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On Alexander polynomials of certain $(2,5)$ torus curves

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Abstract. In this paper, we compute Alexander polynomials of a torus curve C of type $(2,5)$, $C:\ f(x,y)=f_2(x,y)^5+f_5(x,y)^2=0$, under the assumption that the origin O is the unique inner singularity and $f_2=0$ is an irreducible conic. We show that the Alexander polynomial remains the same with that of a generic torus curve as long as C is irreducible.

1. Introduction.

A plane curve $C\subset \boldsymbol{P}^2$ of degree pq is called a *curve of torus type* (p,q) with $p>q\geq 2$, if there is a defining polynomial F of C of the form $F=F_p^q+F_q^p$, where F_p, F_q are homogeneous polynomials of X,Y,Z of degree p and q respectively. A singularity $P\in C$ is called *inner* if $F_p(P)=F_q(P)=0$. Otherwise, P is called an *outer singularity*. A torus curve C is called *tame* if it has no outer singularity. We assume $O=(0,0)$ hereafter. In [**6**], the first author classified the topological types of the germs of inner singularity of curves of $(2,5)$ torus type. In this paper, we are interested in the *Alexander polynomial* of C which is an important topological invariant ([**17**]). In the case of irreducible sextics of torus type $(2,3)$, there are only 3 possible Alexander polynomials: $\Delta_{3,2}^j(t)=(t^2-t+1)^j$, $j=1,2,3$ ([**13**]).

A tame torus curve C of type (p,q) is said to be *generic* if the associated curves $C_p=\{F_p=0\}$ and $C_q=\{F_q=0\}$ intersect transversely at pq distinct points. It is known that the Alexander polynomial of a generic C is equal to $\Delta_{p,q}(t)$ ([**14**]) where

$$\Delta_{p,q}(t):=\frac{(t^{pq/r}-1)^r(t-1)}{(t^p-1)(t^q-1)},\qquad r=\gcd(p,q).$$

Moreover it is also known that the Alexander polynomial of C is still equal to $\Delta_{p,q}(t)$, if C is tame and C_p, C_q intersect at O with intersection multiplicity pq and C_p is smooth ([**2**], [**3**]).

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Let C be a torus curve of type $(2,5)$ such that C has a unique inner singularity, say $O \in C$ (thus $I(C_2, C_5; O) = 10$) and we assume that C has no outer singularity. Then we will show that there are 22 possible singularities for (C, O) under the assumption that C_2 is irreducible ([**6**]). For 8 classes among 22 type of singularities, C can be either irreducible or reducible. We list those 22-singularities below. Throughout this paper, we use the same notations of singularities as in [**6**], [**12**].

(I) Assume that C is irreducible, the possibilities are:

$$B_{50,2}, \quad B_{43,2} \circ B_{2,3}, \quad B_{36,2} \circ B_{4,3}, \quad B_{29,2} \circ B_{6,3}, \quad B_{22,2} \circ B_{8,3}, \quad B_{15,2} \circ B_{10,3}, \quad B_{25,4},$$
$$(B_{4,2}^2)^{B_{32,2}+B_{2,2}}, \quad (B_{4,2}^2)^{B_{32,2}+B_{2,2}}, \quad (B_{6,2}^2)^{B_{23,2}+B_{3,2}}, \quad (B_{8,2}^2)^{B_{14,2}+B_{4,2}}, \quad (B_{10,2}^2)^{2B_{5,2}},$$
$$(B_{11,2}^2)^{B_{6,2}}, \quad (B_{12,2}^2)^{2B_{1,2}}, \quad (B_{6,2}^2)^{B_{16,2}+B_{1,2}} \circ B_{2,1}, \quad (B_{8,2}^2)^{B_{7,2}+B_{2,2}} \circ B_{2,1},$$
$$(B_{9,2}^2)^{B_{5,2}} \circ B_{2,1}, \quad B_{29,2} \circ B_{2,1} \circ (B_{2,1}^2)^{B_{k,2}} \ (k = 1, 2, 3, 5).$$

