J. Math. Soc. Japan
Vol. 62, No. 1 (2010) pp. 321–353
doi: 10.2969/jmsj/06210321

Boundedness of sublinear operators on product Hardy spaces and its application

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(Received Jan. 27, 2009)

Abstract. Let $p \in (0, 1]$. In this paper, the authors prove that a sublinear operator T (which is originally defined on smooth functions with compact support) can be extended as a bounded sublinear operator from product Hardy spaces $H^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ to some quasi-Banach space $\mathscr{B}$ if and only if T maps all $(p, 2, s_1, s_2)$-atoms into uniformly bounded elements of $\mathscr{B}$. Here $s_1 \geq \lfloor n(1/p-1) \rfloor$ and $s_2 \geq \lfloor m(1/p-1) \rfloor$. As usual, $\lfloor n(1/p-1) \rfloor$ denotes the maximal integer no more than $n(1/p-1)$. Applying this result, the authors establish the boundedness of the commutators generated by Calderón-Zygmund operators and Lipschitz functions from the Lebesgue space $L^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ with some $p > 1$ or the Hardy space $H^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ with some $p \leq 1$ but near 1 to the Lebesgue space $L^q(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ with some $q > 1$.

1. Introduction.

The theory of Calderón-Zygmund operators and Hardy spaces on product spaces has been studied by many mathematicians extensively in the past thirty years, see, for example, [**8**], [**9**], [**11**], [**12**], [**18**], [**20**], [**28**], [**29**]. Recently, Ferguson and Lacey [**13**] characterized the product BMO $(\boldsymbol{R}^2_+ \times \boldsymbol{R}^2_+)$ by the nested commutator determined by the one-dimensional Hilbert transform in the jth variable, $j = 1, 2$. Motivated by this, Chen, Han and Miao in [**6**] established the boundedness on $H^1(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ of bi-commutators of fractional integrals with BMO functions. The boundedness on $H^1(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ of the Marcinkiewicz integral and its commutator with Lipschitz function was also established in [**28**].

To establish the boundedness of operators on Hardy spaces on $\boldsymbol{R}^n$ and $\boldsymbol{R}^n \times \boldsymbol{R}^m$, one usually appeals to the atomic decomposition characterization of Hardy spaces, which means that a function or distribution in Hardy spaces can be represented as a linear combination of atoms; see [**7**], [**21**] and [**3**], [**5**] respectively.

2000 *Mathematics Subject Classification.* Primary 42B20; Secondary 42B30, 42B25, 47B47.
Key Words and Phrases. product space, Hardy space, Lebesgue space, sublinear operator, commutator, Calderón-Zygmund operator, Lipschitz function.
The first author is partially supported by a research grant from United States Army Research Office and a competitive research grant at Georgetown University. The second author (the corresponding author) is supported by the National Natural Science Foundation (Grant No. 10871025) of China.

Then, the boundedness of linear operators on Hardy spaces can be deduced from their behavior on atoms in principle.

However, Meyer [**23**, p. 513] (see also [**2**], [**15**]) gave an example of $f \in H^1(\boldsymbol{R}^n)$, whose norm cannot be achieved by its finite atomic decompositions via $(1, \infty)$-atoms. Based on this fact, Bownik [**2**, Theorem 2] constructed a surprising example of a linear functional defined on a dense subspace of $H^1(\boldsymbol{R}^n)$, which maps all $(1, \infty)$-atoms into bounded scalars, but yet cannot extend to a bounded linear functional on the whole $H^1(\boldsymbol{R}^n)$. This implies that it cannot guarantee the boundedness of linear operator T from $H^p(\boldsymbol{R}^n)$ with $p \in (0, 1]$ to some quasi-Banach space $\mathscr{B}$ only proving that T maps all (p, ∞)-atoms into uniformly bounded elements of $\mathscr{B}$. This phenomenon has also essentially already been observed by Y. Meyer in [**22**, p. 19]. Moreover, motivated by this, Yabuta [**27**] gave some sufficient conditions for the boundedness of T from $H^p(\boldsymbol{R}^n)$ with $p \in (0, 1]$ to $L^q(\boldsymbol{R}^n)$ with $q \geq 1$ or $H^q(\boldsymbol{R}^n)$ with $q \in [p, 1]$. However, these conditions are not necessary. In [**29**], a boundedness criterion was established as follows: a sublinear operator T (which is originally defined on smooth functions with compact support) extends to a bounded sublinear operator from $H^p(\boldsymbol{R}^n)$ with $p \in (0,1]$ to some quasi-Banach spaces $\mathscr{B}$ if and only if T maps all $(p,2)$-atoms into uniformly bounded elements of $\mathscr{B}$. This result shows the structure difference between atomic characterization of $H^p(\boldsymbol{R}^n)$ via $(p,2)$-atoms and (p,∞)-atoms. This result is generalized to spaces of homogeneous type in [**30**].

The purpose of this paper is two folds. We first generalize the boundedness criterion on $\boldsymbol{R}^n$ in [**29**] to product Hardy spaces on $\boldsymbol{R}^n \times \boldsymbol{R}^m$. Precisely, we prove that a sublinear operator T (which is originally defined on smooth functions with compact support) extends to a bounded sublinear operator from $H^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ with $p \in (0,1]$ to some quasi-Banach spaces $\mathscr{B}$ if and only if T maps all $(p,2)$-atoms into uniformly bounded elements of $\mathscr{B}$. Invoking this result and motivated by [**6**], [**13**], [**28**], we then establish the boundedness of the commutators generated by Calderón-Zygmund operators and Lipschitz functions from the Lebesgue space $L^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ with some $p > 1$ or the Hardy space $H^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ with some $p \leq 1$ but near 1 to the Lebesgue space $L^q(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ with some $q > 1$.

To state the main results, we first recall some notation and notions on product Hardy spaces. For $n, m \in \boldsymbol{N}$, denote by $\mathscr{S}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ the space of Schwartz functions on $\boldsymbol{R}^n \times \boldsymbol{R}^m$ and by $\mathscr{S}'(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ its dual space. Let $\mathscr{D}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ be the space of all smooth functions on $\boldsymbol{R}^n \times \boldsymbol{R}^m$ with compact support. For $s_1, s_2 \in \boldsymbol{Z}_+$, let $\mathscr{D}_{s_1, s_2}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ be the set of all functions $f \in \mathscr{D}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ with vanishing moments up to order s_1 with respect to the first variable and order s_2 with respect to the second variable. More precisely, if $f \in \mathscr{D}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$, then for $\alpha_1 \in \boldsymbol{Z}_+^n$ and $\alpha_2 \in \boldsymbol{Z}_+^m$ with $|\alpha_1| \leq s_1$ and $|\alpha_2| \leq s_2$, one has

$$\int_{\boldsymbol{R}^n} f(x_1,\, x_2) x_1^{\alpha_1}\, dx_1 = 0 \quad \text{for all} \quad x_2 \in \boldsymbol{R}^m,$$
$$\int_{\boldsymbol{R}^m} f(x_1,\, x_2) x_2^{\alpha_2}\, dx_2 = 0 \quad \text{for all} \quad x_1 \in \boldsymbol{R}^n.$$

For $s_1,\, s_2 \in \boldsymbol{Z}_+$ and $\sigma_1,\, \sigma_2 \in [0, \infty)$, we denote by $\mathscr{D}_{s_1,\, s_2;\, \sigma_1,\, \sigma_2}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ the space $\mathscr{D}_{s_1,\, s_2}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ endowed with the norm

$$\|f\|_{\mathscr{D}_{s_1,\, s_2;\, \sigma_1,\, \sigma_2}(\boldsymbol{R}^n \times \boldsymbol{R}^m)} \equiv \sup_{x_1 \in \boldsymbol{R}^n,\, x_2 \in \boldsymbol{R}^m} (1 + |x_1|)^{\sigma_1} (1 + |x_2|)^{\sigma_2} |f(x_1,\, x_2)|.$$

In articles [**3**], [**4**], [**5**], Chang and Fefferman introduced the following atoms and atomic Hardy spaces on the product space $\boldsymbol{R}^n \times \boldsymbol{R}^m$.

DEFINITION 1.1. Let $p \in (0,1]$, $s_1 \geq \lfloor n(1/p - 1) \rfloor$ and $s_2 \geq \lfloor m(1/p - 1) \rfloor$. A function a supported in an open set $\Omega \subset \boldsymbol{R}^n \times \boldsymbol{R}^m$ with finite measure is said to be a $(p,\, 2,\, s_1,\, s_2)$-atom provided that

(AI) a can be written as $a = \sum_{R \in \mathscr{M}(\Omega)} a_R$, where $\mathscr{M}(\Omega)$ denotes all the maximal dyadic subrectangles of Ω and a_R is a function satisfying that

- (i) a_R is supported on $2R = 2I \times 2J$, which is a rectangle with the same center as R and whose side length is 2 times that of R,
- (ii) a_R satisfies the cancelation conditions that

$$\int_{2I} a_R(x_1,\, x_2) x_1^{\alpha_1}\, dx_1 = 0 \quad \textit{for all } x_2 \in 2J \textit{ and } |\alpha_1| \leq s_1,$$
$$\int_{2J} a_R(x_1,\, x_2) x_2^{\alpha_2}\, dx_2 = 0 \quad \textit{for all } x_1 \in 2I \textit{ and } |\alpha_2| \leq s_2;$$

(AII) a satisfies the size conditions that $\|a\|_{L^2(\boldsymbol{R}^n \times \boldsymbol{R}^m)} \leq |\Omega|^{1/2 - 1/p}$ and

$$\left(\sum_{R \in \mathscr{M}(\Omega)} \|a_R\|_{L^2(\boldsymbol{R}^n \times \boldsymbol{R}^m)}^2 \right)^{1/2} \leq |\Omega|^{1/2 - 1/p}.$$

DEFINITION 1.2. Let $p \in (0,\, 1]$, $s_1 \geq \lfloor n(1/p - 1) \rfloor$ and $s_2 \geq \lfloor m(1/p - 1) \rfloor$. A distribution $f \in \mathscr{S}'(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ is said to be an element in $H^{p,\, 2,\, s_1,\, s_2}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ if there exist a sequence $\{\lambda_k\}_{k \in \boldsymbol{N}} \subset \boldsymbol{C}$ and $(p,\, 2,\, s_1,\, s_2)$-atoms $\{a_k\}_{k \in \boldsymbol{N}}$ such that $f = \sum_{k \in \boldsymbol{N}} \lambda_k a_k$ in $\mathscr{S}'(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ with $\sum_{k \in \boldsymbol{N}} |\lambda_k|^p < \infty$. Moreover, define the quasi-norm of $f \in H^{p,2,s_1,s_2}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ by $\|f\|_{H^{p,2,s_1,s_2}(\boldsymbol{R}^n\times\boldsymbol{R}^m)} \equiv \inf\{(\sum_{k\in\boldsymbol{N}} |\lambda_k|^p)^{1/p}\}$, where the infimum is taken over all the decompositions as above.

It is well known that $H^{p,2,s_1,s_2}(\boldsymbol{R}^n \times \boldsymbol{R}^m) = H^{p,2,t_1,t_2}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ with equivalent norms when $s_1, t_1 \geq \lfloor n(1/p-1)\rfloor$ and $s_2, t_2 \geq \lfloor m(1/p-1)\rfloor$; see [**3**], [**4**], [**5**], [**10**], [**17**]. Thus, we denote $H^{p,2,s_1,s_2}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ simply by $H^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$.

Recall that a quasi-Banach space $\mathscr{B}$ is a vector space endowed with a quasi-norm $\|\cdot\|_{\mathscr{B}}$ which is nonnegative, non-degenerate (i.e., $\|f\|_{\mathscr{B}} = 0$ if and only if $f = 0$), homogeneous, and obeys the quasi-triangle inequality, i.e., there exists a constant $C_0 \geq 1$ such that for all $f, g \in \mathscr{B}$,

$$\|f+g\|_{\mathscr{B}} \leq C_0(\|f\|_{\mathscr{B}} + \|g\|_{\mathscr{B}}). \tag{1.1}$$

DEFINITION 1.3. Let $q \in (0, 1]$. A quasi-Banach spaces $\mathscr{B}_q$ with the quasi-norm $\|\cdot\|_{\mathscr{B}_q}$ is said to be a q-quasi-Banach space if $\|\cdot\|^q_{\mathscr{B}_q}$ satisfies the triangle inequality, i.e., $\|f+g\|^q_{\mathscr{B}_q} \leq \|f\|^q_{\mathscr{B}_q} + \|g\|^q_{\mathscr{B}_q}$ for all $f, g \in \mathscr{B}_q$.

We point out that by the Aoki theorem (see [**1**] or [**16**, p. 66]), any quasi-Banach space with the positive constant C_0 as in (1.1) is essentially a q-quasi-Banach space with $q = \lfloor \log_2(2C_0)\rfloor^{-1}$. From this, any Banach space is a 1-quasi-Banach space. Moreover, ℓ^q, $L^q(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ and $H^q(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ with $q \in (0, 1)$ are typical q-quasi-Banach spaces.

Let $q \in (0, 1]$. For any given q-quasi-Banach space $\mathscr{B}_q$ and linear space $\mathscr{Y}$, an operator T from $\mathscr{Y}$ to $\mathscr{B}_q$ is called to be $\mathscr{B}_q$-sublinear if for any $f, g \in \mathscr{Y}$ and $\lambda, \nu \in \boldsymbol{C}$, we have

$$\|T(\lambda f + \nu g)\|_{\mathscr{B}_q} \leq \left(|\lambda|^q \|T(f)\|^q_{\mathscr{B}_q} + |\nu|^q \|T(g)\|^q_{\mathscr{B}_q}\right)^{1/q}$$

and $\|T(f) - T(g)\|_{\mathscr{B}_q} \leq \|T(f-g)\|_{\mathscr{B}_q}$; see [**29**], [**30**]. Obviously, if T is linear, then T is $\mathscr{B}_q$-sublinear. Moreover, if $\mathscr{B}_q$ is a space of functions, T is sublinear in the classical sense and $T(f) \geq 0$ for all $f \in \mathscr{Y}$, then T is also $\mathscr{B}_q$-sublinear.

The following is one of main results in this paper, which generalizes the main result in [**29**] to product Hardy spaces.

THEOREM 1.1. *Let $p \in (0, 1]$, $q \in [p, 1]$ and $\mathscr{B}_q$ be a q-quasi-Banach space. Suppose that $s_1 \geq \lfloor n(1/p-1)\rfloor$ and $s_2 \geq \lfloor m(1/p-1)\rfloor$. Let T be a $\mathscr{B}_q$-sublinear operator from $\mathscr{D}_{s_1,s_2}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ to $\mathscr{B}_q$. Then T can be extended as a bounded $\mathscr{B}_q$-sublinear operator from $H^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ to $\mathscr{B}_q$ if and only if T maps all $(p, 2, s_1, s_2)$-atoms in $\mathscr{D}_{s_1,s_2}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ into uniformly bounded elements of $\mathscr{B}_q$.*

Theorem 1.1 further complements the proofs of Theorem 1 in [**11**] and a theorem in [**9**], whose proof is presented in Section 2 below. The necessity of Theorem 1.1 is obvious. To prove the sufficiency, for $p \in (0,1]$, $s_1 \geq \lfloor n(1/p-1)\rfloor$, $s_2 \geq \lfloor m(1/p-1)\rfloor$ and $f \in \mathscr{D}_{s_1,\,s_2}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$, we first prove that f has an atomic decomposition which converges in $\mathscr{D}_{s_1,\,s_2;\sigma_1,\,\sigma_2}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ for some $\sigma_1 \in (\max\{n/p,\, n+s\},\, n+s+1)$ and $\sigma_2 \in (\max\{n/p,\, n+s\},\, n+s+1)$ (Lemma 2.3), and then extend T to the whole $\mathscr{D}_{s_1,\,s_2;\sigma_1,\,\sigma_2}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ boundedly (Lemma 2.4). Finally, we continuously extend T to the whole $H^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ by using the density of $\mathscr{D}_{s_1,\,s_2}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ in $H^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$.

