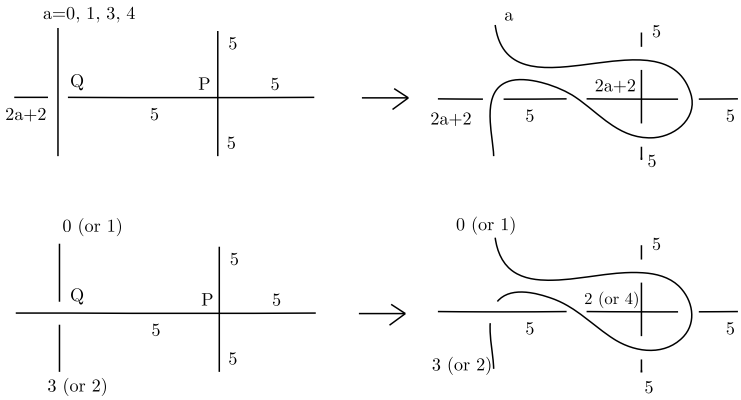


Figure 4. Deformation near a crossing of $\{5|5|5\}$-type.

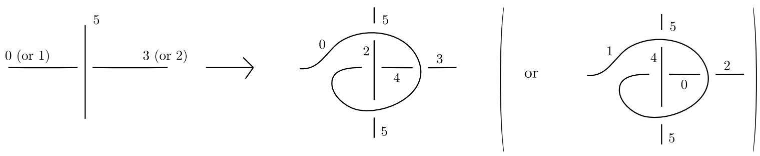


Figure 5. Deformation near a crossing of $\{0|5|3\}$-type or $\{1|5|2\}$-type.

around $P$ passing over the arcs as in Figure 4. Then the color of $P$ changes into $\{2a+2|2a+2|2a+2\}$. And the new crossings are of $\{2a+2|a|5\}$-type. Notice that $2a+2$ equals neither 5 nor 6 in the above conditions. When $Q$ is of $\{0|5|3\}$-type (or of $\{1|5|2\}$-type), deform the arc with the color 0 (or 1) near $Q$ such that the arc detours around $P$ passing over the arcs as in Figure 4. Then the color of $P$ changes into $\{2|2|2\}$ (or $\{4|4|4\}$) and the new crossings are of $\{2|0|5\}$-type (or of $\{4|1|5\}$-type). We repeat the deformation above if the obtained diagram still has a crossing of $\{5|5|5\}$-type.

We may assume that $D$ has no crossing whose color is $\{5|5|5\}$ and no arc colored by 6 by the above deformation and Lemma 2.1. Assume that $D$ has a crossing of $\{a|5|3-a\}$-type for some $a \neq 5, 6$. Since $3-a$ equals neither 5 nor 6, $a \neq 4$, i.e., the crossing is of $\{0|5|3\}$-type or of $\{1|5|2\}$-type. For each crossing of $\{0|5|3\}$-type (or of $\{1|5|2\}$-type), deform the arc colored by 0 (or 1) as in Figure 5. Then the color of the original crossing changes into $\{0|2|4\}$ (or $\{1|4|0\}$), and the new crossings are of $\{2|0|5\}$-type and of $\{3|0|4\}$-type (or of $\{4|1|5\}$-type and of $\{0|1|2\}$-type). We repeat the deformation above if the obtained diagram still has a crossing of $\{a|5|3-a\}$-type for some $a \neq 4, 5, 6$.

We may assume that the upper arc of any crossing of $D$ does not have the color 5 and there exists no arc colored by 6. Then each arc with the color 5

| $(a,b)$ | $(0,4)$ | $(1,0)$ | $(3,4)$ | $(4,1)$ |
|---|---|---|---|---|
| $2a-b$ | 3 | 2 | 2 | 0 |
| $2a-2b-2$ | 4 | 0 | 3 | 4 |

Table 3. Table of $2a-b$ and $2a-2b-2$.

