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A variant of Jacobi type formula for Picard curves

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Abstract. The classical Jacobi formula for the elliptic integrals (Gesammelte Werke I, p. 235) shows a relation between Jacobi theta constants and periods of ellptic curves $E(\lambda): w^2 = z(z-1)(z-\lambda)$. In other words, this formula says that the modular form $\vartheta_{00}^4(\tau)$ with respect to the principal congruence subgroup $\Gamma(2)$ of $PSL(2, \boldsymbol{Z})$ has an expression by the Gauss hypergeometric function $F(1/2, 1/2, 1; 1-\lambda)$ via the inverse of the period map for the family of elliptic curves $E(\lambda)$ (see Theorem 1.1). In this article we show a variant of this formula for the family of Picard curves $C(\lambda_1, \lambda_2): w^3 = z(z-1)(z-\lambda_1)(z-\lambda_2)$, those are of genus three with two complex parameters. Our result is a two dimensional analogy of this context. The inverse of the period map for $C(\lambda_1, \lambda_2)$ is established in [**S**] and our modular form $\vartheta_0^3(u, v)$ (for the definition, see (2.7)) is defined on a two dimensional complex ball $\mathscr{D} = \{2\mathrm{Re}v + |u|^2 < 0\}$, that can be realized as a Shimura variety in the Siegel upper half space of degree 3 by a modular embedding. Our main theorem says that our theta constant is expressed in terms of the Appell hypergeometric function $F_1(1/3, 1/3, 1/3, 1; 1-\lambda_1, 1-\lambda_2)$.

1. Introduction.

Consider the family of elliptic curves

$$E(\lambda): w^2 = z(z-1)(z-\lambda) \quad \lambda(\lambda-1) \neq 0.$$

For a real parameter λ in the interval $(0, 1)$, take the ratio of the periods

$$\tau = \int_1^\infty \frac{dz}{\sqrt{z(z-1)(z-\lambda)}} \Bigg/ \int_{-\infty}^0 \frac{dz}{\sqrt{z(z-1)(z-\lambda)}},$$

here we may suppose that τ is pure imaginary with $\mathrm{Im}(\tau) > 0$. We have the theta representation of the λ-invariant

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$$\lambda(\tau) = \frac{\vartheta_{01}^4(\tau)}{\vartheta_{00}^4(\tau)}. \tag{1.1}$$

Here, ϑ_{jk} indicates the Jacobi theta constant

$$\vartheta_{jk}(\tau) = \sum_{n\in\boldsymbol{Z}} \exp\left[\pi i\left(n+\frac{j}{2}\right)^2\tau + 2\pi i\left(n+\frac{j}{2}\right)\frac{k}{2}\right] \quad \text{for } \tau \in \boldsymbol{H} = \{\operatorname{Im}\tau > 0\}.$$

So (1.1) holds for all $\tau \in \boldsymbol{H}$, and it is the inverse of the period map for the family of $E(\lambda)$. Recall the Jacobi formula relating the elliptic integral and the theta constant:

THEOREM 1.1 (see [**J**], p. 235). *Under the relation* (1.1) *we have*

$$\vartheta_{00}^2(\tau) = F\left(\frac{1}{2},\frac{1}{2},1;1-\lambda\right) = \frac{1}{\pi}\int_1^\infty \frac{dz}{\sqrt{z(z-1)(z-(1-\lambda))}}. \tag{1.2}$$

Note that at first (1.2) is valid only for pure imaginary $\tau \in \boldsymbol{H}$. By making analytic continuation we get the equality on the whole $\boldsymbol{H}$. The classical theorem of arithmetic geometric mean by Gauss says

$$\frac{1}{M(1,x)} = F\left(\frac{1}{2},\frac{1}{2},1;1-x^2\right). \tag{1.3}$$

Here $M(a,b)$ denotes the arithmetic geometric mean with the initial positive values a, b, and $F(\alpha,\beta,\gamma;x)$ indicates the Gauss hypergeometric function.
According to the duplication formula (see [**Ga**])

$$\begin{cases} \vartheta_{00}^2(2\tau) = \dfrac{1}{2}\left(\vartheta_{00}^2(\tau) + \vartheta_{01}^2(\tau)\right), \\ \vartheta_{01}^2(2\tau) = \vartheta_{00}(\tau)\vartheta_{01}(\tau), \end{cases}$$

by putting $x = \vartheta_{01}^2(\tau)/\vartheta_{00}^2(\tau)$ we can derive the Gauss AGM theorem (1.3) from the above Jacobi formula. In fact, we have

$$M(\vartheta_{00}^2(\tau),\vartheta_{01}^2(\tau)) = \lim_{n\to\infty}\frac{1}{2}\left(\vartheta_{00}^2(2^n\tau) + \vartheta_{01}^2(2^n\tau)\right) = \lim_{\tau\to i\infty}\vartheta_{00}^2(\tau) = 1.$$

So we have $\vartheta_{00}^2(\tau)\, M(1,x) = 1$.