Recall that a function a is said to be a rectangular $(p,\, 2,\, s_1,\, s_2)$-atom if

(R1) $\operatorname{supp} a \subset R = I \times J$, where I and J are cubes in $\boldsymbol{R}^n$ and $\boldsymbol{R}^m$, respectively;

(R2) $\int_{\boldsymbol{R}^n} a(x_1,\, x_2) x_1^{\alpha_1}\, dx_1 = 0$ for all $x_2 \in \boldsymbol{R}^m$ and $|\alpha_1| \leq s_1$, and $\int_{\boldsymbol{R}^m} a(x_1,\, x_2) x_2^{\alpha_2}\, dx_2 = 0$ for all $x_1 \in \boldsymbol{R}^n$ and $|\alpha_2| \leq s_2$;

(R3) $\|a\|_{L^2(\boldsymbol{R}^n \times \boldsymbol{R}^m)} \leq |R|^{1/2-1/p}$.

As a consequence of Theorem 1.1, we obtain the following result which includes a fractional version of Theorem 1 in [**11**] and is known to have many applications in harmonic analysis.

Corollary 1.1. *Let $q_0 \in [2,\infty)$ and T be a bounded sublinear operator from $L^2(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ to $L^{q_0}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$. Let $p \in (0,1]$ and $1/q - 1/p = 1/q_0 - 1/2$. If there exist positive constants C and δ such that for all rectangular $(p,\, 2,\, s_1,\, s_2)$-atoms a supported in R and all $\gamma \geq 8\max\{n^{1/2},\, m^{1/2}\}$,*

$$\int_{(\boldsymbol{R}^n \times \boldsymbol{R}^m)\setminus \widetilde{R}_\gamma} |Ta(x_1,\, x_2)|^q\, dx_1\, dx_2 \leq C\gamma^{-\delta},$$

where $\widetilde{R}_\gamma$ denotes the γ-fold enlargement of R, then T can be extended as a bounded sublinear operator from $H^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ to $L^q(\boldsymbol{R}^n \times \boldsymbol{R}^m)$.

The proof of Corollary 1.1 is given in Section 2 below. We point out that if $q_0 = 2$ and T is linear, then Corollary 1.1 is just Theorem 1 in [**11**]. Moreover, there exists a gap in the proof of Theorem 1 in [**11**] (so is the proof of a theorem in [**9**]), namely, it was not clear in [**11**] how to deduce the boundedness of the considered linear operator T on the whole Hardy space $H^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ from its boundedness uniformly on atoms. Our Theorem 1.1 here seals this gap.

Remark 1.1. Using Corollary 1.1, we now give affirmative answers to the questions in Remark 4.2 and Remark 4.3 of [**28**]. We use the same notation and notions as in [**28**]. Particularly, denote by μ_Ω the Marcinkiewicz integral operator on $\boldsymbol{R}^n \times \boldsymbol{R}^m$ with kernel $\Omega \in \mathrm{Lip}\,(\alpha_1, \alpha_2; \boldsymbol{S}^{n-1}, \boldsymbol{S}^{m-1})$, here $\alpha_1, \alpha_2 \in (0,1]$. If $\max\{n/(n+\alpha_1), m/(m+\alpha_2)\} < p \le 1$, then in Remark 4.2 of [**28**], we proved that for all $(p,2,0,0)$ atoms a, $\|\mu_\Omega(a)\|_{L^p(\boldsymbol{R}^n\times\boldsymbol{R}^m)} \lesssim 1$. Moreover, let $b \in \mathrm{Lip}\,(\beta_1, \beta_2; \boldsymbol{R}^n \times \boldsymbol{R}^m)$ with $\beta_1, \beta_2 \in (0,1]$ satisfying $\beta_1/n = \beta_2/m$ and $C_b(\mu_\Omega)$ be the commutator of b and μ_Ω. If $1/q = 1/p - \beta_1/n$ and

$$\max\{n/(n+\alpha_1), m/(m+\alpha_2)\} < p \le 1,$$

then in Remark 4.3 of [**28**], we proved that for all $(p,2,0,0)$ atoms a,

$$\|C_b(\mu_\Omega)(a)\|_{L^q(\boldsymbol{R}^n\times\boldsymbol{R}^m)} \lesssim 1.$$

However, in [**28**], it is not clear how to obtain the boundedness of μ_Ω from $H^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ to $L^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ and boundedness of $C_b(\mu_\Omega)$ from $H^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ to $L^q(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ by these known facts. Applying Theorem 1.1 here, we now obtain these desired boundedness, and hence answer the questions in Remark 4.2 and Remark 4.3 of [**28**].

Now we turn to the boundedness of commutators generated by Lipschitz functions and Calderón-Zygmund operators. We first introduce the notion of Lipschitz functions on $\boldsymbol{R}^n \times \boldsymbol{R}^m$. Let $\alpha \in (0,1]$. A function b on $\boldsymbol{R}^n$ is said to belong to $\mathrm{Lip}\,(\alpha;\, \boldsymbol{R}^n)$ if there exists a positive constant C such that for all $x,\, x' \in \boldsymbol{R}^n$,

$$|b(x) - b(x')| \le C|x - x'|^\alpha.$$

Obviously, a function in the space $\mathrm{Lip}\,(\alpha;\, \boldsymbol{R}^n)$ is not necessary bounded. For example, $|x|^\alpha \in \mathrm{Lip}\,(\alpha;\, \boldsymbol{R}^n)$, but $|x|^\alpha \notin L^\infty(\boldsymbol{R}^n)$.

DEFINITION 1.4. Let $\alpha_1,\, \alpha_2 \in (0,1]$. A function f on $\boldsymbol{R}^n \times \boldsymbol{R}^m$ is said to belong to $\mathrm{Lip}\,(\alpha_1,\, \alpha_2;\, \boldsymbol{R}^n \times \boldsymbol{R}^m)$, if there exists a positive constant C such that for all $x_1,\, y_1 \in \boldsymbol{R}^n$ and $x_2,\, y_2 \in \boldsymbol{R}^m$,

$$|[f(x_1,\, x_2) - f(x_1,\, y_2)] - [f(y_1,\, x_2) - f(y_1,\, y_2)]| \le C|x_1 - y_1|^{\alpha_1}|x_2 - y_2|^{\alpha_2}. \quad (1.2)$$

The minimal constant C satisfying (1.2) is defined to be the norm of f in the space $\mathrm{Lip}\,(\alpha_1,\, \alpha_2;\, \boldsymbol{R}^n \times \boldsymbol{R}^m)$ and denoted by $\|f\|_{\mathrm{Lip}(\alpha_1,\, \alpha_2;\, \boldsymbol{R}^n\times\boldsymbol{R}^m)}$.

We remark that a function in the space $\mathrm{Lip}\,(\alpha_1,\, \alpha_2;\, \boldsymbol{R}^n \times \boldsymbol{R}^m)$ is also not necessary to be bounded. In fact, if $f_1 \in \mathrm{Lip}\,(\alpha_1;\, \boldsymbol{R}^n)$ and $f_2 \in \mathrm{Lip}\,(\alpha_2;\, \boldsymbol{R}^m)$, then it is easy to check $f_1(x_1)f_2(x_2) \in \mathrm{Lip}\,(\alpha_1,\, \alpha_2;\, \boldsymbol{R}^n \times \boldsymbol{R}^m)$.

In this paper, we consider a class of Calderón-Zygmund operators T on $\boldsymbol{R}^n \times \boldsymbol{R}^m$, whose kernel K is a continuous function on $(\boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^m) \setminus \{(x_1, y_1, x_2, y_2):\ x_1 = y_1 \text{ or } x_2 = y_2\}$ and satisfies that there exist positive constants C and $\epsilon_1, \epsilon_2 \in (0, 1]$ such that

(K1) for all $x_1 \neq y_1$ and $x_2 \neq y_2$,

$$|K(x_1, y_1, x_2, y_2)| \le C \frac{1}{|x_1 - y_1|^n} \frac{1}{|x_2 - y_2|^m};$$

(K2) for all $x_1 \neq y_1$, $x_2 \neq y_2$, $z_1 \in \boldsymbol{R}^n$ and $|y_1 - z_1| \le |x_1 - y_1|/2$,

$$|K(x_1, y_1, x_2, y_2) - K(x_1, z_1, x_2, y_2)| \le C \frac{|y_1 - z_1|^{\epsilon_1}}{|x_1 - y_1|^{n+\epsilon_1}} \frac{1}{|x_2 - y_2|^m};$$

(K3) for all $x_1 \neq y_1$, $x_2 \neq y_2$, $z_2 \in \boldsymbol{R}^m$ and $|y_2 - z_2| \le |x_2 - y_2|/2$,

$$|K(x_1, y_1, x_2, y_2) - K(x_1, y_1, x_2, z_2)| \le C \frac{1}{|x_1 - y_1|^n} \frac{|y_2 - z_2|^{\epsilon_2}}{|x_2 - y_2|^{m+\epsilon_2}};$$

(K4) for all $x_1 \neq y_1$, $x_2 \neq y_2$, $z_1 \in \boldsymbol{R}^n$, $z_2 \in \boldsymbol{R}^m$, $|y_1 - z_1| \le |x_1 - y_1|/2$ and $|y_2 - z_2| \le |x_2 - y_2|/2$,

$$\begin{aligned} &|[K(x_1, y_1, x_2, y_2) - K(x_1, z_1, x_2, y_2)] - [K(x_1, y_1, x_2, z_2) - K(x_1, z_1, x_2, z_2)]| \\ &\quad \le C \frac{|y_1 - z_1|^{\epsilon_1}}{|x_1 - y_1|^{n+\epsilon_1}} \frac{|y_2 - z_2|^{\epsilon_2}}{|x_2 - y_2|^{m+\epsilon_2}}. \end{aligned}$$

The minimal constant C satisfying (K1) through (K4) is denoted by $\|K\|$.

Let $\alpha_1, \alpha_2 \in (0, 1]$, $b \in \mathrm{Lip}\,(\alpha_1, \alpha_2; \boldsymbol{R}^n \times \boldsymbol{R}^m)$ and T be any Calderón-Zygmund operator with kernel K satisfying the above conditions from (K1) to (K4). For any suitable function f and $(x_1, x_2) \in \boldsymbol{R}^n \times \boldsymbol{R}^m$, define the commutator $[b, T]$ by

$$\begin{aligned} &[b, T](f)(x_1, x_2) \\ &\quad = \int_{\boldsymbol{R}^n \times \boldsymbol{R}^m} K(x_1, y_1, x_2, y_2) \\ &\qquad \times [b(x_1, x_2) - b(x_1, y_2) - b(y_1, x_2) + b(y_1, y_2)] f(y_1, y_2)\, dy_1 dy_2. \quad (1.3) \end{aligned}$$

The following result gives the boundedness of the commutator $[b, T]$ on Lebesgue spaces.

THEOREM 1.2. *Let* $\epsilon_1, \epsilon_2, \alpha_1, \alpha_2 \in (0,1]$, $\alpha_1/n = \alpha_2/m$, $p \in (1, n/\alpha_1)$ *and* $1/q = 1/p - \alpha_1/n$. *Let* $b \in \mathrm{Lip}\,(\alpha_1, \alpha_2; \boldsymbol{R}^n \times \boldsymbol{R}^m)$, T *be a Calderón-Zygmund operator whose kernel* K *satisfies the conditions from* $(K1)$ *to* $(K4)$, *and* $[b, T]$ *be the commutator as in* (1.3). *Then there exists a positive constant* C *independent of* $\|b\|_{\mathrm{Lip}\,(\alpha_1, \alpha_2; \boldsymbol{R}^n \times \boldsymbol{R}^m)}$ *and* $\|K\|$ *such that for all* $f \in L^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$,

$$\|[b, T](f)\|_{L^q(\boldsymbol{R}^n \times \boldsymbol{R}^m)} \le C\|K\| \|b\|_{\mathrm{Lip}(\alpha_1, \alpha_2; \boldsymbol{R}^n \times \boldsymbol{R}^m)} \|f\|_{L^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)}.$$

Here is another main result of this paper, whose proof depends on Corollary 1.1.

THEOREM 1.3. *Let* $0 < \alpha_1 \le \min\{n/2, 1\}$, $\alpha_1/n = \alpha_2/m$, $\epsilon_1, \epsilon_2 \in (0,1]$,

$$\max\{n/(n+\epsilon_1),\, n/(n+\alpha_1),\, m/(m+\epsilon_2),\, m/(m+\alpha_2)\} < p \le 1 \tag{1.4}$$

and $1/q = 1/p - \alpha_1/n$. *Assume that* $b \in \mathrm{Lip}\,(\alpha_1, \alpha_2; \boldsymbol{R}^n \times \boldsymbol{R}^m)$. *Let* T *be a Calderón-Zygmund operator whose kernel* K *satisfies the conditions* $(K1)$ *through* $(K4)$, *and* $[b, T]$ *be the commutator defined in* (1.3). *Then there exists a positive constant* C *independent of* $\|b\|_{\mathrm{Lip}(\alpha_1, \alpha_2; \boldsymbol{R}^n \times \boldsymbol{R}^m)}$ *and* $\|K\|$ *such that for all* $f \in H^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$,

$$\|[b, T](f)\|_{L^q(\boldsymbol{R}^n \times \boldsymbol{R}^m)} \le C\|K\| \|b\|_{\mathrm{Lip}(\alpha_1, \alpha_2; \boldsymbol{R}^n \times \boldsymbol{R}^m)} \|f\|_{H^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)}.$$

The proofs of Theorem 1.2 and Theorem 1.3 are presented in Section 3.

We finally make some conventions. Throughout this paper, let $\boldsymbol{N} = \{1, 2, \cdots\}$ and $\boldsymbol{Z}_+ = \boldsymbol{N} \cup \{0\}$. We always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use $f \lesssim g$ to denote $f \le Cg$ and $f \sim g$ to denote $f \lesssim g \lesssim f$.

2. Proofs of Theorem 1.1 and Corollary 1.1.

As a matter of convenience, in this section, we denote n and m, respectively, by n_1 and n_2. For $i = 1, 2$ and $s_i \in \boldsymbol{Z}_+$, denote by $\mathscr{D}_{s_i}(\boldsymbol{R}^{n_i})$ the set of all smooth functions with compact support and vanishing moments up to order s_i. Then there exist functions $\psi^{(i)} \in \mathscr{D}_{s_i}(\boldsymbol{R}^{n_i})$ and $\varphi^{(i)} \in \mathscr{S}(\boldsymbol{R}^{n_i})$ such that

(i) $\operatorname{supp} \psi^{(i)} \subset B^{(i)}(0, 1)$, $\widehat{\psi^{(i)}} \ge 0$ and $\widehat{\psi^{(i)}}(\xi_i) \ge \frac{1}{2}$ if $\frac{1}{2} \le |\xi_i| \le 2$, where and in what follows, $B^{(i)}(0, r_i) \equiv \{x_i \in \boldsymbol{R}^{n_i} : |x_i| < r_i\}$ and $\widehat{\psi^{(i)}}$ denotes the Fourier transform of $\psi^{(i)}$;

(ii) $\operatorname{supp} \widehat{\varphi^{(i)}} \subset \{\xi_i \in \boldsymbol{R}^{n_i} : 1/2 \le |\xi_i| \le 2\}$ and $\widehat{\varphi^{(i)}} \ge 0$;

(iii) $\sup\{\widehat{\varphi^{(i)}}(\xi_i) : 3/5 \le |\xi_i| \le 5/3\} > C$ for some positive constant C;

(iv) $\int_0^\infty \widehat{\psi^{(i)}}(t_i\xi_i)\widehat{\varphi^{(i)}}(t_i\xi_i)\,\frac{dt_i}{t_i}=1$ for all $\xi_i \in \boldsymbol{R}^{n_i}\setminus\{0\}$.