connects a pair of crossings directly whose colors are $\{5|a|2a+2\}$ and $\{5|b|2b+2\}$ for some $a,b \neq 5,6$. Notice that since $2a+2$ and $2b+2$ equal neither 5 nor 6, $a \neq 2$ and $b \neq 2$. We denote by $A$ the crossing of $\{5|a|2a+2\}$-type and by $B$ the crossing of $\{5|b|2b+2\}$-type. According to the pair of $a$ and $b$, we deform arcs so that the arc with the color 5 is eliminated. First we examine the cases when the pair $(a,b)$ is equal to $(0,4)$, $(1,0)$, $(3,4)$ or $(4,1)$. Deform the arc colored by $a$ near $A$ as shown in Figure 6 (1). Then we obtain the new arcs which is colored by $a, b, 2a+2, 2b+2, 2a-b$ and $2a-2b-2$. Every value of $a, b, 2a+2, 2b+2, 2a-b$ and $2a-2b-2$ equals neither 5 nor 6 in these conditions, refer to Table 3. Next let us see the pair $(a,b) = (4,0)$, $(0,1)$, $(4,3)$ or $(1,4)$. Deform the arc colored by $b$ near $B$ as shown in Figure 6 (2). Then we obtain the new arcs which is colored by $a, b, 2a+2, 2b+2, 2b-a$ and $2b-2a-2$. Every value of $a, b, 2a+2, 2b+2, 2a-b$ and $2a-2b-2$ equals neither 5 nor 6 in these conditions, refer to Table 3. When the pair $(a,b)$ is equal to $(1,1)$ or $(4,4)$, deform arcs as shown in Figure 6 (3). Then we obtain the new arcs which is colored by $a, 2a+2, 3a+4$ and $4a-1$. Every value of $a, 2a+2, 3a+4$ and $4a-1$ equals neither 5 nor 6 in these conditions. In the case of $(a,b) = (0,0), (3,3), (0,3), (3,0), (1,3)$ or $(3,1)$, deform the arcs as shown in Figure 6 (4), (5), (6), (7), (8) or (9), respectively. Then all of the new arcs which are obtained by the above deformation do not have the color 5 nor 6. By these deformations, we can eliminate the color 5. We repeat the deformation above if the obtained diagram still has an arc whose color is 5. □

Proof of Theorem 1.3. Let $D$ be a non-trivially 7-colored diagram of a knot. By Lemma 2.2, we may assume that there exists no arc colored by 5 or 6. Let us consider deformations to eliminate the color 3. As in the proof of Lemmas 2.1 and 2.2, we need three steps for the proof: The first step is to prove that any 7-colorable knot has a non-trivially 7-colored diagram $D$ with no crossing whose color is $\{3|3|3\}$. The second step is to prove that any 7-colorable knot has a non-trivially 7-colored diagram $D$ with no crossing whose color is $\{*|3|*\}$. The final step is to complete the proof. Notice that for these steps, the resulting diagrams have neither arc colored by 5 nor arc colored by 6.

Assume that $D$ has a crossing of $\{3|3|3\}$-type. There exists an adjacent pair of crossings $P$ and $Q$ such that $P$ is of $\{3|3|3\}$-type and $Q$ is of $\{3|a|2a+4\}$-type or of $\{a|3|6-a\}$-type for some $a \neq 3,5,6$. If $Q$ is of $\{3|a|2a+4\}$-type, then the color