In this article we show a variant of this Jacobi formula for the Picard curves (2.1).

As an application of our main theorem, we give a new proof of the three terms AGM theorem discovered in [**K-S1**, Theorem 2.2]. As a byproduct we show a one variable variant of the Jacobi formula (Theorem 4.1) for the Borweins curves (4.1) which was originally shown by J. M. Borwein and P. B. Borwein in [**B-B**]. Analogous results corresponding to the extended Gauss AGM in [**K-S2**] will be published elsewhere.

2. Jacobi type formula for the Picard curves.

2.1. The Picard modular form revisited.

We consider Picard curves of genus three with projective parameters:

$$C(\xi) : w^3 = z(z-\xi_0)(z-\xi_1)(z-\xi_2), \tag{2.1}$$

where

$$\xi \in \Xi = \{[\xi_0 : \xi_1 : \xi_2] \in \boldsymbol{P}^2(\boldsymbol{C}) : \xi_0\xi_1\xi_2(\xi_0-\xi_1)(\xi_1-\xi_2)(\xi_2-\xi_0) \neq 0\}.$$

The Jacobian variety $Jac(C(\xi))$ of $C(\xi)$ has a generalized complex multiplication by $\sqrt{-3}$ of type $(2,1)$. In fact we have a basis of holomorphic differentials

$$\varphi = \varphi_1 = \frac{dz}{w},\ \varphi_2 = \frac{dz}{w^2},\ \varphi_3 = \frac{zdz}{w^2}.$$

Put $\lambda_1 = \xi_1/\xi_0, \lambda_2 = \xi_2/\xi_0$. For the moment we assume $0 < \lambda_1 < \lambda_2 < 1$. We may choose a canonical basis $\{A_1, A_2, A_3, B_1, B_2, B_3\}$ of $H_1(C, \boldsymbol{Z})$ with the following property (2.2) that is used in [**S**]. We put a graphic configuration for it in Figure 1, here we put cut lines starting from branch points in the lower half z-plane to get simply connected sheets. The real line (resp. the dotted line, the chained line) indicates a path on the first sheet (resp. the second sheet, the third sheet).

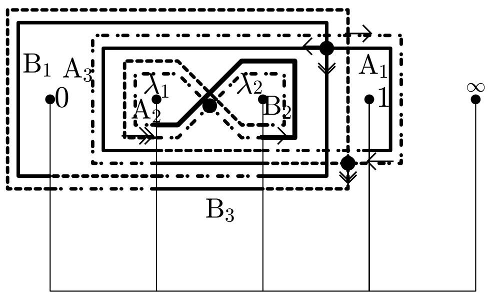

Figure 1. homology basis.

Setting $\rho(z,w)=(z,\omega w)$, we have

$$B_3=\rho(B_1),\ A_3=-\rho^2(A_1),\ B_2=-\rho^2(A_2), \tag{2.2}$$

here ω stands for $\exp[2\pi i/3]$. We have $A_iB_j=\delta_{ij}$. Put

$$\eta_0=\int_{A_1}\varphi,\ \eta_1=-\int_{B_3}\varphi,\ \eta_2=\int_{A_2}\varphi. \tag{2.3}$$

They can be extended as multivalued analytic functions on the (λ_1,λ_2)-space $\boldsymbol{P}^2(\boldsymbol{C})$. It holds

$$\begin{pmatrix}\eta_0\\ \eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}\int_{A_1}\varphi_1\\ -\omega^2\int_{B_1}\varphi_1\\ \int_{A_2}\varphi_1\end{pmatrix}=\begin{pmatrix}-\omega^2\int_{A_3}\varphi_1\\ -\int_{B_3}\varphi_1\\ -\omega^2\int_{B_2}\varphi_1\end{pmatrix},$$
$$\begin{pmatrix}\int_{A_1}\varphi_i\\ -\omega\int_{B_1}\varphi_i\\ \int_{A_2}\varphi_i\end{pmatrix}=\begin{pmatrix}-\omega\int_{A_3}\varphi_i\\ -\int_{B_3}\varphi_i\\ -\omega\int_{B_2}\varphi_i\end{pmatrix}\quad \text{for } i=2,3. \tag{2.4}$$