Such $\psi^{(i)}$ and $\varphi^{(i)}$ can be constructed by a slight modification of Lemma (1.2) of [**14**]; see also Lemma (5.12) in [**14**] for a discrete variant. Then by an argument similar to the proofs of Theorem (1.3) and Theorem 1 in Appendix of [**14**], we have that for all $f\in\mathscr{S}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ and $(x_1,x_2)\in\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}$,

$$f(x_1,\,x_2)=\int_0^\infty\int_0^\infty(\psi_{t_1,\,t_2}*\varphi_{t_1,\,t_2}*f)(x_1,\,x_2)\,\frac{dt_1}{t_1}\frac{dt_2}{t_2} \tag{2.1}$$

in both $L^2(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ and pointwise, where and in what follows, for any $i=1,\,2$, $\phi^{(i)}\in\mathscr{S}(\boldsymbol{R}^{n_i})$, $x_i\in\boldsymbol{R}^{n_i}$ and $t_i\in(0,\,\infty)$, we always let $\phi^{(i)}_{t_i}(x_i)\equiv t_i^{-n_i}\phi^{(i)}(t_i^{-1}x_i)$ and $\phi_{t_1,\,t_2}(x_1,x_2)\equiv\phi^{(1)}_{t_1}(x_1)\phi^{(2)}_{t_2}(x_2)$. For any set $E\subset(\boldsymbol{R}^n\times\boldsymbol{R}^m)$, set $E^{\complement}\equiv(\boldsymbol{R}^n\times\boldsymbol{R}^m)\setminus E$.

Lemma 2.1. *Let $s_i\in\boldsymbol{Z}_+$, $\psi^{(i)}\in\mathscr{D}_{s_i}(\boldsymbol{R}^{n_i})$ and $\varphi^{(i)}\in\mathscr{S}(\boldsymbol{R}^{n_i})$ satisfy the above conditions* (i) *through* (iv)*, where $i=1,\,2$. Let $0<\sigma_i<\sigma_i'<n_i+s_i+1$ for $i=1,\,2$. Then for any $f\in\mathscr{D}_{s_1,\,s_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$, there exists a positive constant C such that for all $\epsilon_1,\,\epsilon_2\in(0,\,1)$ and $L_1,L_2\in(1,\infty)$,*

$$\begin{aligned}&\sup_{(x_1,\,x_2)\in\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}}(1+|x_1|)^{\sigma_1}(1+|x_2|)^{\sigma_2}\\&\quad\times\left(\int_0^{\epsilon_1}\int_0^\infty+\int_{L_1}^\infty\int_0^\infty+\int_0^\infty\int_0^{\epsilon_2}+\int_0^\infty\int_{L_2}^\infty\right)\int_{\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}}|(\varphi_{t_1,\,t_2}*f)(y_1,\,y_2)|\\&\quad\times|\psi_{t_1,\,t_2}(x_1-y_1,\,x_2-y_2)|\,dy_1\,dy_2\,\frac{dt_1}{t_1}\frac{dt_2}{t_2}\\&\le C\Big[\epsilon_1+\epsilon_2+(L_1)^{\sigma_1-n_1-s_1-1}+(L_2)^{\sigma_2-n_2-s_2-1}\Big],\end{aligned}$$

$$\begin{aligned}&\sup_{(x_1,\,x_2)\in\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}}(1+|x_1|)^{\sigma_1}(1+|x_2|)^{\sigma_2}\\&\quad\times\int_0^{L_1}\int_0^\infty\int_{[B^{(1)}(0,\,2L_1)]^{\complement}\times\boldsymbol{R}^{n_2}}|(\varphi_{t_1,\,t_2}*f)(y_1,\,y_2)|\\&\quad\times|\psi_{t_1,\,t_2}(x_1-y_1,\,x_2-y_2)|\,dy_1\,dy_2\,\frac{dt_1}{t_1}\frac{dt_2}{t_2}\le C(L_1)^{\sigma_1-\sigma_1'}\end{aligned} \tag{2.2}$$

and (2.2) *with L_1, σ_1, n_1, s_1 and $B^{(1)}$ replaced, respectively, by L_2, σ_2, n_2, s_2 and $B^{(2)}$.*

In order to prove Lemma 2.1, we need the following technical lemma. For $i=1,\,2$, $u_i\ge0$, let

$$\mathscr{S}_{u_i}(\boldsymbol{R}^{n_i}) \equiv \left\{\varphi \in \mathscr{S}(\boldsymbol{R}^{n_i}) : \int_{\boldsymbol{R}^{n_i}} \varphi(x_i) x_i^{\alpha}\, dx_i = 0,\ |\alpha| \le u_i\right\}.$$

For any $s_1,\, s_2 \in \boldsymbol{Z}_{-1} \equiv \boldsymbol{N} \cup \{0, -1\}$, we denote by $\mathscr{S}_{s_1, s_2}(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2})$ the space of functions in $\mathscr{S}(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2})$ with the vanishing moments up to order s_1 in the first variable and order s_2 in the second variable, where we say that $f \in \mathscr{S}(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2})$ has vanishing moments up to order -1 in the first or second variable, if f has no vanishing moment with respect to that variable.

LEMMA 2.2. *Let $s_i \in \boldsymbol{Z}_{-1}$, $u_i \in \boldsymbol{Z}_{-1}$, $\sigma_i \in [0, \infty)$ and $\varphi^{(i)} \in \mathscr{S}_{u_i}(\boldsymbol{R}^{n_i})$ for $i = 1, 2$. For any $f \in \mathscr{S}_{s_1, s_2}(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2})$, there exists a positive constant C such that*

(i) *if $u_1 > -1$, then for all $t_1 \in (0, 1]$ and $(x_1, x_2) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$,*

$$|(\varphi^{(1)}_{t_1} *_1 f)(x_1, x_2)| \le C t_1^{u_1+1} (1 + |x_1|)^{-\sigma_1} (1 + |x_2|)^{-\sigma_2},$$

where and in what follows, $(\varphi^{(1)}_{t_1} *_1 f)(x_1, x_2) \equiv \int_{\boldsymbol{R}^{n_1}} \varphi^{(1)}_{t_1}(y_1) f(x_1 - y_1, x_2)\, dy_1$;

(ii) *if $s_1 > -1$, then for all $t_1 \in [1, \infty)$ and $(x_1, x_2) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$,*

$$|(\varphi^{(1)}_{t_1} *_1 f)(x_1, x_2)| \le C t_1^{-n_1-s_1-1} \left(1 + \frac{|x_1|}{t_1}\right)^{-\sigma_1} (1 + |x_2|)^{-\sigma_2};$$

(iii) *if $u_1,\, u_2 > -1$, then for all $t_1,\, t_2 \in (0,1]$ and $(x_1, x_2) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$,*

$$|(\varphi_{t_1, t_2} * f)(x_1, x_2)| \le C t_1^{u_1+1} t_2^{u_2+1} (1 + |x_1|)^{-\sigma_1} (1 + |x_2|)^{-\sigma_2};$$

(iv) *if $u_1,\, s_2 > -1$, then for all $t_1 \in (0, 1]$, $t_2 \in [1, \infty)$ and $(x_1, x_2) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$,*

$$|(\varphi_{t_1, t_2} * f)(x_1, x_2)| \le C t_1^{u_1+1} t_2^{-n_2-s_2-1} (1 + |x_1|)^{-\sigma_1} \left(1 + \frac{|x_2|}{t_2}\right)^{-\sigma_2};$$

(v) *if $s_1,\, u_2 > -1$, then for all $t_1 \in [1, \infty)$, $t_2 \in (0, 1]$ and $(x_1, x_2) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$,*

$$|(\varphi_{t_1, t_2} * f)(x_1, x_2)| \le C t_1^{-n_2-s_2-1} t_2^{u_2+1} \left(1 + \frac{|x_1|}{t_1}\right)^{-\sigma_1} (1 + |x_2|)^{-\sigma_2};$$

(vi) *if $s_1,\, s_2 > -1$, then for all $t_1,\, t_2 \in [1, \infty)$ and $(x_1, x_2) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$,*

$$|(\varphi_{t_1, t_2} * f)(x_1, x_2)| \le C t_1^{-n_1-s_1-1} t_2^{-n_2-s_2-1} \left(1 + \frac{|x_1|}{t_1}\right)^{-\sigma_1} \left(1 + \frac{|x_2|}{t_2}\right)^{-\sigma_2}.$$

PROOF. To prove Lemma 2.2, we use some ideas in the proofs of Lemma 2 and Lemma 4 in Appendix (III) of [**14**].

To prove (i), by $\int_{\boldsymbol{R}^{n_1}} \varphi^{(1)}(x_1)x_1^\alpha \, dx_1 = 0$ for $|\alpha| \le u_1$, we have

$$\begin{aligned}(\varphi_{t_1} *_1 f)(x_1, x_2) &= \int_{\boldsymbol{R}^{n_1}} \varphi_{t_1}^{(1)}(y_1)\left[f(x_1 - y_1, x_2) - \sum_{|\gamma|\le u_1} \frac{1}{\gamma!} y_1^\gamma (D_1^\gamma f)(x_1, x_2)\right] dy_1 \\ &= \int_{|y_1|<|x_1|/2} \varphi_{t_1}^{(1)}(y_1)\left[f(x_1 - y_1, x_2) - \sum_{|\gamma|\le u_1} \frac{1}{\gamma!} y_1^\gamma (D_1^\gamma f)(x_1, x_2)\right] dy_1 \\ &\quad + \int_{|y_1|\ge|x_1|/2} \cdots \\ &\equiv I_1 + I_2.\end{aligned}$$

For the estimation of I_1, noticing that $|x_1|/2 \le |x_1 - z_1| \le 2|x_1|$ for $|z_1| \le |x_1|/2$, by $|y_1| < |x_1|/2$ and the mean value theorem, we obtain

$$\begin{aligned}&\left|f(x_1 - y_1, x_2) - \sum_{|\gamma|\le u_1} \frac{1}{\gamma!} y_1^\gamma (D_1^\gamma f)(x_1, x_2)\right| \\ &\quad = \sup_{|\gamma|=u_1+1} \sup_{|z_1|\le|x_1-y_1|} |(D_1^\gamma f)(x_1 - z_1, x_2)||y_1|^{u_1+1} \\ &\quad \lesssim |y_1|^{u_1+1} \sup_{|z_1|\le|x_1|/2} (1 + |x_1 - z_1|)^{-\sigma_1}(1 + |x_2|)^{-\sigma_2} \\ &\quad \lesssim |y_1|^{u_1+1}(1 + |x_1|)^{-\sigma_1}(1 + |x_2|)^{-\sigma_2}, \qquad (2.3)\end{aligned}$$

where $\gamma = (\gamma_1, \cdots, \gamma_{n_1}) \in \boldsymbol{Z}_+^{n_1}$, $x_1 = (x_1^1, \cdots, x_1^{n_1})$ and $D_1^\gamma = (\frac{\partial}{\partial x_1^1})^{\gamma_1} \cdots (\frac{\partial}{\partial x_1^{n_1}})^{\gamma_{n_1}}$. This leads to that

$$\begin{aligned}|I_1| &\lesssim (1 + |x_1|)^{-\sigma_1}(1 + |x_2|)^{-\sigma_2} \int_{|y_1|<|x_1|/2} |y_1|^{u_1+1}|\varphi_{t_1}^{(1)}(y_1)| \, dy_1 \\ &\lesssim t_1^{u_1+1}(1 + |x_1|)^{-\sigma_1}(1 + |x_2|)^{-\sigma_2} \int_{\boldsymbol{R}^{n_1}} |y_1|^{u_1+1}|\varphi^{(1)}(y_1)| \, dy_1 \\ &\lesssim t_1^{u_1+1}(1 + |x_1|)^{-\sigma_1}(1 + |x_2|)^{-\sigma_2}.\end{aligned}$$

To estimate I_2, similarly to (2.3), we have

$$\left|f(x_1-y_1,\,x_2)-\sum_{|\gamma|\le s_1}\frac{1}{\gamma!}\,y_1^{\gamma}(D_1^{\gamma}f)(x_1,\,x_2)\right|\lesssim|y_1|^{u_1+1}(1+|x_2|)^{-\sigma_2}. \qquad (2.4)$$

If $|x_1|\ge 1$ and $\sigma_1>0$, by $|x_1|^{-1}\le 2(1+|x_1|)^{-1}$ and (2.4), for all $t_1\in(0,\,1]$, we have

$$\begin{aligned}|I_2|&\lesssim(1+|x_2|)^{-\sigma_2}\int_{|y_1|\ge|x_1|/2}|y_1|^{u_1+1}|\varphi^{(1)}_{t_1}(y_1)|\,dy_1\\&\lesssim(1+|x_2|)^{-\sigma_2}t_1^{u_1+1}\int_{|y_1|\ge|x_1|/(2t_1)}|y_1|^{u_1+1}|\varphi^{(1)}(y_1)|\,dy_1\\&\lesssim t_1^{u_1+1}(1+|x_2|)^{-\sigma_2}\int_{|x_1|/(2t_1)}^{\infty}r_1^{-\sigma_1-1}\,dr_1\\&\lesssim t_1^{u_1+1}(1+|x_1|)^{-\sigma_1}(1+|x_2|)^{-\sigma_2}.\end{aligned}$$

If $|x_1|\le 1$ or $\sigma_1=0$, by (2.4),

$$|I_2|\lesssim t_1^{u_1+1}(1+|x_2|)^{-\sigma_2}\int_{\boldsymbol{R}^{n_1}}|y_1|^{u_1+1}|\varphi^{(1)}(y_1)|\,dy_1\lesssim t_1^{u_1+1}(1+|x_1|)^{-\sigma_1}(1+|x_2|)^{-\sigma_2}.$$

Thus combining the estimations for I_1 and I_2 yields (i).

