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Intersection sheaves over normal schemes

Dedicated to the memory of Professor Masayoshi Nagata

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Abstract. Intersection sheaves are usually defined for a projective flat surjective morphism of Noetherian schemes of relative dimension d and for $d+1$ invertible sheaves on the ambient scheme. In this article, the construction is generalized to the case of the equi-dimensional projective surjective morphisms to normal separated Noetherian schemes. Applications to the studies on family of effective algebraic cycles and on polarized endomorphisms are also given.

Introduction.

Let $\pi\colon X \to Y$ be a flat projective surjective morphism of Noetherian schemes of relative dimension d. For invertible sheaves $\mathscr{L}_1, \ldots, \mathscr{L}_{d+1}$ on X, we can associate an invertible sheaf $\mathscr{I}_{X/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_{d+1})$ on Y in a canonical way. Roughly speaking, the invertible sheaf satisfies suitable conditions similar to those satisfied by the fiber integral of Chern classes:

$$\int_\pi c_1(\mathscr{L}_1) \wedge \cdots \wedge c_1(\mathscr{L}_{d+1}).$$

Especially, if $\pi\colon X \to Y$ is a morphism of algebraic $\boldsymbol{k}$-schemes smooth over a field $\boldsymbol{k}$, then

$$c_1\bigl(\mathscr{I}_{X/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_{d+1})\bigr) = \pi_*\bigl(c_1(\mathscr{L}_1) \cdots c_1(\mathscr{L}_{d+1})\bigr)$$

in the Chow group $\mathrm{CH}^1(Y)$, where π_* is the push-forward homomorphism $\mathrm{CH}^{d+1}(X) \to \mathrm{CH}^1(Y)$ of Chow groups, and c_1 denotes the first Chern class in $\mathrm{CH}^1(X)$. In particular, $\mathscr{I}_{X/Y}(\mathscr{L})$ is the norm sheaf of $\mathscr{L}$ in case $d = 0$, and

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$$\mathscr{I}_{X/Y}(\mathscr{L}_1, \dots, \mathscr{L}_{d+1}) \simeq \mathscr{I}_{H/Y}(\mathscr{L}_2|_H, \dots, \mathscr{L}_{d+1}|_H)$$

if $\mathscr{L}_1 \simeq \mathscr{O}_X(H)$ for an effective relative Cartier divisor H on X with respect to π. The sheaf $\mathscr{I}_{X/Y}(\mathscr{L}_1, \dots, \mathscr{L}_{d+1})$ is called the *intersection sheaf*, the *intersection bundle*, or the *Deligne pairing* (when $d = 1$). For the Picard groups $\mathrm{Pic}(X)$ and $\mathrm{Pic}(Y)$, we have a homomorphism $\mathrm{Sym}^{d+1} \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ by $\mathscr{I}_{X/Y}$. In [**6**], Problème 2.1.2, Deligne posed a problem of constructing $\mathscr{I}_{X/Y}$ as a functor $\mathit{PIC}(X)_{\mathrm{is}}^{d+1} \to \mathit{PIC}(Y)_{\mathrm{is}}$ having natural properties on multi-additivity and base change. Here $\mathit{PIC}(X)_{\mathrm{is}}$ denotes the Picard category whose 'objects' are invertible sheaves on X and whose 'morphisms' are isomorphisms of invertible sheaves. The intersection sheaf $\mathscr{I}_{X/Y}(\mathscr{L}_1, \dots, \mathscr{L}_{d+1})$ can be defined a priori as a symmetric difference of $\det \boldsymbol{R}\pi_*(\mathscr{L})$ for some invertible sheaves $\mathscr{L}$ (cf. Remark 2.11 below; [**7**, p. 34]), but there is a problem of sign related to 'det.' The problem was solved in [**8**], [**10**], [**33**], and [**7**] by several methods.

The flatness assumption is important for the functorial properties. In this article, we consider not the functoriality but the construction of intersection sheaves for non-flat morphisms. More precisely, we shall construct intersection sheaves for projective surjective equi-dimensional morphisms $\pi \colon X \to Y$ to normal separated Noetherian schemes Y. The following is obtained mainly in Section 3 (cf. Theorems 3.14 and 3.25; Propositions 2.15, 2.32, and 3.20):

Theorem. *Let Y be a normal separated Noetherian integral scheme and $\pi \colon X \to Y$ a projective equi-dimensional surjective morphism of relative dimension d. Let U be a Zariski-open subset of Y such that* $\mathrm{codim}(Y \setminus U) \geq 2$ *and π is flat over U. Then the intersection sheaf*

$$\mathscr{I}_{\pi^{-1}(U)/U}\left(\mathscr{L}_1|_{\pi^{-1}(U)}, \dots, \mathscr{L}_{d+1}|_{\pi^{-1}(U)}\right)$$

defined for invertible sheaves $\mathscr{L}_1, \dots, \mathscr{L}_{d+1} \in \mathrm{Pic}(X)$, naturally extends to an invertible sheaf $\mathscr{I}_{X/Y}(\mathscr{L}_1, \dots, \mathscr{L}_{d+1})$ on Y. In particular, $\mathscr{I}_{X/Y}$ induces a natural homomorphism $\mathrm{Sym}^{d+1} \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$*. Moreover, it has the following properties:*

(1) *Suppose that, for any i, there exists a surjection $\pi^*\mathscr{G}_i \to \mathscr{L}_i$ for a locally free sheaf $\mathscr{G}_i$ on Y of finite rank. Then there is a surjection*

$$\Phi \colon \mathrm{Sym}^{e_1}(\mathscr{G}_1) \otimes \cdots \otimes \mathrm{Sym}^{e_{d+1}}(\mathscr{G}_{d+1}) \to \mathscr{I}_{X/Y}(\mathscr{L}_1, \dots, \mathscr{L}_{d+1}),$$

where e_i is the intersection number

$$i_{\boldsymbol{F}/\boldsymbol{k}}(\mathscr{L}_1|_{\boldsymbol{F}}, \dots, \mathscr{L}_{i-1}|_{\boldsymbol{F}}, \mathscr{L}_{i+1}|_{\boldsymbol{F}}, \dots, \mathscr{L}_{d+1}|_{\boldsymbol{F}}; \boldsymbol{F})$$

for the generic fiber $\boldsymbol{F}$ *of* π *and the function field* $\boldsymbol{k}$ *of* Y (cf. Definition 1.11 below)*, and where* Sym *stands for the symmetric tensor product.*

(2) *Let* $g\colon Y' \to Y$ *be a dominant morphism of finite type from another normal separated Noetherian scheme* Y'*,* $\pi'\colon X' = X \times_Y Y' \to Y'$ *the pullback of* π*, and* $g'\colon X' \to X$ *the pullback of* g*. Then*

$$\mathscr{I}_{X'/Y'}\big(g'^*\mathscr{L}_1, \dots, g'^*\mathscr{L}_{d+1}\big) \simeq g^*\,\mathscr{I}_{X/Y}\big(\mathscr{L}_1, \dots, \mathscr{L}_{d+1}\big).$$

The sheaf $\mathscr{I}_{X/Y}(\mathscr{L}_1, \dots, \mathscr{L}_{d+1})$ is first defined as a *reflexive sheaf* (cf. Definition 1.16 below; [**23**, Section 1]) of rank one, but after certain discussion, it is shown to be invertible (cf. Section 3.2). By the invertibility, we can prove that, for the equi-dimensional morphism $\pi\colon X \to Y$, if X is normal and $\boldsymbol{Q}$-factorial (i.e., every Weil divisor is $\boldsymbol{Q}$-Cartier), then so is Y (cf. Theorem 3.18).

The surjection Φ above can be regarded as a homomorphism giving the resultant: For sections $v_i \in \mathrm{H}^0(Y, \mathscr{G}_i)$ and its images $s_i \in \mathrm{H}^0(X, \mathscr{L}_i)$, $\Phi(v_1^{e_1} \otimes \cdots \otimes v_{d+1}^{e_{d+1}})$ is the resultant of sections $s_1, \dots, s_{d+1}$, up to unit. In particular, $\Phi(v_1^{e_1} \otimes \cdots \otimes v_{d+1}^{e_{d+1}})$ does not vanish at a point $y \in Y$ if and only if $\mathrm{div}(s_1) \cap \cdots \cap \mathrm{div}(s_{d+1}) \cap \pi^{-1}(y) = \emptyset$.

The intersection number $i_{X/\boldsymbol{k}}(\mathscr{L}_1, \mathscr{L}_2, \dots, \mathscr{L}_d; \mathscr{F})$ is defined for invertible sheaves $\mathscr{L}_i$ $(1 \le i \le d)$ and coherent sheaves $\mathscr{F}$ on projective varieties X defined over a field $\boldsymbol{k}$ with $d = \dim \operatorname{Supp} \mathscr{F}$ (cf. Definition 1.11 below). As an analogy, we can define the intersection sheaf $\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \dots, \mathscr{L}_{d+1})$ by replacing X with a coherent sheaf $\mathscr{F}$ such that $\operatorname{Supp} \mathscr{F} \to Y$ is equi-dimensional and $\dim(\operatorname{Supp} \mathscr{F})/Y = d$. Moreover, we can define $\mathscr{I}_{\mathscr{F}/Y}$ as a homomorphism $\mathrm{Gr}_F^{d+1} K^\bullet(X) \to \mathrm{Pic}(Y)$ for the λ-filtration $\{F^i K^\bullet(X)\}$ of the Grothendieck K-group $K^\bullet(X) = K_0(X)$. A similar result to the theorem above also holds for the intersection sheaves $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ for $\eta \in \mathrm{Gr}_F^{d+1} K^\bullet(X)$. For example, for a locally free sheaf $\mathscr{E}$ on X of rank r and for a Chern polynomial $P = P(x_1, \dots, x_r)$ of weighted degree $d+1$, we have the intersection sheaf $\mathscr{I}_{\mathscr{F}/Y}(P(\mathscr{E})) = \mathscr{I}_{\mathscr{F}/Y}(P(c_1(\mathscr{E}), \dots, c_r(\mathscr{E})))$. In Theorem 2.41 and Corollary 3.21 below, we prove that if $\mathscr{E}$ is generated by finitely many global sections and if P is *numerically positive for ample vector bundles* (cf. Definition 2.38) in the sense of [**14**], then $\mathscr{I}_{\mathscr{F}/Y}(P(\mathscr{E}))$ is also generated by finitely many global sections.

Suppose that $\pi\colon X \to Y$ is an equi-dimensional surjective morphism of normal projective varieties over a field such that $d = \dim X/Y$. For invertible sheaves $\mathscr{L}_1, \dots, \mathscr{L}_{d+1}$ on X, let $\mathscr{M}$ be the intersection sheaf $\mathscr{I}_{X/Y}(\mathscr{L}_1, \dots, \mathscr{L}_{d+1})$. If the invertible sheaves $\mathscr{L}_i$ are generated by global sections, then so is $\mathscr{M}$. Similarly, some numerical properties on the invertible sheaves $\mathscr{L}_i$ descend to $\mathscr{M}$. For example, if $\mathscr{L}_i$ are all ample (resp. nef and big), then so is $\mathscr{M}$ (cf. Corollary 4.6). For a Chern polynomial $P(x_1, \dots, x_r)$ of weighted degree $d+1$ which is numerically positive for ample vector bundles in the sense of [**14**], we prove in Theorem 4.7

below that $\mathscr{I}_{X/Y}(P(\mathscr{E}))$ is ample if $\mathscr{E}$ is an ample locally free sheaf of rank r. If $X \subset V \times Y$ for a projective variety V and if $\mathscr{L}_i$ are the pullbacks of very ample invertible sheaves of V by the first projection $X \to V$, then $\mathscr{M}$ is isomorphic to the pullback of an ample invertible sheaf on T for the Stein factorization $Y \to T$ of the morphism $Y \to \mathrm{Chow}(V)$ into the Chow variety of V, which associates a general point $\boldsymbol{y} \in Y$ the algebraic cycle $\mathrm{cyc}(\pi^{-1}(\boldsymbol{y}))$ of V for the fiber $\pi^{-1}(\boldsymbol{y})$ (cf. Section 4.2). In particular, the base point free linear system $|\mathscr{M}^{\otimes m}|$ for some $m > 0$ defines the Stein factorization of $Y \to \mathrm{Chow}(V)$. By the property, we have the notion of Chow reduction (cf. Definition 4.15) for a dominant rational map $X \cdots\to Y$ of normal projective varieties, and also the notion of special MRC fibration (cf. Theorem 4.18) for uniruled complex projective varieties generalizing the notion of maximal rationally connected (= MRC) fibration (cf. [**4**], [**28**], [**16**]) defined for smooth varieties. The following results on endomorphisms are proved in Theorem 4.19 and Corollary 4.20:

(i) If $f\colon X \to X$ is a finite surjective morphism for a normal complex uniruled projective variety X, then f descends to an endomorphism $h\colon Y \to Y$ of the base Y of the special MRC fibration $X \cdots\to Y$.

(ii) Here, if f is a polarized endomorphism, i.e., $f^*\mathscr{A} \simeq \mathscr{A}^{\otimes q}$ for some $q > 0$ and an ample invertible sheaf $\mathscr{A}$, then the endomorphism h is also polarized.

The motivation of this article is a question by D.-Q. Zhang on [**43**, Proposition 2.2.4], which is a similar result to above on the endomorphisms of complex normal projective uniruled varieties and on the MRC fibration, where the intersection sheaf is used for proving (ii), but the notion of intersection sheaf is defined only for flat morphisms in the paper [**43**]. The results (i) and (ii) above solve the question. The results in Section 4.3 are used in a joint paper [**37**] with D.-Q. Zhang.

It is hopeless to give a similar definition of the intersection sheaves $\mathscr{I}_{X/Y}(\eta)$ for a proper equi-dimensional surjective non-flat morphism $X \to Y$ to a non-normal base scheme (cf. Remark 3.8). In order to extend the notion of intersection sheaf to the non-normal case, we must add some additional data. For example, [**1**] studies the 'incidence divisors' for analytic families of cycles parametrized by a reduced complex analytic space, where the invertible sheaf associated with an incidence divisor can be regarded as an intersection sheaf. However, the definition of the analytic family requires more than the equi-dimensionality.

This article is organized as follows: After preparing basics on K-groups in Section 1, we define and study the intersection sheaves $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ in Section 2 for Y-flat coherent sheaves $\mathscr{F}$ on X and $\eta \in \mathrm{Gr}_F^{d+1} K^\bullet(X)$ with $d = (\dim \mathscr{F})/Y$ for a projective morphism $\pi\colon X \to Y$, under a suitable assumption: Assumption 2.1. We use essentially the same argument as in [**8**], [**33**], and the description of the Hilbert-Chow correspondence in [**32**, Chapter 5]. In Section 3, we consider the case where Y is a normal separated Noetherian integral scheme. For a locally projective surjective morphism $\pi\colon X \to Y$, we define the intersection sheaf $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ for a coherent sheaf $\mathscr{F}$ on X and $\eta \in \mathrm{Gr}_F^{d+1} K^\bullet(X)$ with $\dim \mathrm{Supp}\, \mathscr{F} \cap \pi^{-1}(\boldsymbol{y}) \le d$ for any $\boldsymbol{y} \in Y$. The invertibility of $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ and some base change properties are proved in Sections 3.2 and 3.3. We apply these fundamental results obtained in Sections 2 and 3 to the equi-dimensional surjective morphisms of projective varieties over a field in Section 4. We prove some numerical properties of the intersection sheaves in Section 4.1, give a relation to morphisms into Chow varieties in Section 4.2, and finally in Section 4.3 have some of results on polarized endomorphisms of projective varieties answering the question of D.-Q. Zhang.

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1. Grothendieck K-groups.

We recall elementary properties of Grothendieck K-groups (cf. [**22**], [**2**]). Even though all the assertions in this section are well-known for specialists, we introduce here some notions, conventions, and terminologies, used for the main part of this article. Some of definitions and basic properties are explained in Section 1.1, and several pull-back and push-forward homomorphisms associated to morphisms of schemes are explained in Section 1.2. In Section 1.3, the Chern classes in the K-theory are discussed and an expression of the top Chern class of a locally free sheaf is mentioned in Lemma 1.7. The notion of intersection numbers is briefly explained in Section 1.4. In Section 1.5, the first two graded pieces of the coniveau filtration, i.e., $\mathrm{Gr}^0_{\mathrm{con}}$ and $\mathrm{Gr}^1_{\mathrm{con}}$, are discussed.

1.1. K-groups and filtrations.

Let X be a Noetherian scheme. Let $K^\bullet(X)$ (resp. $K_\bullet(X)$) be the Grothendieck group on the locally free sheaves (resp. coherent sheaves) on X. For a locally free (resp. coherent) sheaf $\mathscr{F}$, let $\mathrm{cl}^\bullet(\mathscr{F}) = \mathrm{cl}^\bullet_X(\mathscr{F})$ (resp. $\mathrm{cl}_\bullet(\mathscr{F}) = \mathrm{cl}_{X\bullet}(\mathscr{F})$) denote the corresponding element in $K^\bullet(X)$ (resp. $K_\bullet(X)$). Note that $K^\bullet(X)$ is the K_0 group in the K-theory. The tensor products of locally free sheaves give $K^\bullet(X)$ a ring structure and give $K_\bullet(X)$ a structure of $K^\bullet(X)$-module so that the canonical homomorphism $\boldsymbol{\phi}\colon K^\bullet(X) \to K_\bullet(X)$, which is called the *Cartan homomorphism*, is $K^\bullet(X)$-linear. Here, $\mathrm{cl}^\bullet(\mathscr{O}_X)$ is the unit element 1 of the ring structure of $K^\bullet(X)$, and $\boldsymbol{\phi}\colon K^\bullet(X) \to K_\bullet(X)$ is regarded as the multiplication map by $\boldsymbol{\phi}(1) = \mathrm{cl}_\bullet(\mathscr{O}_X) \in K_\bullet(X)$. If X is a regular separated Noetherian scheme, then $\boldsymbol{\phi}\colon K^\bullet(X) \to K_\bullet(X)$ is isomorphic by the existence of global locally free resolution for coherent sheaves (cf. [**22**, Exp. II, Proposition 2.2.3 and Exp. II, Corollaire 2.2.7.1]).

The ring $\mathrm{H}^0(X, \boldsymbol{Z})$ of locally constant $\boldsymbol{Z}$-valued functions is a direct summand of $K^\bullet(X)$, in which a projection $\varepsilon\colon K^\bullet(X) \to \mathrm{H}^0(X, \boldsymbol{Z})$, called the *augmentation map*, is given by $\mathrm{cl}^\bullet(\mathscr{E}) \mapsto \operatorname{rank}\mathscr{E}$ for locally free sheaves $\mathscr{E}$. The λ-*ring structure* (cf. [**22**, Exp. V], [**27**, Chapter I]) of $K^\bullet(X)$ is introduced by setting $\lambda^p(\mathrm{cl}^\bullet(\mathscr{E})) = \mathrm{cl}^\bullet(\bigwedge^p \mathscr{E})$ for locally free sheaves $\mathscr{E}$: Note that the group homomorphisms $\lambda^n\colon K^\bullet(X) \to K^\bullet(X)$ for non-negative integers n satisfy $\lambda^0(x) = 1$, $\lambda^1(x) = x$, and

$$\lambda^n(x+y) = \sum_{i=0}^{n} \binom{n}{i} \lambda^{n-i}(x)\lambda^i(y)$$

for any x, $y \in K^\bullet(X)$. In particular, $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$ for the formal power series

$$\lambda_t(x) := \sum_{n=0}^{\infty} \lambda^n(x)t^n \in K^\bullet(X)[\![t]\!].$$

The augmentation map ε is a λ-homomorphism with respect to the natural λ-ring structure of $\mathrm{H}^0(X, \boldsymbol{Z})$. The operator $\gamma^p\colon K^\bullet(X) \to K^\bullet(X)$ for an integer $p \geq 0$ associated with the λ-ring structure is defined by $\gamma^p(x) = \lambda^p(x + p - 1)$ for $x \in K^\bullet(X)$, where the integer $p-1$ is regarded as an element of $K^\bullet(X)$ by $p-1 = (p-1)1 = (p-1)\,\mathrm{cl}^\bullet(\mathscr{O}_X)$. The λ-filtration $\{F^pK^\bullet(X)\}$ of $K^\bullet(X)$ is defined as follows: $F^pK^\bullet(X) = K^\bullet(X)$ for $p \leq 0$, $F^1K^\bullet(X) = \mathrm{Ker}(\varepsilon)$, and $F^pK^\bullet(X)$ for $p \geq 2$ is generated by

$$\gamma^{k_1}(x_1)\gamma^{k_2}(x_2)\cdots\gamma^{k_l}(x_l)$$

with $x_i \in \mathrm{Ker}(\varepsilon)$ and $\sum k_i \geq p$. Then $K^\bullet(X)$ is a filtered ring, i.e., $F^pK^\bullet(X)F^qK^\bullet(X) \subset F^{p+q}K^\bullet(X)$ for p, $q \geq 0$. For $\mathrm{Gr}_F^i K^\bullet(X) = F^iK^\bullet(X)/F^{i+1}K^\bullet(X)$, we have

$$\mathrm{Gr}_F^0(X) \simeq \mathrm{H}^0(X, \boldsymbol{Z}) \quad \text{and} \quad \mathrm{Gr}_F^1(X) \simeq \mathrm{Pic}(X),$$

by [**22**, Exp. X, Théorème 5.3.2], where $\mathrm{Pic}(X)$ denotes the Picard group of X.

On the other hand, $K_\bullet(X)$ also has a natural filtration $\{F^p_{\mathrm{con}}K_\bullet(X)\}$, called the *coniveau filtration*, which is defined as follows (cf. [**22**, Exp. X, Remarque 1.4 and Exp. X, Exemple 1.5], [**15**, Definition 32]): $F^p_{\mathrm{con}}K_\bullet(X)$ is generated by

$\mathrm{cl}_\bullet(\mathscr{F})$ for coherent sheaves $\mathscr{F}$ with $\operatorname{codim}\operatorname{Supp}\mathscr{F} \geq p$. We have another natural subgroup $F_pK_\bullet(X) \subset K_\bullet(X)$ for $p \geq 0$, which is generated by $\mathrm{cl}_\bullet(\mathscr{F})$ for the coherent sheaves $\mathscr{F}$ with $\dim\operatorname{Supp}\mathscr{F} \leq p$. Note that $K_\bullet(X) = \bigcup_{p\geq 0} F_pK_\bullet(X)$ does not hold unless $\dim X$ is bounded. If X is of finite type over a field and if X is of pure dimension n, then $F^p_{\mathrm{con}}K_\bullet(X) = F_{n-p}K_\bullet(X)$. The following properties are known for the filtration $F_pK_\bullet(X)$ by [**22**, Exp. X, Corollaire 1.1.4 and Exp. X, Théorème 1.3.2]:

(i) $\operatorname{Gr}^F_p K_\bullet(X) = F_pK_\bullet(X)/F_{p-1}K_\bullet(X)$ is generated by $\mathrm{cl}_\bullet(\mathscr{O}_Z)$ for the closed integral subschemes Z of dimension p.

(ii) $F^pK^\bullet(X)F_qK_\bullet(X) \subset F_{q-p}K_\bullet(X)$ for any p, $q \geq 0$. In particular, if $\dim X \leq n$, then $\boldsymbol{\phi}(F^pK^\bullet(X)) \subset F_{n-p}K_\bullet(X)$.

Similar properties are also satisfied for the filtration $F^p_{\mathrm{con}}K_\bullet(X)$:

(iii) $\operatorname{Gr}^p_{F_{\mathrm{con}}} K_\bullet(X) = F^p_{\mathrm{con}}K_\bullet(X)/F^{p+1}_{\mathrm{con}}K_\bullet(X)$ is generated by $\mathrm{cl}_\bullet(\mathscr{O}_Z)$ for the closed integral subschemes Z of codimension p.

(iv) $F^pK^\bullet(X)F^q_{\mathrm{con}}K_\bullet(X) \subset F^{p+q}_{\mathrm{con}}K_\bullet(X)$ for any p, $q \geq 0$. In particular, we have $\boldsymbol{\phi}(F^pK^\bullet(X)) \subset F^p_{\mathrm{con}}K_\bullet(X)$.

The property (iii) is proved by the same argument as in the proof of (i) in [**22**, Exp. X, Corollaire 1.1.4]. The property (iv) is proved in Proposition 3.2 below.

Convention. For the sake of simplicity, we write

$$\begin{aligned}
&F^p(X) = F^pK^\bullet(X), && F^p_{\mathrm{con}}(X) = F^p_{\mathrm{con}}K_\bullet(X), && F_p(X) = F_pK_\bullet(X),\\
&G^p(X) = \operatorname{Gr}^p_F K^\bullet(X), && G^p_{\mathrm{con}}(X) = \operatorname{Gr}^p_{F_{\mathrm{con}}} K_\bullet(X), && G_p(X) = \operatorname{Gr}^F_p K_\bullet(X),\\
&G^\bullet(X) = \bigoplus_{p\geq 0} G^p(X), && G^\bullet_{\mathrm{con}}(X) = \bigoplus_{p\geq 0} G^p_{\mathrm{con}}(X), && G_\bullet(X) = \bigoplus_{p\geq 0} G_p(X).
\end{aligned}$$

Then, $G^\bullet(X)$ is a graded ring, and $G^\bullet_{\mathrm{con}}(X)$ and $G_\bullet(X)$ have graded $G^\bullet(X)$-module structures by $G^p(X) \otimes G^q_{\mathrm{con}}(X) \to G^{p+q}_{\mathrm{con}}(X)$ and $G^p(X) \otimes G_q(X) \to G_{q-p}(X)$, respectively. We denote by $G(\boldsymbol{\phi})\colon G^p(X) \to G^p_{\mathrm{con}}(X)$ the homomorphism induced from the Cartan homomorphism $\boldsymbol{\phi}\colon K^\bullet(X) \to K_\bullet(X)$.

Remark. Suppose that X is an n-dimensional smooth algebraic variety defined over a field. Then $\boldsymbol{\phi}\colon K^\bullet(X) \to K_\bullet(X)$ is isomorphic and $F^p_{\mathrm{con}}(X) = F_{n-p}(X) \subset K_\bullet(X)$. Moreover, $K_\bullet(X)$ has a structure of filtered ring by $\{F^p_{\mathrm{con}}(X)\}$, i.e., $F^p_{\mathrm{con}}(X)F^q_{\mathrm{con}}(X) \subset F^{p+q}_{\mathrm{con}}(X)$ for any p, $q \geq 0$ (cf. [**22**, Exp. 0, App. II, Théorème 2.12, Corollaire 1]). Since $\boldsymbol{\phi}(F^p(X)) \subset F^p_{\mathrm{con}}(X)$, we have a ring homomorphism $G(\boldsymbol{\phi})\colon G(X) \to G_{\mathrm{con}}(X)$, which is not necessarily isomorphic but $G(\boldsymbol{\phi}) \otimes \boldsymbol{Q}$ is an isomorphism (cf. [**22**, Exp. XIV, Section 4.5 and Exp. VII, Proposition 4.11]).

1.2. Pull-back and push-forward homomorphisms.

Let $f\colon X \to Y$ be a morphism of Noetherian schemes. We discuss several homomorphisms between K-groups induced from f.

Firstly, we have the pullback homomorphism $f^{\star}\colon K^{\bullet}(Y) \to K^{\bullet}(X)$ which is a λ-ring homomorphism and which maps $\mathrm{cl}^{\bullet}_{Y}(\mathscr{E})$ to $\mathrm{cl}^{\bullet}_{X}(f^{*}\mathscr{E})$ for a locally free sheaf $\mathscr{E}$ on Y: In order to avoid confusion with the sheaf pullback f^{*}, we use the symbol $f^{\star}$ instead. Here, $f^{\star}F^{p}(Y) \subset F^{p}(X)$, and hence $G(f^{\star})\colon G^{p}(Y) \to G^{p}(X)$ is induced; sometimes $G(f^{\star})$ is denoted by $f^{\star}$ for simplicity. For a morphism $g\colon Y \to Z$ to another Noetherian scheme Z, we have $(g \circ f)^{\star} = f^{\star} \circ g^{\star}$. If f is an immersion, then $f^{\star}(\eta)$ is denoted by $\eta|_{X}$ for $\eta \in K^{\bullet}(Y)$ or for $\eta \in G^{\bullet}(Y)$.

Secondly, we have the push-forward homomorphism $f_{\star}\colon K_{\bullet}(X) \to K_{\bullet}(Y)$ when f is proper, in which $f_{\star}$ maps $\mathrm{cl}_{X\bullet}(\mathscr{F})$ to $\sum_{i\geq 0}(-1)^{i}\,\mathrm{cl}_{Y\bullet}(\mathrm{R}^{i} f_{*}\mathscr{F})$ for a coherent sheaf $\mathscr{F}$: In order to avoid confusion with f_{*} as the direct image of sheaves, we use the symbol $f_{\star}$ instead. Here, $f_{\star}F_{p}(X) \subset F_{p}(Y)$, since $\dim \operatorname{Supp} \mathrm{R}^{i} f_{*}\mathscr{F} \leq \dim \operatorname{Supp} \mathscr{F}$ for any i for any coherent sheaf $\mathscr{F}$ on X. In particular, $G(f_{\star})\colon G_{p}(X) \to G_{p}(Y)$ is induced; sometimes $G(f_{\star})$ is denoted by $f_{\star}$ for simplicity. We have the following projection formula for $x \in K_{\bullet}(X)$ and $y \in K^{\bullet}(Y)$:

$$f_{\star}(x \cdot f^{\star}y) = f_{\star}(x) \cdot y. \tag{1.1}$$

This follows from the usual projection formula $\mathrm{R}^{i} f_{*}(\mathscr{F} \otimes f^{*}\mathscr{E}) \simeq (\mathrm{R}^{i} f_{*}\mathscr{F}) \otimes \mathscr{E}$ for coherent sheaves $\mathscr{F}$ on X and locally free sheaves $\mathscr{E}$ on Y. As a result, we infer that $f_{\star}\colon K_{\bullet}(X) \to K_{\bullet}(Y)$ is $K^{\bullet}(Y)$-linear and $G(f_{\star})\colon G_{\bullet}(X) \to G_{\bullet}(Y)$ is $G^{\bullet}(Y)$-linear. Note that $(g \circ f)_{\star} = g_{\star} \circ f_{\star}$ as a homomorphism $K_{\bullet}(X) \to K_{\bullet}(Z)$ for a proper morphism $g\colon Y \to Z$ to another Noetherian scheme Z.

Thirdly, we have another natural pullback homomorphism $f^{\star}\colon K_{\bullet}(Y) \to K_{\bullet}(X)$ when f is flat (cf. [**22**, Exp. IV, 2.12]). Here, $f^{\star}(\mathrm{cl}_{\bullet}(\mathscr{G})) = \mathrm{cl}_{\bullet}(f^{*}\mathscr{G})$ for a coherent sheaf $\mathscr{G}$ on Y. This is compatible with $f^{\star}\colon K^{\bullet}(Y) \to K^{\bullet}(X)$ via the Cartan homomorphisms $\boldsymbol{\phi}$. Here, $f^{\star}F^{p}_{\mathrm{con}}(Y) \subset F^{p}_{\mathrm{con}}(X)$ for any p, and hence $G(f^{\star})\colon G^{p}_{\mathrm{con}}(Y) \to G^{p}_{\mathrm{con}}(X)$ is induced. If $g\colon Z \to Y$ is a proper morphism, then for the fiber product $W = Z \times_{Y} X$ and for the natural projections $p_{1}\colon W \to Z$ and $p_{2}\colon W \to X$, we have the base change formula

$$f^{\star}(g_{\star}(z)) = p_{2\star}(p_{1}^{\star}(z)) \tag{1.2}$$

for $z \in K_{\bullet}(Z)$ (cf. [**22**, Exp. IV, Proposition 3.1.1]).

We add some remarks on $f^{\star}$ and $f_{\star}$. If f is proper, flat, and of relative dimension d, then $f_\star F^{p+d}_{\mathrm{con}}(X) \subset F^p_{\mathrm{con}}(Y)$ for $f_\star \colon K_\bullet(X) \to K_\bullet(Y)$ by the formula:

$$\dim \mathscr{O}_{X,\boldsymbol{x}} = \dim \mathscr{O}_{Y,f(\boldsymbol{x})} + \dim \mathscr{O}_{X,\boldsymbol{x}} \otimes_{\mathscr{O}_{Y,f(\boldsymbol{x})}} \boldsymbol{k}(f(\boldsymbol{x})),$$

where $\boldsymbol{k}(\boldsymbol{y})$ denotes the residue field of $\mathscr{O}_{Y,\boldsymbol{y}}$. For an open immersion $j \colon U \hookrightarrow X$ and for the closed immersion $i \colon Z \hookrightarrow X$ from the complement $Z = X \setminus U$, we have the following natural exact sequence (cf. [**22**, Exp. 0, App. II, Proposition 2.10]):

$$K_\bullet(Z) \xrightarrow{i_\star} K_\bullet(X) \xrightarrow{j^\star} K_\bullet(U) \to 0. \tag{1.3}$$

1.3. Algebraic cycles and Chern classes.

Let X be a Noetherian scheme. An *algebraic cycle* $Z = \sum n_i Z_i$ on X is a finite linear combination of closed integral subschemes Z_i of X with integral coefficients n_i. If the coefficients n_i are all non-negative, then Z is called *effective.* If $\dim Z_i = k$ (resp. $\operatorname{codim} Z_i = k$) for all i, then Z is called a cycle of dimension k (resp. of codimension k). The group of algebraic cycles of dimension k (resp. codimension k) is denoted by $\mathscr{Z}_k(X)$ (resp. $\mathscr{Z}^k(X)$).

Definition 1.1. Let $\mathscr{F}$ be a coherent sheaf on X. For an irreducible component W of $\operatorname{Supp}\mathscr{F}$, we define the length $l_W(\mathscr{F})$ of $\mathscr{F}$ along W as the length of the $\mathscr{O}_{X,\boldsymbol{x}}$-module $\mathscr{F}_{\boldsymbol{x}}$ for the generic point $\boldsymbol{x}$ of W. This is well-defined, since $\mathscr{O}_{X,\boldsymbol{x}}$ is Artinian. We define an effective algebraic cycle by

$$\operatorname{cyc}(\mathscr{F}) := \sum_{W \subset \operatorname{Supp}\mathscr{F}} l_W(\mathscr{F}) W,$$

where the summation is taken over all the irreducible components W of $\operatorname{Supp}\mathscr{F}$. If $\dim \operatorname{Supp}\mathscr{F} \le k$, then we set

$$\operatorname{cyc}_k(\mathscr{F}) := \sum_{\dim W = k,\, W \subset \operatorname{Supp}\mathscr{F}} l_W(\mathscr{F}) W \quad \in \mathscr{Z}_k(X).$$

If $\operatorname{codim} \operatorname{Supp}\mathscr{F} \ge k$, then we set

$$\operatorname{cyc}^k(\mathscr{F}) := \sum_{\operatorname{codim} W = k,\, W \subset \operatorname{Supp}\mathscr{F}} l_W(\mathscr{F}) W \quad \in \mathscr{Z}^k(X).$$

We write $\operatorname{cyc}(V) = \operatorname{cyc}(\mathscr{O}_V)$ for closed subschemes V.

We have natural homomorphisms $\mathrm{cl}_\bullet\colon \mathscr{Z}^k(X) \to F^k_{\mathrm{con}}(X) \subset K_\bullet(X)$ and $\mathrm{cl}_\bullet\colon \mathscr{Z}_k(X) \to F_k(X) \subset K_\bullet(X)$ defined by $\mathrm{cl}_\bullet(Z) = \sum n_i \,\mathrm{cl}_\bullet(\mathscr{O}_{Z_i})$, where $Z = \sum n_i Z_i$ for closed integral subschemes Z_i and $n_i \in \boldsymbol{Z}$. Then, $\mathrm{cl}_\bullet(\mathrm{cyc}^k(\mathscr{F})) \equiv \mathrm{cl}_\bullet(\mathscr{F}) \mod F^{k+1}_{\mathrm{con}}(X)$ and $\mathrm{cl}_\bullet(\mathrm{cyc}_k(\mathscr{G})) \equiv \mathrm{cl}_\bullet(\mathscr{G}) \mod F_{k-1}(X)$ for coherent sheaves $\mathscr{F}$ and $\mathscr{G}$ with $\operatorname{codim}\operatorname{Supp}\mathscr{F} = \dim\operatorname{Supp}\mathscr{G} = k$.

Remark. Suppose that X is an n-dimensional smooth quasi-projective variety over a field. The *Chow group* $\mathrm{CH}_k(X)$ (resp. $\mathrm{CH}^k(X)$) is defined as the quotient group of $\mathscr{Z}_k(X)$ (resp. $\mathscr{Z}^k(X)$) by the rational equivalence relation. Here, $\mathrm{CH}^i(X) \simeq \mathrm{CH}_{n-i}(X)$ for $i \geq 0$, since $\mathscr{Z}^i(X) = \mathscr{Z}_{n-i}(X)$. Then $\mathrm{CH}(X) = \bigoplus_{i=0}^n \mathrm{CH}^i(X)$ has a graded ring structure by the intersection theory, which is called the *Chow ring* of X. The map $\mathrm{cl}_\bullet\colon \mathscr{Z}^k(X) \to F^k_{\mathrm{con}}(X)$ above induces $G(\mathrm{cl}_\bullet)\colon \mathrm{CH}^k(X) \to G^k_{\mathrm{con}}(X)$ and a ring homomorphism $G(\mathrm{cl}_\bullet)\colon \mathrm{CH}(X) \to G^\bullet_{\mathrm{con}}(X)$.

Definition 1.2 (Chern class, cf. [**22**, Exp. VI, (6.7.1)]). The p-th Chern class of $x \in K^\bullet(X)$ for $p \geq 0$ in the K-theory is defined to be

$$\boldsymbol{c}^p(x) := \gamma^p(x - \varepsilon(x)) \mod F^{p+1}(X) \quad \in G^p(X),$$

where ε is the augmentation map. For a locally free sheaf $\mathscr{E}$, we write $\boldsymbol{c}^p(\mathscr{E}) = \boldsymbol{c}^p(\mathrm{cl}^\bullet(\mathscr{E}))$.

Remark (cf. [**22**, Exp. 0, App. II, Section 5]). Suppose that X is an n-dimensional smooth quasi-projective variety over a field. Then we have the map of the i-th Chern class $c_i\colon K^\bullet(X) \to \mathrm{CH}^i(X)$ for $0 \leq i \leq n$. The Chern class $c_i(x)$ and the Chern class $\boldsymbol{c}^i(x)$ in the K-theory for $x \in K^\bullet(X)$ are related by

$$G(\mathrm{cl}_\bullet)(c_i(x)) = G(\boldsymbol{\phi})(\boldsymbol{c}^i(x)).$$

Definition 1.3. For an invertible sheaf $\mathscr{L}$ on X, we set

$$\boldsymbol{\delta}(\mathscr{L}) = \boldsymbol{\delta}_X(\mathscr{L}) := 1 - \mathrm{cl}^\bullet(\mathscr{L}^{-1}) \in F^1(X).$$

Furthermore, for invertible sheaves $\mathscr{L}_1, \ldots, \mathscr{L}_l$ on X, we set

$$\boldsymbol{\delta}(\mathscr{L}_1, \ldots, \mathscr{L}_l) = \boldsymbol{\delta}^{(l)}_X(\mathscr{L}_1, \ldots, \mathscr{L}_l) := \boldsymbol{\delta}(\mathscr{L}_1)\,\boldsymbol{\delta}(\mathscr{L}_2)\cdots\boldsymbol{\delta}(\mathscr{L}_l) \in F^l(X).$$

REMARK 1.4.

(1) $\boldsymbol{\delta}(\mathscr{L} \otimes \mathscr{L}') = \boldsymbol{\delta}(\mathscr{L}) + \boldsymbol{\delta}(\mathscr{L}') - \boldsymbol{\delta}(\mathscr{L}, \mathscr{L}')$ for two invertible sheaves $\mathscr{L}$ and $\mathscr{L}'$.

(2) $\boldsymbol{\delta}(\mathscr{L}) \bmod F^2(X) = \boldsymbol{c}^1(\mathscr{L}) \in G^1(X)$ for an invertible sheaf $\mathscr{L}$. In fact,

$$\begin{aligned}\boldsymbol{\delta}(\mathscr{L}) - \gamma^1(\mathrm{cl}^\bullet(\mathscr{L}) - 1) &= 1 - \mathrm{cl}^\bullet(\mathscr{L}^{-1}) - (\mathrm{cl}^\bullet(\mathscr{L}) - 1)\\ &= (\mathrm{cl}^\bullet(\mathscr{L}) - 1)(\mathrm{cl}^\bullet(\mathscr{L}^{-1}) - 1)\\ &= \gamma^1(\mathrm{cl}^\bullet(\mathscr{L}) - 1)\gamma^1(\mathrm{cl}^\bullet(\mathscr{L}^{-1}) - 1) \in F^2(X).\end{aligned}$$

In particular,

$$\boldsymbol{\delta}(\mathscr{L}_1, \dots, \mathscr{L}_l) \bmod F^{l+1}(X) = \boldsymbol{c}^1(\mathscr{L}_1) \cdots \boldsymbol{c}^1(\mathscr{L}_l) = \boldsymbol{c}^l(\mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_l).$$

(3) We have the following explicit expression:

$$\begin{aligned}\boldsymbol{\delta}(\mathscr{L}_1, \dots, \mathscr{L}_l) &= \sum_{k=0}^{l} (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le l} \mathrm{cl}^\bullet \left(\bigotimes_{j=1}^{k} \mathscr{L}_{i_j}^{-1} \right)\\ &= 1 - \sum_{i=1}^{l} \mathrm{cl}^\bullet(\mathscr{L}_i^{-1}) + \cdots + (-1)^l \,\mathrm{cl}^\bullet(\mathscr{L}_1^{-1} \otimes \cdots \otimes \mathscr{L}_l^{-1}).\end{aligned}$$

REMARK 1.5. The determinant map $\det\colon K^\bullet(X) \to \mathrm{Pic}(X)$ is defined by $\det(\mathrm{cl}^\bullet(\mathscr{E})) = \det \mathscr{E}$ for locally free sheaves $\mathscr{E}$. We note that

$$\det(xy) \simeq \det(x)^{\otimes \varepsilon(y)} \otimes \det(y)^{\otimes \varepsilon(x)}$$

for x, $y \in K^\bullet(X)$. Since det is trivial on $F^2(X)$ by [**22**, Exp. X, Lemma 5.3.4], a homomorphism $G^1(X) \to \mathrm{Pic}(X)$ is induced by det. This is an isomorphism and its inverse is the first Chern class map $\boldsymbol{c}^1\colon \mathrm{Pic}(X) \to G^1(X)$, by [**22**, Exp. X, Théorème 5.3.2].

DEFINITION 1.6. Let $\mathscr{F}$ be a coherent sheaf and $\mathscr{E}$ a coherent locally free sheaf on X. Let σ be a section of $\mathscr{E}$ and let $\sigma^\vee\colon \mathscr{E}^\vee = \mathscr{H}om(\mathscr{E}, \mathscr{O}_X) \to \mathscr{O}_X$ denote the dual of $\sigma\colon \mathscr{O}_X \to \mathscr{E}$.

(1) The zero subscheme $V(\sigma)$ of the section σ is a closed subscheme defined by $\mathrm{Coker}(\sigma^\vee) = \mathscr{O}_{V(\sigma)}$.

(2) σ is called *$\mathscr{F}$-regular*, if, for any point $P \in V(\sigma)$ and for a local trivialization

$\mathscr{E}_P \simeq \mathscr{O}_P^{\oplus r}$, the germ $\sigma_P \in \mathscr{E}_P$ corresponds to an $\mathscr{F}_P$-regular sequence. In other words, the natural Koszul complex

$$\left[\cdots \to \bigwedge^{p}(\mathscr{E}^\vee) \to \bigwedge^{p-1}(\mathscr{E}^\vee) \to \cdots \to \mathscr{E}^\vee \xrightarrow{\sigma^\vee} \mathscr{O}_X \to 0\right]$$

defined by $\sigma^\vee$ induces an exact sequence

$$\begin{aligned}\cdots \to \mathscr{F} \otimes \bigwedge^{p}(\mathscr{E}^\vee) \to \mathscr{F} \otimes \bigwedge^{p-1}(\mathscr{E}^\vee) \to \\ \cdots \to \mathscr{F} \otimes \mathscr{E}^\vee \to \mathscr{F} \to \mathscr{F} \otimes \mathscr{O}_{V(\sigma)} \to 0.\end{aligned} \tag{1.4}$$

(3) Suppose that $\mathscr{E} = \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_l$ for invertible sheaves $\mathscr{L}_1, \ldots, \mathscr{L}_l$ and that σ is given by sections σ_i of $\mathscr{L}_i$ for $1 \le i \le l$. Then, we define $V(\sigma_1, \ldots, \sigma_l) := V(\sigma)$. Similarly, $(\sigma_1, \ldots, \sigma_l)$ is called $\mathscr{F}$-regular if so is σ.

Lemma 1.7 (cf. [**17**, Théorème 2]). *Let $\mathscr{E}$ be a locally free sheaf of rank r on X.*

(1) *For the formal power series $\lambda_t(x) = \sum_{p \ge 0} \lambda^p(x) t^p \in K^\bullet(X)[\![t]\!]$ for $x \in K^\bullet(X)$,*

$$\lambda_{-1}(\mathscr{E}) := \lambda_{-1}(\mathrm{cl}^\bullet(\mathscr{E})) = \lambda_t(\mathrm{cl}^\bullet(\mathscr{E}))|_{t=-1}$$

is well-defined as an element of $K^\bullet(X)$, and is equal to $(-1)^r \gamma^r(\mathrm{cl}^\bullet(\mathscr{E}) - r)$. In particular, $\boldsymbol{c}^r(\mathscr{E}) = \lambda_{-1}(\mathscr{E}^\vee) \mod F^{r+1}(X)$.

(2) *Let $\mathscr{F}$ be a coherent sheaf and σ an $\mathscr{F}$-regular section of $\mathscr{E}$. Then,*

$$\lambda_{-1}(\mathscr{E}^\vee)\,\mathrm{cl}_\bullet(\mathscr{F}) = \mathrm{cl}_\bullet(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)}). \tag{1.5}$$

In particular,

$$\begin{aligned}&\boldsymbol{c}^r(\mathscr{E})\big(\mathrm{cl}_\bullet(\mathscr{F}) \bmod F^{k+1}_{\mathrm{con}}(X)\big) \\ &\quad = \mathrm{cl}_\bullet(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)}) \bmod F^{k+r+1}_{\mathrm{con}}(X), \qquad \textit{if } \operatorname{codim} \mathscr{F} \ge k, \\ &\boldsymbol{c}^r(\mathscr{E})\big(\mathrm{cl}_\bullet(\mathscr{F}) \bmod F_{k-1}(X)\big) \\ &\quad = \mathrm{cl}_\bullet(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)}) \bmod F_{k-r-1}(X), \qquad \textit{if } \dim \mathscr{F} \le k.\end{aligned}$$

Proof. For $x \in K^\bullet(X)$, formally, we have

$$\lambda_{-1}(x) = \lambda_t(x)|_{t=-1} = \sum_{p \geq 0} (-1)^p \lambda^p(x).$$

If $x = \mathrm{cl}^{\bullet}(\mathscr{E})$, then $\lambda^p(x) = \mathrm{cl}^{\bullet}(\bigwedge^p \mathscr{E}) = 0$ for $p > r$. Hence, $\lambda_{-1}(\mathscr{E})$ is well-defined. The formal power series $\gamma_t(x) := \sum_{p \geq 0} \gamma^p(x) t^p$ is related to $\lambda_t(x)$ by $\lambda_t(x) = \gamma_{t/(1+t)}(x)$ and $\gamma_t(x) = \lambda_{t/(1-t)}(x)$. Thus, the following equalities hold:

$$\lambda_t(x) = \lambda_t(x-r)\lambda_t(r) = \gamma_{t/(1+t)}(x-r)(1+t)^r = \sum_{p \geq 0} \gamma^p(x-r) t^p (1+t)^{r-p}, \tag{1.6}$$

$$\gamma_t(x-r) = \gamma_t(x)\gamma_t(r)^{-1} = \lambda_{t/(1-t)}(x)(1-t)^r = \sum_{p \geq 0} \lambda^p(x) t^p (1-t)^{r-p}. \tag{1.7}$$

We have $\gamma^p(\mathrm{cl}^{\bullet}(\mathscr{E}) - r) = 0$ for $p > r$ by (1.7), since $\lambda^p(\mathrm{cl}^{\bullet}(\mathscr{E})) = 0$ for $p > r$. Therefore, substituting $t = -1$ in (1.6), we have the expected equality $\lambda_{-1}(\mathscr{E}) = (-1)^r \gamma^r(\mathrm{cl}^{\bullet}(\mathscr{E}) - r)$. The other formula in the assertion (1) follows from the equality $\boldsymbol{c}^r(\mathscr{E}) = (-1)^r \boldsymbol{c}^r(\mathscr{E}^{\vee})$ (cf. Remark 1.8 below). The assertion (2) is derived from the exact sequence (1.4). □

Remark 1.8. For a locally free sheaf $\mathscr{E}$ of rank r on X, suppose that there is a flag $0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_r = \mathscr{E}$ of subbundles such that $\mathscr{L}_i := \mathscr{E}_i / \mathscr{E}_{i-1}$ is an invertible sheaf for any $1 \leq i \leq r$. Then $\mathrm{cl}^{\bullet}(\mathscr{E}) = \sum_{i=1}^{r} \mathrm{cl}^{\bullet}(\mathscr{L}_i)$, $\mathrm{cl}^{\bullet}(\mathscr{E}^{\vee}) = \sum_{i=1}^{r} \mathrm{cl}^{\bullet}(\mathscr{L}_i^{\vee})$, and

$$\gamma_t(\mathrm{cl}^{\bullet}(\mathscr{E}^{\vee}) - r) = \gamma_t\Big(-\sum_{i=1}^{r} \boldsymbol{\delta}(\mathscr{L}_i)\Big) = \prod_{i=1}^{r} \gamma_t(-\boldsymbol{\delta}(\mathscr{L}_i)) = \prod_{i=1}^{r} (1 - \boldsymbol{\delta}(\mathscr{L}_i) t), \tag{1.8}$$

where the last equality follows from (1.7): In fact, $\gamma_t(x-1) = (1-t) + xt = 1 + (x-1)t$ if $\lambda^p(x) = 0$ for any $p > 1$. Comparing the coefficient of t^p in (1.8), we have

$$(-1)^p \gamma^p(\mathrm{cl}^{\bullet}(\mathscr{E}^{\vee}) - r) = \sum_{1 \leq i_1 < \cdots < i_p \leq r} \boldsymbol{\delta}(\mathscr{L}_{i_1}, \ldots, \mathscr{L}_{i_p})$$

for $1 \leq p \leq r$. In particular, $\lambda_{-1}(\mathscr{E}^{\vee}) = \boldsymbol{\delta}(\mathscr{L}_1, \ldots, \mathscr{L}_r)$. Moreover,

$$(-1)^p \boldsymbol{c}^p(\mathscr{E}^{\vee}) = \sum_{1 \leq i_1 < \cdots < i_p \leq r} \boldsymbol{c}^1(\mathscr{L}_{i_1}) \cdots \boldsymbol{c}^1(\mathscr{L}_{i_p}) = \boldsymbol{c}^p(\mathscr{E}) \quad \in G^p(X)$$

from the equality above. Many equalities on the Chern classes related to $\mathscr{E}$ are derived from the same argument on the pullback $q^*(\mathscr{E})$ for the *flag scheme* $q\colon \boldsymbol{F} \to X$ of $\mathscr{E}$. Here, $\boldsymbol{F}$ is a universal scheme parametrizing the flags $0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_r$ of subbundles of the pullback $\mathscr{E}_T$ of $\mathscr{E}$ to an X-scheme T such that $\mathscr{E}_i/\mathscr{E}_{i-1}$ is invertible for any $1 \le i \le r$. Then, $q\colon \boldsymbol{F} \to X$ is a composite of projective space bundles, and $K^\bullet(X)$ is a direct summand of $K^\bullet(\boldsymbol{F})$ by $q^\star\colon K^\bullet(X) \to K^\bullet(\boldsymbol{F})$ and $F^p(\boldsymbol{F}) \cap K^\bullet(X) = F^p(X)$ for any $p \ge 0$ (cf. [**22**, Exp. VI, Proposition 3.1 and Corollaire 5.5]). Hence, we can check some of equalities related to $\boldsymbol{c}^p(\mathscr{E})$ by pulling back $\mathscr{E}$ to $\boldsymbol{F}$. For example, the equality $\boldsymbol{c}^p(\mathscr{E}^\vee) = (-1)^p\boldsymbol{c}^p(\mathscr{E})$ for general $\mathscr{E}$ is proved by the argument.

Notation (Div). Let A be a local Noetherian ring of depth $A = 1$ and M a finitely generated A-module. Assume that $M_{\mathfrak{p}} = 0$ for any associated prime $\mathfrak{p}$ of A and that M has finite Tor-dimension. Then, there is an exact sequence

$$0 \to A^{\oplus r} \xrightarrow{h} A^{\oplus r} \to M \to 0$$

of A-modules for some $r > 0$, by a theorem of Auslander and Buchsbaum. The determinant $\det(h) \in A$ defines an effective Cartier divisor on $\operatorname{Spec} A$, which we denote by $\operatorname{Div}_A(M)$. Note that the divisor $\operatorname{Div}_A(M)$ does not depend on the choice of exact sequences above (cf. [**32**, Chapter 5, Section 3]).