Set

$$\Omega_1=\left(\int_{A_j}\varphi_i\right),\ \Omega_2=\left(\int_{B_j}\varphi_i\right),\quad (1\le i,j\le 3).$$

The normalized period matrix of $C(\xi)$ is given by $\Omega=\Omega_1^{-1}\Omega_2$. By the relations of periods (2.4) together with the symmetricity ${}^t\Omega=\Omega$, we can rewrite

$$\Omega=\Omega_1^{-1}\Omega_2=\begin{pmatrix}\frac{u^2+2\omega^2v}{1-\omega} & \omega^2u & \frac{\omega u^2-\omega^2v}{1-\omega}\\ \omega^2u & -\omega^2 & u\\ \frac{\omega u^2-\omega^2v}{1-\omega} & u & \frac{\omega^2u^2+2\omega^2v}{1-\omega}\end{pmatrix}, \tag{2.5}$$

here we put $u=\eta_2/\eta_0,\ v=\eta_1/\eta_0$. So we set $\Omega=\Omega(u,v)$. The Riemann period relation $\operatorname{Im}\Omega>0$ induces the inequality $2\mathrm{Re}\,(v)+|u|^2<0$. We set

$$\mathscr{D}=\{\eta=[\eta_0:\eta_1:\eta_2]\in\boldsymbol{P}^2:\eta H^t\overline{\eta}<0\}=\{(u,v)\in\boldsymbol{C}^2:2\mathrm{Re}\,(v)+|u|^2<0\},$$

here we put $H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We define our period map $\Phi : \Xi \to \mathscr{D}$ by

$$\Phi(\lambda_1, \lambda_2) = [\eta_0, \eta_1, \eta_2].$$

Set the Picard modular group

$$\Gamma = \{g \in GL_3(\boldsymbol{Z}[\omega]) : {}^t\overline{g}Hg = H\},$$

and set $\Gamma(\sqrt{-3}) = \{g \in \Gamma : g \equiv I_3 \bmod \sqrt{-3}\}$. The element $g = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} \in \Gamma$ acts on $\mathscr{D}$ by

$$g(u, v) = \left(\frac{p_3 + q_3 v + r_3 u}{p_1 + q_1 v + r_1 u}, \frac{p_2 + q_2 v + r_2 u}{p_1 + q_1 v + r_1 u} \right). \tag{2.6}$$

Let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$ be in $\boldsymbol{Q}^3$. Set the Riemann theta constant

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\Omega) = \sum_{n \in \boldsymbol{Z}^3} \exp[\pi i (n + a) \Omega^t (n + a) + 2\pi i (n + a)^t b],$$

here Ω is a variable on the Siegel upper half space of degree 3. We use the following Riemann theta constants and their Fourier expansions (see [**S**, p. 327], also [**K-S1**, formula (1.3)]):

$$\vartheta_k(u, v) = \vartheta \begin{bmatrix} 0 & 1/6 & 0 \\ k/3 & 1/6 & k/3 \end{bmatrix} (\Omega(u, v)) = \sum_{\mu \in \boldsymbol{Z}[\omega]} \omega^{2kTr(\mu)} H(\mu u) q^{N(\mu)} \tag{2.7}$$

with an index $k \in \boldsymbol{Z}$, where $Tr(\mu) = \mu + \bar{\mu}$, $\mathrm{N}(\mu) = \mu\bar{\mu}$ and

$$H(u) = \exp\left[\frac{\pi}{\sqrt{3}} u^2\right] \vartheta \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix} (u, -\omega^2), \quad q = \exp\left[\frac{2\pi}{\sqrt{3}} v\right].$$

Apparently it holds $\vartheta_k(u, v) = \vartheta_{k+3}(u, v)$, so k runs over $\{0, 1, 2\} = \boldsymbol{Z}/3\boldsymbol{Z}$.

The following properties are established.

FACT 2.1.
(i) ([**P**], [**D-M**], [**T**], [**S**, p. 349]) The period map Φ induces a biholomorphic isomorphism from the ξ-space $\boldsymbol{P}^2$ to the Satake compactification $\overline{\mathscr{D}/\Gamma(\sqrt{-3})}$ of $\mathscr{D}/\Gamma(\sqrt{-3})$.