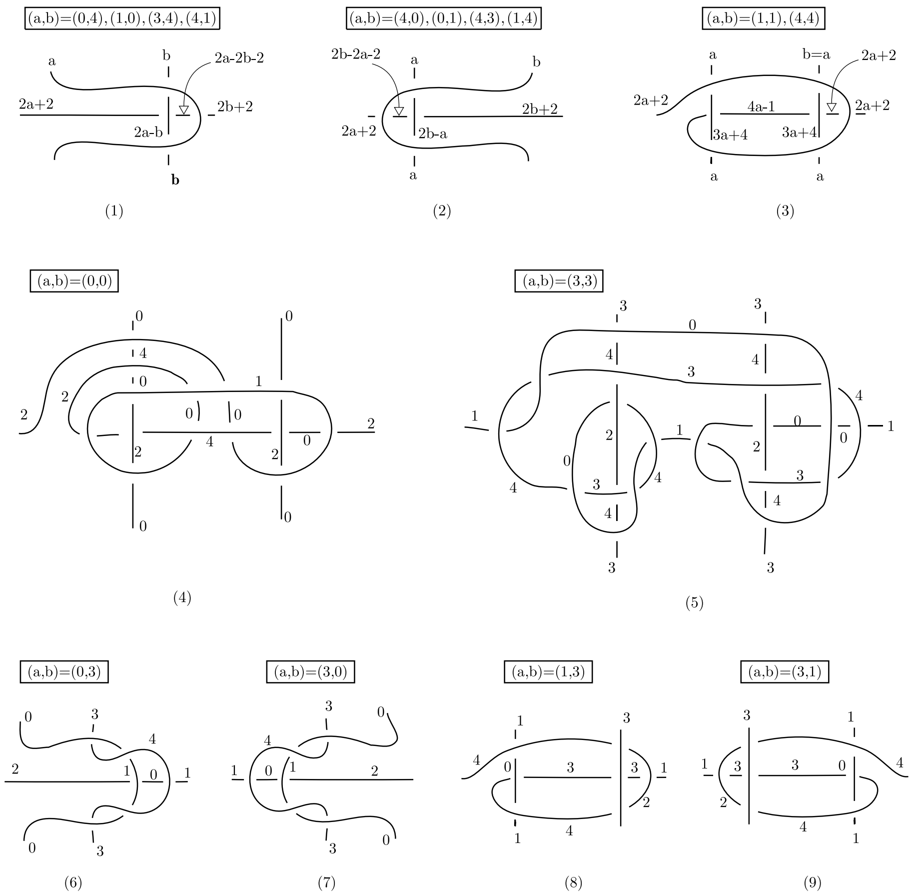


Figure 6. Deformation for eliminating the color 5.

$2a+4$ does not equal 5 nor 6, i.e., $a \neq 1, 4$. It implies that $Q$ is of $\{3|0|4\}$-type or of $\{3|2|1\}$-type. If $Q$ is of $\{a|3|6-a\}$-type, then the color $6-a$ does not equal 5 nor 6, i.e., $a \neq 0, 1$. It implies that $Q$ is of $\{2|3|4\}$-type. When $Q$ is of $\{3|0|4\}$-type, of $\{3|2|1\}$-type or of $\{2|3|4\}$-type, deform the arc colored by 0, 2 or 2 such that the arc detours around $P$ passing over the arcs as in Figure 7, respectively. By the deformation, the color of $P$ changes into $\{4|4|4\}$, $\{1|1|1\}$ or $\{1|1|1\}$ and the new crossings are of $\{3|0|4\}$-type, of $\{1|2|3\}$-type or of $\{1|2|3\}$-type, respectively. We repeat the deformation above if the obtained diagram still has a crossing of $\{3|3|3\}$-type.

We may assume that $D$ has no crossing whose color is $\{3|3|3\}$ and no arc colored by 5 or 6 by the above deformation and Lemma 2.2. Assume that $D$ has

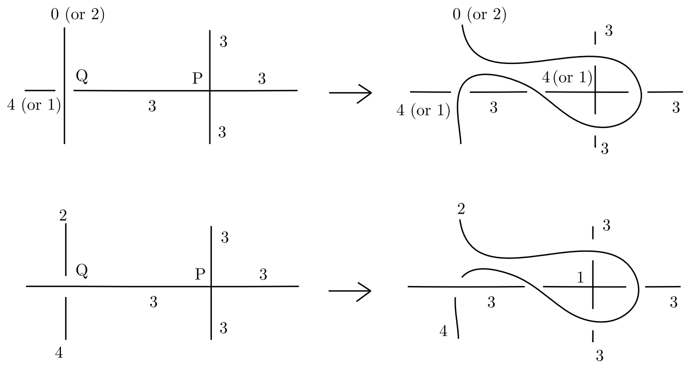


Figure 7. Deformation near a crossing of $\{3|3|3\}$-type.

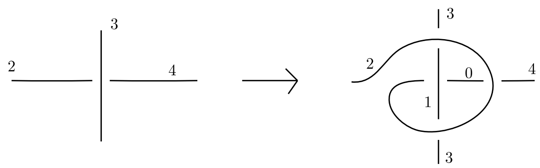


Figure 8. Deformation near a crossing of $\{2|3|4\}$-type.

a crossing of $\{a|3|6-a\}$-type for some $a \neq 3, 5, 6$. Since $6-a$ equals neither 5 nor 6, $a \neq 0, 1$, i.e., each crossing of $\{a|3|6-a\}$-type is of $\{2|3|4\}$-type. Deform the arc colored by 2 such that the arc detours around the crossing of $\{2|3|4\}$-type as in Figure 8. Then the color of the original crossing changes into $\{0|1|2\}$, and the new crossings are of $\{0|2|4\}$-type and of $\{1|2|3\}$-type. We repeat the deformation above if the obtained diagram still has a crossing of $\{2|3|4\}$-type.