To prove (ii), since $\varphi^{(1)}\in\mathscr{S}_0(\boldsymbol{R}^{n_1})$ and $f\in\mathscr{S}_{s_1,\,s_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$, we have

$$\begin{aligned}&(\varphi^{(1)}_{t_1}*_1 f)(x_1,\,x_2)\\&\quad=\int_{\boldsymbol{R}^{n_1}}\left[\varphi^{(1)}_{t_1}(y_1)-\sum_{|\gamma|\le s_1}\frac{1}{\gamma!}\,(y_1-x_1)^{\gamma}(D_1^{\gamma}\varphi^{(1)}_{t_1})(x_1)\right]f(x_1-y_1,\,x_2)\,dy_1\\&\quad=\int_{|x_1-y_1|<|x_1|/2}\left[\varphi^{(1)}_{t_1}(y_1)-\sum_{|\gamma|\le s_1}\frac{1}{\gamma!}\,(y_1-x_1)^{\gamma}(D_1^{\gamma}\varphi^{(1)}_{t_1})(x_1)\right]f(x_1-y_1,\,x_2)\,dy_1\\&\qquad+\int_{|x_1-y_1|\ge|x_1|/2}\cdots\\&\quad\equiv J_1+J_2.\end{aligned}$$

On the estimation for J_1, notice that if $|z_1|\le|x_1-y_1|<|x_1|/2$, then $|x_1|/2\le|x_1-z_1|\le 2|x_1|$. By this and $\varphi^{(1)}\in\mathscr{S}_0(\boldsymbol{R}^{n_1})$, we have

$$\begin{aligned}
&\left|\varphi_{t_1}^{(1)}(y_1) - \sum_{|\gamma|\le s_1} \frac{1}{\gamma!}\,(y_1-x_1)^\gamma (D_1^\gamma \varphi_{t_1}^{(1)})(x_1)\right| \\
&\quad\lesssim \sup_{|\gamma|=s_1+1} \sup_{|z_1|\le|x_1-y_1|} |(D_1^\gamma \varphi_{t_1}^{(1)})(x_1-z_1)||x_1-y_1|^{s_1+1} \\
&\quad\lesssim t_1^{-n_1-s_1-1} \sup_{|z_1|\le|x_1-y_1|} \left(1+\frac{|x_1-z_1|}{t_1}\right)^{-\sigma_1} |x_1-y_1|^{s_1+1} \\
&\quad\lesssim t_1^{-n_1-s_1-1}\left(1+\frac{|x_1|}{t_1}\right)^{-\sigma_1} |x_1-y_1|^{s_1+1}.
\end{aligned}$$

Thus, applying

$$|f(x_1-y_1,\, x_2)| \lesssim (1+|x_1-y_1|)^{-n_1-s_1-2}(1+|x_2|)^{-\sigma_2}, \tag{2.5}$$

we further have

$$\begin{aligned}
|J_1| &\lesssim t_1^{-n_1-s_1-1}(1+|x_2|)^{-\sigma_2} \int_{\boldsymbol{R}^{n_1}} \left(1+\frac{|x_1|}{t_1}\right)^{-\sigma_1} \frac{|x_1-y_1|^{s_1+1}}{(1+|x_1-y_1|)^{n_1+s_1+2}}\, dy_1 \\
&\lesssim t_1^{-n_1-s_1-1}\left(1+\frac{|x_1|}{t_1}\right)^{-\sigma_1}(1+|x_2|)^{-\sigma_2}.
\end{aligned}$$

To estimate J_2, if $|x_1|>1$ and $\sigma_1>0$, using an estimate similar to (2.5) and the estimation that

$$\left|\varphi^{(1)}(y_1) - \sum_{|\gamma|\le s_1} \frac{1}{\gamma!}\,(y_1-x_1)^\gamma (D_1^\gamma \varphi_{t_1}^{(1)})(x_1)\right| \lesssim t_1^{-n_1-s_1-1}|x_1-y_1|^{s_1+1},$$

we obtain

$$\begin{aligned}
|J_2| &\lesssim \int_{|y_1-x_1|\ge|x_1|/2} (1+|x_2|)^{-\sigma_2} t_1^{-n_1-s_1-1} \frac{|x_1-y_1|^{s_1+1}}{(1+|x_1-y_1|)^{\sigma_1+n_1+s_1+1}}\, dy_1 \\
&\lesssim t_1^{-n_1-s_1-1}(1+|x_2|)^{-\sigma_2} \int_{|x_1|/2}^{\infty} r_1^{-\sigma_1-1}\, dr_1 \\
&\lesssim t_1^{-n_1-s_2-1}(1+|x_1|)^{-\sigma_1}(1+|x_2|)^{-\sigma_2},
\end{aligned}$$

where in the last step, we used the fact that $|x_1|^{-\sigma_1} \lesssim (1+|x_1|/t_1)^{-\sigma_1}$ for $t_1\ge 1$. If $|x_1|\le 1$ or $\sigma_1=0$, by (2.5), we then have

$$\begin{aligned}|J_2| &\lesssim (1+|x_2|)^{-\sigma_2} t_1^{-n_1-s_1-1} \int_0^\infty \frac{r_1^{n_1+s_1}}{(1+r_1)^{n_1+s_1+2}}\,dr_1 \\ &\lesssim t_1^{-n_1-s_1-1}\left(1+\frac{|x_1|}{t_1}\right)^{-\sigma_1}(1+|x_2|)^{-\sigma_2}.\end{aligned}$$

This gives (ii).

To prove (iii), by an argument similar to (i), we obtain that for all $t_2 \in (0,1]$,

$$|(\varphi_{t_2}^{(2)} *_2 f)(x_1, x_2)| \lesssim t_2^{u_2+1}(1+|x_1|)^{-\sigma_1}(1+|x_2|)^{-\sigma_2}, \tag{2.6}$$

where and in what follows, $(\varphi_{t_2}^{(2)} *_2 f)(x_1, x_2) \equiv \int_{\boldsymbol{R}^{n_2}} \varphi_{t_2}^{(2)}(y_2) f(x_1, x_2-y_2)\,dy_2$. Thus, if $|y_1| < |x_1|/2$, then by the mean value theorem, (2.6) and the fact that $|x_1 - z_1| \sim |x_1|$ for $|z_1| \le |x_1|/2$, we have

$$\begin{aligned}&\left|(\varphi_{t_2}^{(2)} *_2 f)(x_1-y_1, x_2) - \sum_{|\gamma|\le u_1} \frac{1}{\gamma!}(y_1-x_1)^\gamma \partial_1^\gamma (\varphi_{t_2}^{(2)} *_2 f)(x_1, x_2)\right| \\ &\quad\le |y_1|^{u_1+1} \sup_{|\gamma|=u_1+1}\ \sup_{|z_1|\le|x_1|/2} |(\varphi_{t_2}^{(2)} *_2 (D_1^\gamma f))(x_1-z_1, x_2)| \\ &\quad\lesssim t_2^{u_2+1}|y_1|^{u_1+1}(1+|x_1|)^{-\sigma_1}(1+|x_2|)^{-\sigma_2}.\end{aligned} \tag{2.7}$$

If $|y_1| \ge |x_1|/2$, by the mean value theorem and (2.6), we then have

$$\begin{aligned}&\left|(\varphi_{t_2}^{(2)} *_2 f)(x_1-y_1, x_2) - \sum_{|\gamma|\le s_1} \frac{1}{\gamma!}(y_1-x_1)^\gamma \partial_1^\gamma (\varphi_{t_2}^{(2)} *_2 f)(x_1, x_2)\right| \\ &\quad\lesssim t_2^{u_2+1}|y_1|^{u_1+1}(1+|x_2|)^{-\sigma_2}.\end{aligned} \tag{2.8}$$

Noticing that

$$(\varphi_{t_1,t_2} * f)(x_1, x_2) = (\varphi_{t_1}^{(1)} *_1 (\varphi_{t_2}^{(2)} *_2 f))(x_1, x_2), \tag{2.9}$$

replacing (2.3) and (2.4) respectively by (2.7) and (2.8), and repeating the proof of (i), we obtain (iii).

For (v), by (2.6), we have

$$|(\varphi_{t_2}^{(2)} *_2 f)(x_1-y_1, x_2)| \lesssim t_2^{u_2+1}(1+|x_2|)^{-\sigma_2}(1+|x_1-y_1|)^{-n_1-s_1-2}$$

for all $t_2 \in (0, 1]$. Replacing (2.5) by this estimate, using (2.9) and repeating the proof of (ii) lead to (v). A similar argument to (v) yields (iv).

To obtain (vi), by an argument similar to (ii), we obtain

$$|(\varphi_{t_2}^{(2)} *_2 f)(x_1 - y_1,\, x_2)| \lesssim t_2^{-n_1-s_1-1}(1 + |x_1 - y_1|)^{-n_1-s_1-2}\left(1 + \frac{|x_2|}{t_2}\right)^{-\sigma_2}$$

for all $t_2 \in [1, \infty)$. Replacing (2.5) by this, using (2.9) and repeating the proof of (ii) leads to (vi). This finishes the proof of Lemma 2.2. □

Proof of Lemma 2.1. Let $\epsilon_1 \in (0,\, 1)$. Notice that for all $t_1 \in (0,\, \infty)$, $|y_1| \le t_1$ and $x \in \boldsymbol{R}^{n_1}$, we have $t_1 + |x_1| \le 2(t_1 + |x_1 - y_1|)$. By this and Lemma 2.2 (iii) and (iv), we have that for any $t_1 \in (0, \epsilon_1)$, $t_2 \in (0,\, 1)$, $|y_1| < t_1$, $|y_2| < t_2$ and $(x_1,\, x_2) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$,

$$|(\varphi_{t_1,\,t_2} * f)(x_1 - y_1,\, x_2 - y_2)| \lesssim t_1 t_2 (1 + |x_1|)^{-\sigma_1}(1 + |x_2|)^{-\sigma_2}, \tag{2.10}$$

and that for any $t_1 \in (0, \epsilon_1]$, $t_2 \in [1, \infty)$, $|y_1| < t_1$, $|y_2| < t_2$ and $(x_1,\, x_2) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$,

$$|(\varphi_{t_1,\,t_2} * f)(x_1 - y_1,\, x_2 - y_2)| \lesssim t_1 t_2^{\sigma_2-n_2-s_2-1}(1 + |x_1|)^{-\sigma_1}(1 + |x_2|)^{-\sigma_2}. \tag{2.11}$$

From this and $\sigma_2 < n_2 + s_2 + 1$, it follows that

$$\begin{aligned}
&\sup_{(x_1,\,x_2)\in \boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}} (1 + |x_1|)^{\sigma_1}(1 + |x_2|)^{\sigma_2} \int_0^{\epsilon_1}\int_0^{\infty}\int_{\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}} |\psi_{t_1,\,t_2}(y_1,\, y_2)| \\
&\qquad \times |(\varphi_{t_1,\,t_2} * f)(x_1 - y_1,\, x_2 - y_2)|\, dy_1\, dy_2\, \frac{dt_1}{t_1}\, \frac{dt_2}{t_2} \\
&\lesssim \int_0^{\epsilon_1}\int_0^{\infty}\int_{\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}} \frac{1}{1 + t^{n_2+s_2+2-\sigma_2}}\, |\varphi_{t_1,\,t_2}(y_1,\, y_2)|\, dy_1\, dy_2\, dt_1\, dt_2 \\
&\lesssim \epsilon_1.
\end{aligned}$$

Let $L_1 > 1$. By Lemma 2.1 (v) and (vi), we have that for any $t_1 \in (L_1,\, \infty)$, $t_2 \in (0,\, 1)$, $|y_1| < t_1$, $|y_2| < t_2$ and $(x_1,\, x_2) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$,

$$|(\varphi_{t_1,\,t_2} * f)(x_1 - y_1,\, x_2 - y_2)| \lesssim t_1^{\sigma_1-n_1-s_1-1} t_2 (1 + |x_1|)^{-\sigma_1}(1 + |x_2|)^{-\sigma_2}, \tag{2.12}$$

and that for any $t_1 \in (L_1,\, \infty)$, $t_2 \in [1,\, \infty)$, $|y_1| < t_1$, $|y_2| < t_2$ and $(x_1,\, x_2) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$,

$$|(\varphi_{t_1,t_2}*f)(x_1-y_1,x_2-y_2)|\lesssim t_1^{\sigma_1-n_1-s_1-1}t_2^{\sigma_2-n_2-s_2-1}(1+|x_1|)^{-\sigma_1}(1+|x_2|)^{-\sigma_2}. \quad (2.13)$$

From this, (2.12), $\sigma_1 < n_1+s_1+1$ and $\sigma_2 < n_2+s_2+1$, it follows that

$$\begin{aligned}
&\sup_{(x_1,x_2)\in \boldsymbol{R}^{n_1}\times \boldsymbol{R}^{n_2}}(1+|x_1|)^{\sigma_1}(1+|x_2|)^{\sigma_2}\int_{L_1}^\infty\int_0^\infty\int_{\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}}|\varphi_{t_1,t_2}(y_1,\,y_2)|\\
&\qquad\times|(\varphi_{t_1,t_2}*f)(x_1-y_1,\,x_2-y_2)|\,dy_1\,dy_2\,\frac{dt_1}{t_1}\frac{dt_2}{t_2}\\
&\qquad\lesssim\int_{L_1}^\infty\int_0^\infty\int_{\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}}|\varphi_{t_1,t_2}(y_1,\,y_2)|\,dy_1\,dy_2\,\frac{dt_1}{t_1^{n_1+s_1+2-\sigma_1}}\,\frac{dt_2}{1+t_2^{n_2+s_2+2-\sigma_2}}\\
&\qquad\lesssim(L_1)^{\sigma_1-n_1-s_1-1}.
\end{aligned}$$

Using the symmetry, we then obtain the desired estimates for the cases $\epsilon_2\in(0,\,1)$, $L_2\in(1,\,\infty)$, $(t_1,\,t_2)\in(0,\,\infty)\times(0,\,\epsilon_2)$ or $(t_1,\,t_2)\in(0,\,\infty)\times(L_2,\,\infty)$, which gives the first inequality of Lemma 2.1.

To prove (2.2), notice that if $|y_1|>2L_1>2$ and $|x_1-y_1|<t_1<L_1$, we have $|x_1|>|y_1|-|x_1-y_1|>L_1$. Then by (2.10) through (2.13) with σ_i replaced by $\sigma_i'\in(\sigma_i,\,n_1-s_1-1)$, we have

$$\begin{aligned}
&\sup_{(x_1,x_2)\in \boldsymbol{R}^{n_1}\times \boldsymbol{R}^{n_2}}(1+|x_1|)^{\sigma_1}(1+|x_2|)^{\sigma_2}\int_0^{L_1}\int_0^\infty\int_{[B^{(1)}(0,\,2L_1)]^\complement\times\boldsymbol{R}^{n_2}}|(\varphi_{t_1,t_2}*f)(y_1,\,y_2)|\\
&\quad\times|\psi_{t_1,t_2}(x_1-y_1,\,x_2-y_2)|\,dy_1\,dy_2\,\frac{dt_1}{t_1}\frac{dt_2}{t_2}\\
&\lesssim\sup_{|x_1|>L_1,\,x_2\in\boldsymbol{R}^{n_2}}(1+|x_1|)^{\sigma_1-\sigma_1'}(1+|x_2|)^{\sigma_2-\sigma_2'}\int_0^\infty\int_0^\infty\int_{\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}}|\psi_{t_1,t_2}(y_1,\,y_2)|\\
&\quad\times\frac{1}{1+t^{n_1-s_1+2-\sigma_1'}}\,\frac{1}{1+t^{n_2-s_2+2-\sigma_2'}}\,dy_1\,dy_2\,dt_1\,dt_2\\
&\lesssim(L_1)^{\sigma_1-\sigma_1'},
\end{aligned}$$

which gives (2.2) and hence completes the proof of Lemma 2.1. $\square$

Let $p\in(0,1]$, $s_i\ge\lfloor n_i(1/p-1)\rfloor$ and $\varphi\in\mathscr{S}_{s_i}(\boldsymbol{R}^{n_i})$ such that (2.1) holds for $i=1,\,2$. For $f\in\mathscr{S}'(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ and $(x_1,x_2)\in\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}$, we define

$$\begin{aligned}
&S(f)(x_1,\,x_2)\\
&\quad\equiv\left(\int_0^\infty\int_0^\infty\int_{|y_1-x_1|<t_1}\int_{|y_2-x_2|<t_2}|(\varphi_{t_1t_2}*f)(y_1,\,y_2)|^2\,dy_1\,dy_2\,\frac{dt_1}{t_1^{n_1+1}}\,\frac{dt_2}{t_2^{n_2+1}}\right)^{1/2}.
\end{aligned}$$

It is well-known that $f \in H^p(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2})$ if and only if $f \in \mathscr{S}'(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2})$ and $S(f) \in L^p(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2})$. Moreover,

$$\|f\|_{H^p(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})} \sim \|S(f)\|_{L^p(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})};$$

see [**3**], [**4**], [**5**], [**10**]. Using this fact, Lemma 2.1 and some ideas from [**3**], [**4**], [**5**], [**10**], we obtain the following conclusion.