(ii) ([**S**, p. 327]) The map $\Lambda : \mathscr{D} \longrightarrow \boldsymbol{P}^2$ defined by

$$\Lambda([\eta_0, \eta_1, \eta_2]) = [\vartheta_0(u,v)^3, \vartheta_1(u,v)^3, \vartheta_2(u,v)^3] \tag{2.8}$$

gives the inverse of the period map Φ.

(iii) ([**S**, p. 329]) The projective group $\Gamma(\sqrt{-3})/\{1, \omega, \omega^2\}$ is generated by

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ \omega - \omega^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 & 0 \\ \omega - 1 & 1 & \omega - 1 \\ 1 - \omega^2 & 0 & 1 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} 1 & \omega - \omega^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & \omega - 1 & \omega - 1 \\ 0 & 1 & 0 \\ 0 & 1 - \omega & 1 \end{pmatrix}.$$

Let G denote the group generated by $g_1, \ldots, g_5$.

(iv) ([**S**, p. 346]) We have the automorphic property:

$$\vartheta_k(g(u,v))^3 = (p_1 + q_1 v + r_1 u)^3 \, \vartheta_k(u,v)^3 \tag{2.9}$$

for $g = \begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix} \in G$.

(v) The compactification $\overline{\mathscr{D}/G}$ is obtained by attaching 4 points corresponding to $P_0 = [\xi_0, \xi_1, \xi_2] = [1,0,0], P_1 = [0,1,0], P_2 = [0,0,1], P_3 = [1,1,1]$ to $\mathscr{D}/G$. Put $Q_i = \Phi(P_i)$ for $i = 0,1,2,3$. Here we note that $\Phi(P_3) = Q_3 = \lim_{u \to 0, v \to -\infty}(u,v)$.

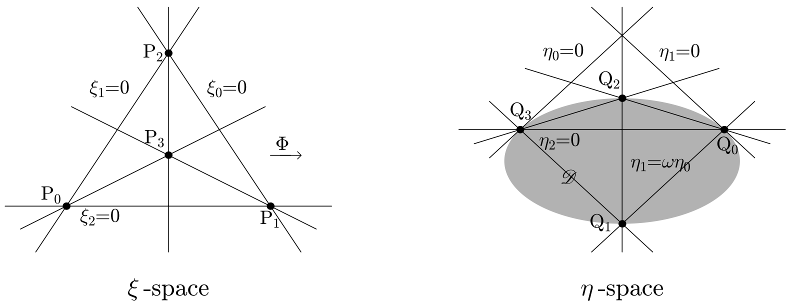

Figure 2. Correspondence between ξ-space and $\mathscr{D}$.

According to Fact 2.1 (ii), we have an isomorphism between $\overline{\mathscr{D}/\Gamma(\sqrt{-3})}$ and ξ-space $\boldsymbol{P}^2$:

$$(\xi_0 : \xi_1 : \xi_2) = (\vartheta_0^3(u,v) : \vartheta_1^3(u,v) : \vartheta_2^3(u,v)). \tag{2.10}$$

PROPOSITION 2.1. *The $\boldsymbol{C}$-vector space of modular forms with the automorphic property as in* (2.9) *is generated by $\vartheta_0^3(u,v), \vartheta_1^3(u,v)$ and $\vartheta_2^3(u,v)$.*

PROOF. This is an easy consequence of Fact 2.1 (ii) and (iv). □

2.2. Main Theorem.

In this subsection we use the Appell hypergeometric function

$$F_1(a,b,b',c;\lambda_1,\lambda_2) = \sum_{m,n\geq 0} \frac{(a,m+n)(b,m)(b',n)}{(c,m+n)m!n!} \lambda_1^m \lambda_2^n \quad \text{for } |\lambda_1|<1, |\lambda_2|<1 \tag{2.11}$$

with

$$(a,n) = \begin{cases} a(a+1)\cdots(a+n-1) & \text{for } n>0, \\ 1 & \text{for } n=0. \end{cases}$$

We have the integral representation

$$F_1(a,b,b',c;\lambda_1,\lambda_2) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_1^\infty z^{b+b'-c}(z-1)^{c-a-1}(z-\lambda_1)^{-b}(z-\lambda_2)^{-b'}dz.$$