Lemma 1.9. *Let $\mathscr{G}^\bullet = [\cdots \to \mathscr{G}^i \to \mathscr{G}^{i+1} \to \cdots]$ be a bounded complex of locally free sheaves $\mathscr{G}^i$ of finite rank on X. For the cohomology sheaves $\mathscr{H}^i = \mathscr{H}^i(\mathscr{G}^\bullet)$, assume the following conditions:*

- *If* $\operatorname{depth}\mathscr{O}_{X,\boldsymbol{x}} = 0$*, then the stalk $\mathscr{H}^i_{\boldsymbol{x}} = 0$ for any $i \in \boldsymbol{Z}$.*
- *If* $\operatorname{depth}\mathscr{O}_{X,\boldsymbol{x}} = 1$*, then the stalk $\mathscr{H}^i_{\boldsymbol{x}} = 0$ for any $i \neq 0$.*

Then, there exists uniquely an effective Cartier divisor $\operatorname{Div}(\mathscr{G}^\bullet)$ *on X such that*

$$\det(\mathscr{G}^\bullet) := \bigotimes_{i\in\boldsymbol{Z}} (\det\mathscr{G}^i)^{\otimes(-i)} \simeq \mathscr{O}_X(\operatorname{Div}(\mathscr{G}^\bullet)), \quad \textit{and}$$

$$\operatorname{Div}(\mathscr{G}^\bullet)|_{\operatorname{Spec}\mathscr{O}_{X,\boldsymbol{x}}} = \operatorname{Div}(\mathscr{H}^0(\mathscr{G}^\bullet)_{\boldsymbol{x}})$$

for any point $\boldsymbol{x} \in X$ with $\operatorname{depth}\mathscr{O}_{X,\boldsymbol{x}} = 1$*. In particular,*

$$\operatorname{cl}^\bullet(\mathscr{G}^\bullet) := \sum_{i\in\boldsymbol{Z}} (-1)^i \operatorname{cl}^\bullet(\mathscr{G}^i) = \boldsymbol{\delta}\left(\mathscr{O}_X(\operatorname{Div}(\mathscr{G}^\bullet))\right).$$

Moreover, if another bounded complex $\mathscr{G}'^{\bullet} = [\cdots \to \mathscr{G}'^{i} \to \mathscr{G}'^{i+1} \to \cdots]$ of locally free sheaves of finite rank on X is quasi-isomorphic to $\mathscr{G}^{\bullet}$, then $\operatorname{Div}(\mathscr{G}^{\bullet}) = \operatorname{Div}(\mathscr{G}'^{\bullet})$.

Proof. We follow the argument of [**32**, Chapter 5, Section 3]. Let $\mathscr{Z}^0 = \mathscr{Z}^0(\mathscr{G}^{\bullet})$ be the kernel of $\mathscr{G}^0 \to \mathscr{G}^1$ and let $\mathscr{B}^0 = \mathscr{B}^0(\mathscr{G}^{\bullet})$ be the image of $\mathscr{G}^{-1} \to \mathscr{G}^0$. Then, we have an exact sequence $0 \to \mathscr{B}^0 \to \mathscr{Z}^0 \to \mathscr{H}^0 \to 0$. Let $U \subset X$ be the maximal open subset such that $\mathscr{H}^i|_U = 0$ for any $i \neq 0$ and $\mathscr{B}^0|_U$ is locally free. Then, $\mathscr{B}^0|_U \to \mathscr{Z}^0|_U$ is an injection between locally free sheaves of the same rank. Taking the determinant of the injection, we have an injection $\det \mathscr{B}^0|_U \to \det \mathscr{Z}^0|_U$. Consequently, we have a global section s_U of $\det \mathscr{Z}^0|_U \otimes (\det \mathscr{B}^0|_U)^{-1} \simeq \det(\mathscr{G}^{\bullet})|_U$. Since U contains all the point $\boldsymbol{x} \in X$ with $\operatorname{depth} \mathscr{O}_{X,\boldsymbol{x}} \leq 1$, we have

$$\mathrm{H}^0(X, \det(\mathscr{G}^{\bullet})) \simeq \mathrm{H}^0(U, \det(\mathscr{G}^{\bullet})|_U)$$

(cf. [**9**, Lemma 2.1]). Let s be the lift of s_U as a global section of $\det(\mathscr{G}^{\bullet})$ on X. Then, s defines an effective Cartier divisor $\operatorname{Div}(\mathscr{G}^{\bullet})$ on X satisfying the expected condition. In order to show the uniqueness of $\operatorname{Div}(\mathscr{G}^{\bullet})$ up to quasi-isomorphism, it is enough to check it locally at points $\boldsymbol{x} \in X$ of $\operatorname{depth} \mathscr{O}_{X,\boldsymbol{x}} = 1$. Then, the uniqueness follows from that of $\operatorname{Div}(\mathscr{H}^0(\mathscr{G}^{\bullet})_{\boldsymbol{x}})$. □

Lemma 1.10. *In the situation of Lemma* 1.9*, let $f \colon Y \to X$ be a morphism from another Noetherian scheme Y and let $\mathscr{G}^{\bullet}_Y$ be the complex $[\cdots \to f^*\mathscr{G}^i \to f^*\mathscr{G}^{i+1} \to \cdots]$ on Y. Assume that*

- *if* $\operatorname{depth} \mathscr{O}_{Y,\boldsymbol{y}} = 0$*, then $\mathscr{H}^p(\mathscr{G}^{\bullet})_{f(\boldsymbol{y})} = 0$ for any $p \in \boldsymbol{Z}$, and*
- *if* $\operatorname{depth} \mathscr{O}_{Y,\boldsymbol{y}} = 1$*, then $\mathscr{H}^p(\mathscr{G}^{\bullet})_{f(\boldsymbol{y})} = \mathscr{H}^p(\mathscr{G}^{\bullet}_Y)_{\boldsymbol{y}} = 0$ for any $p \neq 0$.*

Then, $\mathscr{G}^{\bullet}_Y$ satisfies the conditions of Lemma 1.9*, and* $\operatorname{Div}(\mathscr{G}^{\bullet}_Y) = f^* \operatorname{Div}(\mathscr{G}^{\bullet})$.

Proof. Let U_Y be the maximal open subset $U_Y \subset Y$ such that $\mathscr{H}^i(\mathscr{G}^{\bullet}_Y)|_{U_Y} = 0$ for any $i \neq 0$ and that $\mathscr{B}^0(\mathscr{G}^{\bullet}_Y)|_{U_Y}$ is locally free. It is enough to prove that the intersection $U' := f^{-1}(U) \cap U_Y$ contains all the points $\boldsymbol{y} \in Y$ of $\operatorname{depth} \mathscr{O}_{Y,\boldsymbol{y}} \leq 1$. In fact, if this is true, then $\mathscr{G}^{\bullet}_Y$ satisfies the conditions of Lemma 1.9, and we have an isomorphism

$$\begin{tikzcd}
0 \arrow[r] & f^*\mathscr{B}^0(\mathscr{G}^{\bullet})|_{U'} \arrow[r] \arrow[d, "\simeq"'] & f^*\mathscr{Z}^0(\mathscr{G}^{\bullet})|_{U'} \arrow[r] \arrow[d, "\simeq"'] & f^*\mathscr{H}^0(\mathscr{G}^{\bullet})|_{U'} \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & \mathscr{B}^0(\mathscr{G}^{\bullet}_Y)|_{U'} \arrow[r] & \mathscr{Z}^0(\mathscr{G}^{\bullet}_Y)|_{U'} \arrow[r] & \mathscr{H}^0(\mathscr{G}^{\bullet}_Y)|_{U'} \arrow[r] & 0
\end{tikzcd}$$

of exact sequences, which induces $\mathrm{Div}(\mathscr{G}^\bullet_Y) = f^* \mathrm{Div}(\mathscr{G}^\bullet)$. If $\operatorname{depth} \mathscr{O}_{Y,\boldsymbol{y}} = 0$, then $\mathscr{G}^\bullet$ is exact at $f(\boldsymbol{y})$; thus $\mathscr{G}^\bullet_Y$ is also exact at $\boldsymbol{y}$, and consequently, $\boldsymbol{y} \in U'$. Hence, we may assume that $\operatorname{depth} \mathscr{O}_{Y,\boldsymbol{y}} = 1$. We set $A = \mathscr{O}_{X,f(\boldsymbol{y})}$, $R = \mathscr{O}_{Y,\boldsymbol{y}}$, $N = \mathscr{B}^0(\mathscr{G}^\bullet)_{f(\boldsymbol{y})}$, and $M = \mathscr{Z}^0(\mathscr{G}^\bullet)_{f(\boldsymbol{y})}$. It is enough to prove that N is a free A-module. From the vanishings $\mathscr{H}^i(\mathscr{G}^\bullet)_{f(\boldsymbol{y})} = \mathscr{H}^i(\mathscr{G}^\bullet_Y)_{\boldsymbol{y}} = 0$ for any $i \neq 0$, we infer that M is a free A-module of finite rank, $N \otimes_A R \to M \otimes_A R$ is injective, and $\mathrm{Tor}^A_i(N, R) = 0$ for any $i > 0$. Since $\operatorname{depth} R = 1$ and $(M/N)_{\mathfrak{p}} = 0$ for any associated prime $\mathfrak{p}$ of R, $N \otimes_A R$ is a free B-module of the same rank r as M, by a theorem of Auslander and Buchsbaum. Since $N \otimes_A \boldsymbol{k}(A)$ is r-dimensional for the residue field $\boldsymbol{k}(A)$, we have a surjection $A^{\oplus r} \to N$, in which the kernel K satisfies $K \otimes_A R = 0$, since $\mathrm{Tor}^A_1(N, R) = 0$ and since the surjection $(A^{\oplus r}) \otimes_A R \to N \otimes_A R$ of free R-modules of the same rank is an isomorphism. In particular, $K \otimes_A \boldsymbol{k}(A) = 0$. Thus $K = 0$, and consequently, N is free. This completes the proof. □

1.4. Intersection numbers.

Definition 1.11 (intersection number). Assume that X is a scheme proper over $\operatorname{Spec} \boldsymbol{k}$ for a field $\boldsymbol{k}$. For the structure morphism $p_X \colon X \to \operatorname{Spec} \boldsymbol{k}$, the composition

$$K_\bullet(X) \xrightarrow{p_{X\star}} K_\bullet(\operatorname{Spec} \boldsymbol{k}) \simeq \mathrm{H}^0(\operatorname{Spec} \boldsymbol{k}, \boldsymbol{Z}) = \boldsymbol{Z}$$

maps $\mathrm{cl}_\bullet(\mathscr{F})$ for a coherent sheaf $\mathscr{F}$ on X to the Euler characteristic

$$\chi_{\boldsymbol{k}}(X, \mathscr{F}) = \sum_{i \geq 0} (-1)^i \dim_{\boldsymbol{k}} \mathrm{H}^i(X, \mathscr{F})$$

over $\boldsymbol{k}$. In particular, it induces a homomorphism $\deg_{0,X/\boldsymbol{k}} \colon G_0(X) = F_0(X) \to \boldsymbol{Z}$, which maps $\mathrm{cl}_\bullet(\mathscr{F})$ to $\dim_{\boldsymbol{k}} \mathrm{H}^0(X, \mathscr{F})$ for a skyscraper sheaf $\mathscr{F}$. The intersection number $i_{X/\boldsymbol{k}}(\eta; \xi) \in \boldsymbol{Z}$ for $\eta \in G^l(X)$ and $\xi \in G_l(X)$ is defined to be the image of $\eta \otimes \xi$ by the natural homomorphism

$$G^l(X) \otimes G_l(X) \to G_0(X) \xrightarrow{\deg_{0,X/\boldsymbol{k}}} \boldsymbol{Z}.$$

If $\eta = \boldsymbol{c}^1(\mathscr{L}_1) \cdots \boldsymbol{c}^1(\mathscr{L}_l)$ for invertible sheaves $\mathscr{L}_1, \ldots, \mathscr{L}_l$, then we write $i_{X/\boldsymbol{k}}(\mathscr{L}_1, \ldots, \mathscr{L}_l; \xi) = i_{X/\boldsymbol{k}}(\eta; \xi)$. For a coherent sheaf $\mathscr{F}$ with $\dim \operatorname{Supp} \mathscr{F} = l$ and a closed subscheme V of dimension l, we write

$$i_{X/\boldsymbol{k}}(\eta; \mathscr{F}) = i_{X/\boldsymbol{k}}(\eta; \mathrm{cl}_\bullet(\mathscr{F})) \quad \text{and} \quad i_{X/\boldsymbol{k}}(\eta; V) = i_{X/\boldsymbol{k}}(\eta; \mathrm{cl}_\bullet(\mathscr{O}_V)).$$

REMARK. For $\mathscr{L}_1, \dots, \mathscr{L}_l \in \operatorname{Pic}(X)$ and $\xi \in G_l(X)$, we have

$$i_{X/\boldsymbol{k}}(\mathscr{L}_1, \dots, \mathscr{L}_l; \xi) = \deg_{0,X/\boldsymbol{k}}(\boldsymbol{\delta}(\mathscr{L}_1, \dots, \mathscr{L}_l)\xi)$$

by Remark 1.4. Using the equality, we can prove that, for a coherent sheaf $\mathscr{F}$ with $\dim \operatorname{Supp} \mathscr{F} = l$, $i_{X/\boldsymbol{k}}(\mathscr{L}_1, \dots, \mathscr{L}_l; \mathscr{F})$ is just the coefficient of $x_1 x_2 \dots x_l$ of the Snapper polynomial $P_{\mathscr{F}}(x_1, \dots, x_l) \in \boldsymbol{Q}[x_1, \dots, x_l]$ defined by

$$P_{\mathscr{F}}(m_1, \dots, m_l) = \chi_{\boldsymbol{k}}\big(X, \mathscr{L}_1^{\otimes m_1} \otimes \dots \otimes \mathscr{L}_l^{\otimes m_l} \otimes \mathscr{F}\big)$$

for $m_1, \dots, m_l \in \boldsymbol{Z}$ (cf. [**22**, Exp. X, Corollaire 4.3.3], [**25**, Chapter I, Section 2]). In particular, if $\dim X = 1$, then the degree $\deg_{X/\boldsymbol{k}}(\mathscr{L})$ of an invertible sheaf $\mathscr{L}$ on X over $\boldsymbol{k}$ is nothing but $i_{X/\boldsymbol{k}}(\boldsymbol{c}^1(\mathscr{L}); X)$.

LEMMA 1.12. *In the situation of Definition* 1.11, *let* $\tau \colon Y \to X$ *be a proper morphism from another Noetherian scheme. Then,*

$$i_{Y/\boldsymbol{k}}(\tau^\star \eta; \xi) = i_{X/\boldsymbol{k}}(\eta; \tau_\star(\xi))$$

for any $\eta \in G^l(X)$ *and* $\xi \in G_l(Y)$, *where* $\tau^\star$ *and* $\tau_\star$ *stand for the induced homomorphisms* $G^l(X) \to G^l(Y)$ *and* $G_0(Y) \to G_0(X)$, *respectively. In particular, if* τ *is a closed immersion and* $\dim Y = 1$, *then*

$$i_{X/\boldsymbol{k}}(\boldsymbol{c}^1(\mathscr{L}); Y) = i_{Y/\boldsymbol{k}}(\boldsymbol{c}^1(\mathscr{L})|_Y) = \deg_{Y/\boldsymbol{k}} \mathscr{L}|_Y$$

for any invertible sheaf $\mathscr{L}$ *on* X.

PROOF. The assertion follows from the projection formula $\tau_\star((\tau^\star\eta)\xi) = \eta(\tau_\star\xi)$, which is derived from (1.1). □

LEMMA 1.13. *In the situation of Definition* 1.11, *let* $\boldsymbol{k}'/\boldsymbol{k}$ *be a field extension. Let* X' *be the fiber product* $X \times_{\operatorname{Spec} \boldsymbol{k}} \operatorname{Spec} \boldsymbol{k}'$ *and* $q \colon X' \to X$ *the induced morphism. For* $\eta \in G^l(X)$ *and* $\xi \in G_l(X)$, *let* $\eta' \in G^l(X')$ *be the image of* $G(q^\star) \colon G^l(X) \to G^l(X')$ *and let* ξ' *be the image of* $G(q^\star) \colon G_l(X) \to G_l(X')$ *induced from* $q^\star \colon K_\bullet(X) \to K_\bullet(X')$ (which is defined since q is flat). *Then*

$$i_{X'/\boldsymbol{k}'}(\eta'; \xi') = i_{X/\boldsymbol{k}}(\eta; \xi).$$

PROOF. Let $x \in F_l(X)$ and $y \in F^l(X)$ be representatives of ξ and η, respectively. Then $yx \in F_0(X)$, and $p_{X\star}(yx) \in K_\bullet(\operatorname{Spec} \boldsymbol{k}) \simeq \boldsymbol{Z}$ corresponds to $i_{X/\boldsymbol{k}}(\eta;\xi)$. Let $h\colon \operatorname{Spec}\boldsymbol{k}' \to \operatorname{Spec}\boldsymbol{k}$ be the morphism associated with the extension $\boldsymbol{k}'/\boldsymbol{k}$ and let $p_{X'}\colon X' \to \operatorname{Spec}\boldsymbol{k}'$ be the induced morphism. Then, $h^\star\colon K_\bullet(\operatorname{Spec}\boldsymbol{k}) \to K_\bullet(\operatorname{Spec}\boldsymbol{k}')$ is just the identity map. For the homomorphisms $q^\star\colon K_\bullet(X) \to K_\bullet(X')$ and $p_{X'\star}\colon K_\bullet(X') \to K_\bullet(\operatorname{Spec}\boldsymbol{k}')$, we have

$$p_{X'\star}(q^\star(yx)) = h^\star(p_{X\star}(yx))$$

by (1.2). Thus, the expected equality is obtained. □

Remark. In the situation of Definition 1.11, if $\boldsymbol{k}''$ is a subfield of $\boldsymbol{k}$ with $e = \dim_{\boldsymbol{k}''}\boldsymbol{k} < \infty$, then $\deg_{0,X/\boldsymbol{k}''} = e\deg_{0,X/\boldsymbol{k}}$ as a homomorphism $G_0(X) \to \boldsymbol{Z}$; hence $i_{X/\boldsymbol{k}''}(\eta;\xi) = e\, i_{X/\boldsymbol{k}}(\eta;\xi)$ for any $\eta \in G^l(X)$ and $\xi \in G_l(X)$.

1.5. First two filters of coniveau filtration.

We shall describe $G^0_{\mathrm{con}}(X)$ and $G^1_{\mathrm{con}}(X)$ for suitable Noetherian schemes X.

Lemma 1.14. *Let X be a Noetherian scheme and let $\{X_i\}_{i\in I}$ be the set of irreducible components of X. Then, for any $i \in I$, there exists uniquely a homomorphism $l_i\colon K_\bullet(X) \to \boldsymbol{Z}$ such that $\mathrm{cl}_\bullet(\mathscr{F})$ is mapped to the length $l_{X_i}(\mathscr{F})$ of $\mathscr{F}$ along X_i* (cf. Definition 1.1) *for any coherent sheaf $\mathscr{F}$ on X. Moreover, $\sum_{i\in I} l_i\colon K_\bullet(X) \to \bigoplus_I \boldsymbol{Z}$ induces an isomorphism $G^0_{\mathrm{con}}(X) \simeq \bigoplus_I \boldsymbol{Z}$.*

Proof. If $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ is an exact sequence of coherent sheaves, then $l_{X_i}(\mathscr{F}_2) = l_{X_i}(\mathscr{F}_1) + l_{X_i}(\mathscr{F}_3)$. Thus, l_{X_i} defines uniquely the expected homomorphism $l_i\colon K_\bullet(X) \to \boldsymbol{Z}$. If $\mathscr{F}$ is a coherent sheaf with $\operatorname{codim}\operatorname{Supp}\mathscr{F} > 0$, then $l_{X_i}(\mathscr{F}) = 0$ for any i. Thus $\sum l_i$ induces a homomorphism $G^0_{\mathrm{con}}(X) \to \bigoplus_I \boldsymbol{Z}$. We have a natural surjective homomorphism $\mathscr{Z}^0(X) \xrightarrow{\mathrm{cl}_\bullet} F^0_{\mathrm{con}}(X) \to G^0_{\mathrm{con}}(X)$ such that the composition with $G^0_{\mathrm{con}}(X) \to \bigoplus_I \boldsymbol{Z}$ induces an isomorphism $\mathscr{Z}^0(X) \to \bigoplus_I \boldsymbol{Z}$. In fact, $\mathscr{Z}^0(X)$ is a free abelian group generated by the cycles X_i of codimension zero, and $l_j(\mathrm{cl}_\bullet(X_i)) = \delta_{i,j}$ for any i, $j \in I$. Thus, $G^0_{\mathrm{con}}(X) \simeq \bigoplus_I \boldsymbol{Z}$. □

Definition 1.15. Let X be a Noetherian scheme with a morphism $X \to Y$ to an integral scheme Y. For a coherent sheaf $\mathscr{F}$ of X, we denote by $\mathscr{F}_{\mathrm{tor}/Y}$ the unique maximal coherent subsheaf $\mathscr{F}'$ such that $\operatorname{Supp}\mathscr{F}'$ does not dominate Y. We denote by $\mathscr{F}_{\mathrm{t.f.}/Y}$ the quotient sheaf $\mathscr{F}/(\mathscr{F}_{\mathrm{tor}/Y})$. In case $X = Y$, then we write $\mathscr{F}_{\mathrm{tor}} = \mathscr{F}_{\mathrm{tor}/Y}$ and $\mathscr{F}_{\mathrm{t.f.}} = \mathscr{F}_{\mathrm{t.f.}/Y}$. The sheaf $\mathscr{F}_{\mathrm{tor}}$ is called the *torsion part* of $\mathscr{F}$. If $\mathscr{F}_{\mathrm{tor}} = 0$, then $\mathscr{F}$ is called *torsion free*. For a torsion free sheaf $\mathscr{F}$, the rank $\mathrm{rk}(\mathscr{F})$ is defined as the length $l_X(\mathscr{F})$.

Definition 1.16. Let X be a normal separated Noetherian scheme. A coherent sheaf $\mathscr{F}$ is called *reflexive* if the double-dual $\mathscr{F}^{\vee\vee} := (\mathscr{F}^\vee)^\vee$ is canonically isomorphic to $\mathscr{F}$, where $\mathscr{F}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$ (cf. [**23**, Section 1]). The set of reflexive sheaves of rank one on X has a natural abelian group structure by $(\mathscr{L}_1, \mathscr{L}_2) \mapsto (\mathscr{L}_1 \otimes \mathscr{L}_2)^{\vee\vee}$. The group is denoted by $\mathrm{Ref}^1(X)$.

Remark. $\mathrm{Ref}^1(X)$ contains the Picard group $\mathrm{Pic}(X)$ as a subgroup. Furthermore, $\mathrm{Ref}^1(X)$ is isomorphic to the Weil divisor class group of X by $D \mapsto \mathscr{O}_X(D)$ for Weil divisors D. Here $\mathscr{O}_X(D)$ is a subsheaf of the sheaf of germs of rational functions on X defined by

$$\varphi \in \mathrm{H}^0(U, \mathscr{O}_X(D)) \setminus \{0\} \iff \mathrm{div}(\varphi) + D|_U \geq 0$$

for any open subset U, where $\mathrm{div}(\varphi)$ stands for the principal divisor associated to a non-zero rational function φ.

Lemma 1.17. *Suppose that X is a normal separated Noetherian scheme. Then there is an isomorphism $\widehat{\det}\colon G^1_{\mathrm{con}}(X) \xrightarrow{\simeq} \mathrm{Ref}^1(X)$ with the natural commutative diagram*

$$\begin{tikzcd} G^1(X) \arrow[r, "\det"] \arrow[d, "G(\phi)"'] & \mathrm{Pic}(X) \arrow[d] \\ G^1_{\mathrm{con}}(X) \arrow[r, "\widehat{\det}"] & \mathrm{Ref}^1(X). \end{tikzcd}$$

Proof. We may assume that X is integral. For a coherent sheaf $\mathscr{F}$, we can associate a reflexive sheaf $\mathscr{D}(\mathscr{F})$ of rank one as follows:

- If $\mathscr{F}$ is a torsion sheaf, i.e., $l_X(\mathscr{F}) = 0$, then $\mathscr{D}(\mathscr{F}) := \mathscr{O}_X(\mathrm{Div}(\mathscr{F}))$ for the Weil divisor

$$\mathrm{Div}(\mathscr{F}) := \mathrm{cyc}^1(\mathscr{F}) = \sum_{\text{prime divisors } \Gamma \subset \mathrm{Supp}\,\mathscr{F}} l_\Gamma(\mathscr{F})\Gamma.$$

- If $\mathscr{F}$ is torsion free, then

$$\mathscr{D}(\mathscr{F}) := \left(\bigwedge^{\mathrm{rk}(\mathscr{F})} \mathscr{F} \right)^{\vee\vee}.$$

- For a general coherent sheaf $\mathscr{F}$, we define

$$\mathscr{D}(\mathscr{F}) := \big(\mathscr{D}(\mathscr{F}_{\mathrm{t.f.}}) \otimes \mathscr{D}(\mathscr{F}_{\mathrm{tor}})\big)^{\vee\vee}.$$

We shall show $\mathscr{D}(\mathscr{F}) \simeq (\mathscr{D}(\mathscr{G}) \otimes \mathscr{D}(\mathscr{H}))^{\vee\vee}$ for any exact sequence $0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{H} \to 0$ of coherent sheaves: Let $\mathscr{K}$ be the kernel of $\mathscr{F}_{\text{t.f.}} \to \mathscr{H}_{\text{t.f.}}$ and $\mathscr{C}$ the cokernel of $\mathscr{F}_{\text{tor}} \to \mathscr{H}_{\text{tor}}$; then we have an exact sequence $0 \to \mathscr{G}_{\text{t.f.}} \to \mathscr{K} \to \mathscr{C} \to 0$. Thus, it is enough to show $\mathscr{D}(\mathscr{F}) \simeq (\mathscr{D}(\mathscr{G}) \otimes \mathscr{D}(\mathscr{H}))^{\vee\vee}$ in the case where $\mathscr{H}$ is torsion and $\mathscr{F}$ is torsion free. For the generic point η of a prime divisor Γ, $\mathscr{G}_\eta \to \mathscr{F}_\eta$ is written as a homomorphism $h\colon \mathscr{O}_{X,\eta}^{\oplus r} \to \mathscr{O}_{X,\eta}^{\oplus r}$ for $r = \operatorname{rk} \mathscr{F} = \operatorname{rk} \mathscr{G}$. Thus, $l_\Gamma(\mathscr{H})$ is just the length of $\mathscr{O}_{X,\eta}/\det(h)\mathscr{O}_{X,\eta}$ for the determinant $\det(h)$. Hence, $\mathscr{D}(\mathscr{F}) \simeq (\mathscr{D}(\mathscr{G}) \otimes \mathscr{D}(\mathscr{H}))^{\vee\vee}$.

Therefore, $\mathscr{D}$ gives rise to a homomorphism $K_\bullet(X) \to \operatorname{Ref}^1(X)$, in which $F^2_{\text{con}}(X)$ is contained in the kernel. Thus, a homomorphism $\widehat{\det}\colon G^1_{\text{con}}(X) \to \operatorname{Ref}^1(X)$ is induced from $\mathscr{D}$. The homomorphism $G(\mathrm{cl}_\bullet)\colon \mathscr{Z}^1(X) \xrightarrow{\mathrm{cl}_\bullet} F^1_{\text{con}}(X) \to G^1_{\text{con}}(X)$ is surjective, and the composite $\widehat{\det} \circ G(\mathrm{cl}_\bullet)\colon \mathscr{Z}^1(X) \to \operatorname{Ref}^1(X)$ is the canonical surjection which maps a Weil divisor D to $\mathscr{O}_X(D)$. Hence, in order to prove that $\widehat{\det}$ is an isomorphism, it suffices to show that $\mathrm{cl}_\bullet(Z) = 0 \in F^1_{\text{con}}(X)$ for any divisor Z with $\mathscr{O}_X(Z) \simeq \mathscr{O}_X$. This is shown as follows: For such a divisor Z, let $Z = Z_1 - Z_2$ be the decomposition into effective divisors Z_1, Z_2 without common prime components. From the equality $\mathrm{cl}_\bullet(\mathscr{O}_{Z_i}) = \mathrm{cl}_\bullet(\mathscr{O}_X) - \mathrm{cl}_\bullet(\mathscr{O}_X(-Z_i))$ for $i = 1, 2$, we have

$$\mathrm{cl}_\bullet(Z) = \mathrm{cl}_\bullet(\mathscr{O}_{Z_1}) - \mathrm{cl}_\bullet(\mathscr{O}_{Z_2}) = \boldsymbol{\phi}\left(\mathrm{cl}^\bullet(\mathscr{O}_X(-Z_2)) - \mathrm{cl}^\bullet(\mathscr{O}_X(-Z_1))\right) = 0.$$

Finally, we compare $\widehat{\det}$ with the other isomorphism $\det\colon G^1(X) \to \operatorname{Pic}(X)$. For a locally free sheaf $\mathscr{E}$ of rank r, we have $\det(x) = \det(\mathscr{E})$ for $x := \mathrm{cl}^\bullet(\mathscr{E}) - r \bmod F^2(X) \in G^1(X)$ (cf. Remark 1.5). On the other hand, $\widehat{\det}(G(\boldsymbol{\phi})(x)) = \mathscr{D}(\mathscr{E}) \simeq \det(\mathscr{E})$. Thus, $\widehat{\det}$ is compatible with det. □

Corollary 1.18. *$F^2(X) = 0$ for any one-dimensional regular separated Noetherian scheme X.*

Proof. We have $F^2_{\text{con}}(X) = 0$, since $\dim X = 1$. Lemma 1.17 implies that $F^2(X) = F^1(X) \cap \boldsymbol{\phi}^{-1}(F^2_{\text{con}}(X))$, where $\boldsymbol{\phi}\colon K^\bullet(X) \to K_\bullet(X)$ is an isomorphism, since X is regular. Thus, we are done. □

We discuss the vanishing result $F^{n+1}(X) = 0$ for $n > \dim X$ in Proposition 2.24 in Section 2.3.

2. Intersection sheaves for flat morphisms.

Let $\pi\colon X \to Y$ be a locally projective morphism of Noetherian schemes and let $\mathscr{F}$ be a coherent sheaf on X which is flat over Y. Then, the relative dimension

$$d := \dim(\operatorname{Supp}\mathscr{F})/Y = \dim \operatorname{Supp}(\mathscr{F} \otimes \mathscr{O}_{\pi^{-1}(\boldsymbol{y})}) = \dim(\operatorname{Supp}\mathscr{F} \cap \pi^{-1}(\boldsymbol{y}))$$

for $\boldsymbol{y} \in Y$ is locally constant. We assume here that Y is connected; hence d is constant. In this section, we shall define the intersection sheaf $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ for $\eta \in G^{d+1}(X)$ under Assumption 2.1 below, and we shall study basic properties of the intersection sheaves.

ASSUMPTION 2.1. π is a *projective morphism* in the sense that there is a π-ample invertible sheaf (cf. [**18**, Définition 4.6.1]). Moreover, the following (A) or (B) is satisfied:

(A) X is flat over Y.
(B) Y admits an ample invertible sheaf (cf. [**18**, Définition 4.5.3]).

Section 2 consists of four subsections. We first prepare a key push-forward homomorphism $\pi^{\mathscr{F}}_{\star} \colon K^{\bullet}(X) \to K^{\bullet}(Y)$ in Section 2.1. Using the homomorphism, the intersection sheaves $\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_{d+1})$ for $\mathscr{L}_1, \ldots, \mathscr{L}_{d+1} \in \operatorname{Pic}(X)$ are defined in Section 2.2 (cf. Definition 2.10). By Remark 2.11 or Lemma 2.19, we infer that if $\mathscr{F} = \mathscr{O}_X$, then $\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_{d+1})$ is just the intersection sheaf $\mathscr{I}_{X/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_{d+1})$ discussed in [**8**], [**33**], and [**7**]. By the idea of [**8**, Section V], we shall define the intersection sheaves $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ for $\eta \in G^{d+1}(X)$ in Section 2.3 (cf. Definition 2.26). Propositions 2.15 and 2.32 seem to be important for the theory of intersection sheaves, which are proved by the method of [**32**, Chapter 5, Sections 3–4]. In Section 2.4, we apply the propositions to prove Theorem 2.41 on the intersection sheaf $\mathscr{I}_{\mathscr{F}/Y}(P(\boldsymbol{c}^1(\mathscr{E}), \ldots, \boldsymbol{c}^r(\mathscr{E})))$ for a locally free sheaf $\mathscr{E}$ on X of rank r and for a weighted homogeneous polynomial $P(x_1, \ldots, x_r)$ of degree $d+1$ which is numerically positive for ample vector bundles in the sense of [**14**].

2.1. A push-forward homomorphism.

We shall define a homomorphism $\pi^{\mathscr{F}}_{\star} \colon K^{\bullet}(X) \to K^{\bullet}(Y)$ under Assumption 2.1. The homomorphism is essential for defining the intersection sheaves. First of all, we consider the set $\mathfrak{G}(X, \pi, \mathscr{F})$ of locally free sheaves $\mathscr{G}$ of finite rank on X such that $\mathrm{R}^p\, \pi_*(\mathscr{F} \otimes \mathscr{G}) = 0$ for any $p > 0$.

REMARK 2.2. Let $\mathscr{G}$ be a locally free sheaf belonging to $\mathfrak{G}(X, \pi, \mathscr{F})$. Then $\pi_*(\mathscr{F} \otimes \mathscr{G})$ is locally free. Moreover, it has the following base change properties (cf. [**19**, Théorème 7.7.5]):

(1) For a morphism $\tau \colon Y' \to Y$ of schemes, let $p_1 \colon X' \to X$ and $p_2 \colon X' \to Y'$ be the first and second projections from the fiber product $X' = X \times_Y Y'$. Then, $p_1^*\mathscr{G} \in \mathfrak{G}(X', p_2, p_1^*\mathscr{F})$, and

$$\tau^*(\pi_*(\mathscr{F} \otimes \mathscr{G})) \simeq p_{2*}p_1^*(\mathscr{F} \otimes \mathscr{G}).$$

(2) For a coherent sheaf $\mathscr{M}$ on Y,

$$\mathrm{R}^p\, \pi_*(\mathscr{F} \otimes \mathscr{G} \otimes \pi^*\mathscr{M}) \simeq \mathrm{R}^p\, \pi_*(\mathscr{F} \otimes \mathscr{G}) \otimes \mathscr{M} \simeq \begin{cases} \pi_*(\mathscr{F} \otimes \mathscr{G}) \otimes \mathscr{M}, & \text{if } p = 0; \\ 0, & \text{if } p > 0. \end{cases}$$

Convention ("subbundle" and "strictly injective").

- For a locally free sheaf $\mathscr{E}$, a subsheaf $\mathscr{E}' \subset \mathscr{E}$ is called a *subbundle* if $\mathscr{E}/\mathscr{E}'$ is also locally free.
- A homomorphism $\mathscr{E}_1 \to \mathscr{E}_2$ of locally free sheaves is called *strictly injective* if it is injective and its cokernel is also locally free.

Lemma 2.3. *Under Assumption* 2.1*, every locally free sheaf $\mathscr{E}$ of finite rank on X is realized as a subbundle of certain $\mathscr{G} \in \mathfrak{G}(X, \pi, \mathscr{F})$, i.e., there is a strict injection $\mathscr{E} \hookrightarrow \mathscr{G}$ for some $\mathscr{G} \in \mathfrak{G}(X, \pi, \mathscr{F})$.*

Proof. We have a π-ample invertible sheaf $\mathscr{A}$, since π is projective. Then, there exists a positive integer k such that $\mathrm{R}^p\, \pi_*(\mathscr{F} \otimes \mathscr{A}^{\otimes k}) = 0$ for any $p > 0$ and that the natural homomorphism

$$\pi^*\pi_*(\mathscr{E}^\vee \otimes \mathscr{A}^{\otimes k}) \to \mathscr{E}^\vee \otimes \mathscr{A}^{\otimes k} \tag{2.1}$$

is surjective. We shall construct $\mathscr{E} \hookrightarrow \mathscr{G}$ as follows.

First, we consider the case where π is flat. We can choose the integer k above so that $\mathrm{R}^p\, \pi_*(\mathscr{E}^\vee \otimes \mathscr{A}^{\otimes k}) = 0$ for any $p > 0$. Then $\pi_*(\mathscr{E}^\vee \otimes \mathscr{A}^{\otimes k})$ is locally free. We set

$$\mathscr{G} := \pi^*\big(\pi_*(\mathscr{E}^\vee \otimes \mathscr{A}^{\otimes k})^\vee\big) \otimes \mathscr{A}^{\otimes k}.$$

Then, $\mathscr{G} \in \mathfrak{G}(X, \pi, \mathscr{F})$, since

$$\mathrm{R}^p\, \pi_*(\mathscr{F} \otimes \mathscr{G}) \simeq \pi_*(\mathscr{E}^\vee \otimes \mathscr{A}^{\otimes k})^\vee \otimes \mathrm{R}^p\, \pi_*(\mathscr{F} \otimes \mathscr{A}^{\otimes k}) = 0$$

for any $p > 0$. Moreover, the surjection (2.1) above induces a strict injection $\mathscr{E} \hookrightarrow \mathscr{G}$.

Second, we consider the case where Y admits an ample invertible sheaf $\mathscr{H}$. Then, we have a surjection

$$\mathscr{O}_Y^{\oplus N} \to \pi_*(\mathscr{E}^\vee \otimes \mathscr{A}^{\otimes k}) \otimes \mathscr{H}^{\otimes l} \tag{2.2}$$

for some positive integers l, N. We set

$$\mathscr{G} := \pi^*(\mathscr{H}^{\otimes l})^{\oplus N} \otimes \mathscr{A}^{\otimes k}.$$

Then, $\mathscr{G} \in \mathfrak{G}(X, \pi, \mathscr{F})$, since

$$\mathrm{R}^p\,\pi_*(\mathscr{F} \otimes \mathscr{G}) \simeq (\mathscr{H}^{\otimes l})^{\oplus N} \otimes \mathrm{R}^p\,\pi_*(\mathscr{F} \otimes \mathscr{A}^{\otimes k}) = 0$$

for any $p > 0$. The surjections (2.1) and (2.2) above induce a strict injection $\mathscr{E} \hookrightarrow \mathscr{G}$. □

LEMMA 2.4. *Assume that every locally free sheaf $\mathscr{E}$ of finite rank on X is realized as a subbundle of certain $\mathscr{G} \in \mathfrak{G}(X, \pi, \mathscr{F})$. Then the following hold:*

(1) *For any locally free sheaf $\mathscr{E}$ of finite rank on X, there is an exact sequence*

$$0 \to \mathscr{E} \to \mathscr{G}^0 \to \cdots \to \mathscr{G}^d \to 0$$

such that $\mathscr{G}^i \in \mathfrak{G}(X, \pi, \mathscr{F})$ for any $0 \le i \le d$.

(2) *Let $\mathscr{E}$ be a locally free sheaf of finite rank on X and let $\mathscr{E} \hookrightarrow \mathscr{G}$ be a strict injection to a locally free sheaf $\mathscr{G} \in \mathfrak{G}(X, \pi, \mathscr{F})$. Let $\mathscr{E} \to \mathscr{E}'$ be a homomorphism to a locally free sheaf $\mathscr{E}'$ of finite rank on X. Then there exist a locally free sheaf $\mathscr{G}' \in \mathfrak{G}(X, \pi, \mathscr{F})$, a strict injection $\mathscr{E}' \hookrightarrow \mathscr{G}'$, and a homomorphism $\mathscr{G} \to \mathscr{G}'$ such that the diagram*

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d] & \mathscr{G} \arrow[d] \\ \mathscr{E}' \arrow[r] & \mathscr{G}' \end{tikzcd}$$

is commutative and Cartesian. If $\mathscr{E} \to \mathscr{E}'$ is strictly injective, then $\mathscr{G} \to \mathscr{G}'$ is strictly injective, and moreover, $\mathscr{E}' \cap \mathscr{G} = \mathscr{E}$ as a subsheaf of $\mathscr{G}'$, and $\mathscr{E}' + \mathscr{G}$ is a subbundle of $\mathscr{G}'$.

(3) *Let $\mathscr{E} \to \mathscr{E}'$ be a homomorphism of locally free sheaves on X and let $0 \to \mathscr{E} \to \mathscr{G}^0 \to \cdots \to \mathscr{G}^d \to 0$ be an exact sequence such that $\mathscr{G}^i \in \mathfrak{G}(X, \pi, \mathscr{F})$ for any $0 \le i \le d$. Then, there exists a commutative diagram*

$$\begin{CD} 0 @>>> \mathscr{E} @>>> \mathscr{G}^0 @>>> \cdots @>>> \mathscr{G}^d @>>> 0 \\ @. @VVV @VVV @. @VVV \\ 0 @>>> \mathscr{E}' @>>> \mathscr{G}'^0 @>>> \cdots @>>> \mathscr{G}'^d @>>> 0 \end{CD}$$

of exact sequences such that $\mathscr{G}'^i \in \mathfrak{G}(X, \pi, \mathscr{F})$ for any $0 \le i \le d$. If $\mathscr{E} \to \mathscr{E}'$ is strictly injective, then one can take $\mathscr{G}'^i$ so that $\mathscr{G}^i \to \mathscr{G}'^i$ is also strictly injective for any $0 \le i \le d$.

(4) *For an exact sequence $0 \to \mathscr{E} \to \mathscr{G}^0 \to \cdots \to \mathscr{G}^d \to 0$ of locally free sheaves such that $\mathscr{G}^i \in \mathfrak{G}(X, \pi, \mathscr{F})$ for any $0 \le i \le d$, the element*

$$\pi_\star^{\mathscr{F}}(\mathscr{E}) := \sum_{i=0}^{d} (-1)^i \operatorname{cl}^\bullet(\pi_*(\mathscr{F} \otimes \mathscr{G}^i))$$

of $K^\bullet(Y)$ depends only on $\operatorname{cl}^\bullet(\mathscr{E}) \in K^\bullet(X)$.

(5) *The map $\pi_\star^{\mathscr{F}} : K^\bullet(X) \to K^\bullet(Y)$ defined by $\mathscr{E} \mapsto \pi_\star^{\mathscr{F}}(\mathscr{E})$ is a homomorphism of abelian groups.*

PROOF. (1): By assumption, $\mathscr{E}$ is realized as a subbundle of some $\mathscr{G}^0 \in \mathfrak{G}(X, \pi, \mathscr{F})$. The cokernel $\mathscr{G}^0/\mathscr{E}$ is also realized in a subbundle of some $\mathscr{G}^1 \in \mathfrak{G}(X, \pi, \mathscr{F})$. Thus, we have an exact sequence $0 \to \mathscr{E} \to \mathscr{G}^0 \to \mathscr{G}^1$. Once we have an exact sequence $0 \to \mathscr{E} \to \mathscr{G}^0 \to \mathscr{G}^1 \to \cdots \to \mathscr{G}^k$ for a number k with $\mathscr{G}^i \in \mathfrak{G}(X, \pi, \mathscr{F})$ for any $0 \le i \le k$, we can extend one more by adding $\mathscr{G}^k \to \mathscr{G}^{k+1}$ for a locally free sheaf $\mathscr{G}^{k+1} \in \mathfrak{G}(X, \pi, \mathscr{F})$ containing the cokernel of $\mathscr{G}^{k-1} \to \mathscr{G}^k$ as a subbundle. In this way, we have an exact sequence $0 \to \mathscr{E} \to \mathscr{G}^0 \to \cdots \to \mathscr{G}^{d-1}$. Let $\mathscr{G}^d$ be the cokernel of the last homomorphism $\mathscr{G}^{d-2} \to \mathscr{G}^{d-1}$. Then, $\mathscr{G}^d \in \mathfrak{G}(X, \pi, \mathscr{F})$. In fact, we have

$$\mathrm{R}^p \pi_*(\mathscr{F} \otimes \mathscr{G}^d) \simeq \mathrm{R}^{p+d} \pi_*(\mathscr{F} \otimes \mathscr{E}) = 0$$

for $p > 0$, since $d = \dim(\operatorname{Supp} \mathscr{F})/Y$.

(2): Let $\mathscr{V}$ be the cokernel of the injection $\mathscr{E} \to \mathscr{E}' \oplus \mathscr{G}$ sending $x \in \mathscr{E}$ to $(x, -x)$. Then we have a commutative diagram

$$\begin{CD} 0 @>>> \mathscr{E} @>>> \mathscr{G} @>>> \mathscr{G}/\mathscr{E} @>>> 0 \\ @. @VVV @VVV @VV{\simeq}V \\ 0 @>>> \mathscr{E}' @>>> \mathscr{V} @>>> \mathscr{V}/\mathscr{E}' @>>> 0 \end{CD}$$

of exact sequences of locally free sheaves on X. In particular, $\mathscr{V}$ contains $\mathscr{E}'$ as a subbundle, and the square consisting of $\mathscr{E}$, $\mathscr{E}'$, $\mathscr{G}$, and $\mathscr{V}$ is Cartesian. By assumption, we have a strict injection from $\mathscr{V}$ to some $\mathscr{G}' \in \mathfrak{G}(X, \pi, \mathscr{F})$. Then the induced homomorphisms $\mathscr{E}' \to \mathscr{V} \to \mathscr{G}'$ and $\mathscr{G} \to \mathscr{V} \to \mathscr{G}'$ make an expected Cartesian diagram. If $\mathscr{E} \to \mathscr{E}'$ is strictly injective, then so are $\mathscr{G} \to \mathscr{V}$ and $\mathscr{G} \to \mathscr{G}'$; hence, $\mathscr{E}' \cap \mathscr{G} = \mathscr{E}$ by the Cartesian property and $\mathscr{E}' + \mathscr{G} = \mathscr{V}$ is a subbundle of $\mathscr{G}'$.

(3): We shall construct inductively a commutative diagram

$$(*)_k : \quad \begin{tikzcd} 0 \arrow[r] & \mathscr{E} \arrow[r] \arrow[d] & \mathscr{G}^0 \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathscr{G}^k \arrow[d] \\ 0 \arrow[r] & \mathscr{E}' \arrow[r] & \mathscr{G}'^0 \arrow[r] & \cdots \arrow[r] & \mathscr{G}'^k \end{tikzcd}$$

of exact sequences for any $0 \le k \le d$ such that $\mathscr{G}'^i \in \mathfrak{G}(X, \pi, \mathscr{F})$ for any $0 \le i \le k$. For the first diagram $(*)_0$, we apply (2) to the homomorphism $\mathscr{E} \to \mathscr{E}'$ and the strict injection $\mathscr{E} \hookrightarrow \mathscr{G}^0$. Then we obtain a locally free sheaf $\mathscr{G}'^0 \in \mathfrak{G}(X, \pi, \mathscr{F})$ together with a homomorphism $\mathscr{G} \to \mathscr{G}'$ and a strict injection $\mathscr{E}' \hookrightarrow \mathscr{G}'^0$ which make a commutative diagram $(*)_0$. Suppose that we are given a commutative diagram $(*)_{k-1}$ for an integer $1 \le k \le d$. Let $\mathscr{E}^k$ and $\mathscr{E}'^k$ be the cokernels of $\mathscr{G}^{k-2} \to \mathscr{G}^{k-1}$ and $\mathscr{G}'^{k-2} \to \mathscr{G}'^{k-1}$, respectively. If $k = d$, then $\mathscr{E}^d \simeq \mathscr{G}^d$, and $\mathscr{E}'^d \in \mathfrak{G}(X, \pi, \mathscr{F})$ by the argument in the proof of (1); thus we are done in this case by setting $\mathscr{G}'^d = \mathscr{E}'^d$. In case $k < d$, applying (2) to the induced homomorphism $\mathscr{E}^k \to \mathscr{E}'^k$ and the induced strict injection $\mathscr{E}^k \hookrightarrow \mathscr{G}^k$, we have a locally free sheaf $\mathscr{G}'^k \in \mathfrak{G}(X, \pi, \mathscr{F})$ together with a homomorphism $\mathscr{G}^k \to \mathscr{G}'^k$ and a strict injection $\mathscr{E}'^k \hookrightarrow \mathscr{G}^k$, which make the next diagram $(*)_k$. In this way, we have finally the diagram $(*)_d$ satisfying the required condition. If $\mathscr{E} \to \mathscr{E}'$ is strictly injective, then, by (2) and by the construction above, $\mathscr{G}^i \to \mathscr{G}'^i$ are all strictly injective.

(4) and (5): In the first step, we shall prove that $\pi_\star^{\mathscr{F}}(\mathscr{E})$ depends only on the isomorphism class of $\mathscr{E}$. By setting $\mathscr{B}^i := \pi_*(\mathscr{F} \otimes \mathscr{G}^i)$, we have a complex $\mathscr{B}^\bullet$: $0 \to \mathscr{B}^0 \to \cdots \to \mathscr{B}^d \to 0$ of locally free sheaves on Y which is quasi-isomorphic to $\boldsymbol{R}\pi_*(\mathscr{F} \otimes \mathscr{E})$. Let $0 \to \mathscr{E} \to \mathscr{G}'^0 \to \cdots \to \mathscr{G}'^d \to 0$ be another exact sequence such that $\mathscr{G}'^i \in \mathfrak{G}(X, \pi, \mathscr{F})$ for any $0 \le i \le d$. Then, we have a similar complex $\mathscr{B}'^\bullet$ by setting $\mathscr{B}'^i := \pi_*(\mathscr{F} \otimes \mathscr{G}'^i)$. We denote $\mathrm{cl}^\bullet(\mathscr{K}^\bullet) := \sum (-1)^i \mathrm{cl}^\bullet(\mathscr{K}^i)$ for a bounded complex $\mathscr{K}^\bullet = [\cdots \to \mathscr{K}^{i-1} \to \mathscr{K}^i \to \mathscr{K}^{i+1} \to \cdots]$ of locally free sheaves. For the first step, it suffices to prove: $\mathrm{cl}^\bullet(\mathscr{B}^\bullet) = \mathrm{cl}^\bullet(\mathscr{B}'^\bullet)$.

In order to compare $\mathscr{G}^i$ and $\mathscr{G}'^i$, we shall construct another exact sequences $0 \to \mathscr{E} \to \widetilde{\mathscr{G}}^0 \to \cdots \to \widetilde{\mathscr{G}}^d \to 0$ with $\widetilde{\mathscr{G}}^i \in \mathfrak{G}(X, \pi, \mathscr{F})$ connecting the sequences of $\mathscr{G}^i$ and $\mathscr{G}'^i$ in some sense. Applying (3) to strict injections $\mathscr{E} \hookrightarrow \mathscr{G}^0$ and $\mathscr{E} \hookrightarrow \mathscr{G}'^0$, we have a locally free sheaf $\widetilde{\mathscr{G}}^0 \in \mathfrak{G}(X, \pi, \mathscr{F})$ which contains $\mathscr{E}$, $\mathscr{G}^0$, and $\mathscr{G}'^0$ as subbundles in such a way that $\mathscr{G}^0 \cap \mathscr{G}'^0 = \mathscr{E}$ in $\widetilde{\mathscr{G}}^0$ and $\mathscr{G}^0 + \mathscr{G}'^0$ is also a subbundle of $\widetilde{\mathscr{G}}^0$. Let $\mathscr{E}^1$, $\mathscr{E}'^1$, and $\widetilde{\mathscr{E}}^1$ be the cokernels of $\mathscr{E} \to \mathscr{G}^0$, $\mathscr{E} \to \mathscr{G}'^0$, and $\mathscr{E} \to \widetilde{\mathscr{G}}^0$, respectively. There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E} \oplus \mathscr{E} \arrow[r] \arrow[d] & \mathscr{G}^0 \oplus \mathscr{G}'^0 \arrow[r] \arrow[d] & \mathscr{E}^1 \oplus \mathscr{E}'^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{E} \arrow[r] & \widetilde{\mathscr{G}}^0 \arrow[r] & \widetilde{\mathscr{E}}^1 \arrow[r] & 0
\end{tikzcd}$$

of exact sequences, where the left vertical homomorphism maps (x, x') to $x + x'$. Then, considering the kernel and the cokernel of the middle vertical homomorphism, we infer that the right vertical homomorphism is strictly injective. We can apply (3) to the exact sequence

$$0 \to \mathscr{E}^1 \oplus \mathscr{E}'^1 \to \mathscr{G}^1 \oplus \mathscr{G}'^1 \to \mathscr{G}^2 \oplus \mathscr{G}'^2 \to \cdots \to \mathscr{G}^d \oplus \mathscr{G}'^d \to 0$$

and to the strict injection $\mathscr{E}^1 \oplus \mathscr{E}'^1 \to \widetilde{\mathscr{E}}^1$. As a result, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E} \oplus \mathscr{E} \arrow[r] \arrow[d] & \mathscr{G}^0 \oplus \mathscr{G}'^0 \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathscr{G}^d \oplus \mathscr{G}'^d \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{E} \arrow[r] & \widetilde{\mathscr{G}}^0 \arrow[r] & \cdots \arrow[r] & \widetilde{\mathscr{G}}^d \arrow[r] & 0
\end{tikzcd}$$

of exact sequences of locally free sheaves such that the induced homomorphisms $\mathscr{G}^i \to \mathscr{G}^i \oplus \mathscr{G}'^i \to \widetilde{\mathscr{G}}^i$ and $\mathscr{G}'^i \to \mathscr{G}^i \oplus \mathscr{G}'^i \to \widetilde{\mathscr{G}}^i$ are strictly injective for any $0 \le i \le d$.