(II) If C is reducible, the possibilities are:

(a) with a line component:

$$B_{29,2} \circ B_{6,3}, \quad (B_{6,2}^2)^{B_{16,2}+B_{1,2}} \circ B_{2,1}, \quad (B_{8,2}^2)^{B_{7,2}+B_{2,2}} \circ B_{2,1}, \quad (B_{9,2}^2)^{B_{5,2}} \circ B_{2,1},$$
$$B_{29,2} \circ B_{2,1} \circ (B_{2,1}^2)^{B_{k,2}}, \ k = 1, 2, 3, 5.$$

(b) with five conics: $B_{20,5}$.

We recall some of the notations.

$$B_{p,q} : \quad x^p + y^q = 0,$$
$$B_{p,q} \circ B_{r,s} : (x^p + y^q)(x^r + y^s) = 0, \quad q/p < s/r.$$

The singularities listed below have degenerate faces in their Newton boundaries and we need one more toric modification for their resolutions. See [**6**] for the detail.

$$(B_{4,2}^2)^{B_{32,2}+B_{2,2}}, \quad (B_{4,2}^2)^{B_{32,2}+B_{2,2}}, \quad (B_{6,2}^2)^{B_{23,2}+B_{3,2}}, \quad (B_{8,2}^2)^{B_{14,2}+B_{4,2}}, \quad (B_{10,2}^2)^{2B_{5,2}},$$
$$(B_{11,2}^2)^{B_{6,2}}, \quad (B_{12,2}^2)^{2B_{1,2}}, \quad (B_{6,2}^2)^{B_{16,2}+B_{1,2}} \circ B_{2,1}, \quad (B_{8,2}^2)^{B_{7,2}+B_{2,2}} \circ B_{2,1},$$
$$(B_{9,2}^2)^{B_{5,2}} \circ B_{2,1}, \quad B_{29,2} \circ B_{2,1} \circ (B_{2,1}^2)^{B_{k,2}} \ (k = 1, 2, 3, 5).$$

In this paper, we use the method of Libgober [**8**], Loeser-Vaquié [**9**] and Esnault-Artal ([**1**], [**5**]) for the computation of the Alexander polynomials.

THEOREM 1. *Let* C *be a tame torus curve of type* $(2,5)$*. Suppose that* C *has a unique inner singularity and* C_2 *is irreducible. Then the Alexander polynomial* $\Delta_C(t)$ *of* C *is given as follows.*

(1) *If* C *is irreducible* (*case* (I)), *then*

$$\Delta_C(t) = \Delta_{5,2}(t) \quad \textit{where } \Delta_{5,2}(t) = t^4 - t^3 + t^2 - t + 1.$$

(2) *If* C *is reducible and have a line component* (*case* (II-a)),

$$\Delta_C(t) = (t-1)(t^4 - t^3 + t^2 - t + 1).$$

(3) *If* C *is reducible and* $(C,O) \sim B_{20,5}$ (*case* (II-b)),

$$\Delta_C(t) = (t-1)^4(t+1)^4(t^4 - t^3 + t^2 - t + 1)^4(t^4 + t^3 + t^2 + t + 1)^3.$$

COROLLARY 1. *Let* C *be a torus curve of type* $(2,5)$ *and assume that there is a degeneration family* C_t, $t \in W$ *such that* $C_t \cong C$, $t \neq 0$ *and* C_0 *is an irreducible tame curve with a unique singular point* P *where* W *is an open neighbourhood of the origin in* $\boldsymbol{C}$*. Then the Alexander polynomial* $\Delta_C(t)$ *is given by* $\Delta_{5,2}(t)$.

COROLLARY 2. *Let* C *be a tame irreducible torus curve of type* $(2,5)$ *such that* C_5 *is smooth and* C_2 *is irreducible. Then the Alexander polynomial is given by* $\Delta_{5,2}(t)$.