THEOREM 2.1 (A Jacobi type formula in two variables). *Under the relation*

$$(\lambda_1,\lambda_2) = \left(\frac{\vartheta_1(u,v)^3}{\vartheta_0(u,v)^3}, \frac{\vartheta_2(u,v)^3}{\vartheta_0(u,v)^3}\right) \tag{2.12}$$

stated in Fact 2.1, *we have*

$$\vartheta_0(u,v) = C_0\, F_1\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},1;1-\lambda_1,1-\lambda_2\right), \tag{2.13}$$

$$C_0 = \vartheta\begin{bmatrix}\frac{1}{6}\\ \frac{1}{6}\end{bmatrix}(-\omega^2).$$

Remark 2.1. By using the power series expansion of F_1, we have the equality (2.13) for an arbitrary point in a neighborhood of the set $\{(u,v) \in \mathscr{D} : u = 0, v < 0\}$ in $\mathscr{D}$. By making the analytic continuation of the both sides we have the equality on the whole domain $\mathscr{D}$.

Corollary 2.1. *We have*

$$\vartheta_i(u,v)^3 = C_0^3 \, \lambda_i \left(F_1\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1-\lambda_1, 1-\lambda_2\right)\right)^3, \quad (i = 1,2). \tag{2.14}$$

Remark 2.2. According to some classical literature (also in [**M-T-Y**]), it holds

$$C_0 = \vartheta \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} (-\omega^2) = \frac{3^{3/8}}{2\pi} \exp\left(\frac{5\pi\sqrt{-1}}{72}\right) \Gamma\left(\frac{1}{3}\right)^{3/2}. \tag{2.15}$$

2.3. Application to a three terms AGM theorem.

In [**K-S1**], a new three terms arithmetic geometric mean $M_3(a,b,c)$ is introduced. For three positive numbers a, b, c, set a new triple (a', b', c') with $a' = \frac{1}{3}(a+b+c)$, $b'^3 + c'^3 = \frac{1}{3}(a^2 b + b^2 c + c^2 a + ab^2 + bc^2 + ca^2)$, $b'^3 - c'^3 = \frac{1}{3\sqrt{-3}}(a-b)(b-c)(c-a)$. Define our AGM process by

$$(a', b', c') = \psi(a, b, c).$$

We can take a nice choice of the cubic roots for b', c' so that $\psi^2(a,b,c)$ is a triple of positive numbers again. Thus, we get a unique positive number

$$M_3(a,b,c) := \lim_{n \to \infty} \psi^n(a,b,c).$$

For the proof of the convergence of $\psi^n(a,b,c)$ see [**K-S1**, Theorem 2.1]. As a consequence of Main Theorem we obtain a new proof of the three terms AGM theorem in [**K-S1**, Theorem 2.2, p. 134]:

Corollary 2.2 (Three terms AGM theorem).

$$\frac{1}{M_3(1,x,y)} = F_1\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1-x^3, 1-y^3\right), \quad (|x| < 1, |y| < 1). \tag{2.16}$$

Observing the following isogeny formula (see [**K-S1**, Theorem 1.1, p. 132])

$$\begin{cases} \vartheta_0(\sqrt{-3}u, 3v) = \dfrac{1}{3}(\vartheta_0 + \vartheta_1 + \vartheta_2), \\ \vartheta_1^3(\sqrt{-3}u, 3v) + \vartheta_2^3(\sqrt{-3}u, 3v) = \dfrac{1}{3}(\vartheta_0^2\vartheta_1 + \vartheta_1^2\vartheta_2 + \vartheta_2^2\vartheta_0 + \vartheta_0\vartheta_1^2 + \vartheta_1\vartheta_2^2 + \vartheta_2\vartheta_0^2), \\ \vartheta_1^3(\sqrt{-3}u, 3v) - \vartheta_2^3(\sqrt{-3}u, 3v) = \dfrac{1}{3\sqrt{-3}}(\vartheta_0 - \vartheta_1)(\vartheta_1 - \vartheta_2)(\vartheta_2 - \vartheta_0), \end{cases}$$

we obtain the above corollary by the exactly analogous argument in the introduction.