Let $\widetilde{\mathscr{B}}^\bullet$ be the complex of locally free sheaves on Y defined by $\widetilde{\mathscr{B}}^i = \pi_*(\mathscr{F} \otimes \widetilde{\mathscr{G}}^i)$ as above. The homomorphisms $\mathscr{F} \otimes \mathscr{G}^i \to \mathscr{F} \otimes \widetilde{\mathscr{G}}^i$ define a morphism $\mathscr{B}^\bullet \to \widetilde{\mathscr{B}}^\bullet$ of complexes. Here, $\mathscr{B}^i \to \widetilde{\mathscr{B}}^i$ is strictly injective; in fact, $\mathscr{V}^i := \widetilde{\mathscr{G}}^i / \mathscr{G}^i$ is a locally free sheaf contained in $\mathfrak{G}(X, \pi, \mathscr{F})$ and $\widetilde{\mathscr{B}}^i / \mathscr{B}^i \simeq \pi_*(\mathscr{F} \otimes \mathscr{V}^i)$ for any $0 \le i \le d$. Moreover, the induced complex

$$\widetilde{\mathscr{B}}^\bullet / \mathscr{B}^\bullet = \left[\cdots \to 0 \to \pi_*(\mathscr{F} \otimes \mathscr{V}^0) \to \pi_*(\mathscr{F} \otimes \mathscr{V}^1) \to \cdots \to \pi_*(\mathscr{F} \otimes \mathscr{V}^d) \to 0 \to \cdots \right]$$

is an exact sequence. This implies that $\mathrm{cl}^\bullet(\mathscr{B}^\bullet) = \mathrm{cl}^\bullet(\widetilde{\mathscr{B}}^\bullet)$. Replacing $\mathscr{G}^i$ with $\mathscr{G}'^i$, we also have $\mathrm{cl}^\bullet(\mathscr{B}'^\bullet) = \mathrm{cl}^\bullet(\widetilde{\mathscr{B}}^\bullet)$.

In the second step, we shall prove the following assertion: *Let* $0 \to \mathscr{E}_1 \to \mathscr{E}_2 \to$

$\mathscr{E}_3 \to 0$ *be an arbitrary exact sequence of locally free sheaves of finite rank on* X. *Then,* $\pi_\star^{\mathscr{F}}(\mathscr{E}_2) = \pi_\star^{\mathscr{F}}(\mathscr{E}_1) + \pi_\star^{\mathscr{F}}(\mathscr{E}_3)$. This proves (4) and (5), by a property of the K-group $K^\bullet(X)$ associated with the locally free sheaves on X. By (1) and (3), we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{E}_1 \arrow[r] \arrow[d] & \mathscr{G}_1^0 \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathscr{G}_1^d \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{E}_2 \arrow[r] & \mathscr{G}_2^0 \arrow[r] & \cdots \arrow[r] & \mathscr{G}_2^d \arrow[r] & 0 \end{tikzcd}$$

of exact sequences such that $\mathscr{G}_1^i, \mathscr{G}_2^i \in \mathfrak{G}(X, \pi, \mathscr{F})$, and $\mathscr{G}^i \to \mathscr{G}'^i$ is strictly injective for any $0 \le i \le d$. This induces an exact sequence $0 \to \mathscr{E}_3 \to \mathscr{G}_3^0 \to \cdots \to \mathscr{G}_3^d \to 0$ for $\mathscr{G}_3^i := \mathscr{G}_2^i/\mathscr{G}_1^i \in \mathfrak{G}(X, \pi, \mathscr{F})$. Since $0 \to \pi_*(\mathscr{F} \otimes \mathscr{G}_1^i) \to \pi_*(\mathscr{F} \otimes \mathscr{G}_2^i) \to \pi_*(\mathscr{F} \otimes \mathscr{G}_3^i) \to 0$ is exact for any i, we have the equality $\pi_\star^{\mathscr{F}}(\mathscr{E}_2) = \pi_\star^{\mathscr{F}}(\mathscr{E}_1) + \pi_\star^{\mathscr{F}}(\mathscr{E}_3)$. Thus, we have finished the proof of Lemma 2.4. □

Definition ($\pi_\star^{\mathscr{F}}$). Let $\pi\colon X \to Y$ be a projective morphism satisfying Assumption 2.1 and let $\mathscr{F}$ be a coherent sheaf on X flat over Y. We define a homomorphism $\pi_\star^{\mathscr{F}}\colon K^\bullet(X) \to K^\bullet(Y)$ by the property that it maps $\mathrm{cl}^\bullet(\mathscr{G})$ to $\mathrm{cl}^\bullet(\pi_*(\mathscr{F} \otimes \mathscr{G}))$ for any locally free sheaf $\mathscr{G}$ belonging to $\mathfrak{G}(X, \pi, \mathscr{F})$. This is well-defined by Lemmas 2.3 and 2.4.

Remark 2.5. Let $X' \subset X$ be a closed subscheme such that $\mathscr{F}$ is defined over X', i.e., $\mathscr{F}$ is an $\mathscr{O}_{X'}$-module. Assume that the morphism $\pi' := \pi|_{X'}\colon X' \to Y$ also satisfies Assumption 2.1. Then, $\pi_\star^{\mathscr{F}}(x) = \pi_\star'^{\mathscr{F}}(x|_{X'})$ for any $x \in K^\bullet(X)$. In fact, if $x = \mathrm{cl}^\bullet(\mathscr{E})$ for a locally free sheaf $\mathscr{E}$ on X and if $0 \to \mathscr{E} \to \mathscr{G}^0 \to \cdots \to \mathscr{G}^d \to 0$ is exact with $\mathscr{G}^i \in \mathfrak{G}(X, \pi, \mathscr{F})$, then $0 \to \mathscr{E}|_{X'} \to \mathscr{G}^0|_{X'} \to \cdots \to \mathscr{G}^d|_{X'} \to 0$ is exact with $\mathscr{G}^i|_{X'} \in \mathfrak{G}(X', \pi, \mathscr{F})$ and $\pi_*(\mathscr{F} \otimes \mathscr{G}^i) \simeq \pi'_*(\mathscr{F} \otimes (\mathscr{G}^i|_{X'}))$ for any i, since $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \simeq \mathscr{F} \otimes_{\mathscr{O}_{X'}} (\mathscr{G}|_{X'})$ for any coherent sheaves $\mathscr{G}$ on X. Thus, we have $\pi_\star^{\mathscr{F}}(\mathscr{E}) = \pi_\star'^{\mathscr{F}}(\mathscr{E}|_{X'})$ by Lemma 2.4, (4).

In what follows in Section 2, we assume that $\pi\colon X \to Y$ and $\mathscr{F}$ satisfy Assumption 2.1 unless otherwise stated. The lemma below is proved immediately from Lemma 2.4.

Lemma 2.6.

(1) *For any* $x \in K^\bullet(X)$, *one has*

$$\phi\left(\pi_\star^{\mathscr{F}}(x)\right) = \pi_\star(x\,\mathrm{cl}_\bullet(\mathscr{F}))$$

for $\pi_\star\colon K_\bullet(X) \to K_\bullet(Y)$ *and the Cartan homomorphism* $\boldsymbol{\phi}\colon K^\bullet(Y) \to K_\bullet(Y)$*. In particular, when* $\mathscr{F} = \mathscr{O}_X$*, one has a commutative diagram*

$$\begin{tikzcd} K^\bullet(X) \arrow[r, "\pi_\star^{\mathscr{O}_X}"] \arrow[d, "\boldsymbol{\phi}"'] & K^\bullet(Y) \arrow[d, "\boldsymbol{\phi}"'] \\ K_\bullet(X) \arrow[r, "\pi_\star"] & K_\bullet(Y). \end{tikzcd}$$

(2) *If* $Y = \operatorname{Spec} \boldsymbol{k}$ *for a field* $\boldsymbol{k}$*, then,*

$$\varepsilon\bigl(\pi_\star^{\mathscr{F}}(x)\bigr) = i_{X/\boldsymbol{k}}(\eta; \mathscr{F})$$

for any $x \in F^d(X)$ *and* $\eta = x \bmod F^{d+1}(X) \in G^d(X)$*, where* $\varepsilon\colon K^\bullet(Y) \xrightarrow{\simeq} \boldsymbol{Z}$ *is the augmentation map and* $i_{X/\boldsymbol{k}}$ *denotes the intersection number* (cf. Definition 1.11).

(3) *If* $\mathscr{F}$ *is a locally free sheaf on* X*, then, for any* $x \in K^\bullet(X)$*, one has*

$$\pi_\star^{\mathscr{F}}(x) = \pi_\star^{\mathscr{O}_X}(x \operatorname{cl}^\bullet(\mathscr{F})).$$

Proof. The assertions (1) and (3) follow from Lemma 2.4, (4). The assertion (2) is a consequence of (1). □

Remark 2.7. Since $\mathscr{F}$ is flat over Y, even if Assumption 2.1 is not satisfied, we know that $\boldsymbol{R}\pi_*\mathscr{F}$ is a *perfect complex*, i.e., locally on Y, it is quasi-isomorphic to a bounded complex $L_\bullet = [\cdots \to L_i \to L_{i+1} \to \cdots]$ of sheaves such that L_i are locally free for any i (cf. [**31**, Chapter II, Section 5], [**22**, Exp. III, Proposition 4.8]). Let $\boldsymbol{D}(Y)_{\mathrm{perf}}$ be the category of perfect complexes on Y and let $K^\bullet(Y)_{\mathrm{perf}}$ be the associated K-group; for the definition, see [**22**, Exp. IV, Section 2], where, however, $K^\bullet(Y)_{\mathrm{perf}}$ is denoted by $K^\bullet(Y)$, and $K^\bullet(Y)$ is denoted by $K^\bullet(Y)_{\mathrm{naïf}}$. Then, we have a similar homomorphism $\pi_\star^{\mathscr{F},\mathrm{perf}}\colon K^\bullet(X) \to K^\bullet(Y)_{\mathrm{perf}}$ which maps $\operatorname{cl}^\bullet(\mathscr{E})$ to the class of $\boldsymbol{R}\pi_*(\mathscr{F} \otimes \mathscr{E})$ for a locally free sheaf $\mathscr{E}$ of finite rank on X. When π satisfies Assumption 2.1, $\pi_\star^{\mathscr{F},\mathrm{perf}}$ is just the composition of $\pi_\star^{\mathscr{F}}$ and the canonical homomorphism $K^\bullet(Y) \to K^\bullet(Y)_{\mathrm{perf}}$. Note that the canonical homomorphism $K^\bullet(Y) \to K^\bullet(Y)_{\mathrm{perf}}$ is isomorphic if the Noetherian scheme Y admits an ample invertible sheaf (cf. [**22**, Exp. IV, Section 2.9]). If $\mathscr{F} = \mathscr{O}_X$ and π is projective, then Assumption 2.1 is satisfied. In this case, $\pi_\star^{\mathscr{O}_X}\colon K^\bullet(X) \to K^\bullet(Y)_{\mathrm{perf}}$ is just the composition of the natural homomorphism $K^\bullet(X) \to K^\bullet(X)_{\mathrm{perf}}$ and the homomorphism $\pi_\star\colon K^\bullet(X)_{\mathrm{perf}} \to K^\bullet(Y)_{\mathrm{perf}}$ defined in [**22**, Exp. IV, (2.12.3)].

Notation ($\pi_{\star}^{\mathscr{F}}$, $\pi_{\star}$). The homomorphism $\pi_{\star}^{\mathscr{F},\mathrm{perf}}\colon K^{\bullet}(X) \to K^{\bullet}(Y)_{\mathrm{perf}}$ in Remark 2.7 is written as $\pi_{\star}^{\mathscr{F}}\colon K^{\bullet}(X) \to K^{\bullet}(Y)_{\mathrm{perf}}$ by abuse of notation. When $\mathscr{F} = \mathscr{O}_X$, we denote $\pi_{\star}^{\mathscr{O}_X}$ simply by $\pi_{\star}$ if it causes no confusion with $\pi_{\star}\colon K_{\bullet}(X) \to K_{\bullet}(Y)$ (cf. Lemma 2.6, (1)).

Lemma 2.8. *Let $h\colon Y' \to Y$ be a morphism from another Noetherian scheme Y', $X' = X \times_Y Y'$, and let $q_1\colon X' \to X$ and $q_2\colon X' \to Y'$ be natural projections. Then $\mathscr{F}' = q_1^{*}\mathscr{F}$ is flat over Y' and the equality*

$$(q_2)_{\star}^{\mathscr{F}'}\big(q_1^{\star}(x) \cdot q_2^{\star}(y')\big) = (q_2)_{\star}^{\mathscr{F}'}\big(q_1^{\star}(x)\big) \cdot y' = h^{\star}\big(\pi_{\star}^{\mathscr{F}}(x)\big) \cdot y'$$

holds in $K^{\bullet}(Y')_{\mathrm{perf}}$ for $x \in K^{\bullet}(X)$ and $y' \in K^{\bullet}(Y')$, even if π does not satisfy Assumption 2.1*. If π is flat or if Y and Y' admit ample invertible sheaves, then the same equality holds in $K^{\bullet}(Y')$.*

Proof. The first equality itself is not so related to the base change; indeed, this follows from the following projection formula in $K^{\bullet}(Y)_{\mathrm{perf}}$ (resp. in $K^{\bullet}(Y)$ when π satisfies Assumption 2.1) for $x \in K^{\bullet}(X)$ and $y \in K^{\bullet}(Y)$:

$$\pi_{\star}^{\mathscr{F}}(x \cdot \pi^{\star}(y)) = \pi_{\star}^{\mathscr{F}}(x) \cdot y. \tag{2.3}$$

In order to prove it, we may assume that $x = \mathrm{cl}^{\bullet}(\mathscr{E})$ and $y = \mathrm{cl}^{\bullet}(\mathscr{G})$ for locally free sheaves $\mathscr{E}$ and $\mathscr{G}$ on X and Y, respectively. Then, (2.3) is derived from the quasi-isomorphism

$$\boldsymbol{R}\pi_{*}(\mathscr{F} \otimes \mathscr{E}) \otimes^{\mathrm{L}} \mathscr{G} \simeq_{\mathrm{qis}} \boldsymbol{R}\pi_{*}(\mathscr{F} \otimes \mathscr{E} \otimes \pi^{*}\mathscr{G}).$$

When π satisfies Assumption 2.1, by Lemmas 2.3 and 2.4, we may assume that $\mathscr{E} \in \mathfrak{G}(X, \pi, \mathscr{F})$, and hence, (2.3) is just derived from the usual projection formula:

$$\pi_{*}(\mathscr{F} \otimes \mathscr{E}) \otimes \mathscr{G} \simeq \pi_{*}(\mathscr{F} \otimes \mathscr{E} \otimes \pi^{*}\mathscr{G}).$$

For the second equality of Lemma 2.8, we may assume that $y' = 1$. Since $\mathscr{F}$ is flat over Y, $\mathscr{F}' = q_1^{*}\mathscr{F}$ is also flat over Y' and is quasi-isomorphic to $\boldsymbol{L}q_1^{*}\mathscr{F}$. There is a natural base change morphism

$$\Theta\colon \boldsymbol{L}h^{*}\boldsymbol{R}\pi_{*}(\mathscr{F} \otimes \mathscr{E}) \to \boldsymbol{R}q_{2*}(\boldsymbol{L}q_1^{*}(\mathscr{F} \otimes \mathscr{E})) \simeq_{\mathrm{qis}} \boldsymbol{R}q_{2*}\big(\mathscr{F}' \otimes q_1^{*}\mathscr{E}\big)$$

for a locally free sheaf $\mathscr{E}$ on X. It is enough to prove that Θ is a quasi-isomorphism.

If $\mathscr{E} \in \mathfrak{G}(X, \pi, \mathscr{F})$, then Θ is a quasi-isomorphism by Remark 2.2. Hence, by Lemmas 2.3 and 2.4, Θ is a quasi-isomorphism for any locally free sheaf $\mathscr{E}$ if π is flat or if Y and Y' admit ample invertible sheaves. There exist open coverings $\{Y_\alpha\}$ of Y and $\{Y'_\alpha\}$ of Y', respectively, such that $Y'_\alpha \subset h^{-1}(Y_\alpha)$ and that Y_α and Y'_α admit ample invertible sheaves. Thus, Θ restricted to the derived category of Y'_α is a quasi-isomorphism for any α. Hence, Θ itself is also a quasi-isomorphism. □

Lemma 2.9. *Let $f \colon Z \to X$ be a projective flat morphism. Then, for any $z \in K^\bullet(Z)$,*

$$(\pi \circ f)_\star^{f^*\mathscr{F}}(z) = \pi_\star^{\mathscr{F}}(f_\star(z)).$$

Proof. We may assume that $z = \mathrm{cl}^\bullet(\mathscr{E})$ for a locally free sheaf $\mathscr{E}$ on Z of finite rank. Note that $f^*\mathscr{F}$ is also flat over Y and that $\pi \circ f$ also satisfies Assumption 2.1. The expected equality essentially follows from the quasi-isomorphism

$$\boldsymbol{R}\pi_*\big(\mathscr{F} \otimes^{\mathrm{L}} \boldsymbol{R}f_*\mathscr{E}\big) \simeq_{\mathrm{qis}} \boldsymbol{R}(\pi \circ f)_*(f^*\mathscr{F} \otimes \mathscr{E}).$$

By the proof of Lemma 2.3, there is a strict injection $\mathscr{E} \hookrightarrow \mathscr{G}$ into a locally free sheaf $\mathscr{G}$ belonging to $\mathfrak{G}(Z, \pi \circ f, f^*\mathscr{F})$ and also to $\mathfrak{G}(P, f, \mathscr{O}_Z)$. Thus, by the same argument as in the proof of Lemma 2.4, we may assume that $\mathscr{E} \in \mathfrak{G}(Z, \pi \circ f, f^*\mathscr{F}) \cap \mathfrak{G}(Z, f, \mathscr{O}_Z)$. Then,

$$\mathrm{R}^p f_*(f^*\mathscr{F} \otimes \mathscr{E}) \simeq \begin{cases} \mathscr{F} \otimes f_*\mathscr{E}, & \text{if } p = 0; \\ 0, & \text{if } p > 0, \end{cases}$$

by Remark 2.2, since $\mathscr{E} \in \mathfrak{G}(Z, f, \mathscr{O}_Z)$. Hence, we have

$$0 = \mathrm{R}^q(\pi \circ f)_*(f^*\mathscr{F} \otimes \mathscr{E}) \simeq \mathrm{R}^q \pi_*(\mathscr{F} \otimes f_*\mathscr{E})$$

for $q > 0$ by the degeneration of Leray's spectral sequence, since $\mathscr{E} \in \mathfrak{G}(Z, \pi \circ f, f^*\mathscr{F})$. As a consequence, $f_*\mathscr{E} \in \mathfrak{G}(X, \pi, \mathscr{F})$. Then, by Lemma 2.4, we have

$$(\pi \circ f)_\star^{f^*\mathscr{F}}(\mathscr{E}) = \mathrm{cl}^\bullet(\pi_*(\mathscr{F} \otimes f_*\mathscr{E})) = \pi_\star^{\mathscr{F}}(f_\star(\mathrm{cl}_\bullet(\mathscr{E}))),$$

and the expected equality follows. □

2.2. Primitive definition of intersection sheaves.

Definition 2.10. Let $\mathscr{L}_1, \ldots, \mathscr{L}_k$ be invertible sheaves on X, where $k \geq d+1$. Under Assumption 2.1, we define

$$i_{\mathscr{F}/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_k) := \varepsilon\big(\pi_\star^{\mathscr{F}}\big(\boldsymbol{\delta}_X^{(k)}(\mathscr{L}_1, \ldots, \mathscr{L}_k)\big)\big) \in \mathrm{H}^0(Y, \boldsymbol{Z}) = \boldsymbol{Z} \quad \text{for } k \geq d,$$
$$\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_k) := \det\big(\pi_\star^{\mathscr{F}}\big(\boldsymbol{\delta}_X^{(k)}(\mathscr{L}_1, \ldots, \mathscr{L}_k)\big)\big) \in \mathrm{Pic}(Y) \qquad \text{for } k \geq d+1,$$

where $\varepsilon\colon K^\bullet(Y) \to \mathrm{H}^0(Y, \boldsymbol{Z}) = \boldsymbol{Z}$ is the augmentation map, $\det\colon K^\bullet(Y) \to \mathrm{Pic}(Y)$ is the determinant map, and $\boldsymbol{\delta}_X^{(k)}(\underline{\mathscr{L}})$ is defined in Definition 1.3 for $\underline{\mathscr{L}} = (\mathscr{L}_1, \ldots, \mathscr{L}_k)$. We call $i_{\mathscr{F}/Y}(\underline{\mathscr{L}})$ the *relative intersection number* and $\mathscr{I}_{\mathscr{F}/Y}(\underline{\mathscr{L}})$ the *intersection sheaf*. If $\mathscr{F} = \mathscr{O}_X$, then we write $i_{X/Y} = i_{\mathscr{F}/Y}$ and $\mathscr{I}_{X/Y} = \mathscr{I}_{\mathscr{F}/Y}$.

Remark 2.11. By Remark 1.4, we can write

$$\mathscr{I}_{X/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_{d+1}) = \bigotimes_{I \subset \{1, \ldots, d+1\}} \big(\det(\boldsymbol{R}\pi_*\mathscr{L}_I^{-1})\big)^{(-1)^{\sharp I}},$$

where $\mathscr{L}_I = \mathscr{L}_{i_1} \otimes \cdots \otimes \mathscr{L}_{i_k}$ for $I = \{i_1, \ldots, i_k\}$ with $\sharp I = k > 0$, and $\mathscr{L}_I = \mathscr{O}_X$ for the empty set $I = \emptyset$. A similar but different formula is written in [**7**, p. 34] (cf. [**8**, Section IV.1]).

Remark. There exist also the augmentation map $\varepsilon\colon K^\bullet(Y)_{\mathrm{perf}} \to \mathrm{H}^0(Y, \boldsymbol{Z})$ and the determinant map $\det\colon K^\bullet(Y)_{\mathrm{perf}} \to \mathrm{Pic}(Y)$, which are lifts of the same maps from $K^\bullet(Y)$, respectively. In fact, ε is defined by the ranks of locally free sheaves, and the existence of det is proved by Knudsen–Mumford [**26**]. Therefore, even if π is only a locally projective morphism, one can define the relative intersection number $i_{\mathscr{F}/Y}(\underline{\mathscr{L}})$ and the intersection sheaf $\mathscr{I}_{\mathscr{F}/Y}(\underline{\mathscr{L}})$ by using ε and det from $K^\bullet(Y)_{\mathrm{perf}}$ and by the homomorphism $\pi_\star^{\mathscr{F}}\colon K^\bullet(X) \to K^\bullet(Y)_{\mathrm{perf}}$ in Remark 2.7.

Lemma 2.12. *Assuming that π is only a locally projective morphism, let $\pi_\star^{\mathscr{F}}\colon K^\bullet(X) \to K^\bullet(Y)_{\mathrm{perf}}$ and $\varepsilon\colon K^\bullet(Y)_{\mathrm{perf}} \to \mathrm{H}^0(Y, \boldsymbol{Z}) = \boldsymbol{Z}$ be the homomorphisms as above. Then, $\varepsilon(\pi_\star^{\mathscr{F}}(x)) = 0$ for any $x \in F^{d+1}(X)$. In particular, $i_{\mathscr{F}/Y}$ gives rise to a homomorphism $G^d(X) \to \boldsymbol{Z}$. Furthermore, $i_{\mathscr{F}/Y}(\eta) = i_{\boldsymbol{F}/\boldsymbol{k}}(\eta|_{\boldsymbol{F}}; \mathscr{F} \otimes \mathscr{O}_{\boldsymbol{F}})$ for any $\eta \in G^d(X)$ and for any fiber $\boldsymbol{F} = \pi^{-1}(\boldsymbol{y})$ of π with $\boldsymbol{k}$ as the residue field $\boldsymbol{k}(\boldsymbol{y})$.*

Proof. Let $\boldsymbol{y} \in Y$ be a point and $\boldsymbol{F}$ the fiber $\pi^{-1}(\boldsymbol{y})$. For the canonical morphisms $h\colon \boldsymbol{y} \to Y$ and $q_2\colon \boldsymbol{F} \to \boldsymbol{y}$, we have

$$\varepsilon\bigl(\pi_\star^{\mathscr{F}}(x)\bigr) = \varepsilon\bigl(h^\star(\pi_\star^{\mathscr{F}}(x))\bigr) = \varepsilon\bigl((q_2)_\star^{\mathscr{F}\otimes\mathscr{O}_{\boldsymbol{F}}}(x|_{\boldsymbol{F}})\bigr) = i_{\boldsymbol{F}/\boldsymbol{k}}(\eta|_{\boldsymbol{F}}; \mathscr{F}\otimes\mathscr{O}_{\boldsymbol{F}})$$

for any $x \in F^d(X)$ and $\eta = x \bmod F^{d+1}(X) \in G^d(X)$, where $\boldsymbol{k} = \boldsymbol{k}(\boldsymbol{y})$, by Lemma 2.8 and by Lemma 2.6, (2). Thus, $\varepsilon(\pi_\star^{\mathscr{F}}(x)) = 0$ for any $x \in F^{d+1}(X)$. □

Lemma 2.13. *Let $\mathscr{L}_1, \dots, \mathscr{L}_k$ be invertible sheaves on X for $k \geq d+1$ and let σ_i be a section of $\mathscr{L}_i$ on X for $1 \leq i \leq k$ such that $\sigma = (\sigma_1, \dots, \sigma_k)$ is $\mathscr{F}$-regular. For the zero subscheme $V := V(\sigma) = V(\sigma_1, \dots, \sigma_k)$* (cf. Definition 1.6)*, assume that if* $\operatorname{depth} \mathscr{O}_{Y,\boldsymbol{y}} = 0$ *(resp. $= 1$), then* $\operatorname{Supp}\mathscr{F} \cap V \cap \pi^{-1}(\boldsymbol{y})$ *is empty (resp. finite). Then, there exists uniquely an effective Cartier divisor $D(\sigma)$ on Y with* $\operatorname{Supp} D(\sigma) \subset \pi(\operatorname{Supp} V)$ *satisfying the following properties*:

$$\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \dots, \mathscr{L}_k) \simeq \mathscr{O}_Y(D(\sigma)), \quad \text{and} \quad D(\sigma)|_{\operatorname{Spec}\mathscr{O}_{Y,\boldsymbol{y}}} = \operatorname{Div}_{\mathscr{O}_{Y,\boldsymbol{y}}}(\pi_*(\mathscr{F}\otimes\mathscr{O}_V))$$

for any point $\boldsymbol{y} \in Y$ with $\operatorname{depth}\mathscr{O}_{Y,\boldsymbol{y}} = 1$.

Proof. We set $\mathscr{E} := \bigoplus_{i=1}^k \mathscr{L}_i$. Then σ is regarded as an $\mathscr{F}$-regular section of $\mathscr{E}$. Let us consider the exact sequence (1.4) obtained from the Koszul complex defined by σ. By Lemma 2.4, we have a double-complex $\mathscr{C}^{\bullet,\bullet}$ satisfying the following conditions:

- $\mathscr{C}^{i,j} \in \mathfrak{G}(X, \pi, \mathscr{F})$ for any i, $j \in \boldsymbol{Z}$.
- If $\mathscr{C}^{i,j} \neq 0$, then $0 \leq j \leq d$ and $i \leq 0$.
- For any $p \geq 0$, $\mathscr{H}^0(\mathscr{C}^{-p,\bullet}) \simeq \bigwedge^p \mathscr{E}^\vee$, and $\mathscr{H}^j(\mathscr{C}^{-p,\bullet}) = 0$ for $j \neq 0$.
- The homomorphism $\bigwedge^p \mathscr{E}^\vee \to \bigwedge^{p-1}\mathscr{E}^\vee$ in the Koszul complex defined by σ is isomorphic to $\mathscr{H}^0(\mathscr{C}^{-p,\bullet}) \to \mathscr{H}^0(\mathscr{C}^{-(p-1),\bullet})$ for any $p \geq 1$.

Then, $\mathscr{F}\otimes\mathscr{O}_V$ is quasi-isomorphic to $\mathscr{F}\otimes\mathscr{C}^\bullet = [\cdots \to \mathscr{F}\otimes\mathscr{C}^i \to \mathscr{F}\otimes\mathscr{C}^{i+1} \to \cdots]$ for the total complex $\mathscr{C}^\bullet$ of $\mathscr{C}^{\bullet,\bullet}$, where $\mathscr{C}^m = \bigoplus_{m=i+j}\mathscr{C}^{i,j}$. Therefore, $\boldsymbol{R}\pi_*(\mathscr{F}\otimes\mathscr{O}_V)$ is quasi-isomorphic to the bounded complex $\mathscr{V}^\bullet = [\cdots \to \mathscr{V}^i \to \mathscr{V}^{i+1} \to \cdots]$ for the locally free sheaves $\mathscr{V}^i = \pi_*(\mathscr{F}\otimes\mathscr{C}^i)$ of finite rank (cf. Remark 2.2). In particular,

$$\pi_\star^{\mathscr{F}}(\boldsymbol{\delta}(\mathscr{L}_1, \dots, \mathscr{L}_k)) = \sum_{i\in\boldsymbol{Z}} (-1)^i \operatorname{cl}_\bullet(\mathscr{V}^i).$$

By our assumption, the cohomology sheaves $\mathscr{H}^i(\mathscr{V}^\bullet) \simeq \mathrm{R}^i\pi_*(\mathscr{F}\otimes\mathscr{O}_V)$ satisfy the conditions of Lemma 1.9. Hence, as a consequence of Lemma 1.9, we have an effective Cartier divisor $D = \operatorname{Div}(\mathscr{V}^\bullet) = \operatorname{Div}(\boldsymbol{R}\pi_*(\mathscr{F}\otimes\mathscr{O}_V))$ on Y with $\operatorname{Supp} D \subset \pi(\operatorname{Supp}\mathscr{F}\cap V)$, which is unique by the following properties:

$$\mathscr{O}_X(D) \simeq \det(\mathscr{V}^\bullet) \simeq \det\big(\pi_\star^{\mathscr{F}}(\boldsymbol{\delta}(\mathscr{L}_1, \dots, \mathscr{L}_k))\big) \simeq \mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \dots, \mathscr{L}_k), \quad \text{and}$$

$$D|_{\operatorname{Spec} \mathscr{O}_{Y,\boldsymbol{y}}} = \operatorname{Div}_{\mathscr{O}_{Y,\boldsymbol{y}}}(\mathscr{H}^0(\mathscr{V}^\bullet)) = \operatorname{Div}_{\mathscr{O}_{Y,\boldsymbol{y}}}(\pi_*(\mathscr{F} \otimes \mathscr{O}_V))$$

for any point $\boldsymbol{y} \in Y$ with $\operatorname{depth} \mathscr{O}_{Y,\boldsymbol{y}} = 1$. Thus, we are done. □

Lemma 2.14. *Let $h\colon Y' \to Y$ be a morphism from another Noetherian scheme Y' and let $q\colon X' \to X$ be the first projection from the fiber product $X' = X \times_Y Y'$. Then,*

$$\mathscr{I}_{q^*(\mathscr{F})/Y'}(q^*\mathscr{L}_1, \dots, q^*\mathscr{L}_k) \simeq h^* \mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \dots, \mathscr{L}_k)$$

for any invertible sheaves $\mathscr{L}_1, \dots, \mathscr{L}_k$ on X. Let $\sigma = (\sigma_1, \dots, \sigma_k)$ be as in Lemma 2.13. Then, $q^*\sigma := (q^*\sigma_1, \dots, q^*\sigma_k)$ is $q^*(\mathscr{F})$-regular for the sections $q^*\sigma_i$ of $q^*\mathscr{L}_i$. Assume that if $\operatorname{depth} \mathscr{O}_{Y',\boldsymbol{y}'} = 0$ (resp. $= 1$), then $\operatorname{Supp} \mathscr{F} \cap V(\sigma) \cap \pi^{-1}(h(\boldsymbol{y}'))$ is empty (resp. finite). Then, $V(q^*\sigma) = V(\sigma) \times_Y Y'$ satisfies the required condition in Lemma 2.13 for the existence of $D(q^*\sigma)$ with respect to $q^*(\mathscr{F})/Y'$ and $(q^*\mathscr{L}_1, \dots, q^*\mathscr{L}_k)$. Moreover, $h^*D(\sigma) = D(q^*\sigma)$.

Proof. The first isomorphism follows from Lemma 2.8. Let $\pi'\colon X' \to Y'$ be the second projection from the fiber product X'. Since $\mathscr{F}$ is flat over Y, the pullback

$$\cdots \to q^*\Big(\mathscr{F} \otimes \bigwedge^p (\mathscr{E}^\vee)\Big) \to \cdots \to q^*(\mathscr{F}) \to q^*(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)}) \to 0$$

of the exact sequence (1.4) by q^* is also exact, where $\mathscr{E} = \bigoplus_{i=1}^k \mathscr{L}_i$ as in the proof of Lemma 2.13. Hence, $q^*\sigma$ is a $q^*(\mathscr{F})$-regular section of $q^*(\mathscr{E})$. Let $\mathscr{C}^{\bullet,\bullet}$ be the double-complex in the proof of Lemma 2.13 and let $\mathscr{C}^\bullet = [\cdots \to \mathscr{C}^i \to \mathscr{C}^{i+1} \to \cdots]$ be the total complex. Since $\mathscr{C}^i \in \mathfrak{G}(X, \pi, \mathscr{F})$, we have

$$h^*\big(\pi_*(\mathscr{F} \otimes \mathscr{C}^i)\big) \simeq \pi'_*\big(q^*(\mathscr{F} \otimes \mathscr{C}^i)\big) \simeq \pi'_*\big(q^*(\mathscr{F}) \otimes q^*(\mathscr{C}^i)\big)$$

and $q^*(\mathscr{C}^i) \in \mathfrak{G}(X', \pi', q^*(\mathscr{F}))$, by Remark 2.2. Note that the complex $\mathscr{V}^\bullet$ on Y given by $\mathscr{V}^i = \pi_*(\mathscr{F} \otimes \mathscr{C}^i)$ is quasi-isomorphic to $\boldsymbol{R}\pi_*(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)})$, and the complex $\mathscr{V}'^\bullet$ on Y' given by $\mathscr{V}'^i = \pi'_*(q^*(\mathscr{F}) \otimes q^*(\mathscr{C}^i))$ is quasi-isomorphic to $\boldsymbol{R}\pi'_*(q^*(\mathscr{F}) \otimes \mathscr{O}_{V(q^*\sigma)})$. Thus, $D(\sigma) = \operatorname{Div}(\mathscr{V}^\bullet)$ and $D(q^*\sigma) = \operatorname{Div}(\mathscr{V}'^\bullet)$. Since $q^*\mathscr{V}^i \simeq \mathscr{V}'^i$, it is enough to check the following conditions by Lemma 1.10:

- If $\operatorname{depth} \mathscr{O}_{Y',\boldsymbol{y}'} = 0$, then $\mathscr{H}^i(\mathscr{V}^\bullet)_{h(\boldsymbol{y}')} = 0$ for any $i \in \boldsymbol{Z}$.
- If $\operatorname{depth} \mathscr{O}_{Y',\boldsymbol{y}'} = 1$, then $\mathscr{H}^i(\mathscr{V}^\bullet)_{h(\boldsymbol{y}')} = \mathscr{H}^i(\mathscr{V}'^\bullet)_{\boldsymbol{y}'} = 0$ for any $i \neq 0$.

Since $\boldsymbol{R}\pi_*(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)}) \simeq_{\mathrm{qis}} \mathscr{V}^\bullet$ and $\operatorname{Supp} q^*(\mathscr{F}) \cap V(q^*\sigma) = q^{-1}(\operatorname{Supp} \mathscr{F} \cap V(\sigma))$, these conditions are satisfied by our assumption on the set $\operatorname{Supp} \mathscr{F} \cap V(\sigma) \cap \pi^{-1}(h(\boldsymbol{y}))$. Thus, we are done. $\square$

PROPOSITION 2.15. *Let $\mathscr{L}_1, \dots, \mathscr{L}_k$ be invertible sheaves on X with surjective homomorphisms $\pi^*\mathscr{G}_i \to \mathscr{L}_i$ for locally free sheaves $\mathscr{G}_i$ of finite rank on Y. If $k \geq d+2$, then $\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \dots, \mathscr{L}_k) \simeq \mathscr{O}_X$. If $k = d+1$, then there is a surjection*

$$\Phi\colon \operatorname{Sym}^{e_1}(\mathscr{G}_1) \otimes \cdots \otimes \operatorname{Sym}^{e_{d+1}}(\mathscr{G}_{d+1}) \to \mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \dots, \mathscr{L}_{d+1}),$$

where $e_i = i_{\mathscr{F}/Y}(\mathscr{L}_1, \dots, \mathscr{L}_{i-1}, \mathscr{L}_{i+1}, \dots, \mathscr{L}_{d+1})$ for $1 \leq i \leq d+1$.

PROOF. We may assume that $k \geq d+1$. Let $q^{(i)}\colon \boldsymbol{P}_Y^{(i)} := \boldsymbol{P}_Y(\mathscr{G}_i^\vee) \to Y$ be the projective space bundle associated to $\mathscr{G}_i^\vee$, $\boldsymbol{P}_X^{(i)} := X \times_Y \boldsymbol{P}_Y^{(i)}$, and let $p_1^{(i)}\colon \boldsymbol{P}_X^{(i)} \to X$ and $p_2^{(i)}\colon \boldsymbol{P}_X^{(i)} \to \boldsymbol{P}_Y^{(i)}$ be natural projections. For the tautological line bundle $\mathscr{O}(1)$ of $\boldsymbol{P}_Y^{(i)}$ with respect to $\mathscr{G}_i^\vee$, we have a natural homomorphism

$$p_1^{(i)*}\mathscr{L}_i^\vee \to p_1^{(i)*}\pi^*\mathscr{G}_i^\vee = p_2^{(i)*}q^{(i)*}\mathscr{G}_i^\vee \to p_2^{(i)*}\mathscr{O}(1),$$

and thus a global section $\sigma^{(i)}$ of $p_1^{(i)*}\mathscr{L}_i \otimes p_2^{(i)*}\mathscr{O}(1)$, which defines an effective Cartier divisor $B^{(i)} = \operatorname{div}(\sigma^{(i)})$ on $\boldsymbol{P}_X^{(i)}$. Then $B^{(i)} \to X$ is a projective space bundle isomorphic to $\boldsymbol{P}_X(\mathscr{K}_i^\vee)$ for the kernel $\mathscr{K}_i$ of $\pi^*\mathscr{G}_i \to \mathscr{L}_i$. Thus we have a diagram

$$\begin{tikzcd}
B^{(i)} \arrow[r, equal] & \boldsymbol{P}_X(\mathscr{K}_i^\vee) \arrow[r, "\subset"] & \boldsymbol{P}_X^{(i)} \arrow[r, "p_1^{(i)}"] \arrow[d, "p_2^{(i)}"'] & X \arrow[d, "\pi"] \\
& \boldsymbol{P}_Y(\mathscr{G}_i^\vee) \arrow[r, equal] & \boldsymbol{P}_Y^{(i)} \arrow[r, "q^{(i)}"] & Y.
\end{tikzcd}$$

Let $q\colon \boldsymbol{P}_Y \to Y$ be the fiber product $\boldsymbol{P}_Y = \boldsymbol{P}_Y^{(1)} \times_Y \cdots \times_Y \boldsymbol{P}_Y^{(k)}$ of the projective space bundles over Y, $\boldsymbol{P}_X := X \times_Y \boldsymbol{P}_Y \simeq \boldsymbol{P}_X^{(1)} \times_X \cdots \times_X \boldsymbol{P}_X^{(k)}$, and let $p_1\colon \boldsymbol{P}_X \to X$, $p_2\colon \boldsymbol{P}_X \to \boldsymbol{P}_Y$, and $\pi^{(i)}\colon \boldsymbol{P}_X \to \boldsymbol{P}_X^{(i)}$ for $1 \leq i \leq k$ be natural projections. Then

$$V := \bigcap_{i=1}^{k} {\pi^{(i)}}^{-1}(B^{(i)}) \simeq B^{(1)} \times_X \cdots \times_X B^{(k)}$$

and we have a diagram:

$$\begin{tikzcd} V \arrow[r, equal] & B^{(1)} \times_X \cdots \times_X B^{(k)} \arrow[r, hook] & \boldsymbol{P}_X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "\pi"] \\ & \boldsymbol{P}_Y^{(1)} \times_Y \cdots \times_Y \boldsymbol{P}_Y^{(k)} \arrow[r, equal] & \boldsymbol{P}_Y \arrow[r, "q"] & Y. \end{tikzcd}$$

The sections $\sigma^{(i)}$ give rise to a global section σ of the locally free sheaf

$$\mathscr{E} := \bigoplus_{i=1}^{k} p_1^* \mathscr{L}_i \otimes p_2^* \big(\mathscr{O}(1)^{(i)} \big),$$

where $\mathscr{O}(1)^{(i)}$ is the pullback of $\mathscr{O}(1)$ by $\boldsymbol{P}_Y \to \boldsymbol{P}_Y^{(i)}$. Furthermore, V coincides with the zero subscheme $V(\sigma)$ of σ. Since V is smooth over X, we infer that σ is a regular section of $\mathscr{E}$. Moreover, σ is $p_1^*\mathscr{F}$-regular, since $V \to X$ is flat. Hence, by Lemma 1.7, we have $\boldsymbol{\phi}(\lambda_{-1}(\mathscr{E}^\vee)) = \mathrm{cl}_\bullet(\mathscr{O}_V)$ and $\lambda_{-1}(\mathscr{E}^\vee)\,\mathrm{cl}_\bullet(p_1^*\mathscr{F}) = \mathrm{cl}_\bullet(p_1^*\mathscr{F} \otimes \mathscr{O}_V)$, where

$$\lambda_{-1}(\mathscr{E}^\vee) = \boldsymbol{\delta}^{(k)}_{\boldsymbol{P}_X} \big(p_1^*\mathscr{L}_1 \otimes p_2^*(\mathscr{O}(1)^{(1)}), \dots, p_1^*\mathscr{L}_k \otimes p_2^*(\mathscr{O}(1)^{(k)}) \big). \tag{2.4}$$

Thus, we may define $\mathrm{cl}^\bullet(\mathscr{O}_V) := \lambda_{-1}(\mathscr{E}^\vee) \in K^\bullet(\boldsymbol{P}_X)$. Note that the equalities

$$(p_2)_\star^{p_1^*\mathscr{F}} \big(p_1^\star x \cdot p_2^\star y' \big) = \big((p_2)_\star^{p_1^*\mathscr{F}} (p_1^\star x) \big) \cdot y' = \big(q^\star \pi_\star^{\mathscr{F}} (x) \big) \cdot y'$$

hold for $x \in K^\bullet(X)$ and $y' \in K^\bullet(\boldsymbol{P}_Y)$ by Lemma 2.8. Applying the equalities and using Remark 1.4, (1), we have the following from (2.4):

$$\begin{aligned} &(p_2)_\star^{p_1^*\mathscr{F}} (\mathrm{cl}^\bullet(\mathscr{O}_V)) - q^\star \big(\pi_\star^{\mathscr{F}}\, \boldsymbol{\delta}_X^{(k)}(\mathscr{L}_1, \dots, \mathscr{L}_k) \big) \\ &\quad \equiv \sum_{i=1}^{k} q^\star \big(\pi_\star^{\mathscr{F}}\, \boldsymbol{\delta}_X^{(k-1)}(\mathscr{L}_1, \dots, \mathscr{L}_{i-1}, \mathscr{L}_{i+1}, \dots, \mathscr{L}_k) \big) \cdot \boldsymbol{\delta}(\mathscr{O}(1)^{(i)}) \\ &\qquad + q^\star \big(\pi_\star^{\mathscr{F}}\, \boldsymbol{\delta}_X^{(k)}(\mathscr{L}_1, \dots, \mathscr{L}_k) \big) \cdot \sum_{i=1}^{k} \boldsymbol{\delta}(\mathscr{O}(1)^{(i)}) \mod F^2(\boldsymbol{P}_Y) \\ &\quad \equiv \sum_{i=1}^{k} i_{\mathscr{F}/Y}(\mathscr{L}_1, \dots, \mathscr{L}_{i-1}, \mathscr{L}_{i+1}, \dots, \mathscr{L}_k)\, \boldsymbol{\delta}(\mathscr{O}(1)^{(i)}) \mod F^2(\boldsymbol{P}_Y), \end{aligned}$$

where $\pi_\star^{\mathscr{F}}\,\boldsymbol{\delta}_X^{(k)}(\mathscr{L}_1,\dots,\mathscr{L}_k) \equiv 0 \mod F^1(Y)$ by Lemma 2.12, since $k \geq d+1$. Therefore, we have an isomorphism

$$\begin{aligned}
&\mathscr{I}_{p_1^*\mathscr{F}/\boldsymbol{P}_Y}\big(p_1^*\mathscr{L}_1 \otimes p_2^*(\mathscr{O}(1)^{(1)}),\dots,p_1^*\mathscr{L}_k \otimes p_2^*(\mathscr{O}(1)^{(k)})\big) \\
&\quad \simeq \det\big((p_2)_\star^{p_1^*\mathscr{F}}(\mathrm{cl}^\bullet(\mathscr{O}_V))\big) \simeq q^*\big(\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1,\dots,\mathscr{L}_k)\big) \otimes \bigotimes_{i=1}^k (\mathscr{O}(1)^{(i)})^{\otimes e_i} \qquad (2.5)
\end{aligned}$$

for $e_i = i_{\mathscr{F}/Y}(\mathscr{L}_1,\dots,\mathscr{L}_{i-1},\mathscr{L}_{i+1},\dots,\mathscr{L}_k)$, since $\det\colon K^\bullet(\boldsymbol{P}_Y) \to \mathrm{Pic}(\boldsymbol{P}_Y)$ is trivial on $F^2(\boldsymbol{P}_Y)$. In order to describe the left hand side of (2.5), we want to define the divisor $D(\sigma)$ on $\boldsymbol{P}_Y$ as in Lemma 2.13. Note that

$$\begin{aligned}
\dim(\operatorname{Supp} p_1^*\mathscr{F} \cap V)/Y &= \dim(\operatorname{Supp} p_1^*\mathscr{F} \cap V)/\operatorname{Supp}\mathscr{F} + d \\
&= \dim \boldsymbol{P}_Y/Y - k + d. \qquad (2.6)
\end{aligned}$$

Since $k \geq d+1$, $p_2(\operatorname{Supp} p_1^*\mathscr{F} \cap V) \neq \boldsymbol{P}_Y$ and moreover, $p_2(\operatorname{Supp} p_1^*\mathscr{F} \cap V)$ does not contain any fiber of $q\colon \boldsymbol{P}_Y \to Y$. Let $\boldsymbol{w} \in \boldsymbol{P}_Y$ be a point with $\operatorname{depth}\mathscr{O}_{\boldsymbol{P}_Y,\boldsymbol{w}} \leq 1$. Note that the fiber $\boldsymbol{F} = q^{-1}(q(\boldsymbol{w}))$ is a product of projective spaces and that we have the formula

$$\operatorname{depth}\mathscr{O}_{\boldsymbol{P}_Y,\boldsymbol{w}} = \operatorname{depth}\mathscr{O}_{Y,q(\boldsymbol{w})} + \operatorname{depth}\mathscr{O}_{\boldsymbol{F},\boldsymbol{w}} = \operatorname{depth}\mathscr{O}_{Y,q(\boldsymbol{w})} + \dim\mathscr{O}_{\boldsymbol{F},\boldsymbol{w}},$$

since $q\colon \boldsymbol{P}_Y \to Y$ is flat and $\boldsymbol{F}$ is Cohen-Macaulay. Hence, $\boldsymbol{w}$ is a generic point of $\boldsymbol{F}$ or the generic point of a prime divisor on $\boldsymbol{F}$. In the former case, $\boldsymbol{w} \notin p_2(\operatorname{Supp} p_1^*\mathscr{F} \cap V)$, and in the latter case, $p_2^{-1}(\boldsymbol{w}) \cap \operatorname{Supp} p_1^*\mathscr{F} \cap V$ is a finite set, by (2.6). Thus, $V = V(\sigma)$ satisfies the condition of Lemma 2.13. As a consequence of Lemma 2.13, we have an effective Cartier divisor $D = D(\sigma)$ on $\boldsymbol{P}_Y$ such that $\operatorname{Supp} D \subset p_2(\operatorname{Supp} p_1^*\mathscr{F} \cap V))$ and

$$\mathscr{O}_{\boldsymbol{P}_Y}(D) \simeq \det\big((p_2)_\star^{p_1^*\mathscr{F}}(\mathrm{cl}^\bullet(\mathscr{O}_V))\big). \qquad (2.7)$$

Here, D is a relative Cartier divisor with respect to $q\colon \boldsymbol{P}_Y \to Y$, since $\dim(\operatorname{Supp} D \cap q^{-1}(\boldsymbol{y})) < \dim \boldsymbol{P}_Y/Y$ for any $\boldsymbol{y} \in Y$ by (2.6).

If $k > d+1$, then $D = 0$, and consequently, $\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1,\dots,\mathscr{L}_k) \simeq \mathscr{O}_Y$ by (2.5) and (2.7), since $e_i = i_{\mathscr{F}/Y}(\mathscr{L}_1,\dots,\mathscr{L}_{i-1},\mathscr{L}_{i+1},\dots,\mathscr{L}_k) = 0$ for any i by Lemma 2.12.

Assume that $k = d+1$. Then (2.5) and (2.7) imply that

$$q^*\big(\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_{d+1})\big) \otimes \bigotimes_{i=1}^{d+1} \big(\mathscr{O}(1)^{(i)}\big)^{\otimes e_i}$$

has a non-zero global section defining the divisor D. The section induces a section of

$$\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_{d+1}) \otimes \bigotimes_{i=1}^{d+1} \mathrm{Sym}^{e_i}(\mathscr{G}_i^\vee),$$

and, equivalently, a homomorphism

$$\Phi\colon \bigotimes_{i=1}^{d+1} \mathrm{Sym}^{e_i}(\mathscr{G}_i) \to \mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_{d+1}).$$

It remains to show the surjectivity of Φ. The composition of a natural injection

$$\bigotimes_{i=1}^{k} \mathscr{O}(-1)^{(i)} \to q^*\left(\bigotimes_{i=1}^{d+1} \mathrm{Sym}^{e_i}(\mathscr{G}_i)\right)$$

with $q^*\Phi$ gives an injection between invertible sheaves whose cokernel defines D. In particular, $q^*\Phi$ is surjective on $\boldsymbol{P}_Y \setminus \operatorname{Supp} D$. Since $\operatorname{Supp} D$ does not contain any fiber of q, we conclude that Φ is surjective. □

LEMMA 2.16. *Let $h\colon Y' \to Y$ be a morphism from a Noetherian scheme Y' and let $\varphi\colon X' \to X$ be the first projection from the fiber product $X' = X \times_Y Y'$. In the situation of Proposition* 2.15*, assume that $k = d+1$ and let*

$$\Phi'\colon \mathrm{Sym}^{e_1}(h^*\mathscr{G}_1) \otimes \mathrm{Sym}^{e_{d+1}}(h^*\mathscr{G}_{d+1}) \to \mathscr{I}_{\varphi^*\mathscr{F}/Y'}(\varphi^*\mathscr{L}_1, \ldots, \varphi^*\mathscr{L}_{d+1})$$

be the surjection on Y' obtained by the same method as in the proof of Proposition 2.15*. Then, Φ' and $h^*(\Phi)$ are isomorphic to each other.*

PROOF. We set $\mathscr{F}' = \varphi^*\mathscr{F}$, and let $q'\colon \boldsymbol{P}_{Y'} \to Y'$, $p_1'\colon \boldsymbol{P}_{X'} \to X'$, $p_2'\colon \boldsymbol{P}_{X'} \to \boldsymbol{P}_{Y'}$, and $\pi'\colon X' \to Y'$, respectively, be the pullbacks of q, p_1, p_2, and π by the base change of $h\colon Y' \to Y$. Then we have two commutative Cartesian diagrams

$$\begin{tikzcd} \boldsymbol{P}_{X'} \arrow[r, "p_2'"] \arrow[d, "\mu"'] & \boldsymbol{P}_{Y'} \arrow[r, "q'"] \arrow[d, "\nu"'] & Y' \arrow[d, "h"'] \\ \boldsymbol{P}_X \arrow[r, "p_2"] & \boldsymbol{P}_Y \arrow[r, "q"] & Y, \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{P}_{X'} \arrow[r, "p_1'"] \arrow[d, "\mu"'] & X' \arrow[r, "\pi'"] \arrow[d, "\varphi"'] & Y' \arrow[d, "h"'] \\ \boldsymbol{P}_X \arrow[r, "p_1"] & X \arrow[r, "\pi"] & Y, \end{tikzcd}$$

for the induced morphisms $\nu\colon \boldsymbol{P}_{Y'} \to \boldsymbol{P}_Y$ and $\mu\colon \boldsymbol{P}_{X'} \to \boldsymbol{P}_X$. Let σ be the section of $\mathscr{E}$ in the proof of Proposition 2.15, and let $\mu^*\sigma$ be the pullback of σ as a section of $\mu^*\mathscr{E}$. By the argument of the proof of Proposition 2.15, Φ and Φ' are determined by the divisors $D(\sigma)$ and $D(\mu^*\sigma)$, respectively; thus it is enough to prove: $D(\mu^*\sigma) = \nu^*D(\sigma)$. By Lemma 2.14, the equality holds if the following conditions are satisfied: If $\operatorname{depth} \mathscr{O}_{\boldsymbol{P}_{Y'}, \boldsymbol{w}'} = 0$ (resp. $= 1$) for a point $\boldsymbol{w}' \in \boldsymbol{P}_{Y'}$, then $\operatorname{Supp} p_1^*\mathscr{F} \cap V(\sigma) \cap p_2^{-1}(\nu(\boldsymbol{w}'))$ is empty (resp. finite).