We may assume that the upper arc of any crossing of $D$ does not have the color 3 and there exists no arc colored by 5 or 6. Hence, each arc with the color 3 connects a pair of crossings directly whose colors are $\{3|a|2a-3\}$ and $\{3|b|2b-3\}$ for some $a, b \neq 3, 5, 6$. Notice that since the colors $2a-3$ and $2b-3$ equal neither 5 nor 6, $a$ and $b$ equal 0 or 2. According to the pair of $a$ and $b$, we deform arcs so that the arc with the color 3 is eliminated. In the case of $(a, b) = (0, 2)$, $(2, 0)$, $(0, 0)$ or $(2, 2)$, deform the arc colored by 2, 2, 4 or 1 as shown in Figure 9 (1), (2), (3) or (4), respectively. Then the new arcs which are obtained by the above deformation do not have the color 3, 5 nor 6. By these deformations, we can eliminate the color 3. We repeat the deformation above if the resulting diagram still has an arc whose color is 3. □

REMARK 2.3. The argument as above can be easily applied to the families of 7-colored virtual knot diagrams [**6**] and virtual arc diagrams [**9**].

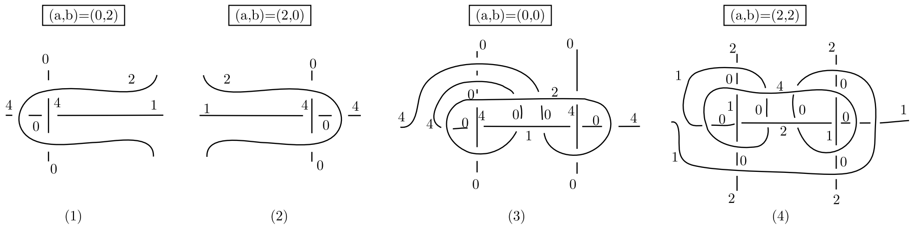


Figure 9. Deformation for eliminating the color 3.

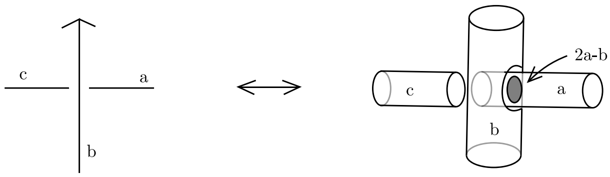


Figure 10. Virtual arc presentation.

## 3. 7-colorable ribbon 2-knots.

A 2-*knot* is a 2-dimensional sphere embedded in $\boldsymbol{R}^4$ smoothly. A *diagram* of a 2-knot $K$ is a projection image $\pi(K)$ under a projection $\pi : \boldsymbol{R}^4 \to \boldsymbol{R}^3$ equipped with crossing information. Any 2-knot diagram is regarded as a disjoint union of compact, connected surfaces, each of which is called a *sheet*, refer to [**1**]. For an odd prime $p$, an assignment of an element of $\boldsymbol{Z}_p$ to each sheet of the diagram is called a *p-coloring* if $a + c = 2b$ in $\boldsymbol{Z}_p$ holds near each double point, where the lower sheets are colored by $a$ and $c$ and the upper sheet is colored by $b$.

A *ribbon* 2-*knot* is obtained by adding 1-handles to a trivial 2-link. It is known that any ribbon 2-knot is presented by a virtual arc diagram, see [**9**]. Given an oriented virtual arc diagram $A$, we construct a diagram $D$ of a ribbon 2-knot Tube($A$). In Figure 10, we show a part of $D$ corresponding to a classical crossing of $A$. Moreover, it is easy to see that there is a one-to-one correspondence between the set of the 7-colorings for $A$ and that for $D$.