EXAMPLE 2.1. We have

$$F_1\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; \frac{1}{2}, \frac{1}{2}\right) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}.$$

(We get it from

$$F\left(a, b, a+b+\frac{1}{2}, 1-x\right) = F\left(2a, 2b, a+b+\frac{1}{2}, \frac{1-\sqrt{x}}{2}\right), \ F\left(a, b, a+b+\frac{1}{2}, 1\right) = \frac{\sqrt{\pi}\Gamma\left(a+b+\frac{1}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)},$$

see [**Go**, p. 115].) Put $x = y = 1/\sqrt[3]{2}$. By three times procedure ψ for them, we get a forty digits approximation

$$\frac{1}{M^3}\left(1, \frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}}\right) = 1.159595266963928365769992051570020881945\cdots$$

of $\dfrac{\sqrt{\pi}}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}$.

With one more step we get 120 digits.

3. Proof of the Main Theorem.

Note that $\vartheta_k(u,v)$ $(k = 0,1,2)$ is holomorphic on $\mathscr{D}$. The period $\eta_0 = \int_{A_1}(x(x-1)(x-\lambda_1)(x-\lambda_2))^{-1/3}dx$ is a single valued holomorphic function on $\mathscr{D}$ via the relation (2.12). So we denote $\eta_0 = \eta_0(u,v)$ in this sense. We compare the behavior of $\vartheta_0(u,v)$ and $\eta_0(u,v)$.

The period η_0 is not equal to zero for any affine parameters λ_1, λ_2. So $\eta_0(u,v)$ has only zeros possibly on $\Phi(\{\xi_0 = 0\})$.

We have

Lemma 3.1. *$\vartheta_k(u,v)$ $(k=0,1,2)$ does not vanish at any point (u,v) on $\Phi(\Xi)$.*

Although this is due to [**S** p. 316], there we used the original expression of theta functions by Picard. And the statement is not so visible. So we give a direct proof in the Appendix.

By the above argument $\vartheta_0(u,v)=0$ on $\Phi(\{\xi_0=0\})$. ϑ_0 may have other divisor components in $\overline{\mathscr{D}-\Phi(\Xi)}$.

For an element $g=\begin{pmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{pmatrix}\in G$, there are actions on $(u,v)\in\mathscr{D}$ given by (2.6) and on the triple (η_0,η_1,η_2) by

$$\begin{aligned} g(\eta) &= (g(\eta_0),g(\eta_1),g(\eta_2)) \\ &= (p_1\eta_0+q_1\eta_1+r_1\eta_2, p_2\eta_0+q_2\eta_1+r_2\eta_2, p_3\eta_0+q_3\eta_1+r_3\eta_2). \end{aligned}$$

Namely $\eta_0(u,v)^3$ has the same automorphic property as $\vartheta_k^3(u,v)$ $(k=0,1,2)$.

Now we claim

$$\vartheta_0(u,v)^3=\alpha\,\eta_0^3(u,v) \tag{3.1}$$

for some constant α.

By Proposition 2.1, we have

$$\eta_0^3=c_0\vartheta_0^3+c_1\vartheta_1^3+c_2\vartheta_2^3 \tag{3.2}$$

for some constants c_0,c_1,c_2. Set $\mathrm{Div}(\eta_0)$ (resp. $\mathrm{Div}(\vartheta_0)$) be the divisor of $\eta_0(u,v)$ (resp. $\vartheta_0(u,v)$). The support of $\mathrm{Div}(\eta_0)$ is contained in that of $\mathrm{Div}(\vartheta_0)$. And the order of $\mathrm{Div}(\eta_0)$ is not smaller than that of $\mathrm{Div}(\vartheta_0)$. Take two different points R_1,R_2 on $\mathrm{Div}(\eta_0)$.

By substituting R_1 and R_2 in (2.10), we have

$$\begin{cases} (0:p:q)=(\vartheta_0^3(R_1):\vartheta_1^3(R_1):\vartheta_2^3(R_1)) \\ (0:r:s)=(\vartheta_0^3(R_2):\vartheta_1^3(R_2):\vartheta_2^3(R_2)). \end{cases}$$

Because these two projective points are different, it holds $ps-qr\neq 0$. On the other hand, by making possible cancellation of zeros in (3.2) we have

$$\begin{cases} 0=0+pc_1+qc_2 \\ 0=0+rc_1+sc_2. \end{cases}$$

So we have $c_1=c_2=0$. Thus we have the equality (3.1).