We shall check this condition. Let $\boldsymbol{w}'$ be a point of $\boldsymbol{P}_{Y'}$ with $\operatorname{depth} \mathscr{O}_{\boldsymbol{P}_{Y'}, \boldsymbol{w}'} \le 1$. We set $\boldsymbol{w} = \nu(\boldsymbol{w}')$. As in the proof of Proposition 2.15, we know that $\boldsymbol{w}'$ is the generic point of the fiber $\boldsymbol{F}' = q'^{-1}q'(\boldsymbol{w}')$ or the generic point of a prime divisor on $\boldsymbol{F}'$. In the former case, $\boldsymbol{w}$ is the generic point of the fiber $\boldsymbol{F} = q^{-1}q(\boldsymbol{w})$, and thus $\operatorname{Supp} p_1^*\mathscr{F} \cap V(\sigma) \cap p_2^{-1}(\boldsymbol{w}) = \emptyset$. In the latter case, $\boldsymbol{w}$ is the generic point of a prime divisor on $\boldsymbol{F}$, and thus $\operatorname{Supp} p_1^*\mathscr{F} \cap V(\sigma) \cap p_2^{-1}(\boldsymbol{w})$ is finite by (2.6). Hence, the condition is satisfied. Thus, we are done. $\square$

Remark 2.17. Let $D = D_{\mathscr{F}, \underline{\mathscr{L}}}$ be the effective Cartier divisor on $\boldsymbol{P}_Y = \boldsymbol{P}(\mathscr{G}_1^\vee) \times_Y \cdots \times_Y \boldsymbol{P}(\mathscr{G}_{d+1}^\vee)$ in the proof of Proposition 2.15. By Lemma 2.16, we infer that, for a point $\boldsymbol{y} \in Y$, the effective divisor $D_{\boldsymbol{y}} := D|_{q^{-1}(\boldsymbol{y})}$ is characterized by the following two conditions:

(1) For $1 \le i \le d$, let $\mathscr{O}(1)^{(i)}$ be the pullback to $\boldsymbol{P}_Y$ of the tautological invertible sheaf of $\boldsymbol{P}_Y(\mathscr{G}_i^\vee)$ with respect to $\mathscr{G}_i^\vee$. Then

$$\mathscr{O}_{q^{-1}(\boldsymbol{y})}(D_{\boldsymbol{y}}) \simeq \bigotimes_{i=1}^{d+1} (\mathscr{O}(1)^{(i)}|_{q^{-1}(\boldsymbol{y})})^{\otimes e_i}.$$

(2) Let v_i be a non-zero element of $\mathscr{G}_i \otimes \boldsymbol{k}(\boldsymbol{y})$ for $1 \le i \le d+1$. For $v = (v_1, \ldots, v_{d+1})$, let $[v]$ be a point of $q^{-1}(\boldsymbol{y})$ corresponding to the surjections $v_i^\vee\colon \mathscr{G}_i^\vee \otimes \boldsymbol{k}(\boldsymbol{y}) \to \boldsymbol{k}(\boldsymbol{y})$. Let v_i^X be the global section of $\mathscr{L}_i \otimes \mathscr{O}_{\pi^{-1}(\boldsymbol{y})}$ defined by $\pi^*\mathscr{G}_i \to \mathscr{L}_i$, and set $v^X := (v_1^X, \ldots, v_{d+1}^X)$ as a global section of $(\mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_{d+1}) \otimes \mathscr{O}_{\pi^{-1}(\boldsymbol{y})}$. Then $[v] \notin \operatorname{Supp} D_{\boldsymbol{y}}$ if and only if $V(v^X) \cap \operatorname{Supp} \mathscr{F} = \emptyset$ for the zero subscheme $V(v^X) \subset \pi^{-1}(\boldsymbol{y})$.

Remark. Assume that $Y = \operatorname{Spec} A$ for a ring A, $X = \boldsymbol{P}^d_A$, $\mathscr{F} = \mathscr{O}_X$, and $\mathscr{L}_i = \mathscr{O}_{\boldsymbol{P}^N}(m_i)$ for some $m_i > 0$. Then, for $\mathscr{G}_i = \mathrm{H}^0(X, \mathscr{L}_i) \simeq \operatorname{Sym}^{m_i}(A^{\oplus(d+1)})$, the homomorphism Φ in Proposition 2.15 defines the *resultants*: An element $v_i \in \mathscr{G}_i$ is regarded as a homogeneous polynomial of degree m_i with coefficients in A. Then

$$\Phi\bigl(v_1^{e_1} \otimes \cdots \otimes v_{d+1}^{e_{d+1}}\bigr)$$

is the *resultant* of $v_1, \dots, v_{d+1}$ up to unit (cf. [**7**, Section 6.1]).

LEMMA 2.18. *If $k > d+1$, then $\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \dots, \mathscr{L}_k) \simeq \mathscr{O}_Y$ for any invertible sheaves $\mathscr{L}_1, \dots, \mathscr{L}_k$ on X.*

PROOF. By Proposition 2.15, this is true if all $\mathscr{L}_i$ satisfy the following condition (†) on invertible sheaves $\mathscr{L}$ on X:

(†) There exist a locally free sheaf $\mathscr{G}$ on Y and a surjection $\pi^*\mathscr{G} \to \mathscr{L}$.

It is enough to prove the following assertion: *For any invertible sheaf $\mathscr{L}$ on X, there exist invertible sheaves $\mathscr{M}_1$ and $\mathscr{M}_2$ satisfying* (†) *and $\mathscr{L} \simeq \mathscr{M}_1 \otimes \mathscr{M}_2^{-1}$*. In fact, by the assertion, we can write $\mathscr{L}_i \simeq \mathscr{M}_{i,1} \otimes \mathscr{M}_{i,2}^{-1}$ for invertible sheaves $\mathscr{M}_{i,1}$, $\mathscr{M}_{i,2}$ satisfying (†). Then, by Remark 1.4, $\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \dots, \mathscr{L}_k)$ is expressed as a tensor product of invertible sheaves $\mathscr{I}_{\mathscr{F}/Y}(\mathscr{N}_1, \dots, \mathscr{N}_l)^{\otimes(\pm 1)}$ with $l \geq k$ such that all $\mathscr{N}_i$ satisfy (†): This implies that $\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \dots, \mathscr{L}_k) \simeq \mathscr{O}_Y$ if $k > d+1$.

The assertion above is shown as follows. Since π is projective, we have a π-ample invertible sheaf $\mathscr{A}$ on X. Then, there is a positive integer n such that the natural homomorphisms

$$\pi^*\pi_*(\mathscr{L} \otimes \mathscr{A}^{\otimes n}) \to \mathscr{L} \otimes \mathscr{A}^{\otimes n} \quad \text{and} \quad \pi^*\pi_*(\mathscr{A}^{\otimes n}) \to \mathscr{A}^{\otimes n}$$

are surjective and that

$$\mathrm{R}^i\,\pi_*(\mathscr{L} \otimes \mathscr{A}^{\otimes n}) = \mathrm{R}^i\,\pi_*(\mathscr{A}^{\otimes n}) = 0$$

for any $i > 0$. We set $\mathscr{M}_1 = \mathscr{L} \otimes \mathscr{A}^{\otimes n}$ and $\mathscr{M}_2 = \mathscr{A}^{\otimes n}$. Then, $\mathscr{L} \simeq \mathscr{M}_1 \otimes \mathscr{M}_2^{-1}$. If π is flat, then, by Remark 2.2, $\mathscr{G}_1 = \pi_*(\mathscr{M}_1)$ and $\mathscr{G}_2 = \pi_*(\mathscr{M}_2)$ are locally free sheaves of finite rank; thus $\mathscr{M}_1$ and $\mathscr{M}_2$ satisfy (†). Assume that π is not flat. Then, by Assumption 2.1, Y admits an ample invertible sheaf $\mathscr{H}$. There is a positive integer m such that $\pi_*(\mathscr{M}_1) \otimes \mathscr{H}^{\otimes m}$ and $\pi_*(\mathscr{M}_2) \otimes \mathscr{H}^{\otimes m}$ are generated by finitely many global sections. Hence, for each $i = 1, 2$, we have a surjection

$$\pi^*\bigl(\mathscr{O}_Y^{\oplus N_i} \otimes \mathscr{H}^{\otimes(-m)}\bigr) \to \pi^*\pi_*(\mathscr{M}_i) \to \mathscr{M}_i$$

for some $N_i > 0$. Thus, $\mathscr{M}_1$ and $\mathscr{M}_2$ satisfy (†). This completes the proof. $\square$

Lemma 2.19 (cf. [**8**, Section III]). *The following hold for invertible sheaves $\mathscr{L}_1, \dots, \mathscr{L}_{d+1}$ on X:*

(1) *For any permutation τ of $\{1, \dots, d+1\}$, one has an isomorphism*

$$\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_{\tau(1)}, \dots, \mathscr{L}_{\tau(d+1)}) \simeq \mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \dots, \mathscr{L}_{d+1}).$$

(2) *If $\mathscr{L}'_1$ is another invertible sheaf, then*

$$\begin{aligned}&\mathscr{I}_{\mathscr{F}/Y}\left(\mathscr{L}_1 \otimes \mathscr{L}'_1, \mathscr{L}_2, \dots, \mathscr{L}_{d+1}\right)\\ &\quad \simeq \mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \mathscr{L}_2, \dots, \mathscr{L}_{d+1}) \otimes \mathscr{I}_{\mathscr{F}/Y}\left(\mathscr{L}'_1, \mathscr{L}_2, \dots, \mathscr{L}_{d+1}\right).\end{aligned}$$

(3) *If σ_1 is an $\mathscr{F}$-regular section of $\mathscr{L}_1$ and if $\mathscr{F} \otimes \mathscr{O}_{B_1}$ is flat over Y for $B_1 = V(\sigma_1)$, then*

$$\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \dots, \mathscr{L}_{d+1}) \simeq \mathscr{I}_{\mathscr{F} \otimes \mathscr{O}_{B_1}/Y}(\mathscr{L}_2|_{B_1}, \dots, \mathscr{L}_{d+1}|_{B_1}).$$

(4) *If $d = 0$ and $\mathscr{F} = \mathscr{O}_X$, then $\mathscr{I}_{X/Y}(\mathscr{L})$ is the norm sheaf of an invertible sheaf $\mathscr{L}$ on X, i.e.,*

$$\mathscr{I}_{X/Y}(\mathscr{L}) \simeq \det(\pi_* \mathscr{O}_X) \otimes \det(\pi_* \mathscr{L}^{-1})^{-1} \simeq \det(\pi_* \mathscr{L}) \otimes \det(\pi_* \mathscr{O}_X)^{-1}.$$

Proof. (1) follows from Definition 1.3, and (2) from Remark 1.4 and Lemma 2.18.

(3): It is enough to show the following equality for any $x \in K^\bullet(X)$:

$$\pi_\star^{\mathscr{F} \otimes \mathscr{O}_{B_1}}(x|_{B_1}) = \pi_\star^{\mathscr{F}}(x) - \pi_\star^{\mathscr{F}}\left(x \, \mathrm{cl}^\bullet(\mathscr{L}_1^{-1})\right) = \pi_\star^{\mathscr{F}}\left((1 - \mathrm{cl}^\bullet(\mathscr{L}_1^{-1}))x\right) = \pi_\star^{\mathscr{F}}(\boldsymbol{\delta}(\mathscr{L}_1)x).$$

In fact, the expected isomorphism are derived from the equality by substituting $x = \boldsymbol{\delta}(\mathscr{L}_2, \dots, \mathscr{L}_{d+1})$ and by taking det. The section σ_1 induces an exact sequence

$$0 \to \mathscr{F} \otimes \mathscr{L}_1^{-1} \to \mathscr{F} \to \mathscr{F} \otimes \mathscr{O}_{B_1} \to 0.$$

If $x = \mathrm{cl}^\bullet(\mathscr{G})$ for a locally free sheaf $\mathscr{G}$ on X belonging to $\mathfrak{G}(X, \pi, \mathscr{F}) \cap \mathfrak{G}(X, \pi, \mathscr{F} \otimes \mathscr{L}_1^{-1})$, then the equality holds, since

$$0 \to \pi_*\left(\mathscr{F} \otimes \mathscr{L}_1^{-1} \otimes \mathscr{G}\right) \to \pi_*(\mathscr{F} \otimes \mathscr{G}) \to \pi_*(\mathscr{F} \otimes \mathscr{O}_{B_1} \otimes \mathscr{G}) \to 0$$

is exact. We may assume that $x = \mathrm{cl}^\bullet(\mathscr{E})$ for a locally free sheaf $\mathscr{E}$ of finite rank on X. Then, there is a strict injection $\mathscr{E} \to \mathscr{G}$ for some $\mathscr{G} \in \mathfrak{G}(X, \pi, \mathscr{F}) \cap \mathfrak{G}(X, \pi, \mathscr{F} \otimes \mathscr{L}_1^{-1})$ by Lemma 2.3. Thus, by the proof of Lemma 2.4, we have an exact sequence $0 \to \mathscr{E} \to \mathscr{G}^0 \to \cdots \to \mathscr{G}^d \to 0$ with $\mathscr{G}^i \in \mathfrak{G}(X, \pi, \mathscr{F}) \cap \mathfrak{G}(X, \pi, \mathscr{F} \otimes \mathscr{L}_1^{-1})$. Therefore, the equality holds for any x.

(4): Since $d = 0$ and $\mathscr{F} = \mathscr{O}_X$, π is a finite flat morphism. Thus, $\mathscr{L} \in \mathfrak{G}(X, \pi, \mathscr{F})$, and we have the expected isomorphisms by Remark 1.4, (2). $\square$

2.3. Refined definition of intersection sheaves.

We recall the following well-known result on Segre classes (cf. [**8**, Section V]):

Lemma 2.20. *Suppose that $X = \boldsymbol{P}_Y(\mathscr{E})$ for a locally free sheaf $\mathscr{E}$ of rank r on Y. For an integer $p \geq -(r-1)$, let us define*

$$\boldsymbol{\sigma}^p(\mathscr{E}) := \pi_\star\big(\boldsymbol{\delta}(\mathscr{O}(1))^{r+p-1}\big) \in K^\bullet(Y),$$

where $\mathscr{O}(1)$ denotes the tautological invertible sheaf on X with respect to $\mathscr{E}$ and $\pi_\star = \pi_\star^{\mathscr{O}_X} \colon K^\bullet(X) \to K^\bullet(Y)$. Then, $\boldsymbol{\sigma}^p(\mathscr{E}) = 1$ for any $p \leq 0$ and $\boldsymbol{\sigma}^1(\mathscr{E}) = \boldsymbol{\delta}(\det \mathscr{E})$. Moreover,

$$\sum_{k=0}^{r} \gamma^k(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r) \cdot \boldsymbol{\sigma}^{i+1-k}(\mathscr{E}) = 0 \tag{2.8}$$

for any $i \geq 0$. In particular, the following hold:

(1) *$\boldsymbol{\sigma}^p(\mathscr{E}) \in F^p(Y)$ for any $p \geq 0$.*
(2) *For a non-negative integer m, let b_m be the sum*

$$\sum_{k=0}^{m} \gamma^{m-k}(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r)\, \boldsymbol{\sigma}^k(\mathscr{E}).$$

Then, $b_0 = 1$, $b_m \in F^{m+1}(X)$ for any $m > 0$, and $b_m = 0$ for any $m \geq r$.
(3) *For any $0 \leq p \leq r$, there is a polynomial $P(x_1, \ldots, x_r, y) \in \boldsymbol{Z}[x_1, \ldots, x_r, y^{\pm}]$ such that*

$$\gamma^p(\mathrm{cl}_\bullet(\mathscr{E}^\vee) - r) = P(\boldsymbol{\sigma}^1(\mathscr{E}), \ldots, \boldsymbol{\sigma}^r(\mathscr{E}), \mathrm{cl}^\bullet(\det \mathscr{E}))$$

and that any monomial in P has weighted degree at least p with respect to the weight w such that $w(y) = 0$ and $w(x_i) = i$ for any $1 \leq i \leq r$.

Proof. If $\mathrm{R}^q\,\pi_*\mathscr{O}(-i) \neq 0$ for integers $0 \le i \le r$ and $q \ge 0$, then $(i,q) = (0,0)$ or $(i,q) = (r, r-1)$. Here, $\mathrm{R}^0\,\pi_*\mathscr{O}_X \simeq \mathscr{O}_Y$ and $\mathrm{R}^{r-1}\,\pi_*\mathscr{O}(-r) \simeq \det \mathscr{E}^\vee$. We know

$$\boldsymbol{\delta}(\mathscr{O}(1))^p = \sum_{i=0}^{l} (-1)^i \binom{p}{i} \operatorname{cl}^\bullet(\mathscr{O}(-i))$$

for $p \ge 0$ by Remark 1.4, (3). Therefore,

$$\begin{aligned} &\boldsymbol{\sigma}^p(\mathscr{E}) = \pi_\star\big(\boldsymbol{\delta}(\mathscr{O}(1))^{r+p-1}\big) = 1 \quad \text{for} \quad -(r-1) \le p \le 0, \qquad \text{and} \\ &\boldsymbol{\sigma}^1(\mathscr{E}) = \pi_\star\big(\boldsymbol{\delta}(\mathscr{O}(1))^{r}\big) = 1 - \operatorname{cl}^\bullet(\det \mathscr{E}^\vee) = \boldsymbol{\delta}(\det \mathscr{E}). \end{aligned}$$

Let $\mathscr{G}$ be the cokernel of the natural injection $\mathscr{O}(-1) \to \pi^*\mathscr{E}^\vee$. Then

$$\begin{aligned} \gamma_t(\operatorname{cl}^\bullet(\mathscr{G}) - (r-1)) &= \gamma_t(\operatorname{cl}^\bullet(\pi^*\mathscr{E}^\vee) - r)\gamma_t(\operatorname{cl}^\bullet(\mathscr{O}(-1)) - 1)^{-1} \\ &= \gamma_t(\operatorname{cl}^\bullet(\pi^*\mathscr{E}^\vee) - r)(1 - \boldsymbol{\delta}(\mathscr{O}(1))t)^{-1}. \end{aligned} \tag{2.9}$$

The left hand side of (2.9) equals the polynomial

$$\lambda_{t/(1-t)}(\operatorname{cl}^\bullet(\mathscr{G}) - (r-1)) = \sum_{p=0}^{r-1} \lambda^p(\mathscr{G}) t^p (1-t)^{r-1-p}$$

of degree $r-1$ (cf. (1.7)). The right hand side of (2.9) equals

$$\Big(\sum_{i=0}^{r} \gamma^i(\pi^\star(\operatorname{cl}^\bullet(\mathscr{E}^\vee)) - r)t^i\Big)\Big(\sum_{j\ge 0} \boldsymbol{\delta}(\mathscr{O}(1))^j t^j\Big),$$

so the coefficient of t^{r+i} for $i \ge 0$ equals

$$0 = \sum_{k=0}^{r} \gamma^k(\pi^\star(\operatorname{cl}^\bullet(\mathscr{E}^\vee)) - r)\,\boldsymbol{\delta}(\mathscr{O}(1))^{i+r-k}.$$

By taking $\pi_\star$, we have the expected equality (2.8) by applying (2.3). Rewriting the equality (2.8) to

$$\boldsymbol{\sigma}^{i+1}(\mathscr{E}) = -\sum_{k=1}^{r} \gamma^k(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r)\, \boldsymbol{\sigma}^{i+1-k}(\mathscr{E}),$$

we infer that $\boldsymbol{\sigma}^p(\mathscr{E}) \in F^p(Y)$ for any $p \geq 0$ by induction on p. Thus, (1) is proved. If $m \geq r$, then b_m in (2) equals

$$\sum_{k=m-r}^{r} \gamma^{m-k}(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r)\, \boldsymbol{\sigma}^k(\mathscr{E}) = \sum_{k=0}^{r} \gamma^k(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r)\, \boldsymbol{\sigma}^{r-k}(\mathscr{E}) = 0$$

by (2.8), since $\gamma^i(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r) = 0$ for $i > r$. If $0 < m < r$, then, by (2.8), we have

$$\begin{aligned} b_m &= \sum_{k=0}^{m} \gamma^k(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r)\, \boldsymbol{\sigma}^{m-k}(\mathscr{E}) = -\sum_{k=m+1}^{r} \gamma^k(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r)\, \boldsymbol{\sigma}^{m-k}(\mathscr{E}) \\ &= -\sum_{k=m+1}^{r} \gamma^k(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r) \in F^{m+1}(X). \end{aligned}$$

Since $b_0 = \gamma^0(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r)\, \boldsymbol{\sigma}^0(\mathscr{E}) = 1$, we have proved (2). Finally, we shall prove (3) by induction on p. If $p = 0$, then it is enough to take $P = 1$. If $p > r$, then $\gamma^p(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r) = 0$. Thus, we may assume that $0 < p \leq r$. Considering the difference of the equalities (2.8) for $i = p$ and for $i = p - 1$, we have

$$\begin{aligned} 0 &= \sum_{k=0}^{r} \gamma^k(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r)(\boldsymbol{\sigma}^{p+1-k}(\mathscr{E}) - \boldsymbol{\sigma}^{p-k}(\mathscr{E})) \\ &= \sum_{k=0}^{p-1} \gamma^k(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r)(\boldsymbol{\sigma}^{p+1-k}(\mathscr{E}) - \boldsymbol{\sigma}^{p-k}(\mathscr{E})) - \gamma^p(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r)\, \mathrm{cl}^\bullet(\det \mathscr{E}^\vee). \end{aligned}$$

Therefore, $\gamma^p(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r)$ is expressed as a polynomial of $\gamma^k(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r)$ for $k < p$, $\boldsymbol{\sigma}^1(\mathscr{E}), \ldots, \boldsymbol{\sigma}^{p+1}(\mathscr{E})$, and $\mathrm{cl}^\bullet(\det \mathscr{E}) = \mathrm{cl}^\bullet(\det \mathscr{E}^\vee)^{-1}$. Thus, by induction, we can prove (3) for $p < r$. For $p = r$, we consider (2.8) for $i = r - 1$. Then,

$$\gamma^r(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r) = -\sum_{k=0}^{r-1} \gamma^k(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r)\, \boldsymbol{\sigma}^{r-k}(\mathscr{E}).$$

Hence, (3) holds also for $p = r$. Thus, we are done. $\square$

Remark. In Lemma 2.20 above, the element $\boldsymbol{s}^p(\mathscr{E}) := \boldsymbol{\sigma}^p(\mathscr{E}) \bmod F^{p+1}(Y) \in G^p(Y)$ for $p \geq 0$ can be regarded as the p-th Segre class of $\mathscr{E}$ (the p-th Segre class of $\mathscr{E}^\vee$ in the sense of [**12**]), by the property Lemma 2.20, (2). In fact, we have

$$\sum_{k=0}^{m} \boldsymbol{c}^{m-k}(\mathscr{E}^\vee)\boldsymbol{s}^k(\mathscr{E}) = \begin{cases} 1 \in G^0(Y), & \text{if } m = 0; \\ 0 \in G^m(Y), & \text{if } m > 0. \end{cases}$$

Fact 2.21. In the situation of Lemma 2.20, the following further properties are known (cf. [**22**, Exp. VI]):

(1) (cf. [**22**, Exp. VI, Théorème 1.1 and Remarque 1.13]): $K^\bullet(X)$ is a free $K^\bullet(Y)$-module of rank $r-1$ and has the decomposition

$$K^\bullet(X) = \bigoplus_{k=0}^{r-1} \pi^\star(K^\bullet(Y))\,\boldsymbol{\delta}(\mathscr{O}(1))^k$$

with a relation

$$\boldsymbol{\delta}(\mathscr{O}(1))^r = -\sum_{k=0}^{r-1} \gamma^{r-k}(\pi^\star(\mathrm{cl}^\bullet(\mathscr{E}^\vee)) - 1)\,\boldsymbol{\delta}(\mathscr{O}(1))^k.$$

(2) (cf. [**22**, Exp. VI, Proposition 5.3]): For $p \geq 0$, we have

$$F^p(X) = \bigoplus_{k=0}^{r-1} \pi^\star(F^{p-k}(Y))\,\boldsymbol{\delta}(\mathscr{O}(1))^k.$$

(3) (cf. [**22**, Exp. VI, Corollaire 5.8]): $\pi_\star(F^{r+i-1}(X)) \subset F^i(Y)$ for any $i \geq 0$, for the homomorphism $\pi_\star = \pi_\star^{\mathscr{O}_X} : K^\bullet(X) \to K^\bullet(Y)$.

Note that (3) is derived from (2), by the projection formula (2.3) and by Lemma 2.20, (2).

Remark 2.22. In the situation of Lemma 2.20, $\pi_\star\colon K^\bullet(X) \to K^\bullet(Y)$ induces a homomorphism $G^{r+i-1}(X) \to G^i(Y)$ for $i \geq 0$, by Fact 2.21, (3). Adding the zero maps from $G^j(X)$ for $j < r-1$, we have a homomorphism $G(\pi_\star)\colon G^\bullet(X) \to G^\bullet(Y)$, which is denoted by $\pi_\star\colon G^\bullet(X) \to G^\bullet(Y)$ for simplicity if it causes no confusion with $\pi_\star\colon K^\bullet(X) \to K^\bullet(Y)$. Let $G(\pi^\star)\colon G^\bullet(Y) \to G^\bullet(X)$ be the homomorphism induced from $\pi^\star\colon K^\bullet(Y) \to K^\bullet(X)$. Then the projection formula

$$G(\pi_\star)(\bar{x} \cdot G(\pi^\star)(\bar{y})) = G(\pi_\star)(\bar{x}) \cdot \bar{y}$$
$$(\text{or} \quad \pi_\star(\bar{x} \cdot \pi^\star(\bar{y})) = \pi_\star(\bar{x}) \cdot \bar{y} \quad \text{for simplicity}) \tag{2.10}$$

holds, where $\bar{x} \in G^\bullet(X)$ and $\bar{y} \in G^\bullet(Y)$. This is derived from the decomposition in Fact 2.21, (2), and from the projection formula (2.3). We note further that

$$G(\pi_\star)(\boldsymbol{c}^{r-1}(\mathscr{G})) = 1 \in G^0(Y) \tag{2.11}$$

for the cokernel $\mathscr{G}$ of $\mathscr{O}(-1) \to \pi^*(\mathscr{E}^\vee)$. In fact, comparing the coefficient of t^{r-1} of the both sides of (2.9), we have

$$\gamma^{r-1}(\mathrm{cl}^\bullet(\mathscr{G}) - (r-1)) = \sum_{k=0}^{r-1} \gamma^k(\pi^\star(\mathrm{cl}^\bullet(\mathscr{E}^\vee)) - r)\, \boldsymbol{\delta}(\mathscr{O}(1))^{r-k-1}.$$

Applying $\pi_\star$, we have the equality (2.11) by (2.3) and by Lemma 2.20 with (2.8) for $i = 0$, since

$$\begin{aligned} &\pi_\star(\gamma^{r-1}(\mathrm{cl}^\bullet(\mathscr{G}) - (r-1))) \\ &\quad = 1 + \sum_{k=1}^{r-1} \gamma^k(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r) = 1 + \sum_{k=1}^{r-1} \gamma^k(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r)\, \boldsymbol{\sigma}^{1-k}(\mathscr{E}) \\ &\quad = 1 - \gamma^r(\mathrm{cl}^\bullet(\mathscr{E}^\vee) - r) - \boldsymbol{\sigma}^1(\mathscr{E}) \equiv 1 \mod F^1(Y). \end{aligned}$$

As an application of Lemma 2.20, we have:

Lemma 2.23 (cf. [**8**, Section V]). *Let X be a connected Noetherian scheme. Then, for any $k \geq 0$, $F^k(X)$ is generated by elements of the form $\boldsymbol{\sigma}^{j_1}(\mathscr{E}_1) \cdots \boldsymbol{\sigma}^{j_l}(\mathscr{E}_l)$ for locally free sheaves $\mathscr{E}_i$ on X of rank r_i, where j_i are positive integers such that $\sum_{i=1}^{l} j_i \geq k$. In other words, $F^k(Y)$ is generated by elements of the form*

$$p_\star\big(\boldsymbol{\delta}(\mathscr{O}(1)^{(1)})^{r_1+j_1-1} \cdots \boldsymbol{\delta}(\mathscr{O}(1)^{(l)})^{r_l+j_l-1}\big)$$

for the fiber product $p\colon P = \boldsymbol{P}(\mathscr{E}_1) \times_X \cdots \times_X \boldsymbol{P}(\mathscr{E}_r) \to X$ of projective space bundles $\boldsymbol{P}_X(\mathscr{E}_i) \to X$ associated with the locally free sheaves $\mathscr{E}_i$, where $p_\star = p_\star^{\mathscr{O}_P} : K^\bullet(P) \to K^\bullet(X)$, and $\mathscr{O}(1)^{(i)}$ denotes the pullback to P of the tautological invertible sheaves

$\mathscr{O}(1)$ on $\boldsymbol{P}_X(\mathscr{E}_i)$ with respect to $\mathscr{E}_i$.

Proof. By definition, $F^k(X)$ is generated by elements of the form $\gamma^{m_1}(x_1)\cdots\gamma^{m_n}(x_n)$ for positive integers m_i with $m_1+\cdots+m_n \geq k$, where $x_i = \mathrm{cl}_\bullet(\mathscr{G}_i) - r_i$ for a locally free sheaf $\mathscr{G}_i$ of rank r_i. By Lemma 2.20, (3), $\gamma^{m_i}(x_i)$ is expressed as a linear combination of certain products of $\boldsymbol{\sigma}^m(\mathscr{G}_i^\vee)$ for $m \geq 0$ and $\boldsymbol{\sigma}^1(\mathscr{G}_i) = \boldsymbol{\delta}(\det \mathscr{G}_i^\vee)$. Thus, by Lemma 2.20, $F^k(X)$ is generated by elements of the form $\boldsymbol{\sigma}^{j_1}(\mathscr{E}_1)\cdots\boldsymbol{\sigma}^{j_l}(\mathscr{E}_l)$ for positive integers j_i with $j_1+\cdots+j_l \geq k$, where $\mathscr{E}_i$ is a locally free sheaf of rank r_i. For the morphism $p\colon P = \boldsymbol{P}_X(\mathscr{E}_1)\times_X\cdots\times_X \boldsymbol{P}_X(\mathscr{E}_l) \to Y$ above, we have

$$\boldsymbol{\sigma}^{j_1}(\mathscr{E}_1)\cdots\boldsymbol{\sigma}^{j_l}(\mathscr{E}_l) = p_\star\big(\boldsymbol{\delta}(\mathscr{O}(1)^{(1)})^{r_1+j_1-1}\cdots\boldsymbol{\delta}(\mathscr{O}(1)^{(l)})^{r_l+j_l-1}\big)$$

by applying Lemma 2.8 successively. Thus, we are done. $\square$

Applying Lemma 2.23, we shall prove the following well-known:

Proposition 2.24 ([**22**, Exp. VI, Théorème 6.9], [**13**, Chapter V, Corollary 3.10]). *Let X be a Noetherian scheme of dimension at most n admitting ample invertible sheaves. Then, $F^{n+1}(X) = 0$.*

Proof. By Lemma 2.23, it is enough to prove that $p_\star(x) = 0$ for a fiber product $p\colon P = \boldsymbol{P}_X(\mathscr{E}_1)\times_X\cdots\times_X \boldsymbol{P}_X(\mathscr{E}_l) \to X$ of projective space bundles $\boldsymbol{P}_X(\mathscr{E}_i)$ associated with a locally free sheaf of rank r_i and for

$$x = \boldsymbol{\delta}(\mathscr{O}(1)^{(1)})^{r_1+j_1-1}\cdots\boldsymbol{\delta}(\mathscr{O}(1)^{(l)})^{r_l+j_l-1},$$

where $\mathscr{O}(1)^{(i)}$ is the pullback to P of the tautological invertible sheaf $\mathscr{O}(1)$ on $\boldsymbol{P}_X(\mathscr{E}_i)$ with respect to $\mathscr{E}_i$ and j_i are positive integers with $\sum_{i=1}^l j_i \geq n+1$. Now, P has also an ample invertible sheaf, since p is a projective morphism. Since $\dim P \leq n + \dim P/X = n + \sum_{i=1}^l (r_i - 1)$, we have $\sum_{i=1}^l (r_i + j_i - 1) \geq \dim P + 1$. It is enough to show that $x = 0$ in $K^\bullet(P)$. Therefore, we are reduced to proving the assertion that

$$\boldsymbol{\delta}(\mathscr{L}_1, \dots, \mathscr{L}_{n+1}) = \boldsymbol{\delta}(\mathscr{L}_1)\cdots\boldsymbol{\delta}(\mathscr{L}_{n+1}) = 0 \in K^\bullet(X)$$

for any invertible sheaves $\mathscr{L}_i$ on X. Since X has an ample invertible sheaf, any invertible sheaf is written as a difference of ample invertible sheaves. In fact, for an invertible sheaf $\mathscr{L}$ and for an ample invertible sheaf $\mathscr{A}$ on X, there is a positive integer b such that $\mathscr{L}\otimes\mathscr{A}^{\otimes b}$ is ample. Hence, $\mathscr{L} \simeq (\mathscr{L}\otimes\mathscr{A}^{\otimes b})\otimes\mathscr{A}^{\otimes(-b)}$. Thus, by Remark 1.4, we may assume that $\mathscr{L}_i$ are all ample. Moreover, by taking

b above too large, we may assume that there is a section $\sigma_1 \in \mathrm{H}^0(X, \mathscr{L}_1)$ such that σ_1 does not vanish at any associated prime of X (cf. [**22**, Exp. VI, Lemma 6.8]). In particular, $\sigma_1^\vee \colon \mathscr{L}_1^\vee \to \mathscr{O}_X$ is injective and the Cartier divisor $V(\sigma_1)$ has codimension at least one. Similarly, we may assume that there exists a section $\sigma_2 \in \mathrm{H}^0(X, \mathscr{L}_2)$ such that

$$\sigma_2^\vee|_{V(\sigma_1)} \colon \mathscr{L}_2^\vee|_{V(\sigma_1)} \to \mathscr{O}_{V(\sigma_1)}$$

is injective. Continuing the same reduction, we may assume finally that there exist sections $\sigma_i \in \mathrm{H}^0(X, \mathscr{L}_i)$ for $1 \le i \le n+1$ such that $\sigma = (\sigma_1, \ldots, \sigma_{n+1})$ is a regular section of $\mathscr{E} := \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_{n+1}$. Here, $V(\sigma) = \emptyset$, since $\dim V(\sigma) = \dim X - (n+1) < 0$. Hence, $\boldsymbol{\delta}(\mathscr{L}_1, \ldots, \mathscr{L}_{n+1}) = \lambda_{-1}(\mathscr{E}^\vee) = 0$ by the exact sequence (1.4) for $\mathscr{E}$ and $\mathscr{F} = \mathscr{O}_X$. Thus, we are done. □

By Lemmas 2.18 and 2.23, we have the following result related to Fact 2.21, (3).

Proposition 2.25.

$$\pi_\star^{\mathscr{F}}(F^{d+2}(X)) \subset F^2(Y).$$

In particular, $\mathscr{I}_{\mathscr{F}/Y}(x) := \det(\pi_\star^{\mathscr{F}}(x))$ *gives rise to a homomorphism* $G^{d+1}(X) \to G^1(Y) \simeq \mathrm{Pic}(Y)$.

Proof. It is enough to prove that $\pi_\star^{\mathscr{F}}(x) \in F^2(Y)$ for $x \in F^{d+2}(X)$ of the form

$$x = p_\star\big(\boldsymbol{\delta}(\mathscr{O}(1)^{(1)})^{r_1+j_1-1} \cdots \boldsymbol{\delta}(\mathscr{O}(1)^{(l)})^{r_l+j_l-1}\big)$$

as in Lemma 2.23, where $j_1 + \cdots + j_l \ge d+2$. Here $p^*\mathscr{F}$ is flat over Y, and $\pi \circ p \colon P \to Y$ is a projective morphism satisfying Assumption 2.1. Then, by Lemma 2.9, we have

$$\mathscr{I}_{p^*\mathscr{F}/Y}(z) = \det\big((\pi \circ p)_\star^{p^*\mathscr{F}}(z)\big) = \det\big(\pi_\star^{\mathscr{F}}(p_\star(z))\big) = \det \pi_\star^{\mathscr{F}}(x)$$

for $z = \boldsymbol{\delta}(\mathscr{O}(1)^{(1)})^{r_1+j_1-1} \cdots \boldsymbol{\delta}(\mathscr{O}(1)^{(l)})^{r_l+j_l-1}$. Note that $\sum_{i=1}^{l}(r_i + j_i - 1) \ge \dim P/X + d + 2 \ge \dim(\operatorname{Supp} p^*\mathscr{F})/Y + 2$. Hence, $\mathscr{I}_{p^*\mathscr{F}/Y}(z) \simeq \mathscr{O}_Y$ by Lemma 2.18. Thus, we are done. □

Definition 2.26. Let $\pi \colon X \to Y$ be a locally projective morphism of

Noetherian schemes and let $\mathscr{F}$ be a coherent sheaf on X which is flat over Y. Assume that the relative dimension $d := \dim(\operatorname{Supp}\mathscr{F})/Y$ is constant. We define the intersection sheaf $\mathscr{I}^{\mathrm{perf}}_{\mathscr{F}/Y}(x)$ for $x \in F^{d+1}(X)$ by

$$\mathscr{I}^{\mathrm{perf}}_{\mathscr{F}/Y} := \det \circ \pi^{\mathscr{F}}_{\star} \colon K^{\bullet}(X) \to K^{\bullet}(Y)_{\mathrm{perf}} \to \mathrm{Pic}(Y).$$

If π satisfies Assumption 2.1, then we define $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ for $\eta \in G^{d+1}(X)$ as in Proposition 2.25.

By Proposition 2.25, we infer that if π satisfies Assumption 2.1, then

$$\mathscr{I}^{\mathrm{perf}}_{\mathscr{F}/Y}(x) \simeq \mathscr{I}_{\mathscr{F}/Y}(\eta)$$

for $x \in F^{d+1}(X)$ and $\eta = x \bmod F^{d+2}(X) \in G^{d+1}(X)$.

Lemma 2.27. *Let $h \colon Y' \to Y$ be a morphism from a Noetherian scheme Y', $X' = X \times_Y Y'$, and $\mathscr{F}' = q_1^*\mathscr{F}$ for the first projection $q_1 \colon X' \to X$. Then*

$$h^* \mathscr{I}^{\mathrm{perf}}_{\mathscr{F}/Y}(x) \simeq \mathscr{I}^{\mathrm{perf}}_{\mathscr{F}'/Y'}(q_1^{\star}x)$$

for any $x \in F^{d+1}(X)$ even if π is only a locally projective morphism. If π and the second projection $q_2 \colon X' \to Y'$ satisfy Assumption 2.1*, then*

$$h^* \mathscr{I}_{\mathscr{F}/Y}(\eta) \simeq \mathscr{I}_{\mathscr{F}'/Y'}(q_1^{\star}\eta)$$

for any $\eta \in G^{d+1}(X)$.

Proof. It is enough to prove the first isomorphism. By Lemmas 2.8 and 2.12, we have

$$(q_2)^{\mathscr{F}'}_{\star}(q_1^{\star}x) = h^{\star}\bigl(\pi^{\mathscr{F}}_{\star}(x)\bigr) \quad \in \mathrm{Ker}(\varepsilon \colon K^{\bullet}(Y)_{\mathrm{perf}} \to \boldsymbol{Z})$$

for any $x \in F^{d+1}(X)$. This induces the first isomorphism by Definition 2.26. □

Lemma 2.28. *Assume that Y is a projective scheme defined over a field $\boldsymbol{k}$ with* $\dim Y = 1$*. Then, for any $\eta \in G^{d+1}(X)$,*

$$\deg_{Y/\boldsymbol{k}} \mathscr{I}_{\mathscr{F}/Y}(\eta) = i_{X/\boldsymbol{k}}(\eta; \mathscr{F}).$$

PROOF. Let $x \in F^{d+1}(X)$ be a representative of $\eta \in G^{d+1}(X)$. We have $\pi_\star^{\mathscr{F}}(x) \in F^1(Y)$ by Lemma 2.12. Hence, $\pi_\star^{\mathscr{F}}(x) = \boldsymbol{\delta}(\mathscr{I}_{\mathscr{F}/Y}(\eta))$, since $F^2(Y) = 0$ by Proposition 2.24. On the other hand, $\deg_{Y/\boldsymbol{k}}(\mathscr{M}) = i_{Y/\boldsymbol{k}}(\boldsymbol{c}^1(\mathscr{M})) = \deg_{0,Y/\boldsymbol{k}}(\boldsymbol{\phi}\,\boldsymbol{\delta}(\mathscr{M}))$ for any invertible sheaf $\mathscr{M}$ on Y (cf. Definition 1.11). For the structure morphisms $p_X : X \to \operatorname{Spec} \boldsymbol{k}$ and $p_Y : Y \to \operatorname{Spec} \boldsymbol{k}$, we have

$$p_{X\star}(x\,\mathrm{cl}_\bullet(\mathscr{F})) = p_{Y\star}(\pi_\star(x\,\mathrm{cl}_\bullet(\mathscr{F}))) = p_{Y\star}\big(\boldsymbol{\phi}\,\pi_\star^{\mathscr{F}}(x)\big) = p_{Y\star}(\boldsymbol{\phi}\,\boldsymbol{\delta}(\mathscr{I}_{\mathscr{F}/Y}(\eta)))$$

by Lemma 2.6, (1). Thus, $i_{X/\boldsymbol{k}}(\eta; \mathscr{F}) = \deg_{Y/\boldsymbol{k}} \mathscr{I}_{\mathscr{F}/Y}(\eta)$. □

The following Lemma 2.29 and Corollary 2.30 are similar to the projection formulas shown in [**8**, Proposition IV.2.2 (b)], and [**33**, Propositions 5.2.1 and 5.2.2].

LEMMA 2.29. *Let $\psi\colon Y \to S$ be a projective surjective flat morphism to a connected Noetherian scheme S with the relative dimension $e = \dim Y/S$, and $\mathscr{G}$ a locally free sheaf on Y of finite rank. Suppose that $\mathscr{F}$ is flat over S and that S admits an ample invertible sheaf when π is not flat. Then, there exist isomorphisms*

$$\mathscr{I}_{\mathscr{F}\otimes\pi^*\mathscr{G}/S}(\eta \cdot \pi^\star\theta) \simeq \mathscr{I}_{\mathscr{G}/S}(\boldsymbol{c}^1(\mathscr{I}_{\mathscr{F}/Y}(\eta)) \cdot \theta),$$
$$\mathscr{I}_{\mathscr{F}\otimes\pi^*\mathscr{G}/S}(\eta' \cdot \pi^\star\theta') \simeq \mathscr{I}_{\mathscr{G}/S}(\theta')^{\otimes i_{\mathscr{F}/Y}(\eta')}$$

for any $\eta \in G^{d+1}(X)$, $\eta' \in G^d(X)$, $\theta \in G^e(Y)$, and $\theta' \in G^{e+1}(Y)$.

PROOF. The assertion follows from the projection formula

$$\pi_\star^{\mathscr{F}\otimes\pi^*\mathscr{G}}(x \cdot \pi^\star y) = \psi_\star^{\mathscr{G}}\big(\pi_\star^{\mathscr{F}}(x) \cdot y\big) \tag{2.12}$$

for any $x \in K^\bullet(X)$ and $y \in K^\bullet(Y)$. This is derived from the quasi-isomorphism

$$\boldsymbol{R}(\psi \circ \pi)_*(\mathscr{F} \otimes \pi^*\mathscr{G} \otimes (\mathscr{E} \otimes \pi^*\mathscr{V})) \simeq_{\mathrm{qis}} \boldsymbol{R}\psi_*((\mathscr{G} \otimes \mathscr{V}) \otimes^{\mathrm{L}} \boldsymbol{R}\pi_*(\mathscr{F} \otimes \mathscr{E}))$$

for any locally free sheaves $\mathscr{E}$ on X and $\mathscr{V}$ on Y of finite rank. □

COROLLARY 2.30. *For $\theta \in G^d(X)$ and an invertible sheaf $\mathscr{M}$ on Y, one has an isomorphism*

$$\mathscr{I}_{\mathscr{F}/Y}(\theta \cdot \boldsymbol{c}^1(\pi^*\mathscr{M})) \simeq \mathscr{M}^{\otimes i_{\mathscr{F}/Y}(\theta)}.$$

Proof. Apply the second isomorphism in Lemma 2.29 to $\theta \in G^d(X)$ and $\boldsymbol{c}^1(\mathscr{M}) \in G^1(Y)$ in the case where ψ is the identity map of Y. $\square$

The following is a combination of variants of Lemmas 2.13 and 2.14. This is proved by the same arguments as in the proofs of the both lemma; so we omit the proof.

Lemma 2.31. *Let $\mathscr{E}$ be a locally free sheaf on X of rank $d+1$ and let σ be an $\mathscr{F}$-regular section of $\mathscr{E}$ on X. For the zero subscheme $V := V(\sigma)$, assume that*

(1) *if* $\operatorname{depth} \mathscr{O}_{Y,\boldsymbol{y}} = 0$ *(resp. $= 1$), then* $\operatorname{Supp} \mathscr{F} \cap V \cap \pi^{-1}(\boldsymbol{y})$ *is empty (resp. finite).*

Then, there exists uniquely an effective Cartier divisor $D(\sigma)$ on Y with $\operatorname{Supp} D(\sigma) \subset \pi(\operatorname{Supp} V)$ *satisfying the following properties:*

$$\mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E})) \simeq \mathscr{O}_Y(D(\sigma)), \quad \textit{and} \quad D(\sigma)|_{\operatorname{Spec} \mathscr{O}_{Y,\boldsymbol{y}}} = \operatorname{Div}_{\mathscr{O}_{Y,\boldsymbol{y}}}(\pi_*(\mathscr{F} \otimes \mathscr{O}_V))$$

for any point $\boldsymbol{y} \in Y$ with $\operatorname{depth} \mathscr{O}_{Y,\boldsymbol{y}} = 1$*. Moreover, $D(\sigma)$ has the following base change property: Let $h \colon Y' \to Y$ be a morphism from another Noetherian scheme Y' and let $q \colon X' \to X$ be the first projection from the fiber product $X' = X \times_Y Y'$. Assume that*

(2) *if* $\operatorname{depth} \mathscr{O}_{Y',\boldsymbol{y}'} = 0$ *(resp. $= 1$), then* $\operatorname{Supp} \mathscr{F} \cap V(\sigma) \cap \pi^{-1}(h(\boldsymbol{y}'))$ *is empty (resp. finite).*

Then, the divisor $D(q^*\sigma)$ on Y' with respect to $q^*\mathscr{F}/Y'$ exists, and $h^*D(\sigma) = D(q^*\sigma)$.

The following is regarded as a generalization of Proposition 2.15 (cf. Lemma 2.37 below):

Proposition 2.32. *Let $\mathscr{E}$ be a locally free sheaf on X of rank $d+1$ admitting a surjection $\pi^*\mathscr{G} \to \mathscr{E}$ for a locally free sheaf $\mathscr{G}$ on Y of finite rank. Then $i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E})) \geq 0$ and there is a natural surjection*

$$\Phi \colon \operatorname{Sym}^{i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))}(\mathscr{G}) \to \mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E})).$$

Proof. We shall prove by essentially the same argument as in Proposition 2.15. Let $q \colon \boldsymbol{P} := \boldsymbol{P}_Y(\mathscr{G}^\vee) \to Y$ be the projective space bundle and $\mathscr{O}(1)$ the tautological invertible sheaf associated with $\mathscr{G}^\vee$. Let $\boldsymbol{P}_X$ be the fiber product $X \times_Y \boldsymbol{P}$, and let $p_1 \colon \boldsymbol{P}_X \to X$ and $p_2 \colon \boldsymbol{P}_X \to \boldsymbol{P}$ be the natural projections. Pulling back the natural injection $\mathscr{O}(-1) \to q^*\mathscr{G}$ to $\boldsymbol{P}_X$, we can consider the composite

$$p_2^*\mathscr{O}(-1) \to p_2^* q^* \mathscr{G} = p_1^* \pi^* \mathscr{G} \to p_1^* \mathscr{E}$$

and hence a section σ of $p_1^*\mathscr{E} \otimes p_2^*\mathscr{O}(1)$. The zero subscheme $V = V(\sigma)$ is isomorphic to $\boldsymbol{P}_X(\mathscr{K}^\vee)$ for the kernel $\mathscr{K}$ of $\pi^*\mathscr{G} \to \mathscr{E}$. Thus, we have a diagram:

$$\begin{tikzcd}
V \arrow[r, equal] & \boldsymbol{P}_X(\mathscr{K}^\vee) \arrow[r, "\subset"] & \boldsymbol{P}_X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "\pi"'] \\
& \boldsymbol{P}_Y(\mathscr{G}^\vee) \arrow[r, equal] & \boldsymbol{P} \arrow[r, "q"] & Y.
\end{tikzcd}$$

Since $V \to X$ is smooth, the section σ is regular and furthermore $p_1^*\mathscr{F}$-regular. We define $\mathrm{cl}^\bullet(\mathscr{O}_V) := \lambda_{-1}(p_1^*\mathscr{E}^\vee \otimes p_2^*\mathscr{O}(-1))$. Then, by Lemma 1.7, we have:

$$\boldsymbol{\phi}(\mathrm{cl}^\bullet(\mathscr{O}_V)) = \mathrm{cl}_\bullet(\mathscr{O}_V), \quad \mathrm{cl}^\bullet(\mathscr{O}_V)\,\mathrm{cl}_\bullet(p_1^*\mathscr{F}) = \mathrm{cl}_\bullet(p_1^*\mathscr{F} \otimes \mathscr{O}_V), \qquad \text{and}$$
$$\mathrm{cl}^\bullet(\mathscr{O}_V) = (-1)^{d+1}\gamma^{d+1}(\mathrm{cl}^\bullet(p_1^*\mathscr{E}^\vee \otimes p_2^*\mathscr{O}(-1)) - (d+1)).$$

We insert here a claim.

Claim 2.33.

$$\det\big((p_2)_\star^{p_1^*\mathscr{F}}(\mathrm{cl}^\bullet(\mathscr{O}_V))\big) \simeq q^*\big(\,\mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E}))\big) \otimes \mathscr{O}(1)^{\otimes i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))}.$$

Proof. We set $l = \mathrm{cl}^\bullet(\mathscr{O}(-1)) \in K^\bullet(\boldsymbol{P})$, $y = \boldsymbol{\delta}(\mathscr{O}(1)) = 1 - l$, and $x = \mathrm{cl}^\bullet(\mathscr{E}^\vee) \in K^\bullet(X)$. Then

$$\begin{aligned}
\lambda_{-1}(p_1^\star x \cdot p_2^\star l) &= \sum_{k \geq 0} (-1)^k p_1^\star(\lambda^k(x)) \cdot (p_2^\star l)^k \\
&= \sum_{k \geq 0} p_1^\star(\lambda^k(x)) \cdot p_2^\star(y-1)^k \\
&= \sum_{0 \leq j \leq k \leq d+1} (-1)^{k-j}\binom{k}{j} p_1^\star(\lambda^k(x)) \cdot p_2^\star y^j \\
&= \sum_{j=0}^{d+1} p_1^\star\bigg(\sum_{k=j}^{d+1}(-1)^{k-j}\binom{k}{j}\lambda^k(x)\bigg) \cdot p_2^\star y^j.
\end{aligned}$$

By Lemma 2.8, we have

$$\begin{aligned}(p_2)_\star^{p_1^*\mathscr{F}}(\mathrm{cl}^\bullet(\mathscr{O}_V)) &= \sum_{j=0}^{d+1} q^\star \pi_\star^{\mathscr{F}}\left(\sum_{k=j}^{d+1}(-1)^{k-j}\binom{k}{j}\lambda^k(x)\right)\cdot y^j \\ &\equiv q^\star\pi_\star^{\mathscr{F}}(\lambda_{-1}(x)) + q^\star\pi_\star^{\mathscr{F}}\left(\sum_{k=1}^{d+1}(-1)^{k-1}k\lambda^k(x)\right)\cdot y \mod F^2(\boldsymbol{P}).\end{aligned}$$

Hence, Claim 2.33 follows from the equality:

$$\varepsilon\left(\pi_\star^{\mathscr{F}}\left(\sum_{k=1}^{d+1}(-1)^{k-1}k\lambda^k(\mathrm{cl}^\bullet(\mathscr{E}^\vee))\right)\right) = \varepsilon\big(\pi_\star^{\mathscr{F}}\boldsymbol{c}^d(\mathscr{E})\big). \tag{2.13}$$

We shall show (2.13) as follows: Comparing the coefficients of t^d on the both sides of the equality

$$\gamma_t(x-(d+1)) = \sum_{k=0}^{d+1}\lambda^k(x)t^k(1-t)^{d+1-k},$$

we have

$$\begin{aligned}\gamma^d(x-(d+1)) &= \sum_{k=0}^{d}(-1)^{d-k}(d+1-k)\lambda^k(x) \\ &= (-1)^d(d+1)\sum_{k=0}^{d+1}(-1)^k\lambda^k(x) - (-1)^d\sum_{k=1}^{d+1}(-1)^k k\lambda^k(x) \\ &= (-1)^d(d+1)\lambda_{-1}(x) + (-1)^d\left(\sum_{k=1}^{d+1}(-1)^{k-1}k\lambda^k(x)\right).\end{aligned}$$

Here $\varepsilon(\pi_\star^{\mathscr{F}}\lambda_{-1}(x)) = \varepsilon(\pi_\star^{\mathscr{F}}\boldsymbol{c}^{d+1}(\mathscr{E})) = 0$ by Lemmas 1.7 and 2.12. Thus, we have the equality (2.13) by

$$\begin{aligned}\varepsilon\big(\pi_\star^{\mathscr{F}}\boldsymbol{c}^d(\mathscr{E})\big) &= (-1)^d\varepsilon\big(\pi_\star^{\mathscr{F}}\boldsymbol{c}^d(\mathscr{E}^\vee)\big) \\ &= (-1)^d\varepsilon\big(\pi_\star^{\mathscr{F}}\gamma^d(x-(d+1))\big) = \varepsilon\left(\pi_\star^{\mathscr{F}}\left(\sum_{k=1}^{d+1}(-1)^{k-1}k\lambda^k(x)\right)\right). \quad\square\end{aligned}$$

Proof of Proposition 2.32 continued. We infer that $p_2(\operatorname{Supp} p_1^*\mathscr{F} \cap V)$ does not contain any fiber of $q\colon \boldsymbol{P} \to Y$ by $\dim(\operatorname{Supp} p_1^*\mathscr{F} \cap V)/Y = N - 1 = \dim \boldsymbol{P}/Y - 1$. Thus, as in the proof of Proposition 2.15, we infer that if $\operatorname{depth} \mathscr{O}_{\boldsymbol{P},\boldsymbol{w}} = 0$ (resp. $= 1$), then $\operatorname{Supp} p_1^*\mathscr{F} \cap V \cap p_2^{-1}(\boldsymbol{w})$ is empty (resp. finite). Hence, by Lemma 2.31, we have an effective Cartier divisor $D = D(\sigma)$ on $\boldsymbol{P}$ such that $\operatorname{Supp} D \subset p_2(\operatorname{Supp} p_1^*\mathscr{F} \cap V)$ and

$$\det(p_2)_\star^{p_1^*\mathscr{F}}(\mathrm{cl}^\bullet(\mathscr{O}_V)) = \mathscr{O}_{\boldsymbol{P}}(D).$$

By Claim 2.33, we have a global section of

$$q^*\bigl(\mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E}))\bigr) \otimes \mathscr{O}(1)^{\otimes i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))}.$$

Restricting it to a fiber of q, we infer that $i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E})) \geq 0$. The global section gives a surjection

$$\Phi\colon \operatorname{Sym}^{i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))}(\mathscr{G}) \to \mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E})).$$

by the same argument as in the proof of Proposition 2.15. □

Lemma 2.34. *Let $h\colon Y' \to Y$ be a morphism from a Noetherian scheme Y'. For the fiber product $X' = X \times_Y Y'$, let $q_1\colon X' \to X$ and $\pi'\colon X' \to Y'$ be the first and second projections. In the situation of Proposition 2.32, let $\psi\colon \pi^*\mathscr{G} \to \mathscr{E}$ be the surjection, and set $\mathscr{F}' := q_1^*(\mathscr{F})$, $\mathscr{E}' := q_1^*\mathscr{E}$, $\mathscr{G}' := h^*\mathscr{G}$, and $\psi' := q_1^*(\psi)\colon \pi'^*(\mathscr{G}') = q_1^*(\pi^*\mathscr{G}) \to \mathscr{E}'$. Let*

$$\Phi\colon \operatorname{Sym}^{i_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E}))}(\mathscr{G}) \to \mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E})) \quad \textit{and}$$
$$\Phi'\colon \operatorname{Sym}^{i_{\mathscr{F}'/Y'}(\boldsymbol{c}^{d+1}(q_1^*\mathscr{E}))}(h^*(\mathscr{G})) \to \mathscr{I}_{\mathscr{F}'/Y'}(\boldsymbol{c}^{d+1}(q_1^*\mathscr{E}))$$

be the surjections defined in Proposition 2.32 for $(\pi, \mathscr{F}, \psi)$ and for $(\pi', \mathscr{F}', \psi')$, respectively. Then, Φ' and $h^*(\Phi)$ are isomorphic to each other.