2. Alexander polynomial.

Let us consider the affine coordinate $\boldsymbol{C}^2 = \boldsymbol{P}^2 \setminus \{Z=0\}$ and let $x = X/Z$, $y = Y/Z$. Let C be a given plane curve of degree d defined by $f(x,y)=0$ and let $O \in C$ be a singular point of C where $O=(0,0)$. We assume that the line at infinity $\{Z=0\}$ is generic with respect to C.

2.1. Loeser-Vaquié formula.

Consider an embedded resolution of $(C,O) \subset (\boldsymbol{C}^2,O)$, $\pi : \tilde{U} \to U$ where U is an open neighborhood of O and let $E_1,\dots,E_s$ be the exceptional divisors. Let (u,v) be a local coordinate system centered at O and k_i and m_i be respective order of zero of the canonical two form $\pi^*(du \wedge dv)$ and $\pi^* f$ along the divisor E_i. *The adjunction ideal* $\mathscr{J}_{O,k,d}$ *of* $\mathscr{O}_O$ is defined by

$$\mathscr{J}_{O,k,d} = \{\phi \in \mathscr{O}_O \mid (\pi^*\phi) \geq \sum_i ([km_i/d] - k_i)E_i\}, \quad k = 1,\dots,d-1$$

where $[r]$ is the largest integer n such that $n \leq r$ for $r \in \boldsymbol{Q}$ ([**1**], [**5**]).

Let $O(j)$ be the set of polynomials in x, y whose degree is less than or equal to j. We consider the canonical mapping $\sigma : \boldsymbol{C}[x, y] \to \mathscr{O}_O$ and its restriction:

$$\sigma_k : O(k-3) \to \mathscr{O}_O.$$

Put $V_k(O) = \mathscr{O}_O / \mathscr{J}_{O,k,d}$ and we denote the composition $O(k-3) \to \mathscr{O}_O \to V_k(O)$ by $\bar{\sigma}_k$. Then the Alexander polynomial is given as follows.

Lemma 1 ([**8**], [**9**], [**1**], [**5**]). *The reduced Alexander polynomial* $\tilde{\Delta}_C(t)$ *is given by the product*

$$\tilde{\Delta}_C(t) = \prod_{k=1}^{d-1} \Delta_k(t)^{\ell_k} \tag{1}$$

where d *is the degree of* f, ℓ_k *is the dimension of* $\operatorname{Coker} \bar{\sigma}_k$ *and*

$$\Delta_k(t) = \left(t - \exp\left(\frac{2k\pi i}{d}\right)\right)\left(t - \exp\left(-\frac{2k\pi i}{d}\right)\right).$$

We use the method of Esnault-Artal ([**1**]) to compute ℓ_k.

Remark 1. The Alexander polynomial $\Delta_C(t)$ is given as

$$\Delta_C(t) = (t-1)^{r-1} \tilde{\Delta}_C(t)$$

where r is the number of irreducible components of C ([**14**]). *Note that for the case of curve of degree* 10.

$$\Delta_5(t) = (t+1)^2, \quad \Delta_6(t)\Delta_8(t) = t^4 + t^3 + t^2 + t + 1,$$
$$\Delta_7(t)\Delta_9(t) = t^4 - t^3 + t^2 - t + 1.$$

2.2. Plücker's formula.

We denote the Milnor number of the singularity of (C, P) by $\mu(C, P)$ and the number of locally irreducible components of (C, P) by $r(C, P)$. We recall the *generalized Plücker's formula*. Let $C_1, \ldots, C_r$ be irreducible components of C and let $\tilde{C}_1, \ldots, \tilde{C}_r$ be their normalizations, let $g(\tilde{C}_i)$ be the genus of $\tilde{C}_i$ and let $\Sigma(C)$ be the singular locus of C. Then

$$\chi(\tilde{C}) = \sum_{i=1}^{r} (2 - 2g(\tilde{C}_i)) = d(3-d) + \sum_{P \in \Sigma(C)} (\mu(C, P) + r(C, P) - 1) \leq 2r$$

For further details, we refer to [**10**], [**11**], [**16**].