Now let us determine the value α. Suppose $0 < \lambda_1 < \lambda_2 < 1$. We have

$$\eta_0 = \int_{A_1} (z(z-1)(z-\lambda_1)(z-\lambda_2))^{-1/3} dz = c_1 \int_0^{-\infty} (z(z-1)(z-\lambda_1)(z-\lambda_2))^{-1/3} dz$$

with some constant c_1. Now recall the integral representation

$$F_1(a,b,b',c;x,y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_1^{\infty} z^{b+b'-c}(z-1)^{c-a-1}(z-x)^{-b}(z-y)^{-b'} dz.$$

By changing the variable $z = 1 - z'$ we have

$$\eta_0 = c_2 F_1\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1-\lambda_1, 1-\lambda_2\right)$$

with some constant c_2. Hence we have

$$\vartheta_0(u,v) = \beta F_1\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; 1-\lambda_1, 1-\lambda_2\right)$$

with some constant β. If we put $\xi = P_3 = [1,1,1]$, it corresponds to $(\xi_0 : \xi_1 : \xi_2) = (1:1:1)$. So the right hand side is equal to $\beta F_1(1/3, 1/3, 1/3, 1; 0, 0) = \beta$. Recall that $\Phi(P_3) = \lim_{u \to 0, v \to -\infty}(u,v)$. According to the Fourier expansion in (2.7) we have

$$\lim_{(u,v)\to(0,-\infty)} \vartheta_0(u,v) = H(0) = \vartheta \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} (0, -\omega^2).$$

So we have $\beta = \vartheta \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \end{bmatrix} (0, -\omega^2)$. Thus, we obtained the required equality. □

4. Degeneration to the case of Borweins' case.

As a degenerate case $\lambda = \lambda_1 = \lambda_2$, we obtain the Jacobi type formula for the Borweins curves (see [**B-B**], [**K-S1**, p. 141])

$$w^3 = z(z-1)(z-\lambda)^2. \tag{4.1}$$

Set

$$\begin{cases} \theta_0(\tau) = \sum_{\mu \in \boldsymbol{Z}[\omega]} q^{N(\mu)} = \sum_{m,n \in \boldsymbol{Z}} \left(e^{2\pi i \tau/3}\right)^{m^2-mn+n^2} \\ \theta_1(\tau) = \sum_{\mu \in \boldsymbol{Z}[\omega]} e^{2\pi i Tr(\mu)/3} q^{N(\mu)} = \sum_{m,n \in \boldsymbol{Z}} e^{2\pi i (m+n)/3} \left(e^{2\pi i \tau/3}\right)^{m^2-mn+n^2} \\ q = \exp[2\pi i \tau/3], N(\mu) = \mu\overline{\mu}, Tr(\mu) = \mu + \overline{\mu}. \end{cases}$$

Putting $u = 0, \tau = -i\sqrt{3}v$ in (2.12) we have the expression of $\lambda = \lambda_1 = \lambda_2$:

$$\lambda = \frac{\theta_1(\tau)^3}{\theta_0(\tau)^3}. \tag{4.2}$$

Theorem 4.1 (Borweins [**B-B**, p. 695]). *Under the relation* (4.2), *we have*

$$\theta_0(\tau) = F\left(\frac{1}{3}, \frac{2}{3}, 1; 1-\lambda\right).$$

This equality is obtained just as the case $\lambda = \lambda_1 = \lambda_2$, namely the case $u = 0$, in the main theorem.

Remark 4.1. Borweins have shown this theorem by using their AGM theorem. But we proved it directly with modular arguments. So our theorem induces their AGM theorem also. This is the context discussed in the proof of Corollary 2.2.

5. Appendix.

We give a direct proof of Lemma 3.1. For it, we need some preparatory propositions. The equalities in first three propositions should be understood as those on the Jacobian variety $Jac(C(\xi))$.

Set $\omega = {}^t(\varphi_1, \varphi_2, \varphi_3)$, and set P_∞ =the point at infinity, $P_0 = (z, w) = (0,0), P_x = (\xi_0, 0), P_y = (\xi_1, 0), P_1 = (\xi_2, 0)$ on $C(\xi)$. According to Proposition I-1 in [**S**, p. 319], we have

Proposition 5.1.

$$\int_{P_\infty}^{P_x} \omega = \frac{1}{3}\left[\Omega\begin{pmatrix}2\\1\\1\end{pmatrix} + \begin{pmatrix}2\\1\\-1\end{pmatrix}\right], \int_{P_\infty}^{P_1} \omega = \frac{1}{3}\Omega\begin{pmatrix}2\\0\\1\end{pmatrix}, \int_{P_\infty}^{P_0} \omega = \frac{1}{3}\begin{pmatrix}2\\0\\-1\end{pmatrix}.$$