Proof. Let $D = D_{\mathscr{F},\mathscr{E}}$ be the effective relative Cartier divisor on $\boldsymbol{P}_Y(\mathscr{G}^\vee)$ in the proof of Proposition 2.32. Let $D' = D_{\mathscr{F}',\mathscr{E}'}$ be the similarly defined effective relative Cartier divisor on $\boldsymbol{P}_{Y'}(\mathscr{G}'^\vee)$. It suffices to check that D' is the pullback of D by the natural morphism $\boldsymbol{P}_{Y'}(\mathscr{G}'^\vee) \simeq \boldsymbol{P}_Y(\mathscr{G}^\vee) \times_Y Y' \to \boldsymbol{P}_Y(\mathscr{G}^\vee)$. Then, it is enough to apply the latter half of Lemma 2.31 and essentially the same argument in the proof of Lemma 2.16. □

Remark 2.35 (cf. Remark 2.17). Let $D = D_{\mathscr{F},\mathscr{E}}$ be the effective relative Cartier divisor on $\boldsymbol{P} = \boldsymbol{P}_Y(\mathscr{G}^\vee)$ in the proof of Proposition 2.32. By Lemma 2.34, we infer that, for a point $\boldsymbol{y} \in Y$, the effective divisor $D_{\boldsymbol{y}} = D|_{q^{-1}(\boldsymbol{y})}$ on the fiber $q^{-1}(\boldsymbol{y})$ of $q\colon \boldsymbol{P} \to Y$ is characterized by the following two conditions:

(1) For the tautological invertible sheaf $\mathscr{O}(1)$ of the projective space $q^{-1}(\boldsymbol{y})$, one has

$$\mathscr{O}_{q^{-1}(\boldsymbol{y})}(D_{\boldsymbol{y}}) \simeq \mathscr{O}(1)^{\otimes i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))}.$$

(2) Let v be a non-zero element of $\mathscr{G} \otimes \boldsymbol{k}(\boldsymbol{y})$. Let $[v]$ be a point of $q^{-1}(\boldsymbol{y})$ corresponding to the surjection $v^\vee\colon \mathscr{G}^\vee \otimes \boldsymbol{k}(\boldsymbol{y}) \to \boldsymbol{k}(\boldsymbol{y})$, and let v^X be the global section of $\mathscr{E} \otimes \mathscr{O}_{\pi^{-1}(\boldsymbol{y})}$ defined by $\pi^*\mathscr{G} \to \mathscr{E}$. Then $[v] \notin \operatorname{Supp} D_{\boldsymbol{y}}$ if and only if $V(v^X) \cap \operatorname{Supp} \mathscr{F} = \emptyset$ for the zero subscheme $V(v^X) \subset \pi^{-1}(\boldsymbol{y})$.

Remark 2.36. In the situation of Proposition 2.32, let $X' \subset X$ be a closed subscheme such that $\mathscr{F}$ is an $\mathscr{O}_{X'}$-module and that $X' \to Y$ also satisfies Assumption 2.1. Then, applying Proposition 2.32 to $X' \to Y$, we have a similar homomorphism

$$\Phi'\colon \operatorname{Sym}^{i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))}(\mathscr{G}) \to \mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E})).$$

Here, Φ' is isomorphic to Φ. In fact, by Remark 2.5, $i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))$ and $\mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E}))$ do not depend on the choice of X and X'. Moreover, by the same reason, the isomorphism in Claim 2.33 also does not depend on the choice. Therefore, $\Phi' \simeq \Phi$ by the proof of Proposition 2.32.

Lemma 2.37. *If $\mathscr{E} = \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_{d+1}$ for invertible sheaves $\mathscr{L}_i$ and if $\mathscr{G} = \mathscr{G}_1 \oplus \cdots \oplus \mathscr{G}_{d+1}$ for locally free sheaves $\mathscr{G}_i$ of finite rank with surjections $\pi^*\mathscr{G}_i \to \mathscr{L}_i$, then the natural surjection*

$$\operatorname{Sym}^e(\mathscr{G}) \to \operatorname{Sym}^{e_1}(\mathscr{G}_1) \otimes \cdots \otimes \operatorname{Sym}^{e_l}(\mathscr{G}_l)$$

to a component is compatible with the surjections Φ in Propositions 2.15 and 2.32, where

$$e_i = i_{\mathscr{F}/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_{i-1}, \mathscr{L}_{i+1}, \ldots, \mathscr{L}_{d+1}) \quad \textit{and} \quad e = i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E})) = \sum_{i=1}^{d+1} e_i.$$

Proof. Let $\mathscr{V}$ be the locally free sheaf $\bigoplus_{i=1}^{d+1} \mathscr{O}(1)^{(i)}$ on $\boldsymbol{P}_Y = \boldsymbol{P}_Y(\mathscr{G}_1^\vee) \times_Y$

$\cdots \times_Y \boldsymbol{P}_Y(\mathscr{G}_{d+1}^\vee)$, where $\mathscr{O}(1)^{(i)}$ is the pullback of the tautological invertible sheaf by $\boldsymbol{P}_Y \to \boldsymbol{P}_Y(\mathscr{G}_i^\vee)$. Then there is a birational morphism $\mu\colon \boldsymbol{P}(\mathscr{V}) \to \boldsymbol{P}_Y(\mathscr{G}^\vee)$ for the projective space bundle $\varpi\colon \boldsymbol{P}(\mathscr{V}) \to \boldsymbol{P}_Y$ such that the tautological invertible sheaf on $\boldsymbol{P}(\mathscr{V})$ associated to $\mathscr{V}$ is just the pullback of the tautological invertible sheaf on $\boldsymbol{P}_Y(\mathscr{G}^\vee)$ by μ. Let $\Gamma_i \subset \boldsymbol{P}(\mathscr{V})$ be the projective subbundle associated with the quotient locally free sheaf $\mathscr{V}/\mathscr{O}(1)^{(i)}$ for $1 \leq i \leq d+1$. Then Γ_i is a Cartier divisor such that

$$\mathscr{O}_{\boldsymbol{P}(\mathscr{V})}(\Gamma_i) \otimes \varpi^*\mathscr{O}(1)^{(i)} \simeq \mu^*\mathscr{O}(1) \quad \text{and} \quad \mu(\Gamma_i) = \boldsymbol{P}_Y(\mathscr{G}^\vee/\mathscr{G}_i^\vee) \subset \boldsymbol{P}_Y(\mathscr{G}^\vee).$$

For a point $\boldsymbol{y} \in Y$, let $v = (v_1, \ldots, v_{d+1})$ be a non-zero element of $\mathscr{G} \otimes \boldsymbol{k}(\boldsymbol{y})$ such that $v_i \in \mathscr{G}_i \otimes \boldsymbol{k}(\boldsymbol{y})$. Then $[v] \in \boldsymbol{P}_Y(\mathscr{G}^\vee) \times_Y \boldsymbol{y}$ is not contained in $\mu(\Gamma_i)$ if and only if $v_i \neq 0$. Let $D_0 = D_{\mathscr{F}, \underline{\mathscr{L}}}$ be the effective relative Cartier divisor on $\boldsymbol{P}_Y$ defining Φ in the proof of Proposition 2.15. Let $D_1 = D_{\mathscr{F}, \mathscr{E}}$ be the effective relative Cartier divisor on $\boldsymbol{P}_Y(\mathscr{G}^\vee)$ defining Φ in the proof of Proposition 2.32. Then,

$$\varpi^* D_0 + \sum_{i=1}^{d+1} e_i \Gamma_i \sim \mu^* D_1,$$

where $\sim$ denotes the linear equivalence relation, and $\mu_*(\varpi^* D_0) = D_1$ over $\boldsymbol{P}_Y(\mathscr{G}^\vee) \setminus \bigcup_{i=1}^{d+1} \mu(\Gamma_i)$, by Remarks 2.17 and 2.35. Hence,

$$\varpi^* D_0 + \sum_{i=1}^{d+1} e_i \Gamma_i = \mu^* D_1,$$

since the invertible sheaves $\mathscr{O}(1)^{(i)}$ are linearly independent in $\mathrm{Pic}(\boldsymbol{P}_Y)$. For the structure morphism $q\colon \boldsymbol{P}_Y \to Y$, the divisor $\varpi^* D_0$ is defined by a section of

$$\varpi^* q^* \mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_{d+1}) \otimes \varpi^* \left(\bigotimes_{i=1}^{d+1} (\mathscr{O}(1)^{(i)})^{\otimes e_i} \right)$$

and $\mu^* D_1$ is defined by a section of

$$\varpi^* q^* \mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_{d+1}) \otimes \mu^* \mathscr{O}(1)^{\otimes e},$$

where the two sections correspond to the surjections Φ in Propositions 2.15 and 2.32. The difference $\sum e_i \Gamma_i$ is defined by the natural injection

$$\varpi^* \left(\bigotimes_{i=1}^{d+1} (\mathscr{O}(1)^{(i)})^{\otimes e_i} \right) \hookrightarrow \bigotimes_{i=1}^{d+1} \mu^* \mathscr{O}(1) = \mu^* \mathscr{O}(1)^{\otimes e}.$$

The push-forward $(q \circ \varpi)_*$ of the injection is just the natural injection

$$\bigotimes_{i=1}^{d+1} \mathrm{Sym}^{e_i}(\mathscr{G}_i^\vee) \hookrightarrow \mathrm{Sym}^e(\mathscr{G}^\vee).$$

Hence, two Φ are related by

$$\Phi \colon \mathrm{Sym}^e(\mathscr{G}) \to \bigotimes_{i=1}^{d+1} \mathrm{Sym}^{e_i}(\mathscr{G}_i) \xrightarrow{\Phi} \mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_{d+1}). \qquad \square$$

2.4. A positivity problem.

Finally in Section 2, we shall consider a kind of positivity problem as an application of Proposition 2.32.

Definition 2.38 ([**14**]). Let $P = P(x_1, x_2, \ldots, x_r) \in \boldsymbol{Q}[x_1, \ldots, x_r]$ be a weighted homogeneous polynomial of degree n such that the weight of x_i is i for any $1 \le i \le r$. If

$$i_{V/\boldsymbol{k}}(P(\boldsymbol{c}^1(\mathscr{E}), \ldots, \boldsymbol{c}^r(\mathscr{E})); V) > 0$$

for any n-dimensional projective variety V defined over a field $\boldsymbol{k}$ and for any ample vector bundle $\mathscr{E}$ of rank r on V, then P is called *numerically positive for ample vector bundles*.

Note that, in [**14**], the intersection number $i_{V/\boldsymbol{k}}(P(\boldsymbol{c}^1(\mathscr{E}), \ldots, \boldsymbol{c}^r(\mathscr{E})); V)$ is denoted by $\int_V P(\boldsymbol{c}^1(\mathscr{E}), \ldots, \boldsymbol{c}^r(\mathscr{E}))$.

Fact 2.39. If $P \in \boldsymbol{Z}[x_1, \ldots, x_r]$ is a weighted homogeneous polynomial of degree n such that the weight of x_i is i for any $1 \le i \le r$, then P is expressed uniquely as $\sum_\lambda a_\lambda P_\lambda$ for the *Schur polynomial* $P_\lambda = P_\lambda(x_1, \ldots, x_r)$ associated with a *partition* λ of n by non-negative integers $\le r$ and for $a_\lambda \in \boldsymbol{Z}$; the partition λ is given by a non-increasing sequence

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge 0$$

of integers with $\sum_{i \ge 1} \lambda_i = n$, $\lambda_1 \le r$, and the Schur polynomial P_λ is defined as the determinant of the $n \times n$-matrix $(b_{i,j})_{1 \le i,j \le n}$ whose (i,j)-component is given by

$$b_{i,j} = \begin{cases} x_{\lambda_i - i + j}, & \text{if } 1 \le \lambda_i - i + j \le r; \\ 1, & \text{if } \lambda_i - i + j = 0; \\ 0, & \text{otherwise.} \end{cases}$$

This is a well-known result in the theory of symmetric functions (cf. [**30**]). Fulton-Lazarsfeld have proved in [**14**, Theorem I], that P is numerically positive for ample vector bundles if and only if $P \neq 0$ and $a_\lambda \ge 0$ for any λ.

Remark 2.40. The Schur polynomial P_λ above corresponds to the usual Schur function $S_{\lambda'}$ associated with the conjugate partition λ'. Here, $\lambda' = (\lambda'_1 \ge \lambda'_2 \ge \cdots \ge 0)$ for $\lambda'_i = \sup(\{0\} \cup \{j \ge 1 \mid \lambda_j \ge i\})$ and

$$S_\lambda(y_1, y_2, \ldots, y_r) = \det\left(y_i^{\lambda_j + r - i}\right)_{1 \le i,j \le r} \Big/ \det\left(y_i^{r-i}\right)_{1 \le i,j \le r},$$

in which we have the equality

$$P_\lambda(e_1(y), \ldots, e_r(y)) = S_{\lambda'}(y_1, \ldots, y_r) \tag{2.14}$$

for the *elementary symmetric polynomials* $e_k(y)$ defined by

$$e_k(y) = \sum_{1 \le i_1 < \cdots < i_k \le r} y_{i_1} \cdots y_{i_k}.$$

The equality (2.14) is called the Jacobi-Trudi formula for elementary symmetric polynomials.

Theorem 2.41. *Let $P \in \boldsymbol{Z}[x_1, \ldots, x_r]$ be a weighted homogeneous polynomial of degree $d+1$ such that the weight of x_i is i for any $1 \le i \le r$. Assume that P is numerically positive for ample vector bundles. Let $\mathscr{E}$ be a locally free sheaf on X of rank r generated by finitely many global sections and $\mathscr{F}$ a coherent sheaf on X flat over Y with* $\dim(\operatorname{Supp}\mathscr{F})/Y = d$. *Then $\mathscr{I}_{\mathscr{F}/Y}(P(\boldsymbol{c}^1(\mathscr{E}), \ldots, \boldsymbol{c}^r(\mathscr{E})))$ is generated by finitely many global sections.*

Proof. We may assume that P is a Schur polynomial P_λ for a partition $\lambda = (\lambda_1 \ge \cdots \ge 0)$ of $d+1$ with $\lambda_i \le r$, by Fact 2.39. We write $P(\mathscr{E}) := P(\boldsymbol{c}^1(\mathscr{E}), \ldots, \boldsymbol{c}^r(\mathscr{E}))$. We shall show that there exist a smooth projective morphism

$q\colon W \to X$ (which is a composition of projective space bundles) and a locally free sheaf $\mathscr{H}$ on W of rank $N := d+1+\dim W - \dim X$ such that $\mathscr{H}$ is generated by finitely many global sections and $q_\star(\boldsymbol{c}^N(\mathscr{H})) = P(\mathscr{E}) \in G^{d+1}(X)$ for the homomorphism $q_\star = G(q_\star)\colon G^\bullet(W) \to G^\bullet(X)$ defined in Remark 2.22. Once this is proved, we have

$$\pi_\star^{\mathscr{F}}(P(\mathscr{E})) = \pi_\star^{\mathscr{F}}(q_\star(\boldsymbol{c}^N(\mathscr{H}))) = (\pi \circ q)_\star^{q^*\mathscr{F}}(\boldsymbol{c}^N(\mathscr{H}))$$

by Lemma 2.9: Thus, $\mathscr{I}_{\mathscr{F}/Y}(P(\mathscr{E})) \simeq \mathscr{I}_{q^*\mathscr{F}/Y}(\boldsymbol{c}^N(\mathscr{H}))$ and this is generated by finitely many global sections by Proposition 2.32.

Therefore, the proof is reduced to constructing $q\colon W \to X$ and $\mathscr{H}$. For the purpose, we follow some arguments in [**24**]. Let $\mathscr{V} \to \mathscr{E}$ be a surjection from a free $\mathscr{O}_X$-module $\mathscr{V}$ of rank $n > r+d+1$. We fix a sequence $\mathscr{V}_\bullet\colon 0 \subset \mathscr{V}_1 \subset \cdots \subset \mathscr{V}_{n-r}$ of subbundles of $\mathscr{V}$ such that, for any i, $\mathscr{V}_i$ is a free $\mathscr{O}_X$-module of rank $v(i) := r+i-\lambda_i$. Note that $v(i) < v(j)$ for any $i < j$ and that if we set $h(i) := n-(n-r)-v(i)+i$, then $h(i) = \lambda_i$. Let $\boldsymbol{F} := F(\mathscr{V}_\bullet)$ be the scheme over X defined in [**24**, Section 1], for the sequence $\mathscr{V}_\bullet$: The scheme $\boldsymbol{F}$ parametrizes flags $\mathscr{C}_1 \subset \mathscr{C}_2 \subset \cdots \subset \mathscr{C}_{n-r}$ of subbundles of $\mathscr{V}$ such that $\mathscr{C}_i$ is a subbundle of $\mathscr{V}_i$ and $\operatorname{rank}\mathscr{C}_i = i$, for any $1 \le i \le n-r$. Let $\mathscr{D}_1 \subset \cdots \subset \mathscr{D}_{n-r}$ be the universal flag on $\boldsymbol{F}$. By an argument just before Lemma 3 of [**24**], we know that the structure morphism $\psi\colon \boldsymbol{F} \to X$ is a composite of the structure morphisms of projective space bundles and

$$\dim \boldsymbol{F}/X = \sum_{i=1}^{n-r}(v(i)-i) = \sum_{i=1}^{n-r}(r-\lambda_i) = (n-r)r-(d+1).$$

Let $\boldsymbol{G} := \mathrm{Grass}_{n-r}(\mathscr{V})$ be the Grassmann scheme over X parametrizing subbundles of $\mathscr{V}$ of rank $n-r$. Let $\mathscr{D}$ be the universal subbundle on $\boldsymbol{G}$ and $\mathscr{Q}$ the universal quotient bundle, i.e., $\mathscr{Q} \simeq \varphi^*(\mathscr{V})/\mathscr{D}$ for the structure morphism $\varphi\colon \boldsymbol{G} \to X$. By the subbundle $\mathscr{D}_{n-r} \subset \psi^*(\mathscr{V})$, we have a morphism $\alpha\colon \boldsymbol{F} \to \boldsymbol{G}$ over X such that $\alpha^*\mathscr{D} = \mathscr{D}_{n-r}$. Let p_1 and p_2 denote the projections from $\boldsymbol{F} \times_X \boldsymbol{G}$ to $\boldsymbol{F}$ and to $\boldsymbol{G}$, respectively. The morphism α defines a section $s\colon \boldsymbol{F} \to \boldsymbol{F} \times_X \boldsymbol{G}$ of the projection p_1. Thus, we have a commutative diagram:

$$\begin{tikzcd}
\boldsymbol{F} \arrow[r, "s"] \arrow[d, "\alpha"'] & \boldsymbol{F} \times_X \boldsymbol{G} \arrow[r, "p_1"] \arrow[d, "p_2"] & \boldsymbol{F} \arrow[d, "\psi"] \\
\boldsymbol{G} \arrow[r, equal] & \boldsymbol{G} \arrow[r, "\varphi"] & X.
\end{tikzcd}$$

By the proof of [**24**, Lemma 2], $s(\boldsymbol{F})$ is the zero subscheme (cf. Definition 1.6) of a regular section of the locally free sheaf $\widehat{\mathscr{H}} := p_1^*\mathscr{D}_{n-r}^\vee \otimes p_2^*\mathscr{Q}$ of rank $r(n-r)$, and hence

$$\begin{aligned}&\mathrm{cl}_\bullet(\mathrm{cyc}(s(\boldsymbol{F}))) \bmod F_{\mathrm{con}}^{r(n-r)+1}(\boldsymbol{F} \times_X \boldsymbol{G})\\ &\quad = G(\boldsymbol{\phi})(\boldsymbol{c}^{r(n-r)}(\widehat{\mathscr{H}})) \in G_{\mathrm{con}}^{r(n-r)}(\boldsymbol{F} \times_X \boldsymbol{G})\end{aligned}$$

for $G(\boldsymbol{\phi})\colon G^\bullet(\boldsymbol{F} \times_X \boldsymbol{G}) \to G^\bullet_{\mathrm{con}}(\boldsymbol{F} \times_X \boldsymbol{G})$. Note that the locally free sheaves $\mathscr{D}^\vee$, $\mathscr{D}_{n-r}^\vee \simeq \alpha^*(\mathscr{D}^\vee)$, and $\mathscr{Q}$ are all generated by finitely many global sections; hence, so is $\widehat{\mathscr{H}}$. By [**24**, Lemma 2], we have

$$\begin{aligned}\boldsymbol{c}^{r(n-r)}(\widehat{\mathscr{H}}) &= \Delta_{r,r,\dots,r}\big(\boldsymbol{c}_t(p_2^*\mathscr{Q} - p_1^*\mathscr{D}_1), \dots, \boldsymbol{c}_t(p_2^*\mathscr{Q} - p_1^*\mathscr{D}_{n-r})\big)\\ &\in G^{r(n-r)}(\boldsymbol{F} \times_X \boldsymbol{G}),\end{aligned}$$

where $\Delta_{\dots}(\cdots)$ is defined in [**24**, Section 1], and $\boldsymbol{c}_t(p_2^*\mathscr{Q} - p_1^*\mathscr{D}_i) := \boldsymbol{c}_t(p_2^*\mathscr{Q})/\boldsymbol{c}_t(p_1^*\mathscr{D}_i)$ for the Chern polynomial $\boldsymbol{c}_t(x) = \sum_{p \geq 0} \boldsymbol{c}^p(x)t^p$.

We have homomorphisms $p_{2\star}\colon F^{r(n-r)+i}(\boldsymbol{F} \times_X \boldsymbol{G}) \to F^{d+1+i}(\boldsymbol{G})$ for $d+1+i \geq 0$, since $d+1 = r(n-r) - \dim(\boldsymbol{F}/X)$. By the proof of Lemmas 3, 4, and Theorem 5 of [**24**], we have

$$p_{2\star}\big(\boldsymbol{c}^{r(n-r)}(\widehat{\mathscr{H}})\big) = P_\lambda(\boldsymbol{c}^1(\mathscr{Q}), \dots, \boldsymbol{c}^r(\mathscr{Q})) = P_\lambda(\mathscr{Q}). \tag{2.15}$$

In fact, [**24**, Lemma 3], corresponds to the equality (2.11) in Remark 2.22 and the argument in the proof of [**24**, Lemma 4], can be applied by the projection formula (2.10) in Remark 2.22. Since $h(i) = n - (n-r) - v(i) + i = \lambda_i$ and $\boldsymbol{c}_t(\mathscr{V}_i) = 1$ for any i, we have

$$\begin{aligned}p_{2\star}(\boldsymbol{c}^{r(n-r)}(\widehat{\mathscr{H}})) &= \Delta_{h(1),\dots,h(n-r)}\big(\boldsymbol{c}_t(\mathscr{Q} - \mathscr{V}_1), \dots, \boldsymbol{c}_t(\mathscr{Q} - \mathscr{V}_{n-r})\big)\\ &= \Delta_{\lambda_1,\dots,\lambda_{d+1},0,\dots,0}(\boldsymbol{c}_t(\mathscr{Q}), \dots, \boldsymbol{c}_t(\mathscr{Q}))\\ &= P_\lambda(\boldsymbol{c}^1(\mathscr{Q}), \dots, \boldsymbol{c}^{n-r}(\mathscr{Q})) = P_\lambda(\mathscr{Q})\end{aligned}$$

by the proof of Lemma 4 and Theorem 5 in [**24**].

Let $\nu\colon X \to \boldsymbol{G}$ be the section of $\varphi\colon \boldsymbol{G} \to X$ corresponding to the surjection $\mathscr{V} \to \mathscr{E}$. Thus, $\nu^*(\mathscr{Q}) = \mathscr{E}$. Let $q\colon W \to X$ be the pullback of $p_2\colon \boldsymbol{F} \times_X \boldsymbol{G} \to \boldsymbol{G}$ by ν and $\mu\colon W \to \boldsymbol{F} \times_X \boldsymbol{G}$ be the other projection from the fiber product W; hence we have a Cartesian diagram:

$$\begin{tikzcd} W \arrow[r, "\mu"] \arrow[d, "q"'] & \boldsymbol{F} \times_X \boldsymbol{G} \arrow[r, "p_1"] \arrow[d, "p_2"'] & \boldsymbol{F} \arrow[d, "\psi"'] \\ X \arrow[r, "\nu"] & \boldsymbol{G} \arrow[r, "\varphi"] & X. \end{tikzcd}$$

Then, $p_1 \circ \mu \colon W \to \boldsymbol{F}$ is an isomorphism over X. Hence, $\dim W/X = \dim \boldsymbol{F}/X = (n-r)r - (d+1)$. The locally free sheaf

$$\mathscr{H} := \mu^*(\widehat{\mathscr{H}}) \simeq \mu^*\bigl(p_1^*\mathscr{D}_{n-r}^{\vee} \otimes p_2^*\mathscr{Q}\bigr) \simeq \mu^* p_1^*\bigl(\mathscr{D}_{n-r}^{\vee}\bigr) \otimes q^*(\mathscr{E})$$

is also generated by finitely many global sections, and

$$\operatorname{rank} \mathscr{H} = \operatorname{rank} \widehat{\mathscr{H}} = r(n-r) = d + 1 + \dim W - \dim X = N.$$

Applying the base change formula $\nu^\star(p_{2\star}(x)) = q_\star(\mu^\star(x))$ for $x \in K^\bullet(\boldsymbol{F} \times_X \boldsymbol{G})$ proved in Lemma 2.8, we have

$$\nu^\star p_{2\star}(\boldsymbol{c}^{r(n-r)}(\widehat{\mathscr{H}})) = q_\star \mu^\star(\boldsymbol{c}^{r(n-r)}(\widehat{\mathscr{H}})).$$

Here, the left hand side equals $\nu^\star(P_\lambda(\mathscr{Q})) = P_\lambda(\mathscr{E})$ by (2.15), and the right hand side equals $q_\star(\boldsymbol{c}^{r(n-r)}(\mathscr{H}))$. Thus, we have the expected equality $q_\star(\boldsymbol{c}^N(\mathscr{H})) = P_\lambda(\mathscr{E})$ in $G^{d+1}(X)$, and the proof has been completed. $\square$

3. Intersection sheaves over normal base schemes.

We generalize the definition of the intersection sheaf $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ to the following situation:

- $\pi \colon X \to Y$ is a locally projective surjective morphism to a *normal separated* Noetherian scheme Y.
- $\eta \in G^{d+1}(X)$ for an integer $d \geq 0$.
- $\mathscr{F}$ is a coherent sheaf on X such that $\dim(\pi^{-1}(\boldsymbol{y}) \cap \operatorname{Supp} \mathscr{F}) \leq d$ for any $\boldsymbol{y} \in Y$.

The new intersection sheaf $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ is an invertible sheaf on Y. If $\mathscr{F}$ is flat over Y and if π satisfies Assumption 2.1, then the new intersection sheaf coincides with the intersection sheaf defined in Section 2 (cf. Definition 2.26).

Recall that a proper surjective morphism $\pi \colon X \to Y$ to an integral scheme Y is called *equi-dimensional* if, for every irreducible component X_α, $\pi(X_\alpha) = Y$ and the function $\boldsymbol{y} \mapsto \dim(\pi^{-1}(\boldsymbol{y}) \cap X_\alpha)$ is constant on Y (cf. [**21**, Sections 13.2, 13.3]). Hence, if $\operatorname{Supp}\mathscr{F} \to Y$ is equi-dimensional and $d = \dim(\operatorname{Supp}\mathscr{F})/Y$, then $\mathscr{F}$ satisfies the condition above.

In Section 3.1, we shall define $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ as a reflexive sheaf of rank one on Y for $\mathscr{F}$ satisfying a weaker condition than above, and in Section 3.2, the invertibility of $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ is proved (cf. Theorem 3.14). Some base change properties and their applications are given in Section 3.3.

3.1. Definition of intersection sheaves.

We begin with a discussion on some condition of the dimension of fibers. The following result is known for proper flat morphisms (cf. Section 1.2).

LEMMA 3.1. *Let $\pi\colon X \to Y$ be a proper morphism of Noetherian schemes and let d be a non-negative integer. For an integer $i \geq 1$, assume that $\dim \pi^{-1}(\boldsymbol{y}) \leq d$ for any point $\boldsymbol{y} \in Y$ with $\dim \mathscr{O}_{Y,\boldsymbol{y}} < i$. Then, $\pi_\star F^{d+i}_{\mathrm{con}}(X) \subset F^{i}_{\mathrm{con}}(Y)$ for the homomorphism $\pi_\star\colon K_\bullet(X) \to K_\bullet(Y)$.*

PROOF. The inclusion is derived from the assertion that, for a closed integral subscheme Z of X, if $\operatorname{codim} Z \geq d+i$, then $\operatorname{codim}\pi(Z) \geq i$. We shall derive a contradiction by assuming $\operatorname{codim} Z \geq d+i$ and $\operatorname{codim}\pi(Z) < i$. Let $\boldsymbol{x}$ be the generic point of Z and set $\boldsymbol{y} = f(\boldsymbol{x})$. Then, for the residue field $\boldsymbol{k}(\boldsymbol{y})$ at $\boldsymbol{y}$, we have:

$$\dim \mathscr{O}_{X,\boldsymbol{x}} \leq \dim \mathscr{O}_{Y,\boldsymbol{y}} + \dim \mathscr{O}_{X,\boldsymbol{x}} \otimes_{\mathscr{O}_{Y,\boldsymbol{y}}} \boldsymbol{k}(\boldsymbol{y}). \tag{3.1}$$

Since $\operatorname{codim} Z = \dim \mathscr{O}_{X,\boldsymbol{x}}$, $\operatorname{codim}\pi(Z) = \dim \mathscr{O}_{Y,\boldsymbol{y}}$, and $\dim_{\boldsymbol{x}} \pi^{-1}(\boldsymbol{y}) = \dim \mathscr{O}_{X,\boldsymbol{x}} \otimes \boldsymbol{k}(\boldsymbol{y})$, we have $\dim_{\boldsymbol{x}} \pi^{-1}(\boldsymbol{y}) > d$ by (3.1). This contradicts our assumption that $\dim_{\boldsymbol{x}} \pi^{-1}(\boldsymbol{y}) \leq \dim \pi^{-1}(\boldsymbol{y}) \leq d$. $\square$

Applying Lemma 2.23, we shall show:

PROPOSITION 3.2 (cf. [**22**, Exp. X, Théorème 1.3.2]). *Let X be a Noetherian scheme. Then $\boldsymbol{\phi}(F^k(X)) = F^k(X)\operatorname{cl}_\bullet(\mathscr{O}_X) \subset F^k_{\mathrm{con}}(X)$. More generally, $F^p(X)F^q_{\mathrm{con}}(X) \subset F^{p+q}_{\mathrm{con}}(X)$ for any p, $q \geq 0$.*

PROOF. Note that $\operatorname{codim}(W_1 \subset W_2) + \operatorname{codim}(W_2 \subset W_3) \leq \operatorname{codim}(W_1 \subset W_3)$ for any closed irreducible subsets $W_1 \subset W_2 \subset W_3$. Thus, by induction on codimension, it is enough to prove the first inclusion: $\boldsymbol{\phi}(F^k(X)) \subset F^k_{\mathrm{con}}(X)$. By the same argument as in the proof of [**22**, Exp. X, Théorème 1.3.2], especially the proof of (1.3.6), we infer that

$$\boldsymbol{\delta}(\mathscr{L})\operatorname{cl}_\bullet(\mathscr{O}_X) \in F^1_{\mathrm{con}}(X)$$

for any invertible sheaf $\mathscr{L}$ on X. Hence, by induction on k, we can prove that

$$\boldsymbol{\delta}(\mathscr{L}_1)\cdots\boldsymbol{\delta}(\mathscr{L}_k)\,\mathrm{cl}_\bullet(\mathscr{O}_X) \in F^k_{\mathrm{con}}(X)$$

for any invertible sheaves $\mathscr{L}_1, \dots, \mathscr{L}_k$ on X. By this property and by Lemma 2.23, we shall prove the proposition. Let $p\colon P = \boldsymbol{P}_X(\mathscr{E}_1) \times_Y \cdots \times_Y \boldsymbol{P}_X(\mathscr{E}_l) \to X$ be the fiber product of the projective space bundles $\boldsymbol{P}_X(\mathscr{E}_i)$ associated with locally free sheaves $\mathscr{E}_i$ on X of rank r_i. We set $e := \dim P/X = \sum_{i=1}^{l}(r_i - 1)$. Since p is smooth, we have $p_\star(F^{k+e}_{\mathrm{con}}(P)) \subset F^k_{\mathrm{con}}(X)$ for any k. Hence, by Lemma 2.23, $\boldsymbol{\phi}(F^k(X)) \subset F^k_{\mathrm{con}}(X)$ is derived from

$$\boldsymbol{\delta}(\mathscr{M}_1)\cdots\boldsymbol{\delta}(\mathscr{M}_{k+e})\,\mathrm{cl}_\bullet(\mathscr{O}_P) \in F^{k+e}_{\mathrm{con}}(P)$$

for invertible sheaves $\mathscr{M}_i$ on P. In fact, $F^k(X)$ is generated by elements of the form $p^{\mathscr{O}_P}_\star(\boldsymbol{\delta}(\mathscr{M}_1)\cdots\boldsymbol{\delta}(\mathscr{M}_{k+e}))$, and $\boldsymbol{\phi}(p^{\mathscr{O}_P}_\star(z)) = p_\star(z\,\mathrm{cl}_\bullet(\mathscr{O}_P))$ for $z = \boldsymbol{\delta}(\mathscr{M}_1)\cdots\boldsymbol{\delta}(\mathscr{M}_{k+e})$ (cf. Lemma 2.6, (1)). Thus, we are done. $\square$

Remark. In [**22**, Exp. X, Remarque 1.4], there is a discussion on a similar property to Proposition 3.2 for a filtration of $K_\bullet(X)$ defined by 'codimension' under the assumption that the scheme is universally catenary.

In what follows in Section 3, let us fix a proper surjective morphism $\pi\colon X \to Y$ to a normal separated Noetherian scheme Y and fix a non-negative integer d. Furthermore, we assume that Y is integral, for the sake of simplicity.

Definition 3.3. Let $\mathscr{V}^{(d)}_\pi(X)$ be the set of closed integral subschemes Z of X such that $\dim(Z \cap \pi^{-1}(\boldsymbol{y})) \leq d$ for any point $\boldsymbol{y} \in Y$ with $\dim \mathscr{O}_{Y,\boldsymbol{y}} \leq 1$. We define $K^{(d)}_\pi(X)$ to be the subgroup of $K_\bullet(X)$ generated by the images of $K_\bullet(Z) \to K_\bullet(X)$ for all the closed integral subschemes $Z \in \mathscr{V}^{(d)}_\pi(X)$. We also define $\mathrm{Coh}^{(d)}_\pi(X)$ to be the set of coherent sheaves $\mathscr{F}$ on X such that any irreducible component of $\operatorname{Supp}\mathscr{F}$ belongs to $\mathscr{V}^{(d)}_\pi(X)$.

Note that, for a closed integral subscheme Z, if $\pi(Z) = Y$ and if $\dim(Z \cap \pi^{-1}(*)) \leq d$ for the generic point $*$ of Y, then $Z \in \mathscr{V}^{(d)}_\pi(X)$. Indeed, $\mathscr{O}_{Y,\boldsymbol{y}}$ is a discrete valuation ring if $\dim \mathscr{O}_{Y,\boldsymbol{y}} = 1$, and hence, $\mathscr{O}_Z$ is flat over the discrete valuation ring if $\pi(Z) = Y$.

Lemma 3.4. (1) *If $\xi \in K^{(d)}_\pi(X)$, then $\pi_\star(F^{d+i}(X)\xi) \subset F^i_{\mathrm{con}}(Y)$ for $i = 1$, 2.*

(2) *If $Z \in \mathscr{V}^{(d)}_\pi(X)$ and $\pi(Z) \neq Y$, then*

$$\pi_\star\bigl(F^{d+1}(X)\,\mathrm{cl}_\bullet(\mathscr{O}_Z)\bigr) \subset (\pi|_Z)_\star\bigl(F^{d+1}_{\mathrm{con}}(Z)\bigr) \subset F^2_{\mathrm{con}}(Y).$$

PROOF. (1): Replacing X with a closed subscheme in $\mathscr{V}^{(d)}_\pi(X)$, we may assume that X is integral and $X \in \mathscr{V}^{(d)}_\pi(X)$. Since $F^{d+i}(X)\xi \subset F^{d+i}_{\mathrm{con}}(X)$ by Proposition 3.2, it suffices to show $\pi_\star F^{d+i}_{\mathrm{con}}(X) \subset F^i_{\mathrm{con}}(Y)$ for $i = 1, 2$. This is done by Lemma 3.1.

(2): The first inclusion is derived from $F^{d+1}(Z)\,\mathrm{cl}_\bullet(\mathscr{O}_Z) \subset F^{d+1}_{\mathrm{con}}(Z)$ (cf. Proposition 3.2). If $\operatorname{codim}\pi(Z) \geq 2$, then the second inclusion follows from

$$(\pi|_Z)_\star(K_\bullet(Z)) \subset \mathrm{Image}(K_\bullet(\pi(Z)) \to K_\bullet(Y)) \subset F^2_{\mathrm{con}}(Y).$$

Thus, we may assume that $\pi(Z)$ is a prime divisor. Then, $\dim Z \cap \pi^{-1}(\boldsymbol{y}) \leq d$ for the generic point $\boldsymbol{y}$ of $\pi(Z)$. Hence, by applying Lemma 3.1 to $Z \to \pi(Z)$, we have

$$(\pi|_Z)_\star\bigl(F^{d+1}_{\mathrm{con}}(Z)\bigr) \subset \mathrm{Image}\bigl(F^1_{\mathrm{con}}(\pi(Z)) \to K_\bullet(X)\bigr) \subset F^2_{\mathrm{con}}(Y).$$

Thus, we are done. □

DEFINITION 3.5. Let $\pi\colon X \to Y$ be a proper surjective morphism from a Noetherian scheme X to a normal separated integral Noetherian scheme Y, and let d be a non-negative integer. For elements $\xi \in K^{(d)}_\pi(X)$, $\theta \in G^d(X)$, and $\eta \in G^{d+1}(X)$, we define the *relative intersection number* $i_{\xi/Y}(\theta)$ and the *intersection sheaf* $\mathscr{I}_{\xi/Y}(\eta)$ by

$$i_{\xi/Y}(\theta) := l_Y(\pi_\star(x\xi)) \in \boldsymbol{Z} \quad \text{and} \quad \mathscr{I}_{\xi/Y}(\eta) := \widehat{\det}(\pi_\star(y\xi)) \in \mathrm{Ref}^1(Y)$$

for representatives $x \in F^d(X)$ and $y \in F^{d+1}(X)$ of θ and η, respectively, where l_Y is the isomorphism $G^0_{\mathrm{con}}(Y) \simeq \boldsymbol{Z}$ in Lemma 1.14, and $\widehat{\det}$ is the isomorphism $G^1_{\mathrm{con}}(Y) \simeq \mathrm{Ref}^1(Y)$ in Lemma 1.17. These are well-defined by Lemma 3.4, (1).

CONVENTION.

(1) If $\theta = \boldsymbol{\delta}_X(\mathscr{L}_1, \ldots, \mathscr{L}_d) \bmod F^{d+1}(X)$ and $\eta = \boldsymbol{\delta}_X(\mathscr{L}_1, \ldots, \mathscr{L}_{d+1}) \bmod F^{d+1}(X)$ for invertible sheaves $\mathscr{L}_1, \ldots, \mathscr{L}_{d+1}$ on X, then we write

$$i_{\xi/Y}(\theta) = i_{\xi/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_d) \quad \text{and} \quad \mathscr{I}_{\xi/Y}(\eta) = \mathscr{I}_{\xi/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_{d+1}).$$

(2) If $\xi = \mathrm{cl}_\bullet(\mathscr{F})$ for a coherent sheaf $\mathscr{F}$ belonging to $\mathrm{Coh}^{(d)}_\pi(X)$, then $\xi \in K^{(d)}_\pi(X)$, and we write $i_{\mathscr{F}/Y}(\cdot) = i_{\xi/Y}(\cdot)$ and $\mathscr{I}_{\mathscr{F}/Y}(\cdot) = \mathscr{I}_{\xi/Y}(\cdot)$.

(3) If $\xi = \mathrm{cl}_\bullet(V)$ for a closed subscheme V whose irreducible components all belong to $\mathscr{V}^{(d)}_\pi(X)$, then $i_{\xi/Y}$ and $\mathscr{I}_{\xi/Y}$ are written by $i_{V/Y}$ and $\mathscr{I}_{V/Y}$, respectively. Similarly, if $\xi = \mathrm{cl}_\bullet(Z)$ for an algebraic cycle Z whose irreducible components all belong to $\mathscr{V}^{(d)}_\pi(X)$, then $i_{\xi/Y}$ and $\mathscr{I}_{\xi/Y}$ are written by $i_{Z/Y}$ and $\mathscr{I}_{Z/Y}$, respectively.

REMARK. For a closed immersion $\iota\colon X \hookrightarrow X'$ into another proper Y-scheme X', and for $\theta' \in G^d(X')$, $\eta' \in G^{d+1}(X')$, we have

$$i_{\xi/Y}(\theta'|_X) = i_{\iota_\star(\xi)/Y}(\theta') \quad \text{and} \quad \mathscr{I}_{\xi/Y}(\eta'|_X) = \mathscr{I}_{\iota_\star(\xi)/Y}(\eta').$$

Thus, the definitions of $i_{\mathscr{F}/Y}$, $i_{V/Y}$, $i_{Z/Y}$, $\mathscr{I}_{\mathscr{F}/Y}$, $\mathscr{I}_{V/Y}$, and $\mathscr{I}_{Z/Y}$ above cause no confusion.

REMARK 3.6. Let $\mathscr{F}$ be a coherent sheaf on X flat over Y and let η be an element of $G^{d+1}(X)$ for $d = \dim(\operatorname{Supp}\mathscr{F})/Y$. If π satisfies Assumption 2.1, then $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ defined in Definition 3.5 coincides with the intersection sheaf $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ defined in Definition 2.26. In fact, this is derived from the equalities

$$\boldsymbol{\phi}\left(\pi^{\mathscr{F}}_\star(x)\right) = \pi_\star(x\,\mathrm{cl}_\bullet(\mathscr{F})) \quad \text{and} \quad \widehat{\det}\,\boldsymbol{\phi}(y) = \det(y)$$

for any $x \in K^\bullet(X)$ and $y \in K^\bullet(Y)$ (cf. Lemma 2.6, (1), and Lemma 1.17). Even if π is only locally projective, $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ is isomorphic to the intersection sheaf $\mathscr{I}^{\mathrm{perf}}_{\mathscr{F}/Y}(x)$ defined in Definition 2.26 for a representative $x \in F^{d+1}(X)$ of $\eta \in G^{d+1}(X)$. This is shown by similar formulas

$$\boldsymbol{\phi}_{\mathrm{perf}}\left(\pi^{\mathscr{F}}_\star(x)\right) = \pi_\star(x\,\mathrm{cl}_\bullet(\mathscr{F})) \quad \text{and} \quad \widehat{\det}(\boldsymbol{\phi}_{\mathrm{perf}}(y)) = \det(y)$$

for the Cartan homomorphism $\boldsymbol{\phi}_{\mathrm{perf}}\colon K^\bullet(Y)_{\mathrm{perf}} \to K_\bullet(Y)$ and for $y \in K^\bullet(Y)_{\mathrm{perf}}$. Here, the latter formula is shown by Lemma 1.17 and by an argument in [**26**, Chapter II] (cf. [**32**, Chapter 5, Section 3]).

EXAMPLE 3.7. Assume that $\mathscr{O}_X \in \mathrm{Coh}^{(d)}_\pi(X)$ and $d = 0$; in other words, $\pi\colon X \to Y$ is generically finite. Then $\mathscr{I}_{X/Y}(\mathscr{L})$ for an invertible sheaf $\mathscr{L}$ on X is nothing but the reflexive sheaf

$$\left(\widehat{\det}\,\pi_*\mathscr{L} \otimes \widehat{\det}(\pi_*\mathscr{O}_X)^\vee\right)^{\vee\vee} \simeq \left(\widehat{\det}(\pi_*\mathscr{O}_X) \otimes \widehat{\det}(\pi_*\mathscr{L}^{-1})^\vee\right)^{\vee\vee}.$$

If X is normal and $\mathscr{L} = \mathscr{O}_X(D)$ for a Cartier divisor D, then $\mathscr{I}_{X/Y}(\mathscr{L}) \simeq$

$\mathscr{O}_Y(\pi_* D)$ for the push-forward $\pi_* D$ as a Weil divisor. In fact, we have an isomorphism

$$\big(\widehat{\det}(\pi_*\mathscr{O}_X) \otimes \widehat{\det}(\pi_*\mathscr{O}_X(-\Delta))^\vee\big)^{\vee\vee} \simeq \mathscr{O}_Y(\pi_*\Delta)$$

for an effective Weil divisor Δ on X, and applying it to effective Weil divisors D_1, D_2 with $D = D_1 - D_2$, we have the isomorphism above (cf. Remark 1.4).

REMARK 3.8. In Section 3, we are assuming that the base scheme Y to be normal. If Y is only a separated integral scheme, then the intersection sheaves $\mathscr{I}_{X/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_{d+1})$ are not naturally defined for an equi-dimensional morphism $\pi\colon X \to Y$ of relative dimension d and invertible sheaves $\mathscr{L}_i$ on X. For example, we consider the following situation: Let Y be a nodal rational cubic plane curve defined over $\boldsymbol{C}$ and $\pi\colon X \to Y$ the normalization. Let $P \in X$ be a point not lying over the node of Y. One can consider the push-forward $\pi_*(P)$ as a divisor on Y. So, the intersection sheaf $\mathscr{I}_{X/Y}(\mathscr{O}(1))$ for the tautological invertible sheaf $\mathscr{O}(1)$ on $X \simeq \boldsymbol{P}^1$ is expected to be the invertible sheaf $\mathscr{O}_Y(\pi_* P)$. However, if $P' \in X$ is not lying over the node, then $\pi_*(P)$ is linearly equivalent to $\pi_*(P')$ if and only if $P = P'$. Hence, we have no natural definition of $\mathscr{I}_{X/Y}(\mathscr{O}(1))$.

LEMMA 3.9. *Let $\mathscr{F}$ be a coherent sheaf belonging to $\mathrm{Coh}^{(d)}_\pi(X)$ and $\eta \in G^{d+1}(X)$. Then*

$$\mathscr{I}_{\mathscr{F}/Y}(\eta) \simeq \mathscr{I}_{(\mathscr{F}_{\mathrm{t.f.}/Y})/Y}(\eta),$$

where $\mathscr{F}_{\mathrm{t.f.}/Y}$ is defined in Definition 1.15*. Let $\{Z_i\}_{i\in I}$ be the set of irreducible components of* $\operatorname{Supp}\mathscr{F}$ *dominating Y and let e_i be the length $l_{Z_i}(\mathscr{F})$ of $\mathscr{F}$ along Z_i (cf. Definition* 1.1*). Then $Z_i \in \mathscr{V}^{(d)}_\pi(X)$ for any $i \in I$, and*

$$\mathscr{I}_{\mathscr{F}/Y}(\eta) \simeq \Bigg(\bigotimes_{i\in I} \mathscr{I}_{Z_i/Y}(\eta)^{\otimes e_i}\Bigg)^{\vee\vee}.$$

PROOF. Let $\{Z'_j\}_{j\in J}$ be the set of irreducible components of $\operatorname{Supp}\mathscr{F}$ not dominating Y. Then, $\bigcup_{j\in J} Z'_j = \operatorname{Supp}\mathscr{F}_{\mathrm{tor}/Y}$ and $\bigcup_{i\in I} Z_i = \operatorname{Supp}\mathscr{F}_{\mathrm{t.f.}/Y}$. Here, Z_i, $Z'_j \in \mathscr{V}^{(d)}_\pi(X)$, since $\mathscr{F} \in \mathrm{Coh}^{(d)}_\pi(X)$. We have

$$\pi_\star\big(F^{d+1}(X)\,\mathrm{cl}_\bullet(\mathscr{F}_{\mathrm{tor}/Y})\big) \subset F^2_{\mathrm{con}}(Y)$$

by Lemma 3.4, (2), since $\mathrm{cl}_\bullet(\mathscr{F}_{\mathrm{tor}/Y}) \in \sum_{j\in J} \mathrm{Image}(K_\bullet(Z'_j) \to K_\bullet(X))$. Thus, the first isomorphism is derived from $\mathrm{cl}_\bullet(\mathscr{F}) = \mathrm{cl}_\bullet(\mathscr{F}_{\mathrm{tor}/Y}) + \mathrm{cl}_\bullet(\mathscr{F}_{\mathrm{t.f.}/Y})$. Hence, we may assume that $\mathscr{F}_{\mathrm{tor}/Y} = 0$ from the beginning. Let X' be a closed subscheme of $\mathscr{F}$ such that $\mathscr{F}$ is an $\mathscr{O}_{X'}$-module. Then, $\mathscr{I}_{\mathscr{F}/Y}(\eta) \simeq \mathscr{I}_{\mathscr{F}/Y}(\eta|_{X'})$. Hence, by replacing X with a closed subscheme, we may assume that $X = \operatorname{Supp}\mathscr{F}$. Then, $\{Z_i\}_{i\in I}$ is the set of irreducible components of X, and $\mathrm{cl}_\bullet(\mathscr{F}) - \sum_{i\in I} e_i \,\mathrm{cl}_\bullet(Z_i) \in F^1_{\mathrm{con}}(X)$ by Lemma 1.14. We have $\pi_\star(F^{d+1}(X)F^1_{\mathrm{con}}(X)) \subset \pi_\star(F^{d+2}_{\mathrm{con}}(X)) \subset F^2_{\mathrm{con}}(Y)$ by Lemma 3.1 and Proposition 3.2. Hence, for a representative $x \in F^{d+1}(X)$ of $\eta \in G^{d+1}(X)$, we have

$$\pi_\star(x\,\mathrm{cl}_\bullet(\mathscr{F})) \equiv \sum_{i\in I} e_i \pi_\star(x\,\mathrm{cl}_\bullet(\mathscr{O}_{Z_i})) \mod F^2_{\mathrm{con}}(Y),$$

which induces the second isomorphism. □

Remark 3.10. In order to study the intersection sheaf $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ for $\mathscr{F} \in \mathrm{Coh}^{(d)}_\pi(X)$, we may assume that $\mathscr{F}_{\mathrm{tor}/Y} = 0$ by Lemma 3.9. Thus, we may remove the irreducible components of X which do not dominate Y, i.e., we may replace $\mathscr{O}_X$ with $(\mathscr{O}_X)_{\mathrm{t.f.}/Y}$. Hence, we may assume that there is an open subset $U \subset Y$ with $\operatorname{codim}(Y \setminus U) \geq 2$ such that π and $\mathscr{F}$ are flat over U. Then, for the sheaf $\mathscr{F}'_U = \mathscr{F}_{\mathrm{t.f.}/Y}|_{\pi^{-1}(U)} = \mathscr{F}|_{\pi^{-1}(U)}$, we have

$$i_{\mathscr{F}/Y}(\theta) = i_{\mathscr{F}'_U/U}(\theta|_{\pi^{-1}(U)}) = i_{\pi^{-1}(\boldsymbol{y})/\boldsymbol{k}(\boldsymbol{y})}\big(\theta|_{\pi^{-1}(\boldsymbol{y})}; \mathscr{F}'_U \otimes \mathscr{O}_{\pi^{-1}(\boldsymbol{y})}\big),$$
$$\mathscr{I}_{\mathscr{F}/Y}(\eta) \simeq j_*\big(\mathscr{I}_{\mathscr{F}'_U/U}(\eta|_{\pi^{-1}(U)})\big)$$

for $\boldsymbol{y} \in U$, $\theta \in G^d(X)$, $\eta \in G^{d+1}(X)$, and for the open immersion $j\colon U \hookrightarrow Y$, where the latter isomorphism follows from a property of reflexive sheaves shown in [**23**, Proposition 1.6]. Note that, by Remark 3.6, $\mathscr{I}_{\mathscr{F}'_U/U}(\eta|_{\pi^{-1}(U)})$ is just the intersection sheaf defined in Definition 2.26 when $\pi|_{\pi^{-1}(U)}\colon \pi^{-1}(U) \to U$ is a projective morphism.