3. Outline of the proof of Theorem 1.

We have to consider the following 22-singularities. We denote a class of a singularity (C,O) which can appear both as an irreducible curve and a reducible curve by ${}^{\sharp}(C,O)$. In the section 3.2, we will use notation ${}^{irr}(C,O)$, ${}^{red}(C,O)$ to distinguish the case of C being irreducible and reducible.

$$B_{50,2},\ B_{43,2}\circ B_{2,3},\ B_{36,2}\circ B_{4,3},\ {}^{\sharp}B_{29,2}\circ B_{6,3},\ B_{22,2}\circ B_{8,3},\ B_{15,2}\circ B_{10,3},\ B_{20,5},\ B_{25,4},$$
$$(B_{4,2}^2)^{B_{32,2}+B_{2,2}},\ (B_{4,2}^2)^{B_{32,2}+B_{2,2}},\ (B_{6,2}^2)^{B_{23,2}+B_{3,2}},\ (B_{8,2}^2)^{B_{14,2}+B_{4,2}},\ (B_{10,2}^2)^{2B_{5,2}},$$
$$(B_{11,2}^2)^{B_{6,2}},\ (B_{12,2}^2)^{2B_{1,2}},\ {}^{\sharp}\,(B_{6,2}^2)^{B_{16,2}+B_{1,2}}\circ B_{2,1},\ {}^{\sharp}\,(B_{8,2}^2)^{B_{7,2}+B_{2,2}}\circ B_{2,1},$$
$${}^{\sharp}(B_{9,2}^2)^{B_{5,2}}\circ B_{2,1},\ {}^{\sharp}\,B_{29,2}\circ B_{2,1}\circ (B_{2,1}^2)^{B_{k,2}}\ (k=1,2,3,5).$$

3.1. Divisibility principle and Sandwich principle.

Suppose we have a degeneration family C_s, $s\in W$ of reducible curves such that C_s, $s\neq 0$ are equisingular family of plane curves. Here W is an open neighbourhood of the origin in $\boldsymbol{C}$. We denote this situation as $C_s\overset{s\to 0}{\longrightarrow}C_0$. Then we have the divisibility $\Delta_{C_s}(t)\mid\Delta_{C_0}(t)$ (Theorem 26 of [**14**]). Suppose that we have two degeneration series $C_s\overset{s\to 0}{\longrightarrow}C_0$ and $D_r\overset{r\to 0}{\longrightarrow}D_0$ such that $C_0\cong D_r\,(r\neq 0)$ and assume that $\Delta_{C_s}(t)=\Delta_{D_0}(t)$. Then the divisibility implies that $\Delta_{C_s}(t)=\Delta_{C_0}(t)$ (the Sandwich principle).

3.2. Degeneration series.

Recall that we have the following degeneration series among the above singularities ([**6**]):

(1) Main sequence:

$$\begin{array}{ccccccccc}
B_{50,2} & \longrightarrow & B_{43,2}\circ B_{2,3} & \longrightarrow & (B_{4,2}^2)^{B_{32,2}+B_{2,2}} & \longrightarrow & B_{36,2}\circ B_{4,3} & & \\
 & & & & & \searrow\,(a) & & & \\
 & \longrightarrow & {}^{\sharp}B_{29,2}\circ B_{6,3} & \longrightarrow & B_{22,2}\circ B_{8,3} & \longrightarrow & B_{15,2}\circ B_{10,3} & \dashrightarrow & B_{20,5} \\
 & & \downarrow(b) & \searrow(c) & & & & &
\end{array}$$

where the branched sequences (a) from $(B_{4,2}^2)^{B_{32,2}+B_{2,2}}$ and (b), (c) from $B_{29,2}\circ B_{6,3}$ in the main sequence are as follows.

(a) $(B_{4,2}^2)^{B_{32,2}+B_{2,2}}\to(B_{6,2}^2)^{B_{23,2}+B_{3,2}}\to(B_{8,2}^2)^{B_{14,2}+B_{4,2}}\to(B_{10,2}^2)^{2B_{5,2}}\to(B_{11,2}^2)^{B_{6,2}}\to(B_{12,2}^2)^{2B_{1,2}}\to B_{25,4}$.