Lemma 2.3. *Let $p \in (0,1]$, $s_i \ge \lfloor n_i(1/p-1)\rfloor$ and $\sigma_i \in (\max\{n_i+s_i,\, n_i/p\},\, n_i+s_i+1)$ for $i=1,\,2$. Then for any $f \in \mathscr{D}_{s_1,\,s_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$, there exist numbers $\{\lambda_k\}_{k\in\boldsymbol{N}} \subset \boldsymbol{C}$ and $(p,\,2,\,s_1,\,s_2)$-atoms $\{a_k\}_{k\in\boldsymbol{N}} \subset \mathscr{D}_{s_1,\,s_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ such that $f = \sum_{k\in\boldsymbol{N}} \lambda_k a_k$ in $\mathscr{D}_{s_1,\,s_2;\,\sigma_1,\,\sigma_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ and $\{\sum_{k\in\boldsymbol{N}}|\lambda_k|^p\}^{1/p} \le C\|f\|_{H^p(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})}$, where C is a positive constant independent of f.*

Proof. We use $\mathscr{R}$ to denote the set of all dyadic rectangles in $\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}$. For $k\in\boldsymbol{Z}$, let

$$\Omega_k \equiv \{(x_1,\,x_2)\in\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}:\ S(f)(x_1,\,x_2) > 2^k\}$$

and

$$\widetilde{\Omega}_k \equiv \{(x_1,\,x_2)\in\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}:\ M_s(\chi_{\Omega_k})(x_1,\,x_2) > 1/2\},$$

where M_s denotes the strong maximal operator on $\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}$. It is easy to see that Ω_k is a bounded set. In fact, observing that $1+|x_i| \le t_i+|x_i| \sim t_i+|y_i|$ for $|x_i-y_i|<t_i$ and $t_i\ge 1$, by Lemma 2.2 and $n_i+s_i+1-\sigma_i>0$, we have

$$\begin{aligned}
&[S(f)(x_1,\,x_2)]^2\\
&\quad\lesssim \int_0^1\int_0^1\int_{|y_1-x_1|<t_1}\int_{|y_2-x_2|<t_2}(1+|y_1|)^{-2\sigma_1}(1+|y_2|)^{-2\sigma_2}\,dy_1\,dy_2\,\frac{dt_1}{t_1^{n_1}}\,\frac{dt_2}{t_2^{n_2}}\\
&\qquad+\int_1^\infty\int_0^1\int_{|y_1-x_1|<t_1}\int_{|y_2-x_2|<t_2}\left(1+\frac{|y_1|}{t_1}\right)^{-2\sigma_1}(1+|y_2|)^{-2\sigma_2}\,dy_1\,dy_2\,\frac{dt_1}{t_1^{3n_1+2s_1+3}}\,\frac{dt_2}{t_2^{n_2}}\\
&\qquad+\int_0^1\int_1^\infty\int_{|y_1-x_1|<t_1}\int_{|y_2-x_2|<t_2}(1+|y_1|)^{-2\sigma_1}\left(1+\frac{|y_2|}{t_2}\right)^{-2\sigma_2}\,dy_1\,dy_2\,\frac{dt_1}{t_1^{n_1}}\,\frac{dt_2}{t_2^{3n_2+2s_2+3}}\\
&\qquad+\int_1^\infty\int_1^\infty\int_{|y_1-x_1|<t_1}\int_{|y_2-x_2|<t_2}\left(1+\frac{|y_1|}{t_1}\right)^{-2\sigma_1}\left(1+\frac{|y_2|}{t_2}\right)^{-2\sigma_2}\,dy_1\,dy_2\\
&\qquad\quad\times\frac{dt_1}{t_1^{2n_1+s_1+2}}\,\frac{dt_2}{t_2^{3n_2+2s_2+3}}\\
&\quad\lesssim (1+|x_1|)^{-2\sigma_1}(1+|x_2|)^{-2\sigma_2}.
\end{aligned}$$

Thus, for any $k \in \boldsymbol{Z}$, Ω_k is a bounded set in $\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$ and so is $\widetilde{\Omega}_k$.

For each dyadic rectangle $R = I \times J$, set

$$\mathscr{A}(R) \equiv \{(y_1, y_2, t_1, t_2) :\ (y_1, y_2) \in R,\\ \sqrt{n_1}|I| < t_1 \le 2\sqrt{n_1}|I|,\ \sqrt{n_2}|J| < t_2 \le 2\sqrt{n_2}|J|\},$$

and

$$\mathscr{R}_k \equiv \{R \in \mathscr{R} :\ |R \cap \Omega_k| \ge 1/2,\ |R \cap \Omega_{k+1}| < 1/2\}.$$

Obviously, for each $R \in \mathscr{R}$, there exists a unique $k \in \boldsymbol{Z}$ such that $R \in \mathscr{R}_k$.

From (2.1), for any $(x_1, x_2) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$, it is easy to see that

$$f(x_1, x_2) = \sum_{k \in \boldsymbol{Z}} \left\{ \sum_{R \in \mathscr{R}_k} \int_{\mathscr{A}(R)} \psi_{t_1, t_2}(x_1 - y_1, x_2 - y_2)(\varphi_{t_1, t_2} * f)(y_1, y_2)\, dy_1\, dy_2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right\}.$$

Let $\lambda_k \equiv \frac{1}{C} 2^k |\Omega_k|^{1/p}$ and

$$a_k(x_1, x_2) \equiv \lambda_k^{-1} \sum_{R \in \mathscr{R}_k} \int_{\mathscr{A}(R)} \psi_{t_1, t_2}(x_1 - y_1, x_2 - y_2)(\varphi_{t_1, t_2} * f)(y_1, y_2)\, dy_1\, dy_2 \frac{dt_1}{t_1} \frac{dt_2}{t_2},$$

where C is a positive constant. By the argument used in [**3**], [**4**], [**5**], [**10**], we see that if we suitably choose the constant C, then $\{a_k\}_{k \in \boldsymbol{Z}}$ are $(p, 2, s_1, s_2)$-atoms and

$$\left\{ \sum_{k \in \boldsymbol{Z}} |\lambda_k|^p \right\}^{1/p} \lesssim \|f\|_{H^p(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2})}.$$

It remains to prove that $f = \sum_{k \in \boldsymbol{Z}} \lambda_k a_k$ converges in $\mathscr{D}_{s_1, s_2; \sigma_1, \sigma_2}(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2})$. Since $\widetilde{\Omega}_k$ is bounded, we may assume that $\widetilde{\Omega}_k \subset B^{(1)}(0, 2^{L_1}) \times B^{(2)}(0, 2^{L_2})$. Then for any $\alpha \in \boldsymbol{Z}_+^{n_1}$ and $\beta \in \boldsymbol{Z}_+^{n_2}$, by Lemma 2.2, we have

$$\begin{aligned} &\sum_{R \in \mathscr{R}_k} \int_{\mathscr{A}(R)} |(\partial_{x_1}^{\alpha} \partial_{x_2}^{\beta} \psi_{t_1, t_2})(x_1 - y_1, x_2 - y_2)| |(\varphi_{t_1, t_2} * f)(y_1, y_2)|\, dy_1\, dy_2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \\ &\quad \lesssim \sum_{R \in \mathscr{R}_k} \int_{\mathscr{A}(R)} |(\varphi_{t_1, t_2} * f)(y_1, y_2)|\, dy_1\, dy_2 \frac{dt_1}{t_1^{1+|\alpha|+n_1}} \frac{dt_2}{t_2^{1+|\beta|+n_2}} \\ &\quad \lesssim \int_{B^{(1)}(0, 2^{L_2})} \int_{B^{(2)}(0, 2^{L_2})} \int_0^{L_1} \int_0^{L_2} dt_1\, dt_2\, dy_1\, dy_2 < \infty, \end{aligned}$$

where $(x_1, x_2) \in \widetilde{\Omega}_k$. This shows that $a_k \in \mathscr{D}_{s_1, s_2, \sigma_1, \sigma_2}(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2})$. Moreover, assume that $\operatorname{supp} f \subset B^{(1)}(0, r_1) \times B^{(2)}(0, r_2)$. For any $N_i > 1 + \log r_i$ with $i = 1, 2$, let

$$E_{N_1, N_2} \equiv B^{(1)}(0, 2^{N_1}) \times B^{(2)}(0, 2^{N_2}) \times [2^{-N_1}, 2^{N_1}] \times [2^{-N_2}, 2^{N_2}].$$

Then there exist finite dyadic rectangles R, whose set is denoted by $\mathscr{R}^{N_1, N_2}$, such that $\mathscr{A}(R) \cap E_{N_1, N_2} \neq \emptyset$. For each $R \in \mathscr{R}^{N_1, N_2}$, there exists a unique $k \in \boldsymbol{Z}$ such that $R \in \Omega_k$. Let K_{N_1, N_2} be the maximal integer of the absolute values of all such k. Then for $K > K_{N_1, N_2}$, by the facts $\mathscr{R}^{N_1, N_2} \subset \cup_{|k| \le K} \mathscr{R}_k$ and Lemma 2.1 together with $\sigma_i < \sigma_i' < n_i + s_i + 1$ for $i = 1, 2$, we then have

$$\begin{aligned}
&\left\| f - \sum_{|k| \le K} \lambda_k a_k \right\|_{\mathscr{D}_{s_1, s_2; \sigma_1, \sigma_2}(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2})} \\
&\quad \lesssim \sup_{(x_1, x_2) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}} (1 + |x_1|)^{\sigma_1} (1 + |x_2|)^{\sigma_2} \\
&\quad \times \left(\int_0^{2^{-N_1}} \int_0^{\infty} + \int_{2^{N_1}}^{\infty} \int_0^{\infty} + \int_0^{\infty} \int_0^{2^{-N_2}} + \int_0^{\infty} \int_{2^{N_2}}^{\infty} \right) \int_{\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}} |(\varphi_{t_1, t_2} * f)(y_1, y_2)| \\
&\quad \times |\varphi_{t_1, t_2}(x_1 - y_1, x_2 - y_2)| \, dy_1 \, dy_2 \, \frac{dt_1}{t_1} \frac{dt_2}{t_2} + \sup_{(x_1, x_2) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}} (1 + |x_1|)^{\sigma_1} (1 + |x_2|)^{\sigma_2} \\
&\quad \times \int_{2^{-N_1}}^{2^{N_1}} \int_0^{\infty} \int_{[B^{(1)}(0, 2^{N_1})]^{\complement} \times \boldsymbol{R}^{n_2}} |(\varphi_{t_1, t_2} * f)(y_1, y_2)| \\
&\quad \times |\psi_{t_1, t_2}(x_1 - y_1, x_2 - y_2)| \, dy_1 \, dy_2 \, \frac{dt_1}{t_1} \frac{dt_2}{t_2} + \sup_{(x_1, x_2) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}} (1 + |x_1|)^{\sigma_1} (1 + |x_2|)^{\sigma_2} \\
&\quad \times \int_0^{\infty} \int_{2^{-N_2}}^{2^{N_2}} \int_{\boldsymbol{R}^{n_1} \times [B^{(2)}(0, 2^{N_2})]^{\complement}} |(\varphi_{t_1, t_2} * f)(y_1, y_2)| \\
&\quad \times |\varphi_{t_1, t_2}(x_1 - y_1, x_2 - y_2)| \, dy_1 \, dy_2 \, \frac{dt_1}{t_1} \frac{dt_2}{t_2} \\
&\quad \lesssim 2^{-N_1} + 2^{-N_2} + 2^{N_1(\sigma_1 - \sigma_1')} + 2^{N_2(\sigma_2 - \sigma_2')}.
\end{aligned}$$

This implies the desired conclusion and hence, finishes the proof of Lemma 2.3. □

The following result plays a key role in the proof of Theorem 1.2. In what follows, for any $f \in \mathscr{D}(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2})$, we set

$$\sup_{x_2 \in \boldsymbol{R}^{n_2}} \operatorname{diam}(\operatorname{supp} f(\cdot, x_2)) \equiv \sup_{x_1, y_1 \in \boldsymbol{R}^{n_1}, x_2 \in \boldsymbol{R}^{n_2}} \{|x_1 - y_1| : f(x_1, x_2) \neq 0, \ f(y_1, x_2) \neq 0\},$$

and $\sup_{x_1\in\boldsymbol{R}^{n_1}} \operatorname{diam}(\operatorname{supp} f(x_1,\cdot))$ is similarly defined by interchanging x_1 and x_2, and y_1 and y_2.

LEMMA 2.4. *Let $p\in(0,1]$, $q\in[p,1]$ and $\mathscr{B}_q$ be a q-quasi-Banach space. Let $s_1, s_2\in\boldsymbol{Z}_+$ and T be a $\mathscr{B}_q$-sublinear operator from $\mathscr{D}_{s_1,s_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ to $\mathscr{B}_q$. If there exists a positive constant C such that for any $f\in\mathscr{D}_{s_1,s_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$,*

$$\begin{aligned}\|Tf\|_{\mathscr{B}_q}\le C&\left[\sup_{x_2\in\boldsymbol{R}^{n_2}}\operatorname{diam}(\operatorname{supp} f(\cdot,x_2))\right]^{n_1/p}\\&\times\left[\sup_{x_1\in\boldsymbol{R}^{n_1}}\operatorname{diam}(\operatorname{supp} f(x_1,\cdot))\right]^{n_2/p}\|f\|_{L^\infty(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})},\end{aligned}$$

then T can be extended as a bounded $\mathscr{B}_q$-sublinear operator from $\mathscr{D}_{s_1,s_2;\sigma_1,\sigma_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ to $\mathscr{B}_q$.

PROOF. Let $\psi\in C^\infty(\boldsymbol{R})$ such that $0\le\psi(x)\le1$ for all $x\in\boldsymbol{R}$, $\psi(x)=1$ if $|x|\le1/2$ and $\psi(x)=0$ if $|x|\ge1$. Let $\phi(x)\equiv\psi(x/2)-\psi(x)$ for all $x\in\boldsymbol{R}$. Then $\operatorname{supp}\phi\subset\{x\in\boldsymbol{R}:\ 1/2\le|x|\le2\}$ and $\sum_{j\in\boldsymbol{Z}}\phi(2^{-j}x)=1$ for all $x\in\boldsymbol{R}\setminus\{0\}$. Let $\Phi_j(x)\equiv\phi(2^{-j}x)$ for all $x\in\boldsymbol{R}$ and $j\in\boldsymbol{N}$, and $\Phi_0(x)\equiv1-\sum_{j=1}^\infty\phi(2^{-j}x)$ for all $x\in\boldsymbol{R}$. Then $\sum_{j\in\boldsymbol{Z}_+}\Phi_j(x)=1$ for all $x\in\boldsymbol{R}$.