Lemma 3.11. *Let $\nu\colon \widehat{X} \to X$ be a proper morphism and let $\widehat{\mathscr{F}}$ be a coherent sheaf on $\widehat{X}$ belonging to $\mathrm{Coh}^{(d)}_{\pi\circ\nu}(\widehat{X})$. Then, the i-th higher direct image sheaf $\mathscr{F}_i = \mathrm{R}^i\,\pi_*\widehat{\mathscr{F}}$ belongs to $\mathrm{Coh}^{(d)}_\pi(X)$ for any $i \geq 0$. If $\dim(\operatorname{Supp}\mathscr{F}_i \cap \pi^{-1}(*)) < d$ for the generic point $*$ of Y for any $i > 0$, then*

$$\mathscr{I}_{\widehat{\mathscr{F}}/Y}(\nu^\star\eta) \simeq \mathscr{I}_{\mathscr{F}_0/Y}(\eta)$$

for any $\eta \in G^{d+1}(X)$.

PROOF. If $\boldsymbol{y} \in Y$ is a point with $\dim \mathscr{O}_{Y,\boldsymbol{y}} \leq 1$, then

$$\dim \big(\nu(\operatorname{Supp} \widehat{\mathscr{F}}\,) \cap \pi^{-1}(\boldsymbol{y})\big) \leq \dim \big(\operatorname{Supp} \widehat{\mathscr{F}} \cap \nu^{-1}\pi^{-1}(\boldsymbol{y})\big) \leq d.$$

Hence, $\mathscr{F}_i \in \mathrm{Coh}_\pi^{(d)}(X)$, since $\operatorname{Supp} \mathscr{F}_i \subset \nu(\operatorname{Supp} \widehat{\mathscr{F}}\,)$. We have

$$(\pi \circ \nu)_\star\big(\nu^\star(x) \cdot \mathrm{cl}_\bullet(\widehat{\mathscr{F}}\,)\big) = \sum_{i \geq 0}(-1)^i \pi_\star(x\, \mathrm{cl}_\bullet(\mathscr{F}_i))$$

for a representative $x \in F^{d+1}(X)$ of $\eta \in G^{d+1}(X)$, by the projection formula (1.1). Assume that $\dim(\operatorname{Supp} \mathscr{F}_i \cap \pi^{-1}(*)) < d$ for any $i > 0$. Let Z be an irreducible component of $\operatorname{Supp} \mathscr{F}_i$ for $i > 0$. If $\pi(Z) \neq Y$, then $\pi_\star(x\, \mathrm{cl}_\bullet(Z)) \in F^2_{\mathrm{con}}(Y)$ by Lemma 3.4, (2), since $Z \in \mathscr{V}_\pi^{(d)}(X)$. If $\pi(Z) = Y$, then $Z \in \mathscr{V}_\pi^{(d-1)}(X)$; thus $\pi_\star(x\, \mathrm{cl}_\bullet(Z)) \in F^2_{\mathrm{con}}(Y)$ by Lemma 3.4, (1). Therefore, $\pi_\star(x\, \mathrm{cl}_\bullet(\mathscr{F}_i)) \in F^2_{\mathrm{con}}(Y)$ for any $i > 0$. Thus,

$$(\pi \circ \nu)_\star\big(\nu^\star(x) \cdot \mathrm{cl}_\bullet(\widehat{\mathscr{F}}\,)\big) \equiv \pi_\star(x\, \mathrm{cl}_\bullet(\mathscr{F}_0)) \mod F^2_{\mathrm{con}}(Y).$$

Hence, we have the expected isomorphism by Definition 3.5. $\square$

LEMMA 3.12. *Let $\tau \colon Y' \to Y$ be a dominant morphism from another normal separated Noetherian integral scheme Y' such that* $\operatorname{codim} \tau^{-1}(B) \geq 2$ *for any closed set $B \subset Y$ of* $\operatorname{codim}(B) \geq 2$. *Let X' be the fiber product $X \times_Y Y'$, and let $p_1 \colon X' \to X$ and $p_2 \colon X' \to Y'$ be the natural projections. For a coherent sheaf $\mathscr{F}$ of X belonging to $\mathrm{Coh}_\pi^{(d)}(X)$ and for $\eta \in G^{d+1}(X)$, one has an isomorphism*

$$\mathscr{I}_{p_1^*\mathscr{F}/Y'}(p_1^\star \eta) \simeq \big(\tau^* \mathscr{I}_{\mathscr{F}/Y}(\eta)\big)^{\vee\vee}.$$

PROOF. We may replace Y with a Zariski open subset U such that $\operatorname{codim}(Y \setminus U) \geq 2$, since the isomorphism of the reflexive sheaves follows from that on $\tau^{-1}(U)$ (cf. [**23**, Proposition 1.6]). Thus, we may assume that Y is regular and τ is flat. Applying the flat base change formula (1.2): $\tau^\star \pi_\star(x) = p_{2\star} p_1^\star(x)$ to $x = y\, \mathrm{cl}_\bullet(\mathscr{F}) \in K_\bullet(X)$ for a representative $y \in F^{d+1}(X)$ of $\eta \in G^{d+1}(X)$, we have the expected isomorphism, since $p_1^\star\, \mathrm{cl}_\bullet(\mathscr{F}) = \mathrm{cl}_\bullet(p_1^*\mathscr{F})$. $\square$

The following corresponds to Corollary 2.30:

LEMMA 3.13. *For $\xi \in K_\pi^{(d)}(X)$, $\theta \in G^d(X)$, and for an invertible sheaf $\mathscr{M}$ on Y, one has an isomorphism*

$$\mathscr{I}_{\xi/Y}(\theta \boldsymbol{c}^1(\pi^*\mathscr{M})) \simeq \mathscr{M}^{\otimes i_{\xi/Y}(\theta)}.$$

Proof. For a representative $x \in F^d(X)$ of θ, and for $y = \boldsymbol{\delta}(\mathscr{M}) \in F^1(Y)$, we have $\pi_\star(x\xi) \mod F^1_{\mathrm{con}}(Y) = i_{\xi/Y}(\theta)$ by Definition 3.5, and $\pi_\star(x\xi\pi^\star(y)) = \pi_\star(x\xi)y$ by the projection formula (1.1). Hence we have the expected isomorphism by

$$\mathscr{I}_{\xi/Y}(\theta \boldsymbol{c}^1(\pi^*\mathscr{M})) \simeq \widehat{\det}\left(\pi_\star(x\xi\pi^\star(y))\right) \simeq \widehat{\det}(i_{\xi/Y}(\theta)y) \simeq \mathscr{M}^{\otimes i_{\xi/Y}(\theta)}. \qquad \square$$

3.2. Invertibility for equi-dimensional morphisms.

We shall show that the intersection sheaf $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ for $\mathscr{F} \in \mathrm{Coh}^{(d)}_\pi(X)$ and $\eta \in G^{d+1}(X)$ is invertible under certain conditions. The following is one of such results:

Theorem 3.14. *Let $\pi\colon X \to Y$ be a proper surjective morphism onto a normal separated Noetherian scheme Y. Let $\mathscr{F}$ be a coherent sheaf on X such that* $\dim(\mathrm{Supp}\,\mathscr{F} \cap \pi^{-1}(\boldsymbol{y})) \le d$ *for any $\boldsymbol{y} \in Y$.*

(1) *If $\mathscr{L}_1, \dots, \mathscr{L}_{d+1}$ are invertible sheaves on X such that $\pi^*\pi_*\mathscr{L}_i \to \mathscr{L}_i$ is surjective for any i, then $\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \dots, \mathscr{L}_{d+1})$ is an invertible sheaf.*
(2) *If π is locally projective, then $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ is invertible for any $\eta \in G^{d+1}(X)$.*

The first proof of this theorem is given after Lemmas 3.15 and 3.17. Theorem 3.18 below on the $\boldsymbol{Q}$-factoriality of Y is obtained by applying Theorem 3.14. The first assertion (1) of Theorem 3.14 is generalized to Proposition 3.20, which gives a second proof. The third proof but in the case where π is projective, is given by the proof of Proposition 3.22 in Section 3.3, which covers Proposition 3.20 in the same case (cf. Remark 3.23).

Lemma 3.15 (cf. Lemma 2.31). *Let $\mathscr{F}$ be a coherent sheaf on X belonging to $\mathrm{Coh}^{(d)}_\pi(X)$ and let $\mathscr{E}$ be a locally free sheaf on X of rank $d+1$. Let σ be an $\mathscr{F}$-regular section of $\mathscr{E}$. Then, $\mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E})) \simeq \mathscr{O}_Y(D)$ for the codimension one part D of the effective algebraic cycle $\pi_*\,\mathrm{cyc}(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)})$. Moreover,*

$$\mathrm{cl}_\bullet(D) \equiv \mathrm{cl}_\bullet(\pi_*(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)})) \equiv \pi_\star\,\mathrm{cl}_\bullet(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)}) \mod F^2_{\mathrm{con}}(Y).$$

Proof. By Lemma 1.7,

$$\boldsymbol{c}^{d+1}(\mathscr{E}) = \lambda_{-1}(\mathscr{E}^\vee) \mod F^{d+2}(X) \quad \text{and} \quad \lambda_{-1}(\mathscr{E}^\vee)\,\mathrm{cl}_\bullet(\mathscr{F}) = \mathrm{cl}_\bullet(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)}).$$

Since $\mathrm{cl}_\bullet(\mathrm{cyc}(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)})) = \mathrm{cl}_\bullet(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)})$ and since $\mathrm{cyc}(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)})$ does not dominate Y, we have

$$\mathrm{cl}_\bullet(D) \equiv \pi_\star \, \mathrm{cl}_\bullet(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)}) \mod F^2_{\mathrm{con}}(Y)$$

for the codimension one part D of $\pi_* \, \mathrm{cyc}(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)})$. Therefore, $\mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E})) \simeq \mathscr{O}_Y(D)$. Since $\operatorname{codim} \operatorname{Supp} \mathrm{R}^i \, \pi_*(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)}) \geq 2$ for $i > 0$, we have

$$\mathrm{cl}_\bullet \left(\pi_*(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)})\right) \equiv \pi_\star \, \mathrm{cl}_\bullet(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)}) \mod F^2_{\mathrm{con}}(Y). \qquad \square$$

Remark 3.16. In the situation of Lemma 3.15, if $\pi^{-1}(\boldsymbol{y}) \cap \operatorname{Supp}(\mathscr{F} \otimes \mathscr{O}_{V(\sigma)}) = \emptyset$ for a point $\boldsymbol{y} \in Y$, then $\mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E}))$ is invertible at $\boldsymbol{y}$, since $\boldsymbol{y} \notin \operatorname{Supp} D$.

Lemma 3.17. *Let V be a Noetherian scheme over a Noetherian local ring A, and $\mathscr{L}$ an invertible sheaf on V generated by finitely many global sections $\sigma_0, \ldots, \sigma_N$. Suppose that the residue field $\boldsymbol{k}(A) = A/\mathfrak{m}_A$ is an infinite field. For coherent sheaves $\mathscr{F}_1, \ldots, \mathscr{F}_m$ on V, there exists a global section σ of $\mathscr{L}$ such that $\sigma \in \sum_{k=0}^N A\sigma_k \subset \mathrm{H}^0(V, \mathscr{L})$ and σ is $\mathscr{F}_i$-regular for any $1 \leq i \leq m$.*

Proof. It is enough to consider the case: $m = 1$. In fact, it is enough to prove for the coherent sheaf $\mathscr{F} = \bigoplus_{i=1}^m \mathscr{F}_i$. Let J be the set of points $\boldsymbol{x} \in V$ with $\operatorname{depth}(\mathscr{F}_{\boldsymbol{x}}) = 0$; in other words, J is the set of associated primes of $\mathscr{F}$. Let $W(\boldsymbol{x})$ be the closure of $\{\boldsymbol{x}\}$ for $\boldsymbol{x} \in J$. Then, a global section σ of $\mathscr{L}$ is $\mathscr{F}$-regular if and only if $\sigma|_{W(\boldsymbol{x})} \neq 0$ as a section of $\mathscr{L}|_{W(\boldsymbol{x})}$ for any $\boldsymbol{x} \in J$.

By the finite global sections $\sigma_0, \ldots, \sigma_N$, we have a morphism $\psi \colon V \to \boldsymbol{P}^N_A$ such that $\psi^*\mathscr{O}(1) \simeq \mathscr{L}$. It is enough to find an element $\sigma \in R^N(A) := \mathrm{H}^0(\boldsymbol{P}^N_A, \mathscr{O}(1))$ such that the divisor $\{\sigma = 0\}$ on $\boldsymbol{P}^N_A$ does not contain $\psi(W(\boldsymbol{x}))$ for any $\boldsymbol{x} \in J$.

We may replace A by the residue field $\boldsymbol{k}(A)$. In fact, if we find a global section $\bar{\sigma} \in R^N(\boldsymbol{k}(A)) = \mathrm{H}^0(\boldsymbol{P}^N_{\boldsymbol{k}(A)}, \mathscr{O}(1)) \simeq R^N(A) \otimes_A \boldsymbol{k}(A)$ which does not vanish along $\psi(W(\boldsymbol{x}))$ for any $\boldsymbol{x}$, then a lift $\sigma \in R^N(A)$ of $\bar{\sigma}$ also does not vanish along $\psi(W(\boldsymbol{x}))$. Thus, we may assume A to be a field $\boldsymbol{k}$.

Let $L(\boldsymbol{x}) \subset R^N(\boldsymbol{k})$ for $\boldsymbol{x} \in J$ be the vector subspace consisting of elements vanishing along $\psi(W(\boldsymbol{x}))$. Then $L(\boldsymbol{x})$ is a proper subspace. Since $\boldsymbol{k}$ is infinite, we can find an expected element σ in $R^N(\boldsymbol{k}) \setminus \bigcup_{\boldsymbol{x} \in J} L(\boldsymbol{x})$. $\square$

We shall prove Theorem 3.14.

Proof of Theorem 3.14.

(1): By a flat base change (cf. Lemma 3.12 and [**23**, Proposition 1.8]), we may assume that $Y = \operatorname{Spec} A$ for a local ring A. If the residue field $\boldsymbol{k}(A)$ is finite, then we replace A with the localization $B = A[x]_{\mathfrak{m}}$ of the polynomial ring $A[x]$ at the maximal ideal $\mathfrak{m} = \mathfrak{m}_A[x] + xA[x]$. Then $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat and the residue field $\boldsymbol{k}(B) = \boldsymbol{k}(A)(x)$ is infinite. Thus, we may assume that $\boldsymbol{k}(A)$ is infinite.

Now, $\mathscr{L}_i$ are all generated by global sections. Applying Lemma 3.17 successively, for the closed point $\boldsymbol{y} \in Y$, we can find global sections $\sigma_i \in \mathrm{H}^0(X, \mathscr{L}_i)$ such that $\sigma = (\sigma_1, \ldots, \sigma_{d+1})$ is $\mathscr{F}$-regular and $\pi^{-1}(\boldsymbol{y}) \cap V(\sigma) = \emptyset$ for the zero subscheme $V(\sigma)$ (cf. Definition 1.6). In fact, σ_i are constructed as follows: By Lemma 3.17, we have a section $\sigma_1 \in \mathrm{H}^0(X, \mathscr{L}_i)$ which is $\mathscr{F}$-regular and also $\mathscr{O}_{\pi^{-1}(\boldsymbol{y})}$-regular. Similarly, for the zero subscheme $V_1 = V(\sigma_1)$, we have a section $\sigma_2 \in \mathrm{H}^0(X, \mathscr{L}_2)$ which is $\mathscr{F} \otimes \mathscr{O}_{V_1}$-regular and $\mathscr{O}_{\pi^{-1}(\boldsymbol{y})} \otimes \mathscr{O}_{V_1}$-regular. Continuing the same process, we have sections $\sigma_i \in \mathrm{H}^0(X, \mathscr{L}_i)$ for $1 \le i \le d+1$ such that $\sigma = (\sigma_1, \ldots, \sigma_{d+1})$ is $\mathscr{F}$-regular and $\mathscr{O}_{\pi^{-1}(\boldsymbol{y})}$-regular. The latter property implies that $\pi^{-1}(\boldsymbol{y}) \cap V(\sigma) = \emptyset$, since $\dim(\pi^{-1}(\boldsymbol{y}) \cap V(\sigma)) = d - (d+1) < 0$. Therefore, $\mathscr{I}_{\mathscr{F}/Y}(\mathscr{L}_1, \ldots, \mathscr{L}_{d+1})$ is invertible at $\boldsymbol{y}$ by Lemma 3.15 and Remark 3.16.

(2): By Lemma 2.23, we may assume that there exist locally free sheaves $\mathscr{E}_1, \ldots, \mathscr{E}_l$ on X with $r_i = \operatorname{rank} \mathscr{E}_i < \infty$ and positive integers $j_1, \ldots, j_l$ with $\sum_{i=1}^{l} j_i = d+1$ such that

$$\eta = p_\star\big(\boldsymbol{\delta}(\mathscr{O}(1)^{(1)})^{r_1+j_1-1} \cdots \boldsymbol{\delta}(\mathscr{O}(1)^{(l)})^{r_l+j_l-1}\big) \bmod F^{d+2}(X)$$

for the fiber product $p\colon P = \boldsymbol{P}_X(\mathscr{E}_1) \times_X \cdots \times_X \boldsymbol{P}(\mathscr{E}_l) \to X$ of the projective space bundles $\boldsymbol{P}_X(\mathscr{E}_i) \to X$, where $\mathscr{O}(1)^{(i)}$ is the pullback to P of the tautological invertible sheaf $\mathscr{O}(1)$ on $\boldsymbol{P}_X(\mathscr{E}_i)$. Then, we have

$$\begin{aligned} &\dim(\operatorname{Supp} p^*\mathscr{F} \cap p^{-1}\pi^{-1}(\boldsymbol{y})) \\ &\qquad\qquad = \dim P/X + \dim(\operatorname{Supp} \mathscr{F} \cap \pi^{-1}(\boldsymbol{y})) \le d + \dim P/X, \\ &\sum_{i=1}^{l}(r_i + j_i - 1) = \dim P/X + 1, \quad \text{and} \\ &\qquad \mathscr{I}_{\mathscr{F}/Y}(\eta) \simeq \mathscr{I}_{p^*\mathscr{F}/Y}\big(\boldsymbol{c}^1(\mathscr{O}(1)^{(1)})^{r_1+j_1-1} \cdots \boldsymbol{c}^1(\mathscr{O}(1)^{l})^{r_l+j_l-1}\big). \end{aligned}$$

Therefore, we may assume that $\eta = \boldsymbol{c}^1(\mathscr{L}_1) \cdots \boldsymbol{c}^1(\mathscr{L}_{d+1})$ for some invertible sheaves $\mathscr{L}_1, \ldots, \mathscr{L}_{d+1}$ on X from the beginning. As in the proof of (1), we can localize Y. Hence, we may assume that X admits a relatively very ample invertible sheaf with respect to π. Thus, by the linearity of $\mathscr{I}_{\mathscr{F}/Y}$, we may assume that $\mathscr{L}_i$ are all relatively very ample. Then the assertion follows from (1). $\square$

As an application of Theorem 3.14, we have:

THEOREM 3.18. *Let $\pi\colon X \to Y$ be an equi-dimensional locally projective surjective morphism between normal separated Noetherian integral schemes. If X is $\boldsymbol{Q}$-factorial, then so is Y.*

PROOF. Let E be a prime divisor on Y. We shall show that some positive multiple of E is Cartier. Thus, we may assume π to be projective by localizing Y. Let $\mathscr{A}$ be a π-ample invertible sheaf on X and set $\theta = \boldsymbol{c}^1(\mathscr{A})^d \in G^d(X)$ for $d = \dim X/Y$. Then $i_{X/Y}(\theta) > 0$. We can take a Zariski open subset $U \subset Y$ such that $\operatorname{codim}(Y \setminus U) \geq 2$ and $E|_U$ is Cartier. Since π is equi-dimensional, $\operatorname{codim}(X \setminus \pi^{-1}(U)) \geq 2$. Therefore, there exists uniquely an effective divisor D on X such that the restriction of D to $\pi^{-1}(U)$ is just the pullback of the Cartier divisor $E|_U$ by $\pi|_{\pi^{-1}(U)}\colon \pi^{-1}(U) \to U$. By assumption, kD is Cartier for some $k > 0$. Thus, $\mathscr{I}_{X/Y}(\theta\boldsymbol{c}^1(\mathscr{O}_X(kD)))$ is an invertible sheaf by Theorem 3.14. On the other hand,

$$\mathscr{I}_{X/Y}\left(\theta\boldsymbol{c}^1(\mathscr{O}_X(kD))\right)|_U \simeq \mathscr{O}_Y(i_{X/Y}(\theta)kE)|_U,$$

by Lemma 3.13. Hence, $i_{X/Y}(\theta)kE$ is Cartier. □

The following is analogous to Lemma 2.29.

LEMMA 3.19. *Let $\psi\colon Y \to S$ be a proper surjective morphism to a normal separated Noetherian integral scheme S of relative dimension $e = \dim Y/S$, and $\mathscr{G}$ a torsion free coherent sheaf on Y. Assume that*

- *π, ψ, and $\psi \circ \pi$ are locally projective morphisms, and*
- *$\dim(\operatorname{Supp}\mathscr{F} \cap \pi^{-1}(\boldsymbol{y})) \leq d$ and $\dim(\operatorname{Supp}\mathscr{F} \cap \pi^{-1}\psi^{-1}(\boldsymbol{s})) \leq d+e$ for any $\boldsymbol{y} \in Y$ and $\boldsymbol{s} \in S$.*

Then there exist isomorphisms

$$\mathscr{I}_{\mathscr{F}\otimes\pi^*\mathscr{G}/S}(\eta\pi^\star\theta) \simeq \mathscr{I}_{\mathscr{G}/S}\left(\boldsymbol{c}^1(\mathscr{I}_{\mathscr{F}/Y}(\eta))\theta\right),$$
$$\mathscr{I}_{\mathscr{F}\otimes\pi^*\mathscr{G}/S}(\eta'\pi^\star\theta') \simeq \mathscr{I}_{\mathscr{G}/S}(\theta')^{\otimes i_{\mathscr{F}/Y}(\eta')}$$

for $\eta \in G^{d+1}(X)$, $\eta' \in G^d(X)$, $\theta \in G^e(Y)$, and $\theta' \in G^{e+1}(Y)$.

PROOF. Let U be an open subset of Y such that $\operatorname{codim}(Y \setminus U) \geq 2$ and that $\mathscr{G}$ is locally free on U. Then,

$$\pi_\star(z\,\mathrm{cl}_\bullet(\mathscr{F}\otimes\pi^*\mathscr{G}))|_U = \pi_\star(z\,\mathrm{cl}_\bullet(\mathscr{F}))|_U \cdot \mathrm{cl}^\bullet(\mathscr{G}|_U)$$

for any $z \in K^\bullet(X)$. Let $x \in F^{d+1}(X)$ be a representative of η. Then

$$\boldsymbol{\phi}(\boldsymbol{\delta}(\mathscr{I}_{\mathscr{F}/Y}(\eta))) \equiv \pi_\star(x\,\mathrm{cl}_\bullet(\mathscr{F})) \mod F^2_{\mathrm{con}}(Y),$$

since $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ is invertible by Theorem 3.14. Hence,

$$\pi_\star(x\,\mathrm{cl}_\bullet(\mathscr{F} \otimes \pi^*\mathscr{G})) \equiv \boldsymbol{\phi}(\boldsymbol{\delta}(\mathscr{I}_{\mathscr{F}/Y}(\eta)))\,\mathrm{cl}_\bullet(\mathscr{G}) \mod F^2_{\mathrm{con}}(Y).$$

Let $x' \in F^d(X)$ be a representative of η'. Then $i_{\mathscr{F}/Y}(\eta') = \varepsilon(\pi_\star(x\,\mathrm{cl}_\bullet(\mathscr{F})))$, and hence

$$\pi_\star(x'\,\mathrm{cl}_\bullet(\mathscr{F} \otimes \pi^*\mathscr{G})) \equiv i_{\mathscr{F}/Y}(\eta')\,\mathrm{cl}_\bullet(\mathscr{G}) \mod F^1_{\mathrm{con}}(Y).$$

Let $y \in F^e(Y)$ and $y' \in F^{e+1}(Y)$ be representatives of θ and θ', respectively. Then,

$$\psi_\star\big(yF^2_{\mathrm{con}}(Y)\big) + \psi_\star\big(y'F^1_{\mathrm{con}}(Y)\big) \subset \psi_\star\big(F^{e+2}_{\mathrm{con}}(Y)\big) \subset F^2_{\mathrm{con}}(S)$$

by Proposition 3.2 and Lemma 3.1, since $Y \in \mathscr{V}^{(e)}_\psi(Y)$. Therefore,

$$\begin{aligned}
\psi_\star\pi_\star(x \cdot (\pi^\star y) \cdot \mathrm{cl}_\bullet(\mathscr{F} \otimes \pi^*\mathscr{G})) &\equiv \psi_\star\big(\boldsymbol{\delta}(\mathscr{I}_{\mathscr{F}/Y}(\eta)) \cdot y \cdot \mathrm{cl}_\bullet(\mathscr{G})\big) \mod F^2_{\mathrm{con}}(S)\\
&\equiv \boldsymbol{\delta}\left(\mathscr{I}_{\mathscr{G}/S}(\boldsymbol{c}^1(\mathscr{I}_{\mathscr{F}/Y}(\eta)) \cdot \theta)\right) \mod F^2_{\mathrm{con}}(S),\\
\psi_\star\pi_\star(x' \cdot (\pi^\star y') \cdot \mathrm{cl}_\bullet(\mathscr{F} \otimes \pi^*\mathscr{G})) &\equiv i_{\mathscr{F}/Y}(\eta')\psi_\star(y'\,\mathrm{cl}_\bullet(\mathscr{G})) \mod F^2_{\mathrm{con}}(S)\\
&\equiv i_{\mathscr{F}/Y}(\eta')\,\boldsymbol{\delta}(\mathscr{I}_{\mathscr{G}/S}(\theta')) \mod F^2_{\mathrm{con}}(S).
\end{aligned}$$

Hence, we have the expected isomorphisms. □

The following is a generalization of Theorem 3.14, (1). This is proved by an argument analogous to Propositions 2.15 and 2.32 in Section 2. In particular, the proof is independent of that of Theorem 3.14.

Proposition 3.20. *Let $\mathscr{G}$ be a locally free sheaf on Y of rank $N+1$ and $\mathscr{E}$ a locally free sheaf on X of rank $d+1$ admitting a surjection $\pi^*\mathscr{G} \to \mathscr{E}$. Let $q\colon \boldsymbol{P} = \boldsymbol{P}(\mathscr{G}^\vee) \to Y$ be the projective space bundle, $\mathscr{O}(1)$ the tautological invertible sheaf on $\boldsymbol{P}$ with respect to $\mathscr{G}^\vee$, and let $p_1\colon \boldsymbol{P}_X \to X$ and $p_2\colon \boldsymbol{P}_X \to \boldsymbol{P}$ be the natural projections from $\boldsymbol{P}_X = X \times_Y \boldsymbol{P}$. Let $\mathscr{F}$ be a coherent sheaf on X such that $\dim(\pi^{-1}(\boldsymbol{y}) \cap \operatorname{Supp}\mathscr{F}) \le d$ for any point $\boldsymbol{y} \in Y$. Then $\mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E}))$ is invertible and $i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E})) \ge 0$. Moreover, there exist an effective relative Cartier* *divisor D on $\boldsymbol{P}$ with respect to $q\colon \boldsymbol{P} \to Y$, an isomorphism*

$$\mathscr{O}_{\boldsymbol{P}}(D) \simeq q^*(\mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E}))) \otimes \mathscr{O}(1)^{\otimes\, i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))}, \tag{3.2}$$

and a surjection

$$\Phi\colon \operatorname{Sym}^{i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))}(\mathscr{G}) \to \mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E})).$$

Proof. As in the proof of Proposition 2.32, from the natural injection $\mathscr{O}(-1) \to q^*\mathscr{G}$, considering the composition

$$p_2^*\mathscr{O}(-1) \to p_2^*q^*\mathscr{G} = p_1^*\pi^*\mathscr{G} \to p_1^*\mathscr{E},$$

we have a global section σ of $p_1^*\mathscr{E} \otimes p_2^*\mathscr{O}(1)$. Then $V(\sigma)$ is isomorphic to $V = \boldsymbol{P}_X(\mathscr{K}^\vee)$ for the kernel $\mathscr{K}$ of $\pi^*\mathscr{G} \to \mathscr{E}$. Thus, we have a diagram:

$$\begin{tikzcd}
V \arrow[r, equal] & \boldsymbol{P}_X(\mathscr{K}^\vee) \arrow[r, "\subset"] & \boldsymbol{P}_X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "\pi"] \\
& \boldsymbol{P}_Y(\mathscr{G}^\vee) \arrow[r, equal] & \boldsymbol{P} \arrow[r, "q"] & Y.
\end{tikzcd}$$

Since $V \to X$ is smooth, the closed immersion $V \hookrightarrow \boldsymbol{P}_X$ is locally of complete intersection. Thus, the section σ is $\mathscr{O}_{\boldsymbol{P}_X}$-regular, and furthermore it is $p_1^*\mathscr{F}$-regular, since V is flat over X. By Lemma 1.7, we have

$$\lambda_{-1}\bigl((p_1^*\mathscr{E} \otimes p_2^*\mathscr{O}(1))^\vee\bigr) \bmod F^{d+2}(\boldsymbol{P}_X) = \boldsymbol{c}^{d+1}\bigl(p_1^*\mathscr{E} \otimes p_2^*\mathscr{O}(1)\bigr), \tag{3.3}$$

$$\lambda_{-1}\bigl((p_1^*\mathscr{E} \otimes p_2^*\mathscr{O}(1))^\vee\bigr)\, \mathrm{cl}_\bullet(p_1^*\mathscr{F}) = \mathrm{cl}_\bullet(p_1^*\mathscr{F} \otimes \mathscr{O}_V). \tag{3.4}$$

Since $\mathscr{O}(1)$ is invertible, we have

$$\begin{aligned}\boldsymbol{c}^{d+1}(p_1^*\mathscr{E} \otimes p_2^*\mathscr{O}(1)) = \boldsymbol{c}^{d+1}(p_1^*\mathscr{E}) + \boldsymbol{c}^d(p_1^*\mathscr{E})\boldsymbol{c}^1(p_2^*\mathscr{O}(1)) \\ + \sum_{i=2}^{d+1} \boldsymbol{c}^{d+1-i}(p_1^*\mathscr{E})\boldsymbol{c}^1(p_2^*\mathscr{O}(1))^i\end{aligned}$$

in $G^{d+1}(\boldsymbol{P}_X)$ (cf. Remark 1.8). Hence, by (3.3),

$$\begin{aligned}&\lambda_{-1}\big((p_1^*\mathscr{E}\otimes p_2^*\mathscr{O}(1))^\vee\big)\\&\quad=p_1^\star(\lambda_{-1}(\mathscr{E}^\vee))+p_1^\star(z)p_2^\star\,\boldsymbol{\delta}(\mathscr{O}(1))+xp_2^\star(\boldsymbol{\delta}(\mathscr{O}(1))^2)+y\end{aligned}\tag{3.5}$$

for a representative $z\in F^d(X)$ of $\boldsymbol{c}^d(\mathscr{E})\in G^d(X)$ and for some $x\in K^\bullet(\boldsymbol{P}_X)$ and $y\in F^{d+2}(\boldsymbol{P}_X)$. Note that $p_{2\star}(y\,\mathrm{cl}_\bullet(p_1^*\mathscr{F}))\in F^2_{\mathrm{con}}(\boldsymbol{P})$ by Lemma 3.4, (1), since every fiber of $\mathrm{Supp}(p_1^*\mathscr{F})\subset\boldsymbol{P}_X\xrightarrow{p_2}\boldsymbol{P}$ has dimension at most d. Furthermore,

$$p_{2\star}\big(xp_2^\star(\boldsymbol{\delta}(\mathscr{O}(1))^2)\,\mathrm{cl}_\bullet(p_1^*\mathscr{F})\big)=\boldsymbol{\delta}(\mathscr{O}(1))^2p_{2\star}\big(x\,\mathrm{cl}_\bullet(p_1^*\mathscr{F})\big)\in F^2_{\mathrm{con}}(\boldsymbol{P}).$$

Therefore, we have the following from (3.4) and (3.5):

$$\begin{aligned}p_{2\star}\big(\,\mathrm{cl}_\bullet(p_1^*\mathscr{F}\otimes\mathscr{O}_V)\big)&=p_{2\star}\big(\lambda_{-1}((p_1^*\mathscr{E}\otimes p_2^*\mathscr{O}(1))^\vee)\cdot\mathrm{cl}_\bullet(p_1^*\mathscr{F})\big)\\&\equiv p_{2\star}p_1^\star(\lambda_{-1}(\mathscr{E}^\vee)\,\mathrm{cl}_\bullet(\mathscr{F}))+\boldsymbol{\delta}(\mathscr{O}(1))\cdot p_{2\star}p_1^\star(z\,\mathrm{cl}_\bullet(\mathscr{F}))\mod F^2_{\mathrm{con}}(\boldsymbol{P})\\&\equiv q^\star\pi_\star(\lambda_{-1}(\mathscr{E}^\vee)\,\mathrm{cl}_\bullet(\mathscr{F}))+\boldsymbol{\delta}(\mathscr{O}(1))\cdot q^\star\pi_\star(z\,\mathrm{cl}_\bullet(\mathscr{F}))\mod F^2_{\mathrm{con}}(\boldsymbol{P}).\end{aligned}\tag{3.6}$$

Since $\boldsymbol{c}^d(\mathscr{E})=z\bmod F^{d+1}(X)$, $\boldsymbol{c}^{d+1}(\mathscr{E})=\lambda_{-1}(\mathscr{E}^\vee)\bmod F^{d+2}(X)$, and since q is flat, we have

$$q^\star(\pi_\star(z\,\mathrm{cl}_\bullet(\mathscr{F})))\equiv i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))\,\mathrm{cl}_\bullet(\mathscr{O}_{\boldsymbol{P}})\mod F^1_{\mathrm{con}}(\boldsymbol{P}),\quad\text{and}$$
$$\widehat{\det}(q^\star\pi_\star(\lambda_{-1}(\mathscr{E}^\vee)\,\mathrm{cl}_\bullet(\mathscr{F})))\simeq q^*\,\widehat{\det}(\pi_\star(\lambda_{-1}(\mathscr{E}^\vee)\,\mathrm{cl}_\bullet(\mathscr{F})))\simeq q^*\,\mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E}))$$

by Definition 3.5, where in the second isomorphism, we use the fact that the pullback of a reflexive sheaf by a flat morphism is also reflexive (cf. [**23**, Proposition 1.8]). Therefore, (3.6) induces an isomorphism

$$\widehat{\det}\,\big(p_{2\star}(\mathrm{cl}_\bullet(p_1^*\mathscr{F}\otimes\mathscr{O}_V))\big)\simeq q^*\,\mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E}))\otimes\mathscr{O}(1)^{\otimes i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))}.\tag{3.7}$$

Let D be the codimension one part of $p_{2*}\,\mathrm{cyc}(p_1^*\mathscr{F}\otimes\mathscr{O}_V)$. Then $\mathrm{Supp}\,D\subset p_2(V\cap p_1^{-1}(\mathrm{Supp}\,\mathscr{F}))$ and

$$\mathscr{O}_{\boldsymbol{P}}(D)\simeq\widehat{\det}\,\big(p_{2\star}(\mathrm{cl}_\bullet(p_1^*\mathscr{F}\otimes\mathscr{O}_V))\big).$$

In particular, the expected isomorphism (3.2) is derived from (3.7). For an arbitrary point $\boldsymbol{y}\in Y$, $\mathrm{Supp}\,D$ does not contain the fiber $q^{-1}(\boldsymbol{y})$, since

$$\begin{aligned}\dim(\operatorname{Supp} D \cap q^{-1}(\boldsymbol{y})) &\le \dim\big(V \cap p_1^{-1}(\operatorname{Supp}\mathscr{F} \cap \pi^{-1}(\boldsymbol{y}))\big) \\ &\le \dim V/X + d = (N-d-1)+d = N-1.\end{aligned}$$

Hence, $q^* \mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E}))$ is invertible along the non-empty set $q^{-1}(\boldsymbol{y}) \setminus \operatorname{Supp} D$ by (3.2). Thus, $\mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E}))$ is invertible at $\boldsymbol{y}$, and D is a relative Cartier divisor with respect to q. Moreover,

$$i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E})) = \deg \mathscr{O}_{\boldsymbol{P}}(D)|_{q^{-1}(\boldsymbol{y})} \ge 0.$$

The effective divisor D defines a global section of

$$\begin{aligned}&q_*\big(q^* \mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E})) \otimes \mathscr{O}(1)^{\otimes i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))}\big) \\ &\quad = \mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E})) \otimes \operatorname{Sym}^{i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))}(\mathscr{G}^\vee),\end{aligned}$$

from which we have an expected homomorphism Φ by considering the natural pairing $\operatorname{Sym}^l(\mathscr{G}) \otimes \operatorname{Sym}^l(\mathscr{G}^\vee) \to \mathscr{O}_Y$. The surjectivity of Φ is shown by the same argument as in the proof of Proposition 2.15. □

Remark. If $\mathscr{F}$ is flat over Y, then, by construction, the surjection Φ in Proposition 3.20 is isomorphic to the surjection Φ in Proposition 2.32.

By Proposition 3.20 and by the proof of Theorem 2.41, we have:

Corollary 3.21. *Let $\mathscr{F}$ be a coherent sheaf with $\dim(\operatorname{Supp}\mathscr{F} \cap \pi^{-1}(\boldsymbol{y})) \le d$ for any $\boldsymbol{y} \in Y$, and let $\mathscr{E}$ be a locally free sheaf on X of rank r generated by finitely many global sections. If $r = d+1$, then $\mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^r(\mathscr{E}))$ is an invertible sheaf generated by finitely many global sections. More generally, if $P(x_1, \dots, x_r)$ is a weighted homogeneous polynomial of degree $d+1$ numerically positive for ample vector bundles, then $\mathscr{I}_{\mathscr{F}/Y}(P(\boldsymbol{c}^1(\mathscr{E}), \dots, \boldsymbol{c}^r(\mathscr{E})))$ is also an invertible sheaf generated by finitely many global sections.*

3.3. Base change properties for equi-dimensional morphisms.

We shall give some of the base change properties of the intersection sheaves $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ by a morphism $h\colon Y' \to Y$ from another normal separated Noetherian scheme Y'. Note that if $\mathscr{F}$ is flat over Y and $\pi\colon X \to Y$ satisfies Assumption 2.1, then by Lemma 2.27, the pullback $h^* \mathscr{I}_{\mathscr{F}/Y}(\eta)$ is isomorphic to the intersection sheaf on Y' associated to the pullbacks of $\mathscr{F}$ and η. However, π and $\mathscr{F}$ are not necessarily flat over Y in the situation of Section 3.

Proposition 3.22 below gives some of fundamental base change properties. The proof uses results in Section 2 but not in Section 3.2.

Proposition 3.22. *Let $\mathscr{F}$ be a coherent sheaf on X such that* $\dim(\operatorname{Supp}\mathscr{F} \cap \pi^{-1}(\boldsymbol{y})) \le d$ *for any point $\boldsymbol{y} \in Y$. Let $h\colon Y' \to Y$ be a proper surjective morphism from a Noetherian integral scheme Y' such that $\mathscr{F}' := (q_1^*\mathscr{F})_{\mathrm{t.f.}/Y'}$ is flat over Y' for the first projection $q_1\colon X \times_Y Y' \to X$. Let X' be a closed subscheme of $X \times_Y Y'$ such that $\mathscr{F}'$ is an $\mathscr{O}_{X'}$-module. Let $\nu\colon X' \to X$ and $\pi'\colon X' \to Y'$ be the restrictions of q_1 and the second projection $q_2\colon X \times_Y Y' \to Y'$ to X', respectively. Suppose that*

- *π is a projective morphism, and*
- *$\pi'\colon X' \to Y'$ satisfies Assumption* 2.1.

Let $\mathscr{I}_{\mathscr{F}'/Y'}(\eta')$ be the intersection sheaf defined in Definition 2.26 *for $\eta' \in G^{d+1}(X')$ with respect to $\pi'\colon X' \to Y'$ and $\mathscr{F}'$. Then, the following assertions hold for any $\eta \in G^{d+1}(X)$:*

(1) *If V is a closed subscheme of a fiber of h, then*

$$\mathscr{I}_{\mathscr{F}'/Y'}(\nu^\star\eta)|_V \simeq \mathscr{O}_V.$$

(2) *Assume that $\eta = \boldsymbol{c}^{d+1}(\mathscr{E})$ for a locally free sheaf $\mathscr{E}$ of rank $d+1$ on X with a surjection $\pi^*\mathscr{G} \to \mathscr{E}$ for a locally free sheaf $\mathscr{G}$ of finite rank on Y. Then $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ is invertible and the surjection*

$$\Phi'\colon \operatorname{Sym}^{i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))}(h^*\mathscr{G}) \to \mathscr{I}_{\mathscr{F}'/Y'}(\boldsymbol{c}^{d+1}(\nu^*\mathscr{E}))$$

on Y' appearing in Proposition 2.32 *descends to a surjection*

$$\Phi\colon \operatorname{Sym}^{i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))}(\mathscr{G}) \to \mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E})) = \mathscr{I}_{\mathscr{F}/Y}(\eta).$$

(3) *$\mathscr{I}_{\mathscr{F}/Y}(\eta)$ is an invertible sheaf. If $\mathscr{O}_Y \simeq h_*\mathscr{O}_{Y'}$ or if π satisfies Assumption* 2.1, *then*

$$\mathscr{I}_{\mathscr{F}'/Y'}(\nu^\star\eta) \simeq h^*\,\mathscr{I}_{\mathscr{F}/Y}(\eta).$$

(4) *There exist a finite birational morphism $\vartheta\colon Y^\sharp \to Y'$ from an integral scheme $Y^\sharp$ and an isomorphism*

$$\vartheta^*\,\mathscr{I}_{\mathscr{F}'/Y'}(\nu^\star\eta) \simeq \vartheta^* h^*\,\mathscr{I}_{\mathscr{F}/Y}(\eta).$$

Proof. (1): Let $\mathscr{F}'_V$ be the pullback of $\mathscr{F}'$ to $X' \times_{Y'} V$ and let W be a closed subscheme of $X' \times_{Y'} V$ such that $\mathscr{F}'_V$ is an $\mathscr{O}_W$-module and $\operatorname{Supp} \mathscr{F}'_V = \operatorname{Supp} W$. Then,

$$\mathscr{I}_{\mathscr{F}'/Y'}(\nu^\star \eta)|_V \simeq \mathscr{I}_{\mathscr{F}'_V/V}(\nu^\star \eta|_W)$$

by Lemma 2.27 and Remark 2.5. Let $\boldsymbol{y}$ be the point $h(V)$. Then, the image Γ of the composite $W \to X' \times_Y Y' \to X' \to X$ is contained in $\operatorname{Supp} \mathscr{F} \cap \pi^{-1}(\boldsymbol{y})$. Thus, $\dim \Gamma \leq d$, and $\eta|_\Gamma \in F^{d+1}(\Gamma) = 0$ by Proposition 2.24. Since $\nu^\star \eta|_W$ is the image of $\eta|_\Gamma$ by $K^\bullet(\Gamma) \to K^\bullet(W)$, we have $\nu^\star \eta|_W = 0$. Therefore, $\mathscr{I}_{\mathscr{F}'/Y'}(\nu^\star \eta)|_V \simeq \mathscr{O}_V$.

(2): The surjection Φ' defines a morphism

$$\varphi \colon Y' \to \boldsymbol{P}_Y\big(\operatorname{Sym}^{i_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E}))}(\mathscr{G})\big)$$

over Y so that $\mathscr{I}_{\mathscr{F}'/Y'}(\boldsymbol{c}^{d+1}(\mathscr{E})) \simeq \varphi^* \mathscr{O}(1)$ for the tautological invertible sheaf $\mathscr{O}(1)$. Then $\varphi(Y') \to Y$ is a finite morphism by (1). By Remark 3.10, we may assume that $\mathscr{F}$ and X are flat over an open subset $U \subset Y$ with $\operatorname{codim}(Y \setminus U) \geq 2$. Then, by Proposition 2.32, $\mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E}))|_U$ is invertible and there is a surjection

$$\Phi_U \colon \operatorname{Sym}^{i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))}(\mathscr{G})|_U \to \mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E}))|_U.$$

Here, $h_U^*(\Phi_U)$ and $\Phi'|_{U'}$ are isomorphic to each other by Lemma 2.34 and Remark 2.36, where $U' = h^{-1}(U)$ and $h_U = h|_{U'} \colon U' \to U$. Thus, $\varphi(Y') \to Y$ is an isomorphism over U. Since Y is normal and $\varphi(Y')$ is integral, we have $\varphi(Y') \simeq Y$. Hence, Φ' descends to a surjection

$$\Phi \colon \operatorname{Sym}^{i_{\mathscr{F}/Y}(\boldsymbol{c}^d(\mathscr{E}))}(\mathscr{G}) \to \mathscr{M}$$

to an invertible sheaf $\mathscr{M}$ with $\mathscr{M}|_U \simeq \mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E}))|_U$. Thus, $\mathscr{M} \simeq \mathscr{I}_{\mathscr{F}/Y}(\boldsymbol{c}^{d+1}(\mathscr{E}))$, since both sides are reflexive sheaves on Y.

(3): First assume that π satisfies Assumption 2.1. Then, as in the proof of Proposition 2.25 and that of Theorem 3.14, (2), we may assume that $\eta = \boldsymbol{c}^1(\mathscr{L}_1) \cdots \boldsymbol{c}^1(\mathscr{L}_{d+1})$ for π-ample invertible sheaves $\mathscr{L}_i$ such that $\pi^*\pi_*\mathscr{L}_i \to \mathscr{L}_i$ is surjective and $\mathrm{R}^p \pi_* \mathscr{L}_i = 0$ for any $p > 0$. If π is flat, then $\pi_*\mathscr{L}_i$ are locally free. If not, then Y admits an ample invertible sheaf, hence there exist surjections $\mathscr{G}_i \to \pi_*\mathscr{L}_i$ from locally free sheaves $\mathscr{G}_i$ of finite rank. Thus, $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ is invertible and we have the base change isomorphism $\mathscr{I}_{\mathscr{F}'/Y'}(\nu^\star \eta) \simeq h^* \mathscr{I}_{\mathscr{F}/Y}(\eta)$ by (2). Since Assumption 2.1 is satisfied locally on Y, $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ is always invertible.

Second, assume that $\mathscr{O}_Y \simeq h_*\mathscr{O}_{Y'}$. By the argument above, we infer that $\mathscr{I}_{\mathscr{F}'/Y'}(\nu^\star \eta)$ is the pullback of an invertible sheaf on Y at least locally on Y.

Thus, $\mathscr{I}_{\mathscr{F}'/Y'}(\nu^\star\eta) \simeq h^*\mathscr{M}$ for an invertible sheaf $\mathscr{M}$ on Y by the assumption: $\mathscr{O}_Y \simeq h_*\mathscr{O}_{Y'}$. Let $U \subset X$ be the open subset in the proof of (2) and let $h_U : U' = h^{-1}(U) \to U$ be the restriction of h. Then,

$$h_U^*(\mathscr{I}_{\mathscr{F}/Y}(\eta)|_U) \simeq \mathscr{I}_{\mathscr{F}'/Y'}(\nu^\star\eta)|_{U'}$$

by Lemma 2.27. Thus, we have an isomorphism $\mathscr{M}|_U \simeq \mathscr{I}_{\mathscr{F}/Y}(\eta)|_U$ by taking h_{U*}. Hence, $\mathscr{M} \simeq \mathscr{I}_{\mathscr{F}/Y}(\eta)$, since both sides are invertible sheaves and $\operatorname{codim}(Y \setminus U) \geq 2$.

(4): Considering the Stein factorization of h, we have a finite surjective morphism $\vartheta_1 : Y_1 \to Y$ and a proper surjective morphism $h_1 : Y' \to Y_1$ such that $h = \vartheta_1 \circ h_1$ and $\vartheta_{1*}\mathscr{O}_{Y_1} \simeq h_*\mathscr{O}_{Y'}$. Then, there is an invertible sheaf $\mathscr{M}_1$ on Y_1 such that $\mathscr{I}_{\mathscr{F}'/Y'}(\nu^\star\eta) \simeq h_1^*\mathscr{M}_1$, since $\mathscr{I}_{\mathscr{F}'/Y'}(\nu^\star\eta)$ is isomorphic to the pullback of an invertible sheaf on Y locally on Y. Let $U \subset Y$ be the open subset in the proof of (2). Then, $\mathscr{M}_1|_{\vartheta_1^{-1}(U)}$ is isomorphic to the pullback of $\mathscr{I}_{\mathscr{F}/Y}(\eta)|_U$ by the proof of (3).

The double-dual of $\vartheta_{1*}\mathscr{O}_{Y_1} = h_*\mathscr{O}_Y$ is a coherent $\mathscr{O}_X$-module having an $\mathscr{O}_Y$-algebra structure. Thus, we have a finite morphism $\nu_1 : Y_1^\sharp \to Y_1$ such that $\nu_{1*}\mathscr{O}_{Y_1^\sharp}$ is isomorphic to the double-dual of $\vartheta_{1*}\mathscr{O}_{Y_1}$. Since Y is normal, so is $Y_1^\sharp$. Consequently, $Y_1^\sharp$ is the normalization of Y_1. Then, there is an isomorphism

$$\nu_1^*\mathscr{M}_1 \simeq \nu_1^*\vartheta_1^*\,\mathscr{I}_{\mathscr{F}/Y}(\eta), \tag{3.8}$$

since both sides are invertible sheaves and we have such an isomorphism over the open subset $\nu_1^{-1}\vartheta_1^{-1}(U)$ whose complement has codimension at least two. Let $Y^\sharp$ be an integral closed subscheme of $Y' \times_{Y_1} Y_1^\sharp$ which dominates Y'. Let $\vartheta : Y^\sharp \to Y'$ be the induced finite surjective morphism, which is a birational morphism, since so is ν_1. Then, (3.8) induces an expected isomorphism

$$\vartheta^*\,\mathscr{I}_{\mathscr{F}'/Y'}(\nu^\star\eta) \simeq \vartheta^* h_1^*\mathscr{M}_1 \simeq \vartheta^* h^*\,\mathscr{I}_{\mathscr{F}/Y}(\eta).$$

Thus, we are done. □

Remark 3.23. By Proposition 3.22, (3), we have another proof of Theorem 3.14, (2), as follows. We may assume that π is projective and that Y is affine by localizing Y. As a flattening (cf. [**38**], [**39**]) of $\mathscr{F}$ over Y, we have a projective birational morphism $h : Y' \to Y$ from an integral scheme Y' such that $\mathscr{F}' = (q_1^*\mathscr{F})_{\mathrm{t.f.}/Y'}$ is flat over Y' for the first projection $q_1 : X \times_Y Y' \to X$. Then, Y and Y' have ample invertible sheaves. Thus, π and the second projection $q_2 : X \times_Y$

$Y' \to Y'$ both satisfy Assumption 2.1. Hence, we can apply Proposition 3.22, (3). In this way, we have a proof of Theorem 3.14, (2).