(b) (i) ${}^{irr}B_{29,2}\circ B_{6,3}\to {}^{irr}(B_{6,2}^2)^{B_{16,2}+B_{1,2}}\circ B_{2,1}\to {}^{irr}(B_{8,2}^2)^{B_{7,2}+B_{2,2}}\circ B_{2,1}\to {}^{irr}(B_{9,2}^2)^{B_{5,2}}\circ B_{2,1}$.

(ii) ${}^{irr}B_{29,2}\circ B_{6,3}\to {}^{irr}B_{29,2}\circ B_{2,1}\circ (B_{2,1}^2)^{B_{1,2}}\to {}^{irr}B_{29,2}\circ B_{2,1}\circ (B_{2,1}^2)^{B_{2,2}}\to {}^{irr}B_{29,2}\circ B_{2,1}\circ (B_{2,1}^2)^{B_{3,2}}\to {}^{irr}B_{29,2}\circ B_{2,1}\circ (B_{2,1}^2)^{B_{5,2}}$.

(c) (i) ${}^{red}B_{29,2}\circ B_{6,3}\to {}^{red}(B_{6,2}^2)^{B_{16,2}+B_{1,2}}\circ B_{2,1}\to {}^{red}(B_{8,2}^2)^{B_{7,2}+B_{2,2}}\circ B_{2,1}\to {}^{red}(B_{9,2}^2)^{B_{5,2}}\circ B_{2,1}$.

(ii) ${}^{red}B_{29,2}\circ B_{6,3}\to {}^{red}B_{29,2}\circ B_{2,1}\circ (B_{2,1}^2)^{B_{1,2}}\to {}^{red}B_{29,2}\circ B_{2,1}\circ (B_{2,1}^2)^{B_{2,2}}\to {}^{red}B_{29,2}\circ B_{2,1}\circ (B_{2,1}^2)^{B_{3,2}}\to {}^{red}B_{29,2}\circ B_{2,1}\circ (B_{2,1}^2)^{B_{5,2}}$.

The main sequence is obtained through the degenerations of the tangent cone of C_5 at O, keeping the irreducibility of C_2. In the last degeneration $B_{15,2}\circ B_{10,3}\dashrightarrow B_{20,5}$ of the main sequence, C degenerates into a reducible curve.

The branched sequence (a) from $(B_{4,2}^2)^{B_{32,2}+B_{2,2}}$ is obtained by degenerating (C_5,O), fixing the tangent cone of C_5 at O. More precisely, the tangent cone of (C_5,O) is a line with multiplicity 2 and the generic singularity of (C_5,O) is A_3 and the corresponding degenerations of (C_5,O) are:

$$(C_5,O):\quad B_{4,2}\to B_{6,2}\to B_{8,2}\to B_{10,2}\to B_{11,2}\to B_{12,2}\to B_{13,2}.$$

The branched sequence (b) (respectively, (c)) from ${}^{irr}B_{29,2}\circ B_{6,3}$ (resp. ${}^{red}B_{29,2}\circ B_{6,3}$) is also obtained by degenerating (C_5,O) fixing the tangent cone of C_5 at O (See Section 3.4).

3.3. Strategy.

Our strategy is the following. The singularity $B_{50,2}$ is obtained when C_2 and C_5 has a maximal contact at O and (C_5,O) is smooth. In this case, it is known that $\Delta_C(t)=t^4-t^3+t^2-t+1$ by Theorem 2 of [**2**]. Hence by virtue of the Sandwich principle, it is enough to show

(1) the irreducibility of C and

(2) $\tilde{\Delta}_C(t)=\Delta_{5,2}(t)$ for the case (C,O) being one of the following singularities which are the end of the degenerations.

$$B_{15,2}\circ B_{10,3},\quad B_{25,4},\quad {}^{\sharp}(B_{9,2}^2)^{B_{5,2}}\circ B_{2,1},\quad {}^{\sharp}B_{29,2}\circ B_{2,1}\circ (B_{2,1}^2)^{B_{5,2}}.$$