Let $i=1,2$. For $j_i\in\boldsymbol{Z}_+$ and $x_i\in\boldsymbol{R}^{n_i}$, let $\Phi^{(i)}_{j_i}(x_i)\equiv\Phi_{j_i}(|x_i|)$. Then for all $x_i\in\boldsymbol{R}^{n_i}$, we have $\sum_{j_i\in\boldsymbol{Z}_+}\Phi^{(i)}_{j_i}(x_i)=1$. Set $R^{(i)}_0\equiv B^{(i)}(0,2)$ and $R^{(i)}_{j_i}\equiv\{x_i\in\boldsymbol{R}^{n_i}:\ 2^{j_i-1}\le|x_i|\le2^{j_i+1}\}$ for $j_i\in\boldsymbol{N}$. Then $\operatorname{supp}\Phi^{(i)}_{j_i}\subset R^{(i)}_{j_i}$ for $j_i\in\boldsymbol{Z}_+$. For $j_i\in\boldsymbol{Z}_+$, let $\{\widetilde{\psi}^{(i)}_{j_i,\alpha_i}:\ |\alpha_i|\le s_i\}\subset C^\infty(\boldsymbol{R}^n)$ be the dual basis of $\{x_i^{\alpha_i}:\ |\alpha_i|\le s_i\}$ with respect to weight $\Phi^{(i)}_{j_i}|R^{(i)}_{j_i}|^{-1}$, namely, for all $\alpha_i,\beta_i\in\boldsymbol{Z}_+$ with $|\alpha_i|\le s_i$ and $|\beta_i|\le s_i$,

$$\frac{1}{|R^{(i)}_{j_i}|}\int_{\boldsymbol{R}^{n_i}}x_i^{\beta_i}\widetilde{\psi}^{(i)}_{j_i,\alpha_i}(x_i)\Phi^{(i)}_{j_i}(x_i)\,dx_i=\delta_{\alpha_i,\beta_i}.$$

Let $\psi^{(i)}_{j_i,\alpha_i}\equiv|R^{(i)}_{j_i}|^{-1}\widetilde{\psi}^{(i)}_{j_i,\alpha_i}\Phi^{(i)}_{j_i}$. Then for $j_i\in\boldsymbol{N}$ and $x_i\in\boldsymbol{R}^{n_i}$, we have

$$\psi^{(i)}_{j_i,\alpha_i}(x_i)=2^{-(j_i-1)(n_i+|\alpha_i|)}\psi^{(i)}_{1,\alpha_i}(2^{-(j_i-1)}x_i).$$

From this, it is easy to see that for all $j_i\in\boldsymbol{Z}_+$ and $|\alpha_i|\le s$,

$$\|\psi^{(i)}_{j_i,\alpha_i}\|_{L^\infty(\boldsymbol{R}^{n_i})}\lesssim2^{-j_i(n_i+|\alpha_i|)}.\tag{2.14}$$

For $f \in \mathscr{D}_{s_1, s_2}(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2})$, assume that $\operatorname{supp} f \subset B^{(1)}(0, 2^{k_1}) \times B^{(2)}(0, 2^{k_2})$ for some $k_1, k_2 \in \boldsymbol{N}$ and $\|f\|_{\mathscr{D}_{s_1, s_2; \sigma_1, \sigma_2}(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2})} = 1$ by the $\mathscr{B}_q$-sublinear property of T. For $j_1, j_2 \in \boldsymbol{Z}_+$, we set $f_{j_1, j_2} \equiv f \Phi^{(1)}_{j_1} \Phi^{(2)}_{j_2}$, and for any $(x_1, x_2) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$,

$$P^{(1)}_{j_1, j_2}(x_1, x_2) \equiv \sum_{|\alpha_1| \le s_1} \psi^{(1)}_{j_1, \alpha_1}(x_1) \int_{\boldsymbol{R}^{n_1}} f_{j_1, j_2}(y_1, x_2) y_1^{\alpha_1}\, dy_1,$$
$$P^{(2)}_{j_1, j_2}(x_1, x_2) \equiv \sum_{|\alpha_2| \le s_2} \psi^{(2)}_{j_2, \alpha_2}(x_2) \int_{\boldsymbol{R}^{n_2}} f_{j_1, j_2}(x_1, y_2) y_2^{\alpha_2}\, dy_2$$

and

$$P_{j_1, j_2}(x_1, x_2) \equiv \sum_{|\alpha_1| \le s_1} \sum_{|\alpha_2| \le s_2} \psi^{(1)}_{j_1, \alpha_1}(x_1) \psi^{(2)}_{j_2, \alpha_2}(x_2) \int_{\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}} f_{j_1, j_2}(y_1, y_2) y_1^{\alpha_1} y_2^{\alpha_2}\, dy_1\, dy_2.$$

Then

$$\begin{aligned} f = & \sum_{j_1=0}^{k_1+1} \sum_{j_2=0}^{k_2+1} \Big(f_{j_1, j_2} - P^{(1)}_{j_1, j_2} - P^{(2)}_{j_1, j_2} + P_{j_1, j_2} \Big) \\ & + \sum_{j_1=0}^{k_1+1} \sum_{j_2=0}^{k_2+1} \Big(P^{(1)}_{j_1, j_2} - P_{j_1, j_2} \Big) + \sum_{j_1=0}^{k_1+1} \sum_{j_2=0}^{k_2+1} \Big(P^{(2)}_{j_1, j_2} - P_{j_1, j_2} \Big) + \sum_{j_1=0}^{k_1+1} \sum_{j_2=0}^{k_2+1} P_{j_1, j_2}. \end{aligned}$$

By the definition of $\mathscr{D}_{s_1, s_2; \sigma_1, \sigma_2}(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2})$, it is easy to see that

$$\|f_{j_1, j_2}\|_{L^\infty(\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2})} \lesssim 2^{-j_1 \sigma_1} 2^{-j_2 \sigma_2}. \tag{2.15}$$

Using $\|\Phi^{(i)}_{j_i}\|_{L^\infty(\boldsymbol{R}^{n_i})} \le 1$, we obtain

$$\left\| \int_{\boldsymbol{R}^{n_1}} f_{j_1, j_2}(y_1, \cdot) y_1^{\alpha_1}\, dy_1 \right\|_{L^\infty(\boldsymbol{R}^{n_2})} \lesssim 2^{j_1(n_1 + |\alpha_1| - \sigma_1)} 2^{-j_2 \sigma_2}, \tag{2.16}$$

$$\left\| \int_{\boldsymbol{R}^{n_2}} f_{j_1, j_2}(\cdot, y_2) y_2^{\alpha_2}\, dy_2 \right\|_{L^\infty(\boldsymbol{R}^{n_1})} \lesssim 2^{-j_1 \sigma_1} 2^{j_2(n_2 + |\alpha_2| - \sigma_2)}, \tag{2.17}$$

and

$$\left| \int_{\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}} f_{j_1, j_2}(y_1, y_2) y_1^{\alpha_1} y_2^{\alpha_2}\, dy_1\, dy_2 \right| \lesssim 2^{j_1(n_1 + |\alpha_1| - \sigma_1)} 2^{j_2(n_2 + |\alpha_2| - \sigma_2)}. \tag{2.18}$$

By the estimates (2.14) through (2.18), we have

$$\left\| f_{j_1,j_2} - P^{(1)}_{j_1,j_2} - P^{(2)}_{j_1,j_2} + P_{j_1,j_2} \right\|_{L^\infty(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})} \lesssim 2^{-j_1\sigma_1} 2^{-j_2\sigma_2}.$$

Since $f_{j_1,j_2} - P^{(1)}_{j_1,j_2} - P^{(2)}_{j_1,j_2} + P_{j_1,j_2} \in \mathscr{D}_{s_1,s_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$, by the assumption of the lemma, we then have

$$\left\| T\Big(f_{j_1,j_2} - P^{(1)}_{j_1,j_2} - P^{(2)}_{j_1,j_2} + P_{j_1,j_2} \Big) \right\|_{\mathscr{B}_q} \lesssim 2^{j_1(n_1/p-\sigma_1)} 2^{j_2(n_2/p-\sigma_2)},$$

and hence, by $\sigma_i > n_i/p$ for $i = 1, 2$,

$$\begin{aligned} &\left\| T\left[\sum_{j_1=0}^{k_1+1} \sum_{j_2=0}^{k_2+1} \Big(f_{j_1,j_2} - P^{(1)}_{j_1,j_2} - P^{(2)}_{j_1,j_2} + P_{j_1,j_2} \Big) \right] \right\|_{\mathscr{B}_q} \\ &\qquad \lesssim \left\{ \sum_{j_1=0}^{k_1+1} \sum_{j_2=0}^{k_2+1} 2^{j_1 q(n_1/p-\sigma_1)} 2^{j_2 q(n_2/p-\sigma_2)} \right\}^{1/q} \lesssim 1. \end{aligned} \tag{2.19}$$

Moreover, we write

$$\begin{aligned} &\sum_{j_1=0}^{k_1+1} \sum_{j_2=0}^{k_2+1} \Big[P^{(1)}_{j_1,j_2}(x_1, x_2) - P_{j_1,j_2}(x_1, x_2) \Big] \\ &\quad = \sum_{|\alpha_1|\le s_1} \sum_{j_1=1}^{k_1+1} \sum_{j_2=0}^{k_2+1} \sum_{\ell_1=j_1}^{k_1+1} \Big[\psi^{(1)}_{j_1,\alpha_1}(x_1) - \psi^{(1)}_{j_1-1,\alpha_1}(x_1) \Big] \Big[\int_{\boldsymbol{R}^{n_1}} f_{\ell_1,j_2}(y_1, x_2) y_1^{\alpha_1}\, dy_1 \\ &\qquad - \sum_{|\alpha_2|\le s_2} \psi^{(2)}_{j_2,\alpha_2}(x_2) \int_{\boldsymbol{R}^{n_1}} \int_{\boldsymbol{R}^{n_2}} f_{\ell_1,j_2}(y_1, y_2) y_1^{\alpha_1} y_2^{\alpha_2}\, dy_1\, dy_2 \Big] \\ &\quad \equiv \sum_{|\alpha_1|\le s_1} \sum_{j_1=1}^{k_1+1} \sum_{j_2=0}^{k_2+1} \sum_{\ell_1=j_1}^{k_1+1} A_{\alpha_1,j_1,\ell_1,j_2}(x_1, x_2). \end{aligned}$$

By (2.14), (2.15) and (2.18), we have

$$\| A_{\alpha_1,j_1,\ell_1,j_2} \|_{L^\infty(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})} \lesssim 2^{-j_1(n_1+|\alpha_1|)} 2^{\ell_1(n_1+|\alpha_1|-\sigma_1)} 2^{-j_2\sigma_2}.$$

Noticing that $A_{\alpha_1,j_1,\ell_1,j_2} \in \mathscr{D}_{s_1,s_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$, by the assumption of the lemma, we obtain

$$\big\| T\big(A_{\alpha_1,j_1,\ell_1,j_2} \big) \big\|_{\mathscr{B}_q} \lesssim 2^{j_1(n_1/p-n_1-|\alpha_1|)} 2^{\ell_1(n_1+|\alpha_1|-\sigma_1)} 2^{j_2(n_2/p-\sigma_2)}.$$

Thus, by $\sigma_i \in (\max\{n_i/p,\, n_i+s_i\},\, n_i+s_i+1)$ for $i=1,\,2$, we further have

$$\begin{aligned}
&\left\|T\left[\sum_{j_1=0}^{k_1+1}\sum_{j_2=0}^{k_2+1}\left(P^{(1)}_{j_1,\,j_2}-P_{j_1,\,j_2}\right)\right]\right\|_{\mathscr{B}_q}\\
&\quad\lesssim\left\{\sum_{|\alpha_1|\le s_1}\sum_{j_1=1}^{k_1+1}\sum_{j_2=0}^{k_2+1}\sum_{\ell_1=j_1}^{k_1+1}2^{j_1q(n_1/p-n_1-|\alpha_1|)}2^{\ell_1q(n_1+|\alpha_1|-\sigma_1)}2^{j_2q(n_2/p-\sigma_2)}\right\}^{1/q}\lesssim 1. \qquad (2.20)
\end{aligned}$$

Similarly, by symmetry, we have

$$\left\|T\left[\sum_{j_1=0}^{k_1+1}\sum_{j_2=0}^{k_2+1}\left(P^{(2)}_{j_1,\,j_2}-P_{j_1,\,j_2}\right)\right]\right\|_{\mathscr{B}_q}\lesssim 1. \qquad (2.21)$$

Finally, we write

$$\begin{aligned}
\sum_{j_1=0}^{k_1+1}\sum_{j_2=0}^{k_2+1}P_{j_1,\,j_2}&=\sum_{|\alpha_1|\le s_1}\sum_{|\alpha_2|\le s_2}\sum_{j_1=1}^{k_1+1}\sum_{j_2=1}^{k_2+1}\sum_{\ell_1=j_1}^{k_1+1}\sum_{\ell_2=j_2}^{k_2+1}\left(\psi^{(1)}_{j_1,\,\alpha_1}-\psi^{(1)}_{j_1-1,\,\alpha_1}\right)\\
&\quad\times\left(\psi^{(2)}_{j_2,\,\alpha_2}-\psi^{(2)}_{j_2-1,\,\alpha_2}\right)\int_{\boldsymbol{R}^{n_1}}\int_{\boldsymbol{R}^{n_2}}f_{\ell_1,\,\ell_2}(y_1,\,y_2)y_1^{\alpha_1}y_2^{\alpha_2}\,dy_1\,dy_2\\
&\equiv\sum_{|\alpha_1|\le s_1}\sum_{|\alpha_2|\le s_2}\sum_{j_1=1}^{k_1+1}\sum_{j_2=1}^{k_2+1}\sum_{\ell_1=j_1}^{k_1+1}\sum_{\ell_2=j_2}^{k_2+1}A_{\alpha_1,\,j_1,\,\ell_1,\,\alpha_2,\,j_2,\,\ell_2}.
\end{aligned}$$

From (2.14) and (2.17), it follows that

$$\left\|A_{\alpha_1,\,j_1,\,\ell_1,\,\alpha_2,\,j_2,\,\ell_2}\right\|_{L^\infty(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})}\lesssim 2^{-j_1(n_1+|\alpha_1|)}2^{\ell_1(n_1+|\alpha_1|-\sigma_1)}2^{-j_2(n_2+|\alpha_2|)}2^{\ell_2(n_2+|\alpha_2|-\sigma_2)}.$$

Since $A_{\alpha_1,\,j_1,\,\ell_1,\,\alpha_2,\,j_2,\,\ell_2}\in\mathscr{D}_{s_1,\,s_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$, by the assumption of the lemma, then

$$\begin{aligned}
&\left\|T(A_{\alpha_1,\,j_1,\,\ell_1,\,\alpha_2,\,j_2,\,\ell_2})\right\|_{\mathscr{B}_q}\\
&\quad\lesssim 2^{j_1(n_1/p-n_1-|\alpha_1|)}2^{\ell_1(n_1+|\alpha_1|-\sigma_1)}2^{j_2(n_2/p-n_2-|\alpha_2|)}2^{\ell_2(n_2+|\alpha_2|-\sigma_2)}.
\end{aligned}$$

From this and $\sigma_i \in (\max\{n_i/p,\, n_i+s_i\},\, n_i+s_i+1)$ for $i=1,\,2$, it follows that

$$\begin{aligned}&\left\|T\left(\sum_{j_1=0}^{k_1+1}\sum_{j_2=0}^{k_2+1}P_{j_1,\,j_2}\right)\right\|_{\mathscr{B}_q}\\&\quad\lesssim\left\{\sum_{|\alpha_1|\le s_1}\sum_{|\alpha_2|\le s_2}\sum_{j_1=1}^{k_1+1}\sum_{j_2=1}^{k_2+1}\sum_{\ell_1=j_1}^{k_1+1}\sum_{\ell_2=j_2}^{k_2+1}2^{j_1q(n_1/p-n_1-|\alpha_1|)}2^{\ell_1q(n_1+|\alpha_1|-\sigma_1)}\right.\\&\qquad\left.\times 2^{j_2q(n_2/p-n_2-|\alpha_2|)}2^{\ell_2q(n_2+|\alpha_2|-\sigma_2)}\right\}^{1/q}\lesssim 1.\end{aligned}$$