REMARK 3.24. In the situation of Proposition 3.22, we have assumed that π is projective. However, in order to prove Proposition 3.22, (3)–(4), we do not need the projectivity assumption on π but the local projectivity. In fact, the same arguments in the proofs work if we replace the intersection sheaf $\mathscr{I}_{*/*}(*)$ with $\mathscr{I}^{\mathrm{perf}}_{*/*}(*)$ (cf. Definition 2.26). For example, Proposition 3.22, (3) is proved as follows when π is only locally projective and $\mathscr{O}_Y \simeq h_*\mathscr{O}_{Y'}$: Let $x \in F^{d+1}(X)$ be a representative of $\eta \in G^{d+1}(X)$. Let $U \subset X$ be the open subset in the proof of Proposition 3.22, (2). Then, we can consider the intersection sheaf $\mathscr{I}^{\mathrm{perf}}_{\mathscr{F}_U/U}(x_U)$ for $\mathscr{F}_U = \mathscr{F}|_{\pi^{-1}(U)}$ and $x_U = x|_{\pi^{-1}(U)}$, since $\mathscr{F}_U$ and $\pi^{-1}(U)$ are flat over U. By Remark 3.6, we have an isomorphism

$$\mathscr{I}^{\mathrm{perf}}_{\mathscr{F}_U/U}(x_U) \simeq \mathscr{I}_{\mathscr{F}/Y}(\eta)|_U,$$

where $\mathscr{I}_{\mathscr{F}/Y}(\eta)$ is the intersection sheaf defined in Definition 3.5. By Lemma 2.27 and Proposition 2.25, we have also an isomorphism

$$h_U^*\big(\mathscr{I}^{\mathrm{perf}}_{\mathscr{F}_U/U}(x_U)\big) \simeq \mathscr{I}_{\mathscr{F}'/Y'}(q_1^\star\eta)|_{U'}$$

for $U' = h^{-1}(U)$ and $h_U = h|_{U'} : U' \to U$. Therefore, $\mathscr{M}|_U \simeq \mathscr{I}_{\mathscr{F}/Y}(\eta)|_U$ for the invertible sheaf $\mathscr{M}$ in the proof of Proposition 3.22, (3), and the rest is done by the same argument.

The base change properties in Lemma 3.12 and Proposition 3.22 are generalized to:

THEOREM 3.25. *Let $\pi : X \to Y$ be a projective surjective morphism to a normal separated Noetherian integral scheme Y, and $\mathscr{F}$ a coherent sheaf on X with* $\dim(\operatorname{Supp}\mathscr{F} \cap \pi^{-1}(\boldsymbol{y})) \leq d$ *for any point $\boldsymbol{y} \in Y$. Let $h : Y' \to Y$ be a dominant morphism of finite type from another normal separated Noetherian integral scheme Y'. Let $q_1 : X' \to X$ be the first projection from the fiber product $X' = X \times_Y Y'$ and let $\mathscr{F}'$ be the sheaf $(q_1^*\mathscr{F})_{\mathrm{t.f.}/Y'}$. Then, for any $\eta \in G^{d+1}(X)$, one has an isomorphism*

$$\mathscr{I}_{\mathscr{F}'/Y'}(q_1^\star\eta) \simeq h^*\,\mathscr{I}_{\mathscr{F}/Y}(\eta).$$

PROOF. We may assume that $\mathscr{F}$ and X are flat over an open subset $U \subset Y$ with $\operatorname{codim}(Y \setminus U) \geq 2$, by Remark 3.10. Let $X'^{\flat} \subset X'$ be the closed subscheme defined by $\mathscr{O}_{X'^\flat} = (\mathscr{O}_{X'})_{\mathrm{t.f.}/Y'}$. We may replace Y' with an open subset whose complement has codimension greater than one, since both sides of the isomorphism in question are invertible sheaves. Hence, we may assume that $\mathscr{F}'$ and $X'^\flat$ are flat over Y'. By Remark 3.10, note that the intersection sheaf $\mathscr{I}_{\mathscr{F}'/Y'}(\eta')$ for $\eta' \in G^{d+1}(X')$ in the sense of Definition 3.5 is just the intersection sheaf $\mathscr{I}_{\mathscr{F}'/Y'}(\eta'|_{X'^\flat})$ in the sense of Definition 2.26 associated with the $\mathscr{O}_{X'^\flat}$-module $\mathscr{F}'$ flat over Y'. By Nagata's completion theorem [**34**], [**35**] (cf. [**29**], [**5**]), Y' is realized as an open subset of an integral scheme $\overline{Y'}$ proper over Y. By taking a flattening (cf. [**38**] and [**39**]), we have a proper birational morphism $\varphi\colon Y'' \to \overline{Y'}$ satisfying the following conditions:

- $\varphi^{-1}(Y') \to Y'$ is an isomorphism.
- $\mathscr{F}'' := (p_1^*\mathscr{F})_{\mathrm{t.f.}/Y''}$ is flat over Y'' for the first projection $p_1\colon X \times_Y Y'' \to X$.
- The restriction of $\mathscr{F}''$ to $X \times_Y \varphi^{-1}(Y')$ is isomorphic to the pullback of $\mathscr{F}'$ by the isomorphism $X \times_Y \varphi^{-1}(Y') \simeq X \times_Y Y' = X'$.
- The closed subscheme $X''^\flat$ of $X \times_Y Y''$ defined by $\mathscr{O}_{X''^\flat} = (\mathscr{O}_{X \times_Y Y''})_{\mathrm{t.f.}/Y''}$ is flat over Y''.
- $X''^\flat \cap (X \times_Y \varphi^{-1}(Y'))$ is isomorphic to $X'^\flat$ by the isomorphism $X \times_Y \varphi^{-1}(Y') \simeq X \times_Y Y' = X'$.

We can apply Proposition 3.22 to $h''\colon Y'' \xrightarrow{\varphi} \overline{Y'} \to Y$, $\mathscr{F}''$, and $X''^\flat \to Y''$, since $\mathscr{F}''$ is an $\mathscr{O}_{X''^\flat}$-module. As a consequence, by Proposition 3.22, (4), we have a finite birational morphism $\vartheta\colon Y^\sharp \to Y''$ from an integral scheme $Y^\sharp$ such that

$$\vartheta^* \mathscr{I}_{\mathscr{F}''/Y''}(p_1^\star\eta|_{X''^\flat}) \simeq \vartheta^* h''^* \mathscr{I}_{\mathscr{F}/Y}(\eta).$$

Since $\varphi^{-1}(Y') \simeq Y'$ is normal, ϑ is an isomorphism over $\varphi^{-1}(Y')$. Thus, restricting the isomorphism to $\varphi^{-1}(Y') \simeq Y'$, we have the expected isomorphism $\mathscr{I}_{\mathscr{F}'/Y'}(q_1^\star\eta) \simeq h^* \mathscr{I}_{\mathscr{F}/Y}(\eta)$. $\square$

The following gives a base change property by morphisms from normal separated Noetherian schemes which are not necessarily dominant.

Proposition 3.26. *Let $\pi\colon X \to Y$ be a projective surjective morphism to a normal separated Noetherian integral scheme Y, and $\mathscr{F}$ a coherent sheaf on X with* $\dim(\operatorname{Supp}\mathscr{F} \cap \pi^{-1}(\boldsymbol{y})) \leq d$ *for any point $\boldsymbol{y} \in Y$. Let $\nu\colon B \to Y$ be a morphism from a normal separated Noetherian integral scheme B. Let $\mu\colon W \to X$ and $\varpi\colon W \to B$ be the first and second projections from the fiber product $W = X \times_Y B$. Then there exist a coherent sheaf $\widehat{\mathscr{F}}$ on W and a positive integer e such that* $\operatorname{Supp}\widehat{\mathscr{F}} \subset \mu^{-1}(\operatorname{Supp}\mathscr{F})$ *and*

$$\nu^* \mathscr{I}_{\mathscr{F}/Y}(\eta)^{\otimes e} \simeq \mathscr{I}_{\widehat{\mathscr{F}}/B}(\mu^\star \eta)$$

for any $\eta \in G^{d+1}(X)$*. Moreover, if* $\dim(\operatorname{Supp}\mathscr{F} \cap \pi^{-1}(\boldsymbol{y})) = d$ *for any point* $\boldsymbol{y} \in Y$*, then one can find the* $\widehat{\mathscr{F}}$ *satisfying also* $\dim(\operatorname{Supp}\widehat{\mathscr{F}} \cap \varpi^{-1}(\boldsymbol{b})) = d$ *for any* $\boldsymbol{b} \in B$*.*

Remark. If $\mathscr{F}$ is flat over Y and π satisfies Assumption 2.1, then one can take $e = 1$ and $\widehat{\mathscr{F}} = \mu^*\mathscr{F}$, by Lemma 2.27.

Proof of Proposition 3.26. We may assume that $X = \operatorname{Supp}\mathscr{F}$, $\dim(\operatorname{Supp}\mathscr{F})/Y = d$, and $\mathscr{F}_{\mathrm{tor}/Y} = (\mathscr{O}_X)_{\mathrm{tor}/Y} = 0$ by Remark 3.10. Let $h\colon Y' \to Y$ be a projective birational morphism from an integral scheme Y' which gives a flattening of both $\mathscr{F}/Y$ and X/Y. Then, $\mathscr{F}' = (p_1^*\mathscr{F})_{\mathrm{t.f.}/Y'}$ for the first projection $p_1\colon X \times_Y Y' \to X'$ and the closed subscheme $X' \subset X \times_Y Y'$ defined by $\mathscr{O}_{X'} = (\mathscr{O}_{X\times_Y Y'})_{\mathrm{t.f.}/Y'}$ are both flat over Y'. Note that $\mathscr{O}_Y \simeq h_*\mathscr{O}_{Y'}$, since Y is normal and h is a proper birational morphism from the integral scheme Y'. Indeed, h is an isomorphism outside a closed subset of Y of codimension at least two, thus $(h_*\mathscr{O}_{Y'})^{\vee\vee} \simeq \mathscr{O}_Y$ by [**23**, Proposition 1.6], which implies that $\mathscr{O}_Y \to h_*\mathscr{O}_{Y'}$ is isomorphic. We set $\pi'\colon X' \to Y'$ to be the flat morphism induced from the second projection $p_2\colon X \times_Y Y' \to Y'$. Then, by Proposition 3.22, (3), we have an isomorphism

$$\mathscr{I}_{\mathscr{F}'/Y'}(p_1^\star \eta|_{X'}) \simeq h^* \mathscr{I}_{\mathscr{F}/Y}(\eta). \tag{3.9}$$

We can find a closed integral subscheme $B' \subset Y' \times_Y B$ such that the second projection induces a surjective and generically finite morphism $\tau\colon B' \to B$. Indeed, it is enough to take $B' = \overline{\{\boldsymbol{b}'\}}$ for a closed point $\boldsymbol{b}'$ of the generic fiber of $Y' \times_Y B \to B$. Note that τ is a finite morphism over $B \setminus \Sigma$ for a closed subset Σ with $\operatorname{codim}\Sigma \geq 2$. In fact, this is true if τ is birational. In the non-birational case, let $B' \to B'' \to B$ be the Stein factorization of τ and let $\widetilde{B}$ be the normalization of B''. Then, $\widetilde{B} \to B$ is a finite morphism, since $\mathscr{O}_{\widetilde{B}}$ is the double-dual of $\mathscr{O}_{B''}$ as a coherent $\mathscr{O}_B$-module. Since $B' \times_{B''} \widetilde{B} \to \widetilde{B}$ is birational, this is an isomorphism outside a closed subset $\widetilde{\Sigma} \subset \widetilde{B}$ with $\operatorname{codim}\widetilde{\Sigma} \geq 2$. Hence, it suffices to set $\Sigma \subset B$ to be the image of $\widetilde{\Sigma}$. As a consequence, we have $\tau_\star F^2_{\mathrm{con}}(B') \subset F^2_{\mathrm{con}}(B)$ for the homomorphism $\tau_\star\colon K_\bullet(B') \to K_\bullet(B)$. Let e be the rank of $\tau_*\mathscr{O}_{B'}$, i.e., the degree of $B' \to B$. Since $\mathrm{cl}_\bullet(\mathrm{R}^i\,\tau_\star(\mathscr{O}_{B'})) \in F^2_{\mathrm{con}}(B)$ for $i > 0$, we have

$$\tau_\star(\mathrm{cl}_\bullet(\mathscr{O}_{B'})) \equiv \mathrm{cl}_\bullet(\tau_*\mathscr{O}_{B'}) \equiv e\,\mathrm{cl}_\bullet(\mathscr{O}_B) \mod F^1_{\mathrm{con}}(B) \tag{3.10}$$

by Lemma 1.14.

Let $\nu' \colon B' \to Y'$ be the restriction of the first projection $Y' \times_Y B \to Y'$ and let W' be the fiber product $X' \times_{Y'} B'$. Let $\rho \colon W' \to W$, $\varpi' \colon W' \to B'$, and $\mu' \colon W' \to X'$ be the induced morphisms. Then, we have commutative diagrams

$$\begin{array}{ccc} W & \xrightarrow{\mu} & X \\ \downarrow{\scriptstyle \varpi} & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{\nu} & Y, \end{array} \qquad \begin{array}{ccc} W' & \xrightarrow{\mu'} & X' \\ \downarrow{\scriptstyle \varpi'} & & \downarrow{\scriptstyle \pi'} \\ B' & \xrightarrow{\nu'} & Y', \end{array} \qquad \begin{array}{ccc} W' & \xrightarrow{\rho} & W \\ \downarrow{\scriptstyle \varpi'} & & \downarrow{\scriptstyle \varpi} \\ B' & \xrightarrow{\tau} & B \end{array} \tag{3.11}$$

in which the first two are Cartesian, and in the last diagram, the induced morphism $W' \to W \times_B B'$ is a closed immersion. We set $\mathscr{F}'_{W'} := \mu'^{*}\mathscr{F}'$. Then

$$\mathscr{I}_{\mathscr{F}'_{W'}/B'}(\mu'^{\star}(p_1^{\star}\eta|_{X'})) \simeq \nu'^{*}\, \mathscr{I}_{\mathscr{F}'/Y'}\, (p_1^{\star}\eta|_{X'})$$

by Lemma 2.27, since $\mathscr{F}'$ and X' are flat over Y'. Combining with (3.9), we have

$$\mathscr{I}_{\mathscr{F}'_{W'}/B'}(\mu'^{\star}(p_1^{\star}\eta|_{X'})) \simeq \nu'^{*}\, \mathscr{I}_{\mathscr{F}'/Y'}(p_1^{\star}\eta|_{X'}) \simeq \tau^{*}\nu^{*}\, \mathscr{I}_{\mathscr{F}/Y}(\eta). \tag{3.12}$$

On the other hand, for a representative $x \in F^{d+1}(X)$ of $\eta \in G^{d+1}(X)$, we have

$$\boldsymbol{\delta}\left(\mathscr{I}_{\mathscr{F}'_{W'}/B'}(\mu'^{\star}(p_1^{\star}\eta|_{X'}))\right) \equiv (\varpi')_{\star}^{\mathscr{F}'_{W'}}(\mu'^{\star}(p_1^{\star}x|_{X'})) \mod F^2(B'), \tag{3.13}$$

by Definition 2.26. For the right hand side of (3.13), we have

$$\tau_{\star}\, \boldsymbol{\phi}\left((\varpi')_{\star}^{\mathscr{F}'_{W'}}(\mu'^{\star}(p_1^{\star}x|_{X'}))\right) = \tau_{\star}\varpi'_{\star}\left((\mu'^{\star}p_1^{\star}x) \cdot \mathrm{cl}_{\bullet}(\mathscr{F}'_{W'})\right) \tag{3.14}$$

for the Cartan homomorphism $\boldsymbol{\phi} \colon K^{\bullet}(B') \to K_{\bullet}(B')$ and the push-forward homomorphism $\tau_{\star} \colon K_{\bullet}(B') \to K_{\bullet}(B)$. Since $W' \to W \times_B B'$ is a closed immersion and $\tau \colon B' \to B$ is a finite morphism over $B \setminus \Sigma$, we have $\mathscr{F}'_{W'} \in \mathrm{Coh}^{(d)}_{\tau\circ\varpi'}(W')$ and $\varpi(\mathrm{Supp}\, \mathrm{R}^i\, \rho_{*}\mathscr{F}'_{W'}) \neq B$ for any $i > 0$. Thus, by Lemma 3.11,

$$\begin{aligned} \widehat{\det}\left(\tau_{\star}\varpi'_{\star}\left((\mu'^{\star}p_1^{\star}x) \cdot \mathrm{cl}_{\bullet}(\mathscr{F}'_{W'})\right)\right) &= \mathscr{I}_{\mathscr{F}'_{W'}/B}(\mu'^{\star}p_1^{\star}\eta) \\ &\simeq \mathscr{I}_{\mathscr{F}'_{W'}/B}(\rho^{\star}\mu^{\star}\eta) \simeq \mathscr{I}_{\widehat{\mathscr{F}}/B}(\mu^{\star}\eta) \end{aligned} \tag{3.15}$$

for the direct image sheaf $\widehat{\mathscr{F}} := \rho_{*}\,(\mathscr{F}'_{W'})$ on W. Note that there is an inclusion $\mathrm{Supp}\, \widehat{\mathscr{F}} \subset \mu^{-1}(\mathrm{Supp}\, \mathscr{F})$ by construction. In fact, by a natural surjection

$\mu'^*(p_1^*\mathscr{F}|_{X'}) = \rho^*\mu^*\mathscr{F} \to \mathscr{F}'_{W'}$, we have $\operatorname{Supp}\mathscr{F}'_{W'} \subset \rho^{-1}\mu^{-1}(\operatorname{Supp}\mathscr{F})$, which induces the inclusion above. In particular, $\dim(\operatorname{Supp}\widehat{\mathscr{F}} \cap \varpi^{-1}(\boldsymbol{b})) \le d$ for any $\boldsymbol{b} \in B$. Therefore, $\mathscr{I}_{\widehat{\mathscr{F}}/B}(\eta)$ is an invertible sheaf by Theorem 3.14. As a consequence of (3.10) and (3.12)–(3.15), we have

$$\begin{aligned}\boldsymbol{\delta}\left(\mathscr{I}_{\widehat{\mathscr{F}}/B}(\mu^\star\eta)\right)\mathrm{cl}_\bullet(\mathscr{O}_B) &\equiv \boldsymbol{\delta}\left(\nu^*\mathscr{I}_{\mathscr{F}/Y}(\eta)\right)\cdot\tau_\star(\mathrm{cl}_\bullet(\mathscr{O}_{B'}))\\ &\equiv e\,\boldsymbol{\delta}\left(\nu^*\mathscr{I}_{\mathscr{F}/Y}(\eta)\right)\mathrm{cl}_\bullet(\mathscr{O}_B) \mod F^2_{\mathrm{con}}(B),\end{aligned}$$

which induces the expected isomorphism

$$\nu^*\mathscr{I}_{\mathscr{F}/Y}(\eta)^{\otimes e} \simeq \mathscr{I}_{\widehat{\mathscr{F}}/B}(\mu^\star\eta).$$

Suppose that $\dim(\operatorname{Supp}\mathscr{F} \cap \pi^{-1}(\boldsymbol{y})) = d$ for any point $\boldsymbol{y} \in Y$. Then, $\dim(\operatorname{Supp}\mathscr{F}'_{W'})/B' = d$. Since we have the inclusion $\operatorname{Supp}\widehat{\mathscr{F}} \subset \mu^{-1}(\operatorname{Supp}\mathscr{F})$, $\dim(\operatorname{Supp}\widehat{\mathscr{F}} \cap \varpi^{-1}(\boldsymbol{b})) = d$ holds for any point $\boldsymbol{b} \in B$, by the upper semi-continuity of dimensions of fibers. Thus, we are done. □

As a corollary of the proof of Proposition 3.26, we have:

Corollary 3.27. *In the situation of Proposition* 3.26, *suppose that* $X = \operatorname{Supp}\mathscr{F}$ *and that* X *and* $\mathscr{F}$ *are flat over the generic point of* $\nu(B)$. *Then, for any* $\eta \in G^{d+1}(X)$, *one has an isomorphism*

$$\nu^*\mathscr{I}_{\mathscr{F}/Y}(\eta) \simeq \mathscr{I}_{\mu^*\mathscr{F}/B}(\mu^\star\eta).$$

Proof. We follow the proof of Proposition 3.26 and use the same notions and the same symbols. We may assume that $\dim X/Y = d$ and $\mathscr{F}_{\mathrm{tor}/Y} = (\mathscr{O}_X)_{\mathrm{tor}/Y} = 0$ by Remark 3.10. Then the flattening $\nu\colon Y' \to Y$ is isomorphic over the generic point of $\nu(B)$, by assumption. Hence, the generically finite morphism $\tau\colon B' \to B$ is birational, i.e., $e = 1$. Moreover, $(\rho^*\mu^*\mathscr{F})_{\mathrm{t.f.}/Y} \simeq \mathscr{F}'_{W'}$ by construction. By applying Proposition 3.22, (4) to $W \to B$ and $B' \to B$, we have a finite birational morphism $\vartheta\colon B^\sharp \to B'$ such that

$$\vartheta^*\mathscr{I}_{\mathscr{F}'_{W'}/B'}(\rho^\star\mu^\star(\eta)) \simeq \vartheta^*\tau^*\mathscr{I}_{\mu^*\mathscr{F}/B}(\mu^\star\eta).$$

Thus, by (3.12),

$$\vartheta^*\tau^*\nu^*\mathscr{I}_{\mathscr{F}/Y}(\eta) \simeq \vartheta^*\tau^*\mathscr{I}_{\mu^*\mathscr{F}/B}(\mu^\star\eta).$$

Taking the direct image sheaves for the birational morphism $\tau \circ \vartheta \colon B^\sharp \to B$, we have the expected isomorphism

$$\nu^* \mathscr{I}_{\mathscr{F}/Y}(\eta) \simeq \mathscr{I}_{\mu^*\mathscr{F}/B}(\mu^\star \eta). \qquad \square$$

As an application of Corollary 3.27, we have:

Proposition 3.28. *Let $\pi \colon X \to Y$ and $\psi \colon Y \to S$ be surjective morphisms of Noetherian schemes, $\boldsymbol{s} \in S$ a closed point, and let d be a non-negative integer in which the following conditions are satisfied*:

(1) *π is a projective morphism.*
(2) *Y and the fiber $Y_{\boldsymbol{s}} := \psi^{-1}(\boldsymbol{s})$ are separated integral normal schemes.*
(3) *π is flat over the generic point of $Y_{\boldsymbol{s}}$.*
(4) *$\dim \pi^{-1}(\boldsymbol{y}) \leq d$ for any point $\boldsymbol{y} \in Y$.*

Then, for any $\eta \in G^{d+1}(X)$, one has an isomorphism

$$\mathscr{I}_{X/Y}(\eta) \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y_{\boldsymbol{s}}} \simeq \mathscr{I}_{X_{\boldsymbol{s}}/Y_{\boldsymbol{s}}}(\eta|_{Y_{\boldsymbol{s}}}).$$

Proof. This follows from Corollary 3.27 applied to $B = Y_{\boldsymbol{s}} \to Y$ and $\mathscr{F} = \mathscr{O}_X$. $\square$

We finish Section 3 by applying Proposition 3.26 to prove the following result, which is an analogue of [**41**, Théorème 2] on Kähler spaces:

Theorem 3.29. *Let $\pi \colon X \to Y$ and $\psi \colon Y \to S$ be proper morphisms of Noetherian schemes. Assume that Y is a normal separated Noetherian scheme and π is an equi-dimensional surjective morphism. If $\psi \circ \pi \colon X \to S$ is a projective morphism, then so is $\psi \colon Y \to S$.*

Proof. Let $\mathscr{A}$ be a relatively ample invertible on X with respect to $\psi \circ \pi$. We set $\eta = \boldsymbol{c}^1(\mathscr{A})^{d+1} \in G^{d+1}(X)$ for $d = \dim X/Y$. It is enough to prove that the invertible sheaf $\mathscr{I}_{X/Y}(\eta) = \mathscr{I}_{X/Y}(\mathscr{A}, \dots, \mathscr{A})$ is relatively ample with respect to ψ. For the purpose, we may replace S with its open subset. Thus, we may assume that S is affine. Also we may replace $\mathscr{A}$ by suitable power $\mathscr{A}^{\otimes k}$. Hence, we may assume that there is a surjection $\pi^*\psi^*\mathscr{G} \to \mathscr{A}$ for a free $\mathscr{O}_S$-module $\mathscr{G}$ of finite rank; in other words, $\mathscr{A}$ is generated by finitely many global sections. Then, $\mathscr{I}_{X/Y}(\eta)$ is generated by finitely many global sections by Corollary 3.21. Thus, there is a morphism $h \colon Y \to \boldsymbol{P}^N_S$ over S for some N such that $h^*\mathscr{O}(1) \simeq \mathscr{I}_{X/Y}(\eta)$. In order to show $\mathscr{I}_{X/Y}(\eta)$ to be ample, it is enough to prove that

$$\deg_{C/\boldsymbol{k}} \mathscr{I}_{X/Y}(\eta)|_C > 0$$

for any closed irreducible curve C on an arbitrary fiber $\psi^{-1}(\boldsymbol{s})$ of ψ, where $\boldsymbol{k} = \boldsymbol{k}(\boldsymbol{s})$. Let $\nu\colon B \to C$ be the normalization and $\mu\colon X \times_Y B \to X$ the induced finite morphism. Then there exist a coherent sheaf $\widehat{\mathscr{F}}$ on $X \times_Y B$ and a positive integer e such that $\dim \operatorname{Supp} \widehat{\mathscr{F}} = d + 1$ and

$$\nu^* \mathscr{I}_{X/Y}(\eta)^{\otimes e} \simeq \mathscr{I}_{\widehat{\mathscr{F}}/B}(\mu^\star \eta)$$

by Proposition 3.26. Since B is a non-singular curve, we may assume that $\widehat{\mathscr{F}}$ is flat over B by Remark 3.10. Hence,

$$\begin{aligned} e \deg_{C/\boldsymbol{k}} \mathscr{I}_{X/Y}(\eta)|_C &= \deg_{B/\boldsymbol{k}} \mathscr{I}_{\widehat{\mathscr{F}}/B}(\mu^\star \eta) = i_{X \times_Y B/\boldsymbol{k}}(\mu^\star \eta; \widehat{\mathscr{F}}) \\ &= i_{X \times_Y B/\boldsymbol{k}}(\mu^* \mathscr{A}, \dots, \mu^* \mathscr{A}; \widehat{\mathscr{F}}) > 0 \end{aligned}$$

by Lemma 2.28, since $\mu^* \mathscr{A}$ is ample. Thus, we are done. □

4. Intersection sheaves for varieties over a field.

In what follows, we shall work in the category of $\boldsymbol{k}$-schemes for a fixed field $\boldsymbol{k}$. A variety (over $\boldsymbol{k}$) is by definition an integral separated scheme of finite type over $\operatorname{Spec} \boldsymbol{k}$. We shall study the intersections sheaves for surjective morphisms $X \to Y$ of normal projective varieties. In Section 4.1, we study some numerical properties of $\mathscr{I}_{X/Y}$. In Section 4.2, for a family Z of effective algebraic cycles of pure dimension on X parametrized by Y (hence, Z is a cycle on $X \times Y$) and for an ample invertible sheaf $\mathscr{A}$ on X, we show that the intersection sheaf $\mathscr{I}_{Z/Y}(p_1^* \mathscr{A}, \dots, p_1^* \mathscr{A})$ is just the pullback of an ample invertible sheaf by the morphism to the Chow variety of X determined by Z/Y. An application to the study of endomorphisms of complex projective normal varieties is given in Section 4.3.

4.1. Numerical properties of intersection sheaves.

Let $\pi\colon X \to Y$ be a proper surjective equi-dimensional morphism from a projective variety X to a normal variety Y. Then Y is also projective by Theorem 3.29. We set $d = \dim X/Y$ and $m = \dim Y$. Then the intersection sheaf $\mathscr{I}_{X/Y}(\eta)$ for $\eta \in G^{d+1}(X)$ is also defined as $\pi_\star G(\boldsymbol{\phi})(\eta)$ modulo $F^2_{\mathrm{con}}(Y) = F_{m-2}(Y)$ for $G(\boldsymbol{\phi})\colon G^{d+1}(X) \to G^{d+1}_{\mathrm{con}}(X) = G_{m-1}(X)$ and $\pi_\star\colon G_{m-1}(X) \to G_{m-1}(Y)$.

Remark. In order to calculate $\mathscr{I}_{X/Y}(\eta)$, we may replace X with its normalization by Lemmas 3.9 and 3.11. In fact, the normalization $\nu\colon \widehat{X} \to X$ is a finite birational morphism. Hence, $\pi \circ \nu$ is also equi-dimensional, $\mathrm{R}^i \nu_* \mathscr{O}_{\widehat{X}} = 0$ for $i > 0$, and $l_X(\nu_* \mathscr{O}_{\widehat{X}}) = 1$. Thus,

$$\mathscr{I}_{\widehat{X}/Y}(\nu^\star \eta) \simeq \mathscr{I}_{(\nu_* \mathscr{O}_{\widehat{X}})/Y}(\eta) \simeq \mathscr{I}_{X/Y}(\eta)$$

by Lemmas 3.11 and 3.9.

In Lemma 4.1 and Theorem 4.2 below, we shall give sufficient conditions for an intersection sheaf $\mathscr{I}_{X/Y}(\eta)$ to be ample or nef.

Convention ("nef"). An invertible sheaf $\mathscr{M}$ on a projective variety Y is called *nef* if the intersection number

$$\mathscr{M} \cdot C := i_{Y/\boldsymbol{k}}(\boldsymbol{c}^1(\mathscr{M}); C) = \deg_{C/\boldsymbol{k}} \mathscr{M}|_C$$

is non-negative for any irreducible closed curve C on Y. Note that $\mathscr{M}$ is nef (resp. ample) if and only if its pullback to $\overline{Y} = Y \times_{\operatorname{Spec} \boldsymbol{k}} \operatorname{Spec} \overline{\boldsymbol{k}}$ is nef (resp. ample) for the algebraic closure $\overline{\boldsymbol{k}}$ of $\boldsymbol{k}$. The notion of nef is introduced by Reid (cf. [**40**, (0.12) (f)]), but formerly, it was called *numerically effective*, *arithmetically effective*, or *numerically semi-positive* (cf. [**25**, Chapter I, Section 4], [**11**, Section 2]). The following property is known by Nakai's criterion of ampleness (cf. [**36**], [**25**, Chapter III]): *An invertible sheaf $\mathscr{M}$ on a projective variety is nef if and only if $\mathscr{M}^{\otimes a} \otimes \mathscr{A}$ is ample for any ample invertible sheaf $\mathscr{A}$ and for any positive integer a.*

Lemma 4.1. *Let η be an element of $G^{d+1}(X)$ such that $i_{X/\boldsymbol{k}}(\eta; W) \geq 0$ for any closed irreducible subset $W \subset X$ of dimension $d+1$. Then $\mathscr{I}_{X/Y}(\eta)$ is nef.*

Proof. By assumption, $i_{X/\boldsymbol{k}}(\eta; \mathscr{G}) \geq 0$ for any coherent sheaf $\mathscr{G}$ with $\dim \operatorname{Supp} \mathscr{G} \leq d+1$, since $\mathrm{cl}_\bullet(\mathscr{G}) \equiv \sum m_i \, \mathrm{cl}_\bullet(W_i) \mod F_d(X)$ for some closed irreducible subsets W_i with integers $m_i \geq 0$. It is enough to prove that $\deg_{C/\boldsymbol{k}} \mathscr{I}_{X/Y}(\eta)|_C \geq 0$ for any closed irreducible curve C on Y. Let $B \to C$ be the normalization. Then, by Proposition 3.26, there exist a positive integer e and a coherent sheaf $\widehat{\mathscr{F}}$ of $X \times_Y B$ such that $\dim(\operatorname{Supp} \widehat{\mathscr{F}}) = d+1$ and

$$e \deg_{C/\boldsymbol{k}} \mathscr{I}_{X/Y}(\eta)|_C = \deg_{B/\boldsymbol{k}} \mathscr{I}_{\widehat{\mathscr{F}}/B}(\mu^\star \eta)$$

for the induced finite morphism $\mu \colon X \times_Y B \to X$. We may assume that $\widehat{\mathscr{F}}$ is flat over B by Remark 3.10. Thus the right hand side of the equality above is equal to

$$i_{X \times_Y B/\boldsymbol{k}}(\mu^\star \eta; \widehat{\mathscr{F}}) = i_{X/\boldsymbol{k}}(\eta; \mu_* \widehat{\mathscr{F}}) \geq 0$$

by Lemmas 2.28 and 1.12, since $\dim(\operatorname{Supp} \mu_* \widehat{\mathscr{F}}) \leq \dim(\operatorname{Supp} \widehat{\mathscr{F}}) = d+1$. Thus, we are done. □

Convention ("algebraic equivalence" and "numerical equivalence"). Let $\mathscr{M}$, $\mathscr{M}_1$, and $\mathscr{M}_2$ be invertible sheaves on a projective variety Y.

- $\mathscr{M}_1$ is called *algebraically equivalent to* $\mathscr{M}_2$ if there exist a connected algebraic scheme S, two $\boldsymbol{k}$-rational points $\boldsymbol{s}_1$, $\boldsymbol{s}_2$ of S, and an invertible sheaf $\widetilde{\mathscr{M}}$ on $X \times S$ such that

$$\mathscr{M}_1 \simeq \widetilde{\mathscr{M}}|_{X \times \{\boldsymbol{s}_1\}} \quad \text{and} \quad \mathscr{M}_2 \simeq \widetilde{\mathscr{M}}|_{X \times \{\boldsymbol{s}_2\}}.$$

  If $\mathscr{M}$ is algebraically equivalent to $\mathscr{O}_Y$, then $\mathscr{M}$ is called *algebraically equivalent to zero.*
- $\mathscr{M}$ is called *numerically trivial* if $\mathscr{M} \cdot C (= \deg \mathscr{M}|_C) = 0$ for any closed irreducible curve C on Y. If $\mathscr{M}_1 \otimes \mathscr{M}_2^{-1}$ is numerically trivial, then $\mathscr{M}_1$ is called *numerically equivalent to* $\mathscr{M}_2$.

Note that if $\mathscr{M}$ is algebraically equivalent to zero, then $\mathscr{M}$ is numerically trivial.

Theorem 4.2. *Let $\pi\colon X \to Y$ be an equi-dimensional proper surjective morphism of normal projective varieties defined over a field. Let θ be an element of $G^d(X)$ for $d = \dim X/Y$. For an invertible sheaf $\mathscr{L}$ of X, the intersection sheaf $\mathscr{M} := \mathscr{I}_{X/Y}(\theta \mathbf{c}^1(\mathscr{L}))$ has the following properties*:

(1) *If $\mathscr{L}$ is algebraically equivalent to zero, then so is $\mathscr{M}$.*
(2) *If $\mathscr{L}$ is numerically trivial, then so is $\mathscr{M}$.*

Assume that $i_{X/\boldsymbol{k}}(\theta; W) \geq 0$ for any closed subscheme $W \subset X$ with $\dim W = d$. Then the following hold:

(3) *If $\mathscr{L}$ is nef, then so is $\mathscr{M}$.*
(4) *If $\mathscr{L}$ is ample and if $i_{X/Y}(\theta) > 0$, then $\mathscr{M}$ is ample.*

Proof. (1): By assumption, we may assume that there exist a variety S with two $\boldsymbol{k}$-rational points $\boldsymbol{s}_0$, $\boldsymbol{s}_1 \in S$ and an invertible sheaf $\widetilde{\mathscr{L}}$ on $X \times S$ such that $\widetilde{\mathscr{L}}_{\boldsymbol{s}_0} \simeq \mathscr{O}_X$ and $\widetilde{\mathscr{L}}_{\boldsymbol{s}_1} \simeq \mathscr{L}$, where $\widetilde{\mathscr{L}}_{\boldsymbol{s}}$ for a $\boldsymbol{k}$-rational point $\boldsymbol{s} \in S$ denotes the invertible sheaf on X isomorphic to the restriction of $\widetilde{\mathscr{L}}$ to $X \times \{\boldsymbol{s}\} \simeq X$. We apply Proposition 3.28 to $\pi \times \mathrm{id}_S \colon X \times S \to Y \times S$ and the second projection $Y \times S \to S$. Since π is flat over $Y \setminus \Sigma$ for a proper closed subset Σ, $\pi \times \mathrm{id}_S$ is also flat over $(Y \setminus \Sigma) \times S$; thus, $\pi \times \mathrm{id}_S$ is flat over the generic point of $Y \times \{\boldsymbol{s}\}$ for any

$\boldsymbol{s} \in Y$. We set

$$\widetilde{\mathscr{M}} := \mathscr{I}_{(X\times S)/(Y\times S)}\left(p_1^{\star}(\theta)\boldsymbol{c}^1(\widetilde{\mathscr{L}})\right)$$

for the first projection $p_1\colon X \times S \to X$. Then, we have isomorphisms $\widetilde{\mathscr{M}}_{\boldsymbol{s}_0} \simeq \mathscr{O}_Y$ and $\widetilde{\mathscr{M}}_{\boldsymbol{s}_1} \simeq \mathscr{M}$ by Proposition 3.28 applied to $Y \times \{\boldsymbol{s}\} \to Y \times S$ for $\boldsymbol{s} = \boldsymbol{s}_0$ and $\boldsymbol{s}_1$. Therefore, $\mathscr{M}$ is algebraically equivalent to zero.

(2): $i_{X/\boldsymbol{k}}(\theta\boldsymbol{c}^1(\mathscr{L}); W') = 0$ for any closed subscheme $W' \subset X$ of dimension $d+1$, since $\mathscr{L}$ is numerically trivial and $\theta\,\mathrm{cl}_{\bullet}(W') \in F^d(X)F_{d+1}(X) \subset F_1(X)$. Hence, $\mathscr{M}$ and $\mathscr{M}^{\vee}$ are both nef by Lemma 4.1. Thus, $\mathscr{M}$ is numerically trivial.

(3): By Lemma 4.1, it suffices to show

$$i_{X/\boldsymbol{k}}(\theta\boldsymbol{c}^1(\mathscr{L}); W^{\sim}) \geq 0 \tag{4.1}$$

for any closed subvariety $W^{\sim} \subset X$ of dimension $d+1$. If $\mathscr{L}$ is ample, then there exist a finite field extension $\boldsymbol{k}' \supset \boldsymbol{k}$, an ample divisor A' on $X' = X \times_{\operatorname{Spec}\boldsymbol{k}} \operatorname{Spec}\boldsymbol{k}'$, and a positive integer k such that

- $\mathscr{O}_{X'}(A') \simeq q^*\mathscr{L}^{\otimes k}$, and
- $\dim q^{-1}(W^{\sim}) \cap A' = d$ (cf. Lemma 3.17),

for the induced morphism $q\colon X' \to X$. In this situation, for $e := [\boldsymbol{k}' : \boldsymbol{k}]$, we have

$$\begin{aligned} ki_{X/\boldsymbol{k}}(\theta\boldsymbol{c}^1(\mathscr{L}); W^{\sim}) &= i_{X'/\boldsymbol{k}'}\left(q^{\star}(\theta)\cdot\boldsymbol{c}^1(q^*\mathscr{L}^{\otimes k}); q^{-1}(W^{\sim})\right) \\ &= i_{X'/\boldsymbol{k}'}(q^{\star}(\theta); q^{-1}(W^{\sim}) \cap A') \\ &= (1/e)i_{X'/\boldsymbol{k}}(q^{\star}(\theta); q^{-1}(W^{\sim}) \cap A') \\ &= (1/e)i_{X/\boldsymbol{k}}(\theta; q_{\star}\,\mathrm{cl}_{\bullet}(q^{-1}(W^{\sim}) \cap A')) \geq 0 \end{aligned}$$

by Lemmas 1.12 and 1.13, since $q_{\star}\,\mathrm{cl}_{\bullet}(q^{-1}(W^{\sim}) \cap A')$ is expressed by an effective algebraic cycle of dimension d on X. Thus, (4.1) holds if $\mathscr{L}$ is ample. Even if $\mathscr{L}$ is only nef, $\mathscr{L}^{\otimes N} \otimes \mathscr{A}$ is ample for any ample invertible sheaf $\mathscr{A}$ of X and for any $N > 0$. Thus

$$0 \leq i_{X/\boldsymbol{k}}(\theta\boldsymbol{c}^1(\mathscr{L}^{\otimes N} \otimes A); W^{\sim}) = Ni_{X/\boldsymbol{k}}(\theta\boldsymbol{c}^1(\mathscr{L}); W^{\sim}) + i_{X/\boldsymbol{k}}(\theta\boldsymbol{c}^1(\mathscr{A}); W^{\sim})$$

for any $N > 0$. Hence, (4.1) holds for any nef invertible sheaf $\mathscr{L}$.

(4): Let $\mathscr{H}$ be an ample invertible sheaf on Y. Then $\mathscr{L}^{\otimes b} \otimes \pi^*\mathscr{H}^{-1}$ is ample for some $b > 0$. By Lemma 3.13, we have an isomorphism

$$\begin{aligned}\mathscr{I}_{X/Y}(\theta\boldsymbol{c}^1(\mathscr{L}^{\otimes b}\otimes\pi^*\mathscr{H}^{-1})) &\simeq \mathscr{M}^{\otimes b}\otimes\mathscr{I}_{X/Y}(\theta\boldsymbol{c}^1(\pi^*\mathscr{H}))^{-1}\\ &\simeq \mathscr{M}^{\otimes b}\otimes\mathscr{H}^{\otimes(-i_{X/Y}(\theta))},\end{aligned}\tag{4.2}$$

where the left hand side is nef by (3). Hence, $\mathscr{M}$ is ample. □

In Lemma 4.3 and Proposition 4.5 below, we shall give sufficient conditions for an intersection sheaf $\mathscr{I}_{X/Y}(\eta)$ to be *effective*, *big*, or *pseudo-effective*.

Convention ("effective", "big", and "pseudo-effective"). Assume that the base field $\boldsymbol{k}$ is algebraically closed. Let $\mathscr{M}$ be an invertible sheaf on a normal projective variety X.

- $\mathscr{M}$ is called *effective* if $\mathrm{H}^0(X,\mathscr{M})\neq 0$, or equivalently, $\mathscr{M}\simeq\mathscr{O}_X(D)$ for an effective Cartier divisor D.
- $\mathscr{M}$ is called *big* if $\mathscr{M}^{\otimes b}\otimes\mathscr{A}^{-1}$ is effective for some ample invertible sheaf $\mathscr{A}$ and a positive integer b.
- $\mathscr{M}$ is called *pseudo-effective* if $\mathscr{M}^{\otimes n}\otimes\mathscr{A}$ is big for any positive integer n and for any ample invertible sheaf $\mathscr{A}$.

From the definition, we have the following properties:

(1) If $\mathscr{M}$ is effective or big, then $\mathscr{M}$ is pseudo-effective.
(2) Ample invertible sheaves are big, and nef invertible sheaves are pseudo-effective.
(3) $\mathscr{M}$ is pseudo-effective if and only if, for any ample invertible sheaf $\mathscr{A}$ and for any positive integer n, there is a positive integer k such that $(\mathscr{M}^{\otimes n}\otimes\mathscr{A})^{\otimes k}$ is effective.

Lemma 4.3. *Let $B\subset Y$ be a closed subset of* $\operatorname{codim}(B)\geq 2$ *and Z an effective algebraic cycle on $X\setminus\pi^{-1}(B)$ of codimension d such that any irreducible component of Z dominates $Y\setminus B$. Let $\theta\in G^d(X)$ be an element such that*

$$G(\boldsymbol{\phi})(\theta|_{X\setminus\pi^{-1}(B)}) = \mathrm{cl}_\bullet(Z) \bmod F^{d+1}_{\mathrm{con}}(X\setminus\pi^{-1}(B)) \quad \in G^\bullet_{\mathrm{con}}(X\setminus\pi^{-1}(B)).$$

If D is an effective Cartier divisor on X which does not contain any irreducible component of Z, then the intersection sheaf $\mathscr{I}_{X/Y}(\theta\boldsymbol{c}^1(\mathscr{O}_X(D)))$ is effective.

Proof. Let $Z=\sum n_iZ_i$ be the irreducible decomposition. Since $\dim Z_i = \dim Y$, the restriction morphism $Z_i\to Y$ of π is generically finite and dominant. By replacing B with a closed subset $B'\subset X$ with $B'\supset B$ and $\operatorname{codim}B'\geq 2$, we may assume from the beginning that $Z_i\to Y\setminus B$ is a finite surjective morphism for any i. Then

$$\begin{aligned}\mathscr{I}_{X/Y}(\theta \boldsymbol{c}^1(\mathscr{O}_X(D)))|_{Y\setminus B} &\simeq \mathscr{I}_{Z/(Y\setminus B)}(\mathscr{O}_X(D)) \\ &\simeq \bigotimes \mathscr{I}_{Z_i/(Y\setminus B)}(\mathscr{O}_{Z_i}(D|_{Z_i}))^{\otimes n_i}\end{aligned}$$

by Lemma 3.9, since the appearing intersection sheaves are all invertible by Theorem 3.14.

If $\mathscr{M}$ is an invertible sheaf on Y, then $\mathrm{H}^0(Y, \mathscr{M}) \simeq \mathrm{H}^0(Y \setminus B', \mathscr{M})$ for a closed subset B' with $\mathrm{codim}(B') \geq 2$. Thus, by replacing Y with $Y \setminus B'$ for a closed subset $B' \supset B$ with $\mathrm{codim}(B') \geq 2$, and by replacing X with Z, we are reduced to proving the existence of a non-zero global section of $\mathscr{M} = \mathscr{I}_{X/Y}(\mathscr{O}_X(D))$ for a finite surjective morphism $\pi\colon X \to Y$ of not necessarily projective varieties, where Y is normal, and for an effective Cartier divisor D on X. We may also assume that X is normal by Lemmas 3.9 and 3.11 as above. Therefore, the assertion follows from the property that the push-forward $\pi_* D$ is effective and from the isomorphism $\mathscr{O}_Y(\pi_* D) \simeq \mathscr{I}_{X/Y}(\mathscr{O}_X(D))$ (cf. Example 3.7). □

Definition 4.4 (cf. [**42**]). Let $\mathscr{N}$ be an invertible sheaf on a normal projective variety X defined over an algebraically closed field, and $W \subset X$ a closed subset. If the following condition is satisfied, then $\mathscr{N}$ is called *weakly positive outside* W:

- For an ample invertible sheaf $\mathscr{A}$ on X, an arbitrary point $\boldsymbol{x} \in X \setminus W$, and for any positive rational number ε, there exist a positive integer m with $m\varepsilon \in \boldsymbol{Z}$ and an effective divisor D such that $\mathscr{O}_X(D) \simeq \mathscr{N}^{\otimes m} \otimes \mathscr{A}^{\otimes m\varepsilon}$ and $\boldsymbol{x} \notin \mathrm{Supp}\, D$.

Proposition 4.5. *Let $\pi\colon X \to Y$ be an equi-dimensional proper surjective morphism of normal projective varieties defined over an algebraically closed field with $d = \dim X/Y$. Let $\mathscr{N}_1, \ldots, \mathscr{N}_d$ be invertible sheaves on X which are weakly positive outside $\pi^{-1}(B)$ for a closed subset $B \subset Y$ of $\mathrm{codim}(B) \geq 2$. For an invertible sheaf $\mathscr{L}$ of X, the intersection sheaf $\mathscr{M} := \mathscr{I}_{X/Y}(\mathscr{N}_1, \ldots, \mathscr{N}_d, \mathscr{L})$ has the following properties*:

(1) *If $\mathscr{L}$ is pseudo-effective, then so is $\mathscr{M}$.*
(2) *If $\mathscr{L}$ is big and if $i_{X/Y}(\mathscr{N}_1, \ldots, \mathscr{N}_d) > 0$, then $\mathscr{M}$ is big.*

Proof.
(1): Let $\mathscr{A}$ be an ample invertible sheaf on X and ε a positive rational number. Then there is an effective divisor Δ such that $\mathscr{O}_X(\Delta) \simeq \mathscr{L}^{\otimes l} \otimes \mathscr{A}^{\otimes l\varepsilon}$ for some $l > 0$ with $l\varepsilon \in \boldsymbol{Z}$. By the weak positivity, there exist also positive integers $m_1, \ldots, m_d$ and effective divisors $D_1, \ldots, D_d$ such that

- $m_i\varepsilon \in \boldsymbol{Z}$ and $\mathscr{O}_X(D_i) \simeq \mathscr{N}_i^{\otimes m_i} \otimes \mathscr{A}^{\otimes m_i\varepsilon}$ for any $1 \leq i \leq d$,

- $\operatorname{codim}(V \cap \Delta \cap \pi^{-1}(Y \setminus B)) = d+1$ for the intersection $V = D_1 \cap \cdots \cap D_d$, and
- every irreducible component of V dominates Y.

Hence, $\mathscr{I}_{X/Y}(\mathscr{O}_X(D_1), \dots, \mathscr{O}_X(D_d), \mathscr{O}_X(\Delta))$ has a non-zero global section by Lemma 4.3. We set $x_i = \boldsymbol{c}^1(\mathscr{N}_i)$ for $1 \le i \le d$, $x_{d+1} = \boldsymbol{c}^1(\mathscr{L})$, and $a = \boldsymbol{c}^1(\mathscr{A})$. Then

$$\mathscr{I}_{X/Y}(\mathscr{O}_X(D_1), \dots, \mathscr{O}_X(D_d), \mathscr{O}_X(\Delta)) \simeq \mathscr{I}_{X/Y}(m\eta)$$

for $m = lm_1m_2 \cdots m_d$ and for

$$\eta = \prod_{i=1}^{d+1}(x_i + \varepsilon a) = \sum_{i=1}^{d+1} e_{d+1-j}(x_1, \dots, x_{d+1})\varepsilon^j a^j \in G^{d+1}(X) \otimes \boldsymbol{Q},$$

where $e_j(x_1, \dots, x_{d+1}) \in \boldsymbol{Z}[x_1, \dots, x_{d+1}]$ is the elementary symmetric polynomial of degree j (cf. Remark 2.40). Thus,

$$\mathscr{I}_{X/Y}(m\eta) \simeq \bigotimes_{j=0}^{d+1} \mathscr{M}_j^{\otimes(m\varepsilon^j)} \tag{4.3}$$

for the invertible sheaves

$$\mathscr{M}_j := \mathscr{I}_{X/Y}(e_{d+1-j}(x_1, \dots, x_{d+1})a^j).$$

Note that $\mathscr{M}_0 = \mathscr{I}_{X/Y}(\mathscr{N}_1, \dots, \mathscr{N}_d, \mathscr{L}) = \mathscr{M}$. For an ample invertible sheaf $\mathscr{H}$ on Y and for a positive integer b, we can take the positive rational number ε so that, for any $0 \le j \le d$,

$$\mathscr{H}_j^{(k)} := \mathscr{H}^{\otimes k} \otimes \mathscr{M}_j^{\otimes(-bk(d+1)\varepsilon^j)}$$

is ample for a positive integer k with $k\varepsilon^j \in \boldsymbol{Z}$. Indeed, $\mathscr{H}^{\otimes N} \otimes \mathscr{M}_j^{-1}$ is ample for $N \gg 0$ and it is possible to take $b(d+1)\varepsilon^j < 1/N$. We can take k so that $\mathscr{H}_j^{(k)}$ above are all effective. By (4.3), we infer that

$$(\mathscr{M}^{\otimes b} \otimes \mathscr{H})^{\otimes mk(d+1)} \simeq \mathscr{I}_{X/Y}(m\eta)^{\otimes bk(d+1)} \otimes \bigotimes_{j=0}^{d} \mathscr{H}^{\otimes km} \otimes \mathscr{M}_j^{\otimes(-bkm(d+1)\varepsilon^j)}$$

$$\simeq \mathscr{I}_{X/Y}(m\eta)^{\otimes bk(d+1)} \otimes \bigotimes_{j=0}^{d} \bigl(\mathscr{H}_j^{(k)}\bigr)^{\otimes m}$$

is effective, since so is $\mathscr{I}_{X/Y}(m\eta)$. We can take b to be an arbitrary positive integer. Therefore, $\mathscr{M}$ is pseudo-effective.