According to Proposition I-2 in [**S**, p. 326], we have

Proposition 5.2. *The Riemann constant Δ with respect to the homology basis $\{A_1, \dots, B_3\}$ and the terminal point P_∞ is given by*

$$\Delta = \frac{1}{2}\left[\Omega \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right].$$

We have

$$2\Delta = \int_{P_\infty}^{R_1} \omega + \dots + \int_{P_\infty}^{R_4} \omega$$

for a positive canonical divisor $K = R_1 + \dots + R_4$. According to Corollary 3.6 in [**M**, p. 160], we have

Proposition 5.3. $\vartheta(z, \Omega) = 0$ *if and only if it holds*

$$z = \Delta - \left(\int_{P_\infty}^{Q_1} \omega + \int_{P_\infty}^{Q_2} \omega\right)$$

for some positive divisor $Q_1 + Q_2$.

So we have

Proposition 5.4. *For a given positive divisor $Q_1 + Q_2 + Q_3$, we have $f(P) = \vartheta\Big(\sum_{i=1}^3 \int_{P_\infty}^{Q_i} \omega - \int_{P_\infty}^{P} \omega - \Delta, \Omega\Big) \equiv 0$ if and only if $K - (Q_1 + Q_2 + Q_3)$ is represented by a positive divisor.*

Proof. If we have $Q_1 + Q_2 + Q_3 + R_1 = K$ for a point R_1, put $R_2 = P$. Then we have

$$\sum_{i=1}^{3} \int_{P_\infty}^{Q_i} \omega + \int_{P_\infty}^{R_1} \omega + \int_{P_\infty}^{R_2} \omega - \int_{P_\infty}^{P} \omega = 2\Delta. \tag{5.1}$$

By Proposition 5.3, it holds $f(P) = 0$.

In case $f(P) \equiv 0$, we must have (5.1). It means

$$Q_1 + Q_2 + Q_3 + R_1 + R_2 - P = K.$$

Considering the residue of a meromorphic differential, we know that it happens only if the left hand side is a positive divisor. So we are forced to have a cancellation $R_2 = P$. Consequently $K - (Q_1 + Q_2 + Q_3)$ is a positive divisor. □

By the theorem of Riemann (see [**M**, Theorem 3.1, p. 149]), we have:

REMARK 5.1. If $f(P)$ is not identically zero, it has exactly three zeros $P = Q_1, Q_2, Q_3$.

Now we can consider our theta functions. We have

$$\begin{aligned}\vartheta_0(u,v) &= \vartheta\left(\frac{1}{6}\left[\Omega\begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}\right], \Omega\right) \times \text{some unit}\\ &= \vartheta\left(\Omega\begin{pmatrix}0\\2/3\\0\end{pmatrix} + \begin{pmatrix}0\\2/3\\0\end{pmatrix} - \Delta, \Omega\right) \times \text{some unit}\\ &= \vartheta\left(\int_{P_\infty}^{P_1}\omega + \int_{P_\infty}^{P_0}\omega + \int_{P_\infty}^{P_\infty}\omega - \int_{P_\infty}^{P_x}\omega - \Delta, \Omega\right) \times \text{some unit.}\end{aligned}$$

We have $(\varphi_1) = P_0 + P_1 + P_x + P_y$, $(\varphi_2) = 4P_\infty$, $(\varphi_3) = 3P_0 + P_\infty$. So $K - (P_0 + P_1 + P_\infty)$ cannot be realized by a positive divisor. So it holds $\vartheta_0(u,v) \neq 0$ for any point $(u,v) \in \Phi(\Xi)$.

We have

$$\begin{aligned}\vartheta_1(u,v) &= \vartheta\left(\int_{P_\infty}^{P_\infty}\omega + \int_{P_\infty}^{P_\infty}\omega + \int_{P_\infty}^{P_x}\omega - \int_{P_\infty}^{P_1}\omega - \Delta, \Omega\right) \times \text{some unit,}\\ \vartheta_2(u,v) &= \vartheta\left(\int_{P_\infty}^{P_\infty}\omega + \int_{P_\infty}^{P_0}\omega + \int_{P_\infty}^{P_x}\omega - \int_{P_\infty}^{P_1}\omega - \Delta, \Omega\right) \times \text{some unit.}\end{aligned}$$

So we show that $\vartheta_1(u,v)\vartheta_2(u,v) \neq 0$ for any point $(u,v) \in \Phi(\Xi)$ in an analogous way. Thus we have proved Lemma 3.1. □

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