By this together with the estimates (2.19) through (2.21) and the $\mathscr{B}_q$-sublinear property of T, we obtain that $\|Tf\|_{\mathscr{B}_q}\lesssim\|f\|_{\mathscr{D}_{s_1,\,s_2;\,\sigma_1,\,\sigma_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})}$, which implies that T is bounded from $\mathscr{D}_{s_1,\,s_2;\,\sigma_1,\,\sigma_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ to $\mathscr{B}_q$. This finishes the proof of Lemma 2.4. □

PROOF OF THEOREM 1.1. The necessity is obvious. In fact, if T extends to a bounded $\mathscr{B}_q$-sublinear operator from $H^p(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ to $\mathscr{B}_q$, then for any $(p,\,2,\,s_1,\,s_2)$-atom a,

$$\|Ta\|_{\mathscr{B}_q}\lesssim\|a\|_{H^p(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})}\lesssim 1.$$

To prove the sufficiency, for any $f\in\mathscr{D}_{s_1,\,s_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$, let

$$\ell_1\equiv\sup_{x_2\in\boldsymbol{R}^{n_2}}\operatorname{diam}(\operatorname{supp}f(\cdot,\,x_2))$$

and $\ell_2\equiv\sup_{x_1\in\boldsymbol{R}^{n_1}}\operatorname{diam}(\operatorname{supp}f(x_1,\,\cdot))$. Then there exists a positive constant C independent of f such that $C(\ell_1)^{-n_1/p}(\ell_2)^{-n_2/p}\|f\|^{-1}_{L^\infty(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})}f$ is a $(p,2,s_1,s_2)$-atom, and thus, by the assumption of the theorem,

$$\|Tf\|_{\mathscr{B}_q}\lesssim(\ell_1)^{n_1/p}(\ell_2)^{n_2/p}\|f\|_{L^\infty(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})},$$

which shows that T satisfies the assumptions of Lemma 2.4. For $i=1,\,2$, choose $\sigma_i\in(\max\{n_i+s_i,\,n_i/p\},\,n_i+s_i+1)$. By Lemma 2.4, T is bounded from $\mathscr{D}_{s_1,\,s_2;\,\sigma_1,\,\sigma_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ to $\mathscr{B}_q$.

On the other hand, for any $f\in\mathscr{D}_{s_1,\,s_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$, by Lemma 2.3, there exist numbers $\{\lambda_j\}_{j\in\boldsymbol{N}}\subset\boldsymbol{C}$ and $(p,\,2,\,s_1,\,s_2)$-atoms $\{a_j\}_{j\in\boldsymbol{N}}\subset\mathscr{D}_{s_1,\,s_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ such that $f=\sum_{j\in\boldsymbol{N}}\lambda_ja_j$ in $\mathscr{D}_{s_1,\,s_2;\,\sigma_1,\,\sigma_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ and $\{\sum_{j\in\boldsymbol{N}}|\lambda_j|^p\}^{1/p}\lesssim\|f\|_{H^p(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})}$. From this and Lemma 2.4, it follows that $Tf=\sum_{j\in\boldsymbol{N}}\lambda_jTa_j$ in $\mathscr{B}_q$. Thus, $Tf\in\mathscr{B}_q$, and by the monotonicity of the sequence space ℓ^q,

$$\|Tf\|_{\mathscr{B}_q} \le \left\{\sum_{j\in\boldsymbol{N}} |\lambda_j|^q \|Ta_j\|_{\mathscr{B}_q}^q\right\}^{1/q} \lesssim \left\{\sum_{j\in\boldsymbol{N}} |\lambda_j|^p\right\}^{1/p} \lesssim \|f\|_{H^p(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})}.$$

This together with the density of $\mathscr{D}_{s_1,s_2}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ in $H^p(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ implies that T can be extended as a bounded $\mathscr{B}_q$-sublinear operator from $H^p(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ to $\mathscr{B}_q$, which completes the proof of Theorem 1.1. □

Using Theorem 1.1, we can now prove Corollary 1.1.

Proof of Corollary 1.1. By Theorem 1.1, it suffices to prove that for all smooth atoms a, $\|T(a)\|_{L^{q_0}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})} \lesssim 1$. To prove this, we follow the procedure used in the proof of Theorem 1 in [**10**] (see also [**11**]). Assume that a is a smooth $(p, 2, s_1, s_2)$-atom supported in open set Ω. Let $\widetilde{\Omega} \equiv \{(x_1, x_2)\in\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}:\ M_s(\chi_\Omega)(x_1, x_2) > 1/2\}$ and

$$\Omega_0 \equiv \{(x_1, x_2)\in\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2}:\ M_s(\chi_{\widetilde{\widetilde{\Omega}}})(x_1, x_2) > 1/16\}.$$

Then $|\Omega_0| + |\widetilde{\Omega}| \lesssim |\Omega|$. By the boundedness of T from $L^2(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ to $L^{q_0}(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})$ and the Hölder inequality, we have

$$\begin{aligned}\left\{\int_{\Omega_0} |T(a)(x_1, x_2)|^q\,dx_1\,dx_2\right\}^{1/q} &\lesssim \left\{\int_{\Omega_0} |T(a)(x_1, x_2)|^{q_0}\,dx_1\,dx_2\right\}^{1/q_0} |\Omega|^{1/q-1/q_0}\\ &\lesssim \|a\|_{L^2(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})} |\Omega|^{1/p-1/2} \lesssim 1.\end{aligned}$$

We still need to prove that $\int_{(\Omega_0)^\complement} |T(a)(x_1, x_2)|^q\,dx_1\,dx_2 \lesssim 1$. Without loss of generality, we may assume that $q \le 1$. The proof of the case $q\in(1,2)$ is similar and we omit the details. To this end, for each $R\in\mathscr{M}(\Omega)$, assume that $R = I\times J$. Denote by $\mathscr{M}^{(1)}(\widetilde{\Omega})$ the set of all maximal subrectangles in the first direction in Ω. Let $\hat{R} \equiv \hat{I}\times J \in \mathscr{M}^{(1)}(\widetilde{\Omega})$ and $\hat{\hat{R}} \equiv \hat{I}\times\hat{J} \in \mathscr{M}^{(1)}(\widetilde{\widetilde{\Omega}})$, and define $\gamma_1(R, \Omega) \equiv |\hat{I}|/|I|$ and $\gamma_2(\hat{R}, \widetilde{\Omega}) \equiv |\hat{J}|/|J|$. Then $16\hat{\hat{R}} \subset \Omega_0$. Notice that by the Journé covering lemma (see [**24**]), for any fixed $\delta' > 0$, we have

$$\sum_{R\in\mathscr{M}(\Omega)} [\gamma_1(R, \Omega)]^{-\delta'} |R| \lesssim |\Omega| \tag{2.22}$$

and

$$\sum_{\hat{R}\in\mathscr{M}^{(1)}(\widetilde{\Omega})} [\gamma_2(\hat{R}, \widetilde{\Omega})]^{-\delta'} |R| \lesssim |\Omega|. \tag{2.23}$$

Since $q \le 1$, we write

$$\begin{aligned}
&\int_{(\Omega_0)^\complement} |T(a_R)(x_1,\, x_2)|^q\, dx_1\, dx_2 \\
&\quad \le \sum_{R \in \mathscr{M}(\Omega)} \int_{(\Omega_0)^\complement} |T(a_R)(x_1,\, x_2)|^q\, dx_1\, dx_2 \\
&\quad \le \sum_{R \in \mathscr{M}(\Omega)} \int_{(\boldsymbol{R}^{n_1}\setminus 16\hat{I})\times \boldsymbol{R}^{n_2}} |T(a_R)(x_1,\, x_2)|^q\, dx_1\, dx_2 + \sum_{R \in \mathscr{M}(\Omega)} \int_{\boldsymbol{R}^{n_1}\times(\boldsymbol{R}^{n_2}\setminus 16\hat{J})} \cdots \\
&\quad \equiv L_1 + L_2.
\end{aligned}$$

Noticing that $a_R|R|^{1/2-1/p}\|a_R\|^{-1}_{L^2(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})}$ is a rectangle atom, we have

$$\int_{(\boldsymbol{R}^{n_1}\setminus 16\hat{I})\times \boldsymbol{R}^{m_2}} |T(a_R)(x_1,\, x_2)|^q\ dx_1\, dx_2 \lesssim [\gamma_1(R,\,\Omega)]^{-\delta}|R|^{1-q/q_0}\|a_R\|^q_{L^2(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})}.$$

By $1/q_0 - 1/q = 1/2 - 1/p$ and $p \le 1$ and (2.22), we obtain

$$\begin{aligned}
L_1 &\lesssim \left\{ \sum_{R \in \mathscr{M}(\Omega)} \|a_R\|^2_{L^2(\boldsymbol{R}^{n_1}\times\boldsymbol{R}^{n_2})} \right\}^{q/2} \\
&\qquad \times \left\{ \sum_{R \in \mathscr{M}(\Omega)} [\gamma_1(R,\,\Omega)]^{-2\delta/(2-q)}|R|^{[2(q_0-q)]/[q_0(2-q)]} \right\}^{1-q/2} \\
&\lesssim |\Omega|^{q(1/2-1/p)}|\Omega|^{q(1/2-1/q_0)} \left\{ \sum_{R \in \mathscr{M}(\Omega)} [\gamma_1(R,\,\Omega)]^{-2\delta/(2-q)}|R| \right\}^{1-q/2} \\
&\lesssim |\Omega|^{q(1/2-1/q)}|\Omega|^{1-q/2} \lesssim 1.
\end{aligned}$$

Similarly, by (2.23), we have $L_2 \lesssim 1$. This finishes the proof of Corollary 1.1. □

3. Proofs of Theorem 1.2 and Theorem 1.3.

To prove Theorem 1.2, we recall the well-known boundedness of fractional integrals on $\boldsymbol{R}^n$; see [**25**, p. 117].

LEMMA 3.1. *Let $\alpha \in (0,1)$, $p \in (1, n/\alpha)$ and $1/q = 1/p - \alpha/n$. Let I_α be the fractional integral operator on $\boldsymbol{R}^n$ defined by*

$$I_\alpha(f)(x) = \int_{\boldsymbol{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy$$

for $f \in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ and $x \in \boldsymbol{R}^n$. Then I_α is bounded from $L^p(\boldsymbol{R}^n)$ to $L^q(\boldsymbol{R}^n)$, namely, there exists a positive constant C such that for all $f \in L^p(\boldsymbol{R}^n)$,

$$\|I_\alpha(f)\|_{L^q(\boldsymbol{R}^n)} \le C\|f\|_{L^p(\boldsymbol{R}^n)}.$$

Proof of Theorem 1.2. Since $[b, T]$ is linear with respect to b and T, then it suffices to prove Theorem 1.2 for $b \in \mathrm{Lip}\,(\alpha_1, \alpha_2; \boldsymbol{R}^n \times \boldsymbol{R}^m)$ with $\|b\|_{\mathrm{Lip}(\alpha_1, \alpha_2; \boldsymbol{R}^n \times \boldsymbol{R}^m)} = 1$ and T with $\|K\| = 1$. By (K1) and Definition 1.4, we have

$$\begin{aligned} |[b, T](f)(x_1, x_2)| &\lesssim \int_{\boldsymbol{R}^n \times \boldsymbol{R}^m} \frac{1}{|x_1-y_1|^{n-\alpha_1}} \frac{1}{|x_2-y_2|^{m-\alpha_2}} |f(y_1, y_2)|\, dy_1\, dy_2 \\ &\lesssim I^{(1)}_{\alpha_1}\Big[I^{(2)}_{\alpha_2}(|f|)\Big](x_1, x_2), \end{aligned}$$

where $I^{(1)}_{\alpha_1}$ and $I^{(2)}_{\alpha_2}$ are the fractional integral operators with respect to x_1 or x_2, respectively. By Lemma 3.1, for all $f \in L^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$, we have

$$\begin{aligned} \|[b, T](f)\|_{L^q(\boldsymbol{R}^n \times \boldsymbol{R}^m)} &\lesssim \Big\| \Big\| I^{(1)}_{\alpha_1}\Big[I^{(2)}_{\alpha_2}(|f|)\Big] \Big\|_{L^q(\boldsymbol{R}^m,\, dx_2)} \Big\|_{L^q(\boldsymbol{R}^n,\, dx_1)} \\ &\lesssim \Big\| \big\| I^{(2)}_{\alpha_2}(|f|) \big\|_{L^q(\boldsymbol{R}^m,\, dx_2)} \Big\|_{L^p(\boldsymbol{R}^n,\, dx_1)} \\ &\lesssim \|f\|_{L^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)}, \end{aligned}$$

where and in the sequel, we use $\|\cdot\|_{L^p(\boldsymbol{R}^n,\, dx_1)}$ and $\|\cdot\|_{L^p(\boldsymbol{R}^m,\, dx_2)}$ to denote the $L^p(\boldsymbol{R}^n)$-norm with respect to the variable x_1 and x_2 respectively. This finishes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Since $[b, T]$ is linear with respect to b and T, then it suffices to prove Theorem 1.3 for $b \in \mathrm{Lip}\,(\alpha_1, \alpha_2; \boldsymbol{R}^n \times \boldsymbol{R}^m)$ with $\|b\|_{\mathrm{Lip}(\alpha_1, \alpha_2; \boldsymbol{R}^n \times \boldsymbol{R}^m)} = 1$ and T with $\|K\| = 1$. By Theorem 1.1 and Corollary 1.1, it suffices to prove that there exists a positive δ such that for all rectangular $(p, 2, s_1, s_2)$-atoms a supported on $R = I \times J$ and $\gamma \ge 8\max\{n^{1/2}, m^{1/2}\}$,

$$\int_{(\boldsymbol{R}^n \times \boldsymbol{R}^m)\setminus \widetilde{R}_\gamma} |[b, T](a)(x_1, x_2)|^q\, dx_1\, dx_2 \lesssim \gamma^{-\delta}. \tag{3.1}$$