Proof of Theorem 1.5. Let $K$ be a 7-colorable ribbon 2-knot. We may assume that $K = \mathrm{Tube}(A)$ for some virtual arc diagram $A$. Since $K$ is 7-colorable, so is $A$. As mentioned in Remark 2.3, we may assume that $A$ has a non-trivial 7-coloring with exactly four colors 0, 1, 2 and 4. Consider the 7-colored diagram $D$ of $K = \mathrm{Tube}(A)$ corresponding to $A$. By the assumption for $A$, if $D$ has a sheet colored by 3, 5 or 6, then the sheet is the small one colored by $2a - b$ $(= 3, 5$ or $6)$ in Figure 10, respectively. Notice that $a$ and $b$ are colored by 0, 1, 2 or 4. If $(a, b) = (0, 1), (0, 2), (0, 4), (2, 1), (4, 2)$ or $(1, 4)$, then the small sheet has the

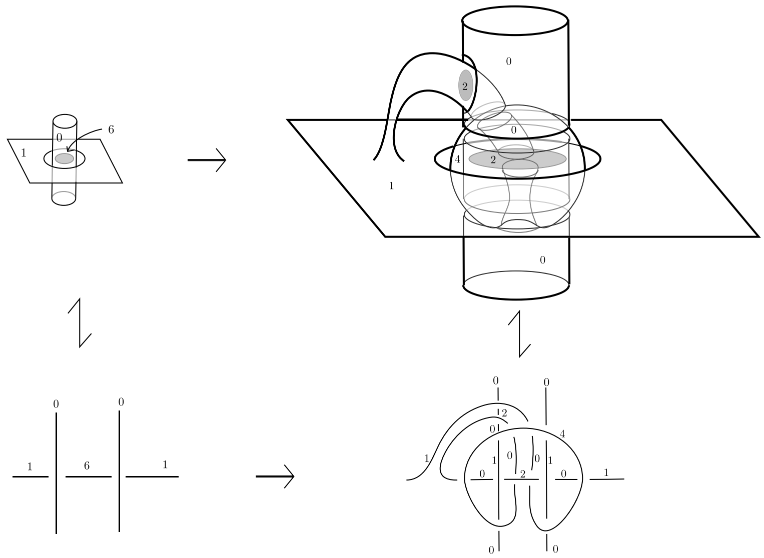

Figure 11. Deformation for eliminating the color 6 the small sheet has.

color $2a-b=3$, 5 or 6, otherwise $2a-b=0$, 1, 2 or 4. Let us consider the case of $(a,b)=(0,1)$. Then the small sheet in Figure 10 has the color 6. In the neighborhood of the sheet with the color 6, deform the sheet colored by 1 as shown in Figure 11 so that the color 6 is eliminated. We express the deformation by using the deformation of 1-dimensional arcs, which is a cross-section of the diagram, as shown below in Figure 11: Each arc shown in the lower figures corresponds to a sheet shown in the upper figures and each crossing shown in the lower figures corresponds to a double point shown in the upper figures. None of the new sheets which are obtained by the deformation have the color 3, 5 or 6. In the case of $(a,b)=(0,2)$ (or $(0,4)$), multiplying each color shown in Figure 11 by 2 (or 4) gives the deformation which eliminates the color 5 (or 3) as in Figure 10. Let us consider the case of $(a,b)=(2,1)$. Then the small sheet as in Figure 10 has the color 3. In the neighborhood of the sheet with the color 3, deform the sheet colored by 1 as shown in Figure 12 so that the color 3 is eliminated. None of the new sheets which are obtained by the deformation have the color 3, 5 or 6. In the case of $(a,b)=(4,2)$ (or $(1,4)$), the multiplication by 2 (or 4) for each color shown in Figure 12 gives the deformation so that the color 6 (or 5) which the small sheet as in Figure 10 has is eliminated. We repeat the deformation above if the obtained diagram still has a small sheet whose color is 3, 5 or 6. $\square$

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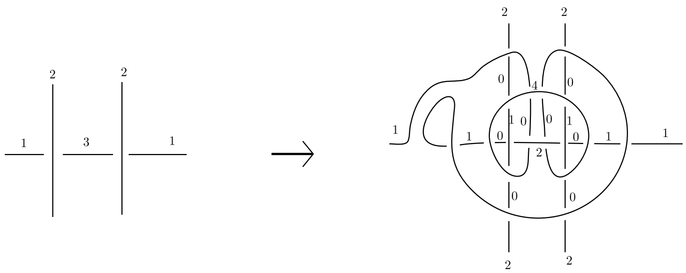


Figure 12. Deformation for eliminating the color 3 the small sheet has.

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