(2): $\mathscr{L}^{\otimes b} \otimes \pi^*\mathscr{H}^{-1}$ is effective for an ample invertible sheaf $\mathscr{H}$ on Y and a positive integer b. By (1) above and by (4.2), we infer that $\mathscr{M}^{\otimes b} \otimes \mathscr{H}^{\otimes(-i)}$ is pseudo-effective for $i = i_{X/Y}(\mathscr{N}_1, \dots, \mathscr{N}_d) > 0$. Thus, $\mathscr{M}$ is big. □

COROLLARY 4.6. *Let $\pi\colon X \to Y$ be an equi-dimensional proper surjective morphism of normal projective varieties defined over an algebraically closed field with $d = \dim X/Y$. Let $\mathscr{L}_1, \dots, \mathscr{L}_{d+1}$ be invertible sheaves on X. If all of $\mathscr{L}_i$ have one of the following three properties* (i)–(iii) *at the same time, then $\mathscr{I}_{X/Y}(\mathscr{L}_1, \dots, \mathscr{L}_{d+1})$ also has the same property*:

(i) *ample*; (ii) *nef*; (iii) *nef and big.*

PROOF. We set $\theta := \boldsymbol{c}^1(\mathscr{L}_1) \cdots \boldsymbol{c}^1(\mathscr{L}_d) \in G^d(X)$. Let W be a closed integral subscheme $W \subset X$ of dimension d. If $\mathscr{L}_i$ are all ample, then $i_{X/\boldsymbol{k}}(\theta; W) > 0$; in particular, $i_{X/Y}(\theta) > 0$, since

$$i_{X/Y}(\theta) = i_{X/\boldsymbol{k}}(\theta; \boldsymbol{F}) = i_{\boldsymbol{F}/\boldsymbol{k}}(\theta|_{\boldsymbol{F}})$$

for a general closed fiber $\boldsymbol{F}$ of π (cf. Remark 3.10). If $\mathscr{L}_i$ are all nef, then $i_{X/\boldsymbol{k}}(\theta; W) \geq 0$ by [**25**, Chapter III, Section 2, Theorem 1]. Thus, the assertions for the properties (i) and (ii) follow from (4) and (3), respectively, of Theorem 4.2 applied to $\mathscr{L} = \mathscr{L}_{d+1}$. For the rest of the proof, we may assume that $\mathscr{L}_i$ are all nef and big. Then, $\mathscr{L}_i$ are all weakly positive on the whole space X. Hence, by applying Proposition 4.5, (2) to $\mathscr{L} = \mathscr{L}_{d+1}$, the proof is reduced to showing $i_{X/Y}(\theta) > 0$. This well-known inequality is shown as follows: Since $\mathscr{L}_1$ is big, we have an isomorphism $\mathscr{L}_1^{\otimes k} \simeq \mathscr{A} \otimes \mathscr{O}_X(D)$ for a positive integer k, an ample invertible sheaf $\mathscr{A}$ on X, and for an effective Cartier divisor D on X. Then, $\overline{D} := D|_{\boldsymbol{F}}$ is also an effective divisor on $\boldsymbol{F}$ for a general closed fiber $\boldsymbol{F}$ of π. Hence,

$$\begin{aligned} k i_{X/Y}(\theta) - i_{X/Y}(\mathscr{A}, \mathscr{L}_2, \dots, \mathscr{L}_d) &= i_{X/Y}(D, \mathscr{L}_2, \dots, \mathscr{L}_d) \\ = i_{\boldsymbol{F}/\boldsymbol{k}}(D|_{\boldsymbol{F}}, \mathscr{L}_2|_{\boldsymbol{F}}, \dots, \mathscr{L}_d|_{\boldsymbol{F}}) &= i_{\overline{D}/\boldsymbol{k}}(\mathscr{L}_2|_{\overline{D}}, \dots, \mathscr{L}_d|_{\overline{D}}) \geq 0, \end{aligned}$$

since $\mathscr{L}_i|_{\overline{D}}$ are all nef. Therefore, we may replace $\mathscr{L}_1$ with $\mathscr{A}$ for the proof. Similarly, we may replace $\mathscr{L}_i$ with an ample invertible sheaf. Then, $i_{X/Y}(\theta) > 0$

is shown as above. □

For the numerically positive polynomials for ample vector bundles discussed in Section 2.4, we have the following:

THEOREM 4.7. *Let $\pi\colon X \to Y$ be an equi-dimensional proper surjective morphism of normal projective varieties defined over a field such that $d = \dim X/Y$. Let $\mathscr{E}$ be an ample locally free sheaf of rank r on X and let $P \in \boldsymbol{Z}[x_1, \dots, x_r]$ be a weighted homogeneous polynomial of degree $d+1$ such that the weight of x_i is i for any $1 \le i \le r$. If P is numerically positive for ample vector bundles, then $\mathscr{I}_{X/Y}(P(\boldsymbol{c}^1(\mathscr{E}), \dots, \boldsymbol{c}^r(\mathscr{E})))$ is ample.*

PROOF. We write $P(\mathscr{E}) := P(\boldsymbol{c}^1(\mathscr{E}), \dots, \boldsymbol{c}^r(\mathscr{E})) \in G^{d+1}(X)$ for short. Let $p\colon \boldsymbol{P}_X(\mathscr{E}) \to X$ be the projective space bundle associated with $\mathscr{E}$ and let $\mathscr{O}(1)$ be the tautological invertible sheaf on $\boldsymbol{P}_X(\mathscr{E})$ associated with $\mathscr{E}$. Let $\mathscr{H}$ be an ample invertible sheaf on Y. Then, $\mathscr{O}(a) \otimes p^*\pi^*\mathscr{H}^{-1}$ is ample for a positive integer a, since $\mathscr{O}(1)$ is ample. There exist a finite surjective morphism $\tau\colon X' \to X$ from a normal projective variety X' and an invertible sheaf $\mathscr{H}'$ on X' such that $\tau^*\pi^*(\mathscr{H}) \simeq \mathscr{H}'^{\otimes a}$ (cf. [**14**, Lemma 1.1]). Then, the locally free sheaf $\tau^*(\mathscr{E}) \otimes \mathscr{H}'^{-1}$ is ample. We set $\eta = P(\tau^*(\mathscr{E}) \otimes \mathscr{H}'^{-1}) \in G^{d+1}(X')$. Then, $i_{X'/\boldsymbol{k}}(\eta; W) > 0$ for any closed subvariety $W \subset X'$ of dimension $d+1$, since P is numerically positive for ample vector bundles. Therefore, $\mathscr{I}_{X'/Y}(\eta)$ is nef on Y by Lemma 4.1. It is enough to prove that

$$\mathscr{I}_{X/Y}(P(\mathscr{E}))^{\otimes b} \simeq \mathscr{I}_{X'/Y}(\eta)^{\otimes c} \otimes \mathscr{H}^{\otimes k} \tag{4.4}$$

for some positive integers b, c, and k. We define

$$S(y) = S(y_1, \dots, y_r) := P(e_1(y), \dots, e_r(y))$$

for the elementary symmetric polynomials $e_k(y)$ (cf. Remark 2.40). Then $S(y)$ is a symmetric polynomial in $\boldsymbol{Z}[y_1, \dots, y_r]$. There exist weighted homogeneous polynomials $P^{(i)}(x_1, \dots, x_r) \in \boldsymbol{Z}[x_1, \dots, x_r]$ of weighted degree $0 \le i \le d$ such that

$$S(y_1 + t, \dots, y_r + t) = P(e_1(y), \dots, e_r(y)) + \sum_{i=1}^{d+1} t^i P^{(d+1-i)}(e_1(y), \dots, e_r(y)) \tag{4.5}$$

as a polynomial in $\boldsymbol{Z}[y_1, \dots, y_r, t]$. Therefore, we have

$$\begin{aligned}P(\tau^*(\mathscr{E})\otimes \mathscr{H}'^{-1}) &= P(\tau^*\mathscr{E}) + \sum_{i=1}^{d+1}(-1)^i \boldsymbol{c}^1(\mathscr{H}')^i(P^{(d+1-i)}(\tau^*\mathscr{E}))\\ &= \tau^\star(P(\mathscr{E})) + \sum_{i=1}^{d+1}(-1)^i \boldsymbol{c}^1(\mathscr{H}')^i \tau^\star(P^{(d+1-i)}(\mathscr{E})),\end{aligned}$$

for $\tau^\star\colon G^{d+1}(X)\to G^{d+1}(X')$. Let l be the degree of τ; in other words, l is the rank of $\tau_*\mathscr{O}_{X'}$. Then $\mathscr{I}_{X'/Y}(\tau^\star\xi) = \mathscr{I}_{X/Y}(\xi)^{\otimes l}$ for any $\xi\in G^{d+1}(X)$ by Lemmas 3.9 and 3.11. Let m be a positive integer divisible by a^{d+1}. Since $\mathscr{H}'^a \simeq \tau^*\pi^*(\mathscr{H})$, the m-th power $\mathscr{I}_{X'/Y}(\eta)^{\otimes m}$ is isomorphic to

$$\mathscr{I}_{X/Y}(P(\mathscr{E}))^{\otimes ml} \otimes \mathscr{I}_{X/Y}(\boldsymbol{c}^1(\pi^*\mathscr{H})P^{(d)}(\mathscr{E}))^{\otimes(-(m/a)l)} \otimes \mathscr{I}_{X/Y}(\boldsymbol{c}^1(\pi^*\mathscr{H})^2\theta)$$

for some $\theta\in G^{d-1}(X)$, where

$$\mathscr{I}_{X/Y}(\boldsymbol{c}^1(\pi^*\mathscr{H})^2\theta)\simeq \mathscr{O}_Y \quad\text{and}\quad \mathscr{I}_{X/Y}(\boldsymbol{c}^1(\pi^*\mathscr{H})P^{(d)}(\mathscr{E}))\simeq \mathscr{H}^{\otimes s}$$

for $s := i_{X/Y}(P^{(d)}(\mathscr{E}))$ by Lemma 3.13. Thus, we have the isomorphism (4.4) for $(b,c,k) = (ml, m, (m/a)ls)$, where the remaining inequality $s = i_{X/Y}(P^{(d)}(\mathscr{E})) > 0$ is a consequence of Lemma 4.8 below. In fact,

$$P^{(d)}(e_1(y),\dots,e_r(y)) = \frac{\partial S}{\partial t}(y_1+t,\dots,y_r+t)\,\Big|_{t=0}$$

by (4.5), and hence $P^{(d)}(x_1,\dots,x_r)$ is numerically positive for ample vector bundles by Lemma 4.8; thus $i_{X/Y}(P^{(d)}(\mathscr{E})) = i_{\boldsymbol{F}}(P^{(d)}(\mathscr{E}|_{\boldsymbol{F}})) > 0$ for a general fiber $\boldsymbol{F}$ of π (cf. Lemma 2.12 and Remark 3.10). □

Lemma 4.8. *Let $S(y) = S(y_1,\dots,y_r)\in \boldsymbol{Z}[y_1,\dots,y_r]$ be a symmetric homogeneous polynomial of degree $d+1$. Suppose that $S(y)$ is expressed as a numerically positive polynomial for ample vector bundles of weighted degree $d+1$, i.e., there is a weighted homogeneous polynomial $P(x_1,\dots,x_r)\in\boldsymbol{Z}[x_1,\dots,x_r]$ such that*

(1) *$P(x_1,\dots,x_r)$ is numerically positive for ample vector bundles,*
(2) *$P(e_1(y),\dots,e_r(y)) = S(y)$ for the elementary symmetric polynomials $e_k(y)$.*

Then, the symmetric polynomial

$$\frac{\partial S}{\partial t}(y_1+t,\dots,y_r+t)\,\Big|_{t=0}$$

is also expressed as a numerically positive polynomial for ample vector bundles of weighted degree d.

PROOF. By [**14**, Theorem I], together with Fact 2.39 and Remark 2.40, we may assume that S is the Schur function S_λ for a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ of $d+1$ such that $\lambda_{r+1} = 0$. Then, P above equals the Schur polynomial $P_{\lambda'}$ in the sense of Fact 2.39, where λ' is the conjugate partition of λ. By definition,

$$S_\lambda(y_1, \ldots, y_r) = \frac{\det\left(y_i^{\lambda_j + r - j}\right)_{1 \leq i,j \leq r}}{\det\left(y_i^{r-j}\right)_{1 \leq i,j \leq r}} = \det\left(y_i^{\lambda_j + r - j}\right)_{1 \leq i,j \leq r} \Delta(y_1, \ldots, y_r)^{-1} \tag{4.6}$$

for the Vandermonde polynomial

$$\Delta(y_1, \ldots, y_r) := \prod_{1 \leq i < j \leq r} (y_i - y_j).$$

For an integral vector $\boldsymbol{m} = (m_1, \ldots, m_r) \in \boldsymbol{Z}^r$, we write

$$A_{\boldsymbol{m}}(y_1, \ldots, y_r) := \det(y_i^{m_j})_{1 \leq i,j \leq r} \in \boldsymbol{Z}\left[y_1^{\pm 1}, \ldots, y_r^{\pm 1}\right].$$

Note that $A_{\boldsymbol{m}}(y) = 0$ if $m_i = m_j$ for some $i \neq j$. For $1 \leq k \leq r$, let $\boldsymbol{\varepsilon}[k]$ be the unit vector $(0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 lies at the k-th place. Then,

$$\frac{\partial A_{\boldsymbol{m}}}{\partial t}(y_1 + t, \ldots, y_r + t)\bigg|_{t=0} = \sum_{k=1}^{r} m_k A_{\boldsymbol{m} - \boldsymbol{\varepsilon}[k]}(y_1, \ldots, y_r).$$

We introduce $\boldsymbol{\delta} := (r-1, r-2, \ldots, 1, 0) = \sum_{k=1}^{r} (r-k)\boldsymbol{\varepsilon}[k] \in \boldsymbol{Z}^r$. Regarding the partition λ as a vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \boldsymbol{Z}^r$, we have

$$\Delta(y_1, \ldots, y_r) S(y_1 + t, \ldots, y_r + t) = A_{\boldsymbol{\lambda} + \boldsymbol{\delta}}(y_1 + t, \ldots, y_r + t)$$

by (4.6). Hence,

$$\begin{aligned} &\Delta(y_1, \ldots, y_r) \frac{\partial S}{\partial t}(y_1 + t, \ldots, y_r + t)\bigg|_{t=0} \\ &\quad = \sum_{k=1}^{r} (\lambda_k + r - k) A_{\boldsymbol{\lambda} + \boldsymbol{\delta} - \boldsymbol{\varepsilon}[k]}(y_1, \ldots, y_r). \end{aligned} \tag{4.7}$$

Note that if $\lambda_k = \lambda_{k+1}$, then $A_{\boldsymbol{\lambda}+\boldsymbol{\delta}-\boldsymbol{\varepsilon}[k]}(y) = 0$ or $\lambda_k + r - k = 0$. Indeed, $(\lambda + \boldsymbol{\delta} - \boldsymbol{\varepsilon}[k])_k = (\lambda + \boldsymbol{\delta} - \boldsymbol{\varepsilon}[k])_{k+1}$ if $k \leq r-1$, and $\lambda_k + r - k = 0$ for $k = r$ if $\lambda_r = \lambda_{r+1} (= 0)$. Thus, the right hand side of (4.7) is regarded as the sum for integers k with $\lambda_k > \lambda_{k+1}$. Let k be such an integer. Then, the coefficient $\lambda_k + r - k$ is positive. Moreover, we have a partition $\mu = (\mu_1 \geq \cdots \geq \mu_r \geq \mu_{r+1} = 0)$ of the integer d by setting $\mu_i := \lambda_i$ for $i \neq k$ and $\mu_k := \lambda_k - 1$, and μ satisfies

$$S_\mu(y_1, \ldots, y_r) = A_{\boldsymbol{\lambda}+\boldsymbol{\delta}-\boldsymbol{\varepsilon}[k]}(y_1, \ldots, y_r)\Delta(y_1, \ldots, y_r)^{-1}.$$

Therefore $(\partial/\partial t)S(y_1 + t, \ldots, y_r + t)|_{t=0}$ is expressed as a positive linear combination of some Schur polynomials associated with partitions of d. This is numerically positive for ample vector bundles by [**14**, Theorem I]. $\square$

4.2. Morphisms into Chow varieties.

Let X be a projective variety, Y a normal variety, and let $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ be the natural projections. Let us fix a non-negative integer d.

Definition 4.9. Let $Z = \sum n_i Z_i$ be an effective algebraic cycle on $X \times Y$, where $n_i > 0$ and Z_i is a closed integral subscheme of $X \times Y$. The cycle Z is called a *family of effective algebraic cycles on X of dimension d parametrized by Y* if $p_2|_{Z_i} \colon Z_i \to Y$ is an equi-dimensional surjective morphism of relative dimension d for any i. We denote by $\operatorname{Supp} Z$ the reduced scheme $\bigcup_i Z_i$.

Let $Z = \sum n_i Z_i$ be a family of effective algebraic cycles on X of dimension d parametrized by Y. For a point $\boldsymbol{y} \in Y$, the fiber $Z_i \times_Y \boldsymbol{y}$ is a closed subscheme of $X_{\boldsymbol{y}} := X \times_{\operatorname{Spec} \boldsymbol{k}} \operatorname{Spec} \boldsymbol{k}(\boldsymbol{y})$ of pure dimension d, where $\boldsymbol{k}(\boldsymbol{y})$ denotes the residue field of $\mathscr{O}_{Y,\boldsymbol{y}}$. Thus, for the associated cycles $\operatorname{cyc}(Z_i \times_Y \boldsymbol{y})$, we can define the algebraic cycle $Z(\boldsymbol{y})$ on $X_{\boldsymbol{y}}$ to be $\sum n_i \operatorname{cyc}(Z_i \times_Y \boldsymbol{y})$.

Let $\tau \colon Y' \to Y$ be a surjective morphism from another normal variety Y'. Then one can consider the pullback $\tau^* Z$ as follows: Let $\{Z'_{i,j}\}$ be the set of irreducible components of $Z_i \times_Y Y'$ such that $Z'_{i,j}$ dominates Y'. Let $l_{i,j}$ be the length of $Z_i \times_Y Y'$ along $Z'_{i,j}$, i.e.,

$$l_{i,j} = l_{Z'_{i,j}}(\mathscr{O}_{Z_i \times_Y Y'}).$$

We set $\tau^* Z$ to be the cycle $\sum_{i,j} n_i l_{i,j} Z'_{i,j}$. Then, $\tau^* Z$ is a family of effective algebraic cycles on X of dimension d parametrized by Y'. For any $\eta \in G^{d+1}(X)$, we have an isomorphism

$$\mathscr{I}_{\tau^* Z/Y'}(p_1^{\star}\eta) \simeq \tau^* \mathscr{I}_{Z/Y}(p_1^{\star}\eta) \tag{4.8}$$

by Lemma 3.9 and Theorem 3.25 (cf. Convention after Definition 3.5), where p_1 denotes the first projection $X \times Y \to X$ or $X \times Y' \to X$.

THEOREM 4.10. *Let X be a projective variety, Y a normal projective variety, and Z a family of algebraic cycles on X of dimension d parametrized by Y. Then, there exists uniquely up to isomorphism a proper surjective morphism $\varphi\colon Y \to T$ into a normal projective variety T with connected fibers such that, for a closed subset B of Y, $\varphi(B)$ is a point if and only if $\dim p_1(\operatorname{Supp} Z \cap (X \times B)) \le d$. Moreover, there exist a family Z_T of algebraic cycles on X of dimension d parametrized by T, and a positive integer m such that*

(1) $mZ = \varphi^* Z_T$,
(2) $\varphi^* \mathscr{I}_{Z_T/T}(p_1^{\star}\eta) \simeq \mathscr{I}_{Z/Y}(p_1^{\star}\eta)^{\otimes m}$ *for any* $\eta \in G^{d+1}(X)$,
(3) $\mathscr{I}_{Z_T/T}(p_1^{*}\mathscr{A}_1, \dots, p_1^{*}\mathscr{A}_{d+1})$ *is ample for any ample invertible sheaves* $\mathscr{A}_i$ *on* X,

where p_1 denotes the first projection $X \times Y \to X$ or $X \times T \to X$. Here, one can take $m = 1$ if the function field $\boldsymbol{k}(Y)$ is separable, or equivalently, separably generated, over $\boldsymbol{k}(T)$.

The proof is given after Lemmas 4.11 and 4.12.

LEMMA 4.11. *Let Z be a family of effective algebraic cycles on X of dimension d parametrized by Y. Let $B \subset Y$ be a connected closed algebraic subset and let F be the image $p_1(\operatorname{Supp} Z \cap (X \times B)) \subset X$. Suppose that $\dim F \le d$. Then* $\operatorname{Supp} Z \cap (X \times B) = F \times B$ *as an algebraic subset of $X \times Y$.*

PROOF. We write $S = \operatorname{Supp} Z$. By construction, there is a natural inclusion $S \cap (X \times B) \subset F \times B$. Note that the equality $S \cap (X \times B) = F \times B$ holds if and only if $S \cap (X \times \{\boldsymbol{b}\}) = F \times \{\boldsymbol{b}\}$ for any $\boldsymbol{b} \in B$. Hence, in order to show the equality, we may assume B to be irreducible, since B is connected. Furthermore, we can reduce to the case where $Z_i \to Y$ is flat for any i as follows: We can find a birational morphism $Y' \to Y$ from a normal projective variety Y' which gives a flattening of $Z_i \to Y$ for any i. Let Z'_i be the irreducible component of $Z_i \times_Y Y'$ flat over Y'. Then $Z'_i \to Z_i$ is surjective, since it is birational. Let S' be the union $\bigcup_i Z'_i$. Then $S' \cap (X \times B') \subset F' \times B'$ for $B' = B \times_Y Y'$ and for the image $F' \subset X$ of $S' \cap (X \times B')$ by the first projection $X \times Y' \to X$. Here, $F = F'$, since $S' \to S$ is surjective. Thus, if $S' \cap (X \times B') = F \times B'$, then we have $S \cap (X \times B) = F \times B$ by considering the image by $X \times Y' \to X \times Y$.

Therefore, we may assume that B is irreducible and $Z_i \to Y$ is flat for any i.

Let $\{V_{i,j}\}$ be the set of irreducible components of $Z_i \cap (X \times B)$. Then $p_2(V_{i,j}) = B$ and $\dim V_{i,j} = \dim B + d$, since $V_{i,j} \to B$ is flat at the generic point of $V_{i,j}$. Let $F_{i,j}$ be the image $p_1(V_{i,j})$. Then the natural inclusion $V_{i,j} \subset F_{i,j} \times B$ is just the equality, since the both sides are irreducible subvarieties of $X \times Y$ of the same dimension. Therefore, $Z_i \cap (X \times B) = F_i \times B$ for the union $F_i = \bigcup_j F_{i,j}$, and finally, $S \cap (X \times B) = F \times B$ by $F = \bigcup F_i$. □

Let $\mathscr{A}_1, \dots, \mathscr{A}_{d+1}$ be very ample invertible sheaves on X. Then we can consider the intersection sheaf $\mathscr{M} := \mathscr{I}_{Z/Y}(p_1^*\mathscr{A}_1, \dots, p_1^*\mathscr{A}_{d+1})$. Here, $\mathscr{M}$ is generated by global sections by Corollary 3.21. Let $\varphi\colon Y \to T$ be the Stein factorization of the morphism

$$\Phi_{|\mathscr{M}|}\colon Y \to |\mathscr{M}|^\vee = \boldsymbol{P}(\mathrm{H}^0(Y, \mathscr{M})) = \mathrm{Proj}(\mathrm{Sym}\, \mathrm{H}^0(Y, \mathscr{M}))$$

associated with the linear system $|\mathscr{M}|$. In other words, φ is the canonical morphism

$$Y \to T = \mathrm{Proj} \bigoplus_{l \geq 0} \mathrm{H}^0(Y, \mathscr{M}^{\otimes l}).$$

Lemma 4.12. *For an integral closed subscheme $B \subset Y$, $\varphi(B)$ is a point if and only if $\dim p_1(Z \cap (X \times B)) \leq d$. In particular, the morphism φ does not depend on the choice of very ample invertible sheaves $\mathscr{A}_i$.*

Proof. Let $\tau\colon Y' \to Y$ be a projective birational morphism from a normal projective variety Y' which gives a flattening of $Z_i \to Y$ for any i. Then $\tau^*\mathscr{M} \simeq \mathscr{I}_{\tau^*Z/Y'}(p_1^\star \eta)$ by (4.8), where $\eta = \boldsymbol{c}^1(\mathscr{A}_1) \cdots \boldsymbol{c}^1(\mathscr{A}_{d+1})$. Thus $\varphi \circ \tau$ is associated with the family τ^*Z of algebraic cycles parametrized by Y'. Hence, we can replace Y with Y' in order to prove the lemma. Therefore, we assume from the beginning that $Z_i \to Y$ is flat for any i. Then,

$$\mathscr{M}|_B \simeq \mathscr{I}_{Z \times_Y B/B}(p_1^\star \eta)$$

by Lemma 2.27. Assume that $\dim p_1(\mathrm{Supp}\, Z \cap (X \times B)) = d$. Then, by Lemma 4.11, there is a closed subscheme $W \subset X$ with $\dim W = d$ such that $Z_i \times_Y B$ is a subscheme of $X \times W$ for any i. For a representative $x \in F^{d+1}(X)$ of $\eta \in G^{d+1}(X)$, we have

$$p_1^\star(x)\, \mathrm{cl}_\bullet(Z_i \times_Y B) = p_1^\star(x|_W)\, \mathrm{cl}_\bullet(Z_i \times_Y B) = 0 \quad \in K_\bullet(Z_i \times_Y B),$$

since $p_1^\star(x|_W) \in F^{d+1}(W) = 0$ by Proposition 2.24. Hence, $\mathscr{M}|_B \simeq \mathscr{O}_B$, and

$\varphi(B)$ is a point. Assume next that $\dim p_1(\operatorname{Supp} Z \cap (X \times B)) \geq d+1$. Then there is a closed irreducible curve $C \subset B$ such that $\dim(\operatorname{Supp} Z \cap (X \times C)) = \dim p_1(\operatorname{Supp} Z \cap (X \times C)) = d+1$ by Lemma 4.11. Hence

$$\deg \mathscr{M}|_C = i_{X \times C/\boldsymbol{k}}(p_1^{\star}\eta; Z \times_Y C) > 0$$

by Lemma 2.28, since $\mathscr{A}_1, \ldots, \mathscr{A}_{d+1}$ are ample on X. This implies that $\varphi(B)$ is not a point. Thus, we are done. □

We are ready to prove Theorem 4.10:

Proof of Theorem 4.10. The existence and the uniqueness of $\varphi \colon X \to T$ is proved in Lemma 4.12. Let $Z_{T,i} \subset X \times T$ be the image of Z_i by $\mathrm{id}_X \times \varphi \colon X \times Y \to X \times T$. Then the natural inclusion $Z_i \subset Z_{T,i} \times_T Y$ is an equality of algebraic sets by Lemmas 4.11 and 4.12. In fact, for any closed point $\boldsymbol{t} \in T$ and the fiber $B = \varphi^{-1}(\boldsymbol{t})$, we have $\dim p_1(Z_i \cap (X \times B)) \leq d$ by Lemma 4.12, and thus $Z_i \cap (X \times B) = F_i \times B$ for a subset F_i of X by Lemmas 4.11. Hence, for any closed point $\boldsymbol{y} \in B = \varphi^{-1}(\boldsymbol{t})$,

$$\begin{aligned}(Z_{T,i} \times_T Y) \cap (X \times \{\boldsymbol{y}\}) &= p_1(Z_{T,i} \cap (X \times \{\boldsymbol{t}\})) \times \{\boldsymbol{y}\} \subset F_i \times \{\boldsymbol{y}\} \\ &= Z_i \cap (X \times \{\boldsymbol{y}\})\end{aligned}$$

as a subset of $X \times Y$. This implies that $Z_i = Z_{T,i} \times_T Y$ as a subset.

As a consequence, we infer that the morphism $Z_{T,i} \to T$ induced from the second projection $X \times T \to T$ is a surjective equi-dimensional morphism of relative dimension d. Thus $Z_{T,i}$ is a family of algebraic cycles on X of dimension d parametrized by T. Therefore, $\varphi^* Z_{T,i} = m_i Z_i$ as a family of algebraic cycles on Y for the length m_i of $Z_{T,i} \times_T Y$ along Z_i. Note that $m_i = 1$ if $\boldsymbol{k}(Y)$ is separable over $\boldsymbol{k}(T)$. In fact, in this situation, $Z_{T,i} \times_T Y$ is reduced at the generic point of Z_i (cf. [**20**, Proposition (4.2.4)]). We set $m = \operatorname{lcm}\{m_i\}$ and $Z_T := \sum (m/m_i) Z_{T,i}$. Then, $mZ = \varphi^*(Z_T)$. Thus, (1) and the last assertion on m in Theorem 4.10 have been proved.

The isomorphism (2) follows from (1) and (4.8). By (2), we have an isomorphism

$$\mathscr{M}^{\otimes m} \simeq \varphi^* \mathscr{I}_{Z_T/T}(p_1^*\mathscr{A}_1, \ldots, p_1^*\mathscr{A}_{d+1}).$$

On the other hand, $\mathscr{M}$ is the pullback of an ample invertible sheaf on T by the construction of φ. Thus, (3) is derived. □

Remark 4.13. The morphism $\varphi\colon Y \to T$ is regarded as the Stein factorization of the morphism $Y \to \mathrm{Chow}(X)$ to the Chow variety of X corresponding to $\boldsymbol{y} \mapsto Z(\boldsymbol{y})$. This is show as follows. We fix a closed immersion $X \hookrightarrow \boldsymbol{P}^n$ into an n-dimensional projective space $\boldsymbol{P}^n$ and set $\mathscr{A} = \mathscr{O}(1)|_X$. Let R_n be the vector space $\mathrm{H}^0(\boldsymbol{P}^n, \mathscr{O}(1))$. We set $\theta = \boldsymbol{c}^1(\mathscr{A})^d \in G^d(X)$ and $\eta = \boldsymbol{c}^1(\mathscr{A})^{d+1} \in G^{d+1}(X)$. Furthermore, we set

$$\begin{aligned} e &:= i_{Z/Y}(p_1^\star\theta) = i_{Z/Y}(p_1^*\mathscr{A}, \dots, p_1^*\mathscr{A}) = i_{Z/Y}\bigl(p_1^*\mathscr{O}(1), \dots, p_1^*\mathscr{O}(1)\bigr), \\ \mathscr{M} &:= \mathscr{I}_{Z/Y}(p_1^\star\eta) = \mathscr{I}_{Z/Y}(p_1^*\mathscr{A}, \dots, p_1^*\mathscr{A}) = \mathscr{I}_{Z/Y}\bigl(p_1^*\mathscr{O}(1), \dots, p_1^*\mathscr{O}(1)\bigr). \end{aligned}$$

Then, by Propositions 2.15 and 3.20 (cf. Lemma 2.37), we have a natural surjection

$$\Phi\colon \bigotimes^{d+1} \mathrm{Sym}^e(R_n) \otimes_{\boldsymbol{k}} \mathscr{O}_Y \to \mathscr{M}.$$

By construction (cf. [**32**, Chapter 5, Section 4]), the associated morphism

$$\psi\colon Y \to \boldsymbol{P}\Bigl(\bigotimes^{d+1} \mathrm{Sym}^e(R_n)\Bigr)$$

is just the morphism to the Chow variety $\mathrm{Chow}_{d,e}(X) \subset \mathrm{Chow}_{d,e}(\boldsymbol{P}^n)$ of d-dimensional algebraic cycles of degree e corresponding to $\boldsymbol{y} \mapsto Z(\boldsymbol{y})$. Therefore, $\varphi\colon Y \to T$ is just the Stein factorization of ψ, by the definition of φ given just before Lemma 4.12.

Proposition 4.14. *Let $\pi\colon X \dashrightarrow Y$ be a dominant rational map from a projective variety X to a normal projective variety Y with $d = \dim X/Y$. Then there exist a normal projective variety T and a birational map $\mu\colon Y \dashrightarrow T$ satisfying the following two conditions*:

(1) *The graph $\Gamma_T \subset X \times T$ of the composite $\mu \circ \pi\colon X \dashrightarrow Y \dashrightarrow T$ is equidimensional over T by the second projection $X \times T \to T$, i.e., $\dim \Gamma_T \cap (X \times \{\boldsymbol{t}\}) = d$ for any $\boldsymbol{t} \in T$.*
(2) *There is an ample invertible sheaf $\mathscr{A}$ on X such that $\mathscr{I}_{\Gamma_T/T}(\boldsymbol{c}^1(p_1^*\mathscr{A})^{d+1}) = \mathscr{I}_{\Gamma_T/T}(p_1^*\mathscr{A}, \dots, p_1^*\mathscr{A})$ is ample.*

The map $\mu\colon Y \dashrightarrow T$ is unique up to isomorphism, and the following conditions are also satisfied:

(3) *For any ample invertible sheaves $\mathscr{A}_1, \dots, \mathscr{A}_{d+1}$ on X, the intersection sheaf $\mathscr{I}_{\Gamma_T/T}(p_1^*\mathscr{A}_1, \dots, p_1^*\mathscr{A}_{d+1})$ is ample.*

(4) *For any birational map* $\mu' \colon Y \cdots\!\to T'$ *satisfying* (1), *there exists a birational morphism* $\nu \colon T' \to T$ *such that* $\mu = \nu \circ \mu'$.

Definition 4.15. For the birational map $\mu \colon Y \cdots\!\to T$ satisfying the conditions (1) and (2) in the proposition above, the composite $\mu \circ \pi \colon X \cdots\!\to Y \cdots\!\to T$ is called the *Chow reduction* of $\pi \colon X \cdots\!\to Y$.

Proof of Proposition 4.14. First, we shall construct a birational map $\mu \colon Y \cdots\!\to T$ satisfying the conditions (1), (2), and (3). We can find a birational morphism $Y' \to Y$ from a normal projective variety Y' which gives a flattening of $\Gamma_Y \to Y$ for the graph Γ_Y of $\pi \colon X \cdots\!\to Y$. Then, an irreducible component X' of $\Gamma_Y \times_Y Y'$ is flat over Y' and is birational to Γ_Y. Thus, we have a morphism $X' \to X \times Y'$ such that the first projection gives a birational morphism $X' \to X$ and the second projection gives a flat morphism $X' \to Y'$. The image of $X' \to X \times Y'$ is just the graph $\Gamma_{Y'}$ of $X \cdots\!\to Y \cdots\!\to Y'$, which is equi-dimensional over Y', since so is X'. Therefore, the rational map $Y \cdots\!\to Y'$ satisfies (1). Since $\Gamma_{Y'}$ is regarded as a family of algebraic cycles on X of dimension d parametrized by Y', we have a morphism $\varphi \colon Y' \to T$ by applying Theorem 4.10. Here, T is a normal projective variety, φ has only connected fibers, and there is a family Z_T of algebraic cycles on X of dimension d parametrized by T such that $m\Gamma_{Y'} = \varphi^*(Z_T)$ for some positive integer m. Since Z_T is the image of $\Gamma_{Y'}$ by $X \times Y' \to X \times T$, the first projection $X \times T \to X$ induces a birational morphism $Z_T \to X$. Hence, Z_T is just the graph of the rational map $X \cdots\!\to Y' \to T$, and $\varphi \colon Y' \to T$ is birational, since $\dim Y' = \dim X - d = \dim Z_T - d = \dim T$. By Theorem 4.10, we infer that $m = 1$ and that the rational map $\mu \colon Y \cdots\!\to Y' \xrightarrow{\varphi} T$ satisfies the conditions (1), (2), and (3).

Second, we compare the birational map $\mu \colon Y \cdots\!\to T$ above with any other birational map $\mu' \colon Y \cdots\!\to T'$ satisfying the condition (1). For such μ', there exist birational morphisms $\nu_0 \colon T'' \to T$ and $\nu_1 \colon T'' \to T'$ from a normal projective variety T'' such that $\nu_0^{-1} \circ \mu = \nu_1^{-1} \circ \mu'$. Here, $\mu'' := \nu_0^{-1} \circ \mu \colon Y \cdots\!\to T''$ also satisfies (1), since the graph $\Gamma_{T''}$ of μ'' is contained in $\Gamma_T \times_T T'' \subset X \times T''$. Let $\mathscr{A}$ be an ample invertible sheaf on X. We define

$$\mathscr{M} := \mathscr{I}_{\Gamma_T/T}\big(p_1^\star \eta\big), \quad \mathscr{M}' := \mathscr{I}_{\Gamma_{T'}/T'}\big(p_1^\star \eta\big), \quad \text{and} \quad \mathscr{M}'' := \mathscr{I}_{\Gamma_{T''}/T''}\big(p_1^\star \eta\big)$$

for $\eta := \boldsymbol{c}^1(\mathscr{A})^{d+1}$, where p_1 denotes the first projections $X \times T \to X$, $X \times T' \to X$, and $X \times T'' \to X$, respectively. Then, $\mathscr{M}'' \simeq \nu_0^* \mathscr{M} \simeq \nu_1^* \mathscr{M}'$ by (4.8). Since $\mathscr{M}$ is ample, we infer that every fiber of $\nu_1 \colon T'' \to T'$ is contracted to a point by $\nu_0 \colon T'' \to T$. Hence, $\nu = \nu_0 \circ \nu_1^{-1} \colon T' \to T$ is a birational morphism. Consequently, μ satisfies the condition (4). In this situation, assume further that $\mu' \colon Y \cdots\!\to T'$ satisfies also the condition (2) for the ample invertible sheaf $\mathscr{A}$. Then, $\mathscr{M}'$ is also ample, and thus $\nu\colon T' \to T$ is an isomorphism by the same reason. Therefore, the uniqueness of the rational map $Y \cdots\!\to T$ satisfying (1) and (2) has been proved. Thus, we are done. $\square$

4.3. Endomorphisms of complex normal projective varieties.

In the last subsection, we shall study surjective endomorphisms $f\colon X \to X$ of a normal projective variety X, mainly over the complex number field $\boldsymbol{C}$.

Lemma 4.16. *Let $\pi\colon X \to Y$, $\pi'\colon X' \to Y'$, $\tau\colon Y' \to Y$, and $\tau'\colon X' \to X$ be surjective morphisms for projective varieties X, X', Y, and Y' such that*

(1) $\pi \circ \tau' = \tau \circ \pi'$,
(2) *Y and Y' are normal,*
(3) *π is equi-dimensional of relative dimension d,*
(4) *for an open dense subset $U' \subset Y'$, the induced morphism $\pi'^{-1}(U') \to X \times_Y U'$ is a surjective generically finite morphism of degree e.*

Then, for any $\eta \in G^{d+1}(X)$, one has an isomorphism

$$\mathscr{I}_{X'/Y'}(\tau'^{\star}\eta) \simeq \tau^{*}\,\mathscr{I}_{X/Y}(\eta)^{\otimes e}.$$

Proof. Considering the fiber product $X \times_Y Y'$, we have the following commutative diagram:

$$\begin{tikzcd} X' \arrow[r, "\nu"] \arrow[d, "\pi'"'] & X \times_Y Y' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "\pi"'] \\ Y' \arrow[r, equal] & Y' \arrow[r, "\tau"] & Y, \end{tikzcd}$$

where p_1 and p_2 denote the natural projections $X \times_Y Y' \to X$ and $X \times_Y Y' \to Y'$, respectively, and ν is the induced morphism $(\tau', \pi')\colon X' \to X \times_Y Y'$. By the condition (4), $Z := \nu(X')$ is a unique irreducible component of $X \times_Y Y'$ dominating Y'. Thus, we have an isomorphism

$$\mathscr{I}_{Z/Y'}(p_1^{\star}\eta) \simeq \mathscr{I}_{X\times_Y Y'/Y'}(p_1^{\star}\eta) \simeq \tau^{*}\,\mathscr{I}_{X/Y}(\eta)$$

by Lemma 3.9 and Theorem 3.25. Let $\mathscr{F}_i$ be the i-th higher direct image sheaf $\mathrm{R}^i\,\nu_*\mathscr{O}_{X'}$. Then, by the condition (4), $\dim(\operatorname{Supp}\mathscr{F}_i \cap p_2^{-1}(*)) < d$ for the generic point $*$ of Y for any $i > 0$, and $l_Z(\mathscr{F}_0) = e$. Therefore,

$$\mathscr{I}_{X'/Y'}(\tau'^{\star}\eta) \simeq \mathscr{I}_{\mathscr{F}/Y'}(p_1^{\star}\eta) \simeq \mathscr{I}_{Z/Y'}(p_1^{\star}\eta)^{\otimes e}$$

by Lemmas 3.11 and 3.9. Thus, we have the expected isomorphism. □

PROPOSITION 4.17. *Let $f\colon X \to X$ be a surjective endomorphism of a normal projective variety X. Let $\pi\colon X \to Y$ be an equi-dimensional surjective morphism of relative dimension d to a normal projective variety Y such that π has connected fibers and that $\pi \circ f = h \circ \pi$ for a surjective endomorphism $h\colon Y \to Y$. Let $\mathscr{A}$ be a nef and big invertible sheaf on X and let $\mathscr{M}$ be the intersection sheaf $\mathscr{I}_{X/Y}(\boldsymbol{c}^1(\mathscr{A})^{d+1}) = \mathscr{I}_{X/Y}(\mathscr{A}, \dots, \mathscr{A})$. Then, $\mathscr{M}$ is a nef and big invertible sheaf. If $\mathscr{A}$ is ample, then so is $\mathscr{M}$. The endomorphisms f and h have the following properties*:

(1) *If $f^*\mathscr{A} \simeq \mathscr{A}^{\otimes q}$ for an integer q, then $h^*\mathscr{M}^{\otimes e} \simeq \mathscr{M}^{\otimes qe}$ for some $e > 0$.*
(2) *If $f^*\mathscr{A}$ is numerically equivalent to $\mathscr{A}^{\otimes q}$ for an integer q, then $h^*\mathscr{M}$ is numerically equivalent to $\mathscr{M}^{\otimes q}$.*

PROOF. The first two assertions on the numerical properties on $\mathscr{M}$ are shown in Corollary 4.6. Hence, it is enough to prove the assertions (1) and (2) for f and h. Note that f and h are finite morphisms. In fact, f^* induces an automorphism of $\mathrm{NS}(X) \otimes \boldsymbol{Q}$ for the *Néron-Severi group* $\mathrm{NS}(X)$, the group of Cartier divisors on X modulo the algebraic equivalence relation: It is well-known that $\mathrm{NS}(X)$ is a finitely generated abelian group. Thus, for any ample divisor A on X, some positive multiple mA is linearly equivalent to the pullback of an ample divisor on X. This implies that f is finite. The finiteness of h is derived from the same argument. The induced morphism $(f, \pi)\colon X \to X \times_{Y,h} Y$ is a finite surjective morphism, since π has connected fibers. Thus, $\deg f = e \deg h$ for the mapping degree e of (f, π). Therefore, by Lemma 4.16, we have an isomorphism

$$\mathscr{I}_{X/Y}(f^*\mathscr{A}, \dots, f^*\mathscr{A}) = \mathscr{I}_{X/Y}(f^{\star}\eta) \simeq h^*\,\mathscr{I}_{X/Y}(\eta)^{\otimes e} = h^*\mathscr{M}^{\otimes e} \tag{4.9}$$

for $\eta := \boldsymbol{c}^1(\mathscr{A})^{d+1} \in G^{d+1}(X)$. In both cases (1) and (2), $f^*\mathscr{A}$ is numerically equivalent to $\mathscr{A}^{\otimes q}$. Note that $q > 0$, since $f^*\mathscr{A}$ is nef and big. Thus, $\mathscr{I}_{X/Y}(f^{\star}\eta)$ is numerically equivalent to

$$\mathscr{I}_{X/Y}(\mathscr{A}^{\otimes q}, \dots, \mathscr{A}^{\otimes q}) \simeq \mathscr{M}^{\otimes q^{d+1}}$$

by Theorem 4.2, (2). Hence, $\mathscr{M}^{\otimes q^{d+1}}$ is numerically equivalent to $h^*\mathscr{M}^{\otimes e}$ by (4.9). Here, we have $\deg f = q^{\dim X} = q^{d+m}$ for $m = \dim Y$, and $\deg h = q_1^m$ for $q_1 = q^{d+1}e^{-1}$ from the calculations

$$\begin{aligned} q^{d+m} i_{X/\boldsymbol{k}}\big(\boldsymbol{c}^1(\mathscr{A})^{d+m}; X\big) &= i_{X/\boldsymbol{k}}\big(\boldsymbol{c}^1(f^*\mathscr{A})^{d+m}; X\big) = i_{X/\boldsymbol{k}}\big(\boldsymbol{c}^1(\mathscr{A})^{d+m}; \mathrm{cl}_\bullet(f_*\mathscr{O}_X)\big) \\ &= (\deg f) i_X\big(\boldsymbol{c}^1(\mathscr{A})^{d+m}; X\big) > 0, \\ (q^{d+1})^m i_{Y/\boldsymbol{k}}\big(\boldsymbol{c}^1(\mathscr{M})^m; Y\big) &= i_{Y/\boldsymbol{k}}\big(\boldsymbol{c}^1(h^*\mathscr{M}^{\otimes e})^m; Y\big) = i_{Y/\boldsymbol{k}}\big(\boldsymbol{c}^1(\mathscr{M}^{\otimes e})^m; \mathrm{cl}_\bullet(h_*\mathscr{O}_Y)\big) \\ &= (\deg h) e^m i_{Y/\boldsymbol{k}}\big(\boldsymbol{c}^1(\mathscr{M})^m; Y\big) > 0, \end{aligned}$$

where we use Lemma 1.12. Furthermore, $e = q^d$ and $\deg h = q^m$ by $\deg f = e \deg h$. In particular, $h^*\mathscr{M}$ is numerically equivalent to $\mathscr{M}^{\otimes q}$. In case (1), from (4.9), we have

$$h^*\mathscr{M}^{\otimes e} \simeq \mathscr{M}^{\otimes q^{d+1}} \simeq \mathscr{M}^{\otimes qe}.$$

Thus, we are done. □

In what follows, we assume the base field $\boldsymbol{k}$ to be the complex number field $\boldsymbol{C}$. Recall that a complex projective variety X is called *uniruled* if there is a dominant rational map $\boldsymbol{P}^1 \times Y \cdots\to X$ from a projective variety Y with $\dim Y = \dim X - 1$. Note that X is uniruled if and only if X contains a dense subset which is a union of rational curves. A complex projective variety X is called *rationally connected* if, for arbitrary two closed points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ of X, there is an irreducible rational curve C which contains $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ (cf. [**28**, Theorem (2.1) and Definition-Remark (2.2)]).

Fact (MRC fibration). Let M be a non-singular complex projective variety. Then, there is a dominant rational map $f\colon M \cdots\to S$, called the *maximal rationally connected fibration*, *MRC fibration* for short, satisfying the following conditions:

(1) S is a non-uniruled variety.
(2) There exist open dense subsets $U \subset M$ and $V \subset S$ such that $f|_U\colon U \to V$ is regular and proper.
(3) A general fiber of f is a rationally connected submanifold of M.

The MRC fibration is unique up to birational equivalence, i.e., if $\mu\colon M' \cdots\to M$ is a birational map from a non-singular projective variety M' and $f'\colon M' \cdots\to S'$ is an MRC fibration of M', then $f \circ \mu = \nu \circ f'$ for a birational map $\nu\colon S' \cdots\to S$. The existence and the uniqueness of the rational map $f\colon M \cdots\to S$ satisfying (2), (3), and the following condition (4), has been proved by [**4**, Théorème 2.3] and [**28**, Theorem (2.7)]:

(4) A sufficiently general fiber of f is a maximal rationally connected manifold.

Note that (1) and (3) imply (4). Later, it was proved in [**16**] that the rational map $f\colon M \cdots\to S$ satisfying (2)–(4) also satisfies the condition (1).

Applying the Chow reduction (cf. Proposition 4.14), we have the notion of *special MRC fibration* for a projective variety, as follows.

THEOREM 4.18. *Let X be a complex projective variety. Then, there exists a dominant rational map $\pi\colon X \cdots\to Y$ uniquely up to isomorphism satisfying the following conditions*:

(1) *Y is a non-uniruled normal projective variety.*
(2) *The graph $\Gamma_Y \subset X \times Y$ of π is equi-dimensional over Y.*
(3) *A general fiber of $\Gamma_Y \to Y$ is rationally connected.*
(4) *If $\pi'\colon X \cdots\to Y'$ is a dominant rational map satisfying* (1)–(3), *then there is a birational morphism $\nu\colon Y' \to Y$ such that $\pi = \nu \circ \pi'$.*

We call the rational map $\pi\colon X \cdots\to Y$ above the *special MRC fibration* of X.

PROOF. Let $M \to X$ be a resolution of singularities and let $M \cdots\to S$ be an MRC fibration. Then, by Proposition 4.14, the Chow reduction $\pi\colon X \cdots\to Y$ of the rational map $X \cdots\to M \cdots\to S$ satisfies the required conditions. □

REMARK. For the special MRC fibration $\pi\colon X \cdots\to Y$, it is not always possible to find an open dense subset $U \subset X$ such that $\pi|_U\colon U \to Y$ is regular and is a proper surjective morphism to an open subset of Y. For example, let $p\colon M \to C$ be a $\boldsymbol{P}^1$-bundle over a non-singular projective curve C of genus ≥ 1 having a section Γ with negative self-intersection number, and let $\mu\colon M \to X$ be the blow-down of the section Γ. Then the special MRC fibration $\pi\colon X \cdots\to Y$ is isomorphic to the composite $p \circ \mu^{-1}\colon X \cdots\to M \to C$. If $\pi|_U$ is regular for an open subset $U \subset X$, then $\mu^{-1}(U) \cap \Gamma = \emptyset$; hence π does not induces a proper morphism from U. Note that the surface X is not rationally connected, but it is rationally connected in the sense of [**4**, Definition 2.1].

THEOREM 4.19. *Let $f\colon X \to X$ be a surjective endomorphism of a normal complex projective variety X. Let $\pi\colon X \cdots\to Y$ be the special MRC fibration. Then there is an endomorphism $h\colon Y \to Y$ such that $\pi \circ f = h \circ \pi$.*

PROOF. Note that f is a finite surjective morphism (cf. the proof of Proposition 4.17). Let $X \cdots\to Y_1 \to Y$ be the Stein factorization of the composite $\pi \circ f\colon X \cdots\to Y$; we set $\pi_1\colon X \cdots\to Y_1$ and $\tau\colon Y_1 \to Y$. Let Γ_Y and Γ_{Y_1} be the graphs of rational maps π and π_1, respectively. Then, we have a commutative diagram:

$$\begin{tikzcd} \Gamma_{Y_1} \arrow[r] \arrow[d] & \Gamma_Y \times_Y Y_1 \arrow[r] \arrow[d] & \Gamma_Y \arrow[d] \\ X \times Y_1 \arrow[r, "f \times \mathrm{id}_{Y_1}"] & X \times Y_1 \arrow[r, "\mathrm{id}_X \times \tau"] & X \times Y, \end{tikzcd} \tag{4.10}$$

where the vertical arrows are all closed immersions and the horizontal arrows are all finite morphisms. In particular, the graph Γ_{Y_1} is equi-dimensional over Y_1, since Γ_Y is so over Y. Let $W \subset Y \times Y_1$ be the image of the rational map $(\pi, \pi_1)\colon X \cdots\to Y \times Y_1$. Then, a general fiber of the first projection $W \to Y$ is rationally connected by a property of special MRC fibration. On the other hand, Y_1 is not uniruled, since $\tau\colon Y_1 \to Y$ is a finite surjective morphism to a non-uniruled variety Y. Hence, the first projection $W \to Y$ is birational. The second projection $W \to Y_1$ is also birational, since a general fiber of $W \to Y_1$ is also connected and since $\dim W = \dim Y_1$. Then, by Proposition 4.14, we infer that π is the Chow reduction of π_1, since Γ_{Y_1} is equi-dimensional over Y_1. Consequently, there is a birational morphism $\varphi\colon Y_1 \to Y$ such that $\pi = \varphi \circ \pi_1$. We set $\eta = \boldsymbol{c}^1(\mathscr{A})^{d+1} \in G^{d+1}(X)$ for an ample invertible sheaf $\mathscr{A}$ of X and for $d = \dim X/Y$. A general fiber of $\Gamma_Y \times_Y Y_1 \to Y_1$ is irreducible, since this is isomorphic to a general fiber of $\Gamma_Y \to Y$. Thus, the morphism $\Gamma_{Y_1} \to \Gamma_Y \times_Y Y_1$ in the diagram (4.10) is a finite surjective morphism over an open dense subset of Y_1. Then, by Lemma 4.16, we have an isomorphism

$$\mathscr{I}_{\Gamma_{Y_1}/Y_1}(p_1^\star f^\star \eta) \simeq \tau^* \mathscr{I}_{\Gamma_Y/Y}(p_1^\star \eta)^{\otimes b}$$

for some $b > 0$, where p_1 denotes the first projection $X \times Y \to Y$ or $X \times Y_1 \to Y_1$. Here, $\mathscr{I}_{\Gamma_Y/Y}(p_1^\star \eta)$ is ample by Proposition 4.14. Thus, $\mathscr{I}_{\Gamma_{Y_1}/Y_1}(p_1^\star f^\star \eta)$ is also ample, since τ is finite. Therefore, $\pi_1\colon X \cdots\to Y_1$ is the Chow reduction of itself by Proposition 4.14. As a consequence, φ is an isomorphism. Then, the endomorphism $h = \tau \circ \varphi^{-1}\colon Y \to Y$ satisfies $\pi \circ f = h \circ \pi$. □

REMARK. In Theorem 4.19, if f is étale, then h is induced from the push-forward morphism $[f_*]\colon \mathrm{Chow}(X) \to \mathrm{Chow}(X)$ given by $Z \mapsto f_* Z$. In fact, for the special MRC fibration $\pi\colon X \cdots\to Y$ and its graph Γ_Y, we have a finite morphism $\psi\colon Y \to \mathrm{Chow}(X)$ which maps a general point $\boldsymbol{y}$ to the point $[\Gamma_Y(\boldsymbol{y})]$ corresponding to the cycle $\Gamma_Y(\boldsymbol{y}) = \mathrm{cyc}(\Gamma_Y \times_Y \{\boldsymbol{y}\})$ (cf. Remark 4.13). For a general point $\boldsymbol{y}$, $\Gamma_Y(\boldsymbol{y})$ and also $f(\Gamma_Y(\boldsymbol{y})) = \Gamma_Y(h(\boldsymbol{y}))$ are rationally connected. Since a non-singular model of $\Gamma_Y(h(\boldsymbol{y}))$ is simply connected (cf. [**3**, Theorem 3.5]), every irreducible component of $f^{-1}(\Gamma_Y(h(\boldsymbol{y})))$ is birational to $\Gamma_Y(h(\boldsymbol{y}))$ by f. Hence, $f_*(\Gamma_Y(\boldsymbol{y})) = \Gamma_Y(h(\boldsymbol{y}))$ as a cycle. Therefore, $\psi \circ h = [f_*] \circ \psi$.

Corollary 4.20. *Let X be a normal complex projective variety admitting a surjective endomorphism $f\colon X \to X$ such that $f^*\mathscr{A} \simeq \mathscr{A}^{\otimes q}$ for a nef and big invertible sheaf $\mathscr{A}$ and a positive integer q. Let $\pi\colon X \dashrightarrow Y$ be the special MRC fibration. Then there exist an endomorphism $h\colon Y \to Y$ and a nef and big invertible sheaf $\mathscr{M}$ on Y such that $\pi \circ f = h \circ \pi$ and $h^*\mathscr{M} \simeq \mathscr{M}^{\otimes q}$. Here, if $\mathscr{A}$ is ample, then one can take $\mathscr{M}$ to be ample.*

Proof. We have h by Theorem 4.19. The intersection sheaf $\mathscr{M}' = \mathscr{I}_{\Gamma_Y/Y}(p_1^\star \boldsymbol{c}^1(\mathscr{A})^{d+1})$ is nef and big by Corollary 4.6. If $\mathscr{A}$ is ample, then so is $\mathscr{M}'$ by Theorem 4.18. Then a suitable power $\mathscr{M} = (\mathscr{M}')^{\otimes e}$ satisfies the required condition by Proposition 4.17. $\square$

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