Without loss of generality, we may assume that $R = I \times J = [0,1]^n \times [0,1]^m$. In fact, if letting $b_{x_1^0, x_2^0, \ell_1, \ell_2}(x_1, x_2) \equiv \ell_1^{-\alpha_1} \ell_2^{-\alpha_2} b(x_1^0 + \ell_1 x_1, x_2^0 + \ell_2 x_2)$,

$$K_{x_1^0, x_2^0, \ell_1, \ell_2}(x_1, y_1, x_2, y_2) = \ell_1^n \ell_2^m K(x_1^0 + \ell_1 x_1, x_1^0 + \ell_1 y_1, x_2^0 + \ell_2 x_2, x_2^0 + \ell_1 y_2)$$

and $T_{x_1^0, x_2^0, \ell_1, \ell_2}$ be a Calderón-Zygmund operator with kernel $K_{x_1^0, x_2^0, \ell_1, \ell_2}$ for some $x_1^0 \in \boldsymbol{R}^n, x_2^0 \in \boldsymbol{R}^m$ and some $\ell_1, \ell_2 > 0$, then it is easy to check that

$$\|b_{x_1^0, x_2^0, \ell_1, \ell_2}\|_{\mathrm{Lip}(\alpha_1, \alpha_2; \boldsymbol{R}^n \times \boldsymbol{R}^m)} = \|b\|_{\mathrm{Lip}(\alpha_1, \alpha_2; \boldsymbol{R}^n \times \boldsymbol{R}^m)} = 1$$

and $K_{x_1^0, x_2^0, \ell_1, \ell_2}$ also satisfies (K1) through (K4) with $\|K_{x_1^0, x_2^0, \ell_1, \ell_2}\| = \|K\| = 1$. Moreover, if let $\widetilde{a}$ be a rectangular $(p, 2, s_1, s_2)$-atom supported in $R' = I' \times J' = \{x_1^0 + \ell_1 I\} \times \{x_2^0 + \ell_2 J\}$, and $a(x_1, x_2) \equiv \ell_1^n \ell_2^m \widetilde{a}(x_1^0 + \ell_1 x_1, x_2^0 + \ell_2 x_2)$, then a is a rectangular $(p, 2, s_1, s_2)$-atom supported in $R = [0, 1]^n \times [0, 1]^m$, where $x_0^1 + \ell_1 I = \{x_0^1 + \ell_1 x_1 : x_1 \in I\}$ and $x_2^0 + \ell_2 J$ is similarly defined. By setting $x_i' = x_i^0 + \ell_i x_i$ and $y_i' = y_i^0 + \ell_i y_i$ for $i = 1, 2$, we have

$$\begin{aligned}
&[b, T](\widetilde{a})(x_1', x_2')\\
&\quad = \int_{\boldsymbol{R}^n \times \boldsymbol{R}^m} K(x_1', y_1', x_2', y_2')\\
&\qquad \times [b(x_1', x_2') - b(x_1', x_2') - b(y_1', y_2') + b(y_1', y_2')]a'(y_1', y_2')\, dy_1'\, dy_2'\\
&\quad = \ell_1^{\alpha_1 - n} \ell_2^{\alpha_2 - m} \int_{\boldsymbol{R}^n \times \boldsymbol{R}^m} K_{x_1^0, x_2^0, \ell_1, \ell_2}(x_1, y_1, x_2, y_2)[b_{x_1^0, x_2^0, \ell_1, \ell_2}(x_1, x_2)\\
&\qquad - b_{x_1^0, x_2^0, \ell_1, \ell_2}(x_1, y_2) - b_{x_1^0, x_2^0, \ell_1, \ell_2}(y_1, x_2) + b_{x_1^0, x_2^0, \ell_1, \ell_2}(y_1, y_2)]a(y_1, y_2)\, dy_1\, dy_2\\
&\quad = \ell_1^{\alpha_1 - n} \ell_2^{\alpha_2 - m} [b_{x_1^0, x_2^0, \ell_1, \ell_2}, T_{x_1^0, x_2^0, \ell_1, \ell_2}](a)(x_1, x_2),
\end{aligned}$$

which together with $1/q = 1 - \alpha_1/n = 1 - \alpha_2/m$ yields

$$\begin{aligned}
&\int_{(\boldsymbol{R}^n \times \boldsymbol{R}^m) \setminus \widetilde{R}'_\gamma} |[b, T](\widetilde{a})(x_1', x_2')|^q\, dx_1'\, dx_2'\\
&\quad = \ell_1^n \ell_2^m \int_{(\boldsymbol{R}^n \times \boldsymbol{R}^m) \setminus \widetilde{R}_\gamma} |[b, T](\widetilde{a})(x_1', x_2')|^q\, dx_1'\, dx_2'\\
&\quad = \int_{(\boldsymbol{R}^n \times \boldsymbol{R}^m) \setminus \widetilde{R}_\gamma} |[b_{x_1^0, x_2^0, \ell_1, \ell_2}, T_{x_1^0, x_2^0, \ell_1, \ell_2}](a)(x_1, x_2)|^q\, dx_1\, dx_2,
\end{aligned}$$

where $\widetilde{R}'$ denotes the γ fold enlargement of R'. Then by this, (1.3) and the facts that $K_{x_1^0, x_2^0, \ell_1, \ell_2}$ and $b_{x_1^0, x_2^0, \ell_1, \ell_2}$ satisfy the same conditions as K and b respectively, we may assume that $R = I \times J = [0, 1]^n \times [0, 1]^m$.

Let a be a rectangular $(p,\, 2,\, s_1,\, s_2)$-atom supported in $R = I \times J = [0,\, 1]^n \times [0,\, 1]^m$. Let $\gamma_1 \equiv 8n^{1/2}$, $\gamma_2 \equiv 8m^{1/2}$ and $\gamma \geq \max\{\gamma_1,\, \gamma_2\}$. Then

$$\begin{aligned}
&\int_{(\boldsymbol{R}^n \times \boldsymbol{R}^m)\setminus \widetilde{R}_\gamma} |[b,\, T](a)(x_1,\, x_2)|^q \, dx_1 dx_2 \\
&\quad \leq \int_{x_1 \notin \gamma I} \int_{x_2 \in \gamma_2 J} |[b,\, T](a)(x_1,\, x_2)|^q \, dx_1 dx_2 + \int_{x_1 \notin \gamma I} \int_{x_2 \notin \gamma_2 J} \cdots + \int_{x_1 \in \gamma_1 I} \int_{x_2 \notin \gamma J} \cdots \\
&\quad \equiv G_1 + G_2 + G_3.
\end{aligned}$$

By symmetry, it suffices to estimate G_1 and G_2.

The Hölder inequality implies that

$$G_1 \lesssim \int_{x_1 \notin \gamma I} \|[b,\, T]a(x_1,\, \cdot)\|^q_{L^{q_1}(\boldsymbol{R}^m,\, dx_2)} dx_1.$$

By $\int_{\boldsymbol{R}^n} a(x_1,\, x_2)\, dx_1 = 0$ for all $x_2 \in \boldsymbol{R}^m$, we have

$$\begin{aligned}
&[b,\, T](a)(x_1,\, x_2) \\
&\quad = \int_{\boldsymbol{R}^n \times \boldsymbol{R}^m} [K(x_1,\, y_1,\, x_2,\, y_2) - K(x_1,\, 0,\, x_2,\, y_2)] \\
&\qquad \times [b(x_1,\, x_2) - b(x_1,\, x_2) - b(x_1,\, y_2) + b(y_1,\, y_2)] a(y_1,\, y_2)\, dy_1\, dy_2 \\
&\qquad + \int_{\boldsymbol{R}^n \times \boldsymbol{R}^m} K(x_1,\, 0,\, x_2,\, y_2) \\
&\qquad \times [b(0,\, x_2) - b(0,\, y_2) - b(y_1,\, x_2) + b(y_1,\, y_2)] a(y_1,\, y_2)\, dy_1\, dy_2 \\
&\quad \equiv L_1 + L_2.
\end{aligned}$$

Notice that if $x_1 \notin \gamma I$ and $y_1 \in I$, then $|y_1| \leq |x_1|/2$ and $|x_1 - y_1| \lesssim 2|x_1|$. Thus, for any $x_1 \notin \gamma I$ and $x_2 \in \boldsymbol{R}^m$, by Definition 1.4, (K1), (K2) and the Hölder inequality, we obtain

$$\begin{aligned}
|L_1| &\lesssim \int_I \int_J \frac{|y_1|^{\epsilon_1}}{|x_1|^{n+\epsilon_1-\alpha_1}} \frac{1}{|x_2 - y_2|^{m-\alpha_2}} |a(y_1,\, y_2)| dy_1\, dy_2 \\
&\lesssim \frac{1}{|x_1|^{n+\epsilon_1-\alpha_1}} \int_J \frac{1}{|x_2 - y_2|^{m-\alpha_2}} \left(\int_I |a(y_1,\, y_2)|^2 dy_1 \right)^{1/2} dy_2 \\
&\lesssim \frac{1}{|x_1|^{n+\epsilon_1-\alpha_1}} I^{(2)}_{\alpha_2} \left[\|a\|_{L^2(\boldsymbol{R}^n,\, dy_1)} \right] (x_2)
\end{aligned}$$

and

$$|L_2| \lesssim \int_I \int_J \frac{|y_1|^{\alpha_1}}{|x_1|^n} \frac{1}{|x_2 - y_2|^{m-\alpha_2}} |a(y_1, y_2)| dy_1\, dy_2$$
$$\lesssim \frac{1}{|x_1|^n} I^{(2)}_{\alpha_2}\left[\|a\|_{L^2(\boldsymbol{R}^n,\, dy_1)}\right](x_2).$$

Since (1.4) implies that $n - (n + \epsilon_1 - \alpha_1)q < 0$ and $n - nq < 0$, then by (R3) and Lemma 3.1, we obtain

$$G_1 \lesssim \int_{x_1 \notin \gamma I} \left(\|L_1\|^q_{L^{q_1}(\boldsymbol{R}^m,\, dx_2)} + \|L_2\|^q_{L^{q_1}(\boldsymbol{R}^m,\, dx_2)}\right) dx_1$$
$$\lesssim \int_{x_1 \notin \gamma I} \left(\frac{1}{|x_1|^{(n+\epsilon_1-\alpha_1)q}} + \frac{1}{|x_1|^{nq}}\right) dx_1$$
$$\lesssim \gamma^{n-(n+\epsilon_1-\alpha_1)q} + \gamma^{n-nq}.$$

Choosing $\delta \equiv -\max\{n - nq,\, n - (n + \epsilon_1 - \alpha_1)q\} > 0$, we have $G_1 \lesssim \gamma^{-\delta}$.

To estimate G_2, by the vanishing moments of a, we have

$$\begin{aligned}
&[b,\, T](a)(x_1, x_2)\\
&= \int_{\boldsymbol{R}^n \times \boldsymbol{R}^m} [K(x_1, y_1, x_2, y_2) - K(x_1, 0, x_2, y_2) - K(x_1, y_1, x_2, 0) + K(x_1, 0, x_2, 0)]\\
&\quad \times [b(x_1, x_2) - b(x_1, x_2) - b(y_1, x_2) + b(y_1, y_2)]a(y_1, y_2)\, dy_1\, dy_2\\
&\quad + \int_{\boldsymbol{R}^n \times \boldsymbol{R}^m} [K(x_1, y_1, x_2, 0) - K(x_1, 0, x_2, 0)]\\
&\quad \times [b(x_1, 0) - b(x_1, y_2) - b(y_1, 0) + b(y_1, y_2)]a(y_1, y_2)\, dy_1\, dy_2\\
&\quad + \int_{\boldsymbol{R}^n \times \boldsymbol{R}^m} [K(x_1, 0, x_2, y_2) - K(x_1, 0, x_2, 0)]\\
&\quad \times [b(0, x_2) - b(y_1, x_2) - b(0, y_2) + b(y_1, y_2)]a(y_1, y_2)\, dy_1\, dy_2\\
&\quad + \int_{\boldsymbol{R}^n \times \boldsymbol{R}^m} K(x_1, 0, x_2, 0)\\
&\quad \times [b(0, 0) - b(y_1, 0) - b(0, y_2) + b(y_1, y_2)]a(y_1, y_2)\, dy_1\, dy_2\\
&\equiv L_3 + L_4 + L_5 + L_6.
\end{aligned}$$

Notice that if $x_1 \notin \gamma I$ and $y_1 \in I$, then $|y_1| \le |x_1|/2$ and $|x_1 - y_1| \le 2|x_1|$; if $x_2 \notin \gamma_2 J$ and $y_2 \in J$, then $|y_2| \le |x_2|/2$ and $|x_2 - y_2| \le 2|x_2|$. Thus, for $x_1 \notin \gamma I$ and

$x_2 \notin \gamma_2 J$, by Definition 1.4, (K1) through (K4), (R3) and the Hölder inequality, we obtain

$$|L_3| \lesssim \int_I \int_J \frac{|y_1|^{\epsilon_1}}{|x_1|^{n+\epsilon_1-\alpha_1}} \frac{|y_2|^{\epsilon_2}}{|x_2|^{m+\epsilon_2-\alpha_2}} |a(y_1, y_2)|\, dy_1\, dy_2 \lesssim \frac{1}{|x_1|^{n+\epsilon_1-\alpha_1}} \frac{1}{|x_2|^{m+\epsilon_2-\alpha_2}};$$

$$|L_4| \lesssim \int_I \int_J \frac{|y_1|^{\epsilon_1}}{|x_1|^{n+\epsilon_1-\alpha_1}} \frac{|y_2|^{\alpha_2}}{|x_2|^{m}} |a(y_1, y_2)|\, dy_1\, dy_2 \lesssim \frac{1}{|x_1|^{n+\epsilon_1-\alpha_1}} \frac{1}{|x_2|^{m}};$$

$$|L_5| \lesssim \int_I \int_J \frac{|y_1|^{\alpha_1}}{|x_1|^{n}} \frac{|y_2|^{\epsilon_2}}{|x_2|^{m+\epsilon_2-\alpha_2}} |a(y_1, y_2)|\, dy_1\, dy_2 \lesssim \frac{1}{|x_1|^{n}} \frac{1}{|x_2|^{m+\epsilon_2-\alpha_2}};$$

and

$$|L_6| \lesssim \int_I \int_J \frac{|y_1|^{\alpha_1}}{|x_1|^{n}} \frac{|y_2|^{\alpha_2}}{|x_2|^{m}} |a(y_1, y_2)|\, dy_1\, dy_2 \lesssim \frac{1}{|x_1|^{n}} \frac{1}{|x_2|^{m}}.$$

From this together with $n-(n+\epsilon_1-\alpha_1)q<0$, $n-nq<0$, $m-(m+\epsilon_2-\alpha_2)q<0$ and $m-mq<0$, it follows that

$$\begin{aligned} G_2 &\lesssim \int_{x_1\notin\gamma I}\int_{x_2\notin\gamma_2 J} (|L_3|^q+|L_4|^q+|L_5|^q+|L_6|^q)\, dx_1\, dx_2 \\ &\lesssim \int_{x_1\notin\gamma I}\int_{x_2\notin\gamma_2 J} \bigg[\frac{1}{|x_1|^{(n+\epsilon_1-\alpha_1)q}} \frac{1}{|x_2|^{(m+\epsilon_2-\alpha_2)q}} + \frac{1}{|x_1|^{(n+\epsilon_1-\alpha_1)q}} \frac{1}{|x_2|^{mq}} \\ &\quad + \frac{1}{|x_1|^{nq}} \frac{1}{|x_2|^{(m+\epsilon_2-\alpha_2)q}} + \frac{1}{|x_1|^{nq}} \frac{1}{|x_2|^{mq}}\bigg]\, dx_1\, dx_2 \\ &\lesssim \gamma^{n-(n+\epsilon_1-\alpha_1)q} + \gamma^{n-nq}. \end{aligned}$$

This shows $G_2 \lesssim \gamma^{-\delta}$, which together with $G_1 \lesssim \gamma^{-\delta}$ gives (3.1) and the proof of Theorem 1.3 is therefore complete. □

Remark 3.1. The restriction $\alpha_1 \le \min\{n/2, 1\}$ is to guarantee the boundedness of the commutator $[b, T]$ from $L^2(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ to $L^{q_1}(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ with $1/q_1 = 1/p - \alpha_1/n$; see Theorem 1.2. Since the $L^2(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ norm appears in the definition of $H^p(\boldsymbol{R}^n \times \boldsymbol{R}^m)$ rectangular atoms, we need this boundedness of the commutator $[b, T]$ in the proof of Theorem 1.3; see Corollary 1.1.

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