By virtue of Lemma 1, to show $\tilde{\Delta}_C(t)=\Delta_{5,2}(t)$ is equivalent to show that

($\sharp$): *$\bar{\sigma}_k: O(k-3)\to V_k(O)$ has one-dimensional cokernel for $k=7,9$ and surjective for other cases.*

REMARK 2. Let C be a reduced curve of degree d and assume that the line at infinity is generic with respect to C. It is known that *the Alexander polynomial* $\Delta_C(t)$ *is a product of cyclotomic polynomial.* This follows by the following observation. First, $\Delta_C(t)$ is a monic polynomial with $\boldsymbol{Z}$ coefficients by the result of Randell ([**15**]). On the other hand, the divisibility result of Libgober ([**7**]) says that $\Delta_C(t)$ divides $\Delta_{d,d}(t) = (t^d - 1)^{d-2}(t-1)$. As cyclotomic polynomials are irreducible, the assertion follows.

Therefore for the calculation of the Alexander polynomial of our curves, it is enough to compute the cokernel for $k = 7$ and $k = 6$ (and $k = 5$).

So for the proof of the assertions (1) and (2) of Theorem 1, we will actually show the above property ($\sharp$).

The last singularity $B_{20,5}$ of the main sequence appears when C consists of five conics. We treat this case separately in the later section.

3.4. Irreducibility of C.

Now we will discuss the irreducibility of C using the generalized Plücker's formula. First we show that C is irreducible if (C,O) is one of 2 singularities $B_{15,2} \circ B_{10,3}$ and $B_{25,4}$.

CASE $(C,O) \sim B_{15,2} \circ B_{10,3}$: Note that the singularities $B_{15,2}$ and $B_{10,3}$ are locally irreducible singularities. As $\mu(B_{15,2}) = 14$, $\mu(B_{10,3}) = 18$ and each singularity appears for sextics or higher degree curves. Thus C must be irreducible, as the degree of C is 10.

CASE $(C,O) \sim B_{25,4}$: The singularity $B_{25,4}$ is a locally irreducible singularity and thus C is irreducible.

CASE $(C,O) \sim B_{29,2} \circ B_{6,3}$: Next we consider the case $(C,O) \sim B_{29,2} \circ B_{6,3}$ and we will show that C can be either irreducible or reducible. Recall that the singularity $B_{29,2} \circ B_{6,3}$ appears in the case that C_2 and C_5 satisfies following three conditions ([**6**]):

(1) C_2 is irreducible and $I(C_2, C_5; O) = 10$.

(2) (C_5, O) has the multiplicity 3 and the tangent cone consists of a multiple line L_1 of the multiplicity 2 and a single line L_2.

(3) The conic C_2 is tangent to the line L_1 at O.

Under the condition $I(C_2, C_5; O) = 10$, we have generically $(C,O) \sim B_{29,2} \circ B_{6,3}$. The singularity $B_{29,2}$ is locally irreducible and $B_{29,2}$ appears for curves of degree $d \geq 7$ as $\mu(B_{29,2}) = 28$. Hence we have four possibilities:

(1) C: irreducible, (2) $C = D_9 \cup D_1$, (3) $C = D_8 \cup D_2$, (4) $C = D_7 \cup D_3$

where D_d is a curve of degree d. But the cases (3) and (4) are impossible. Indeed, if $C = D_7 \cup D_3$, then either (a) $(D_7, O) \sim B_{29,2}$, $(D_3, O) \sim B_{6,3}$ or (b) $(D_7, O) \sim B_{29,2} \circ B_{2,1}$, $(D_3, O) \sim B_{4,2}$ or (c) $(D_7, O) \sim B_{29,2} \circ B_{4,2}$, $(D_3, O) \sim B_{2,1}$. We observe that $\mu(D_3, O) = 10$ in the case (a) and $\mu(D_7, O) = 35$ in the case (b). Thus $\mu(B_{29,2} \circ B_{4,2}) > 35$ and neither cases are possible by the generalized Plücker's formula. By the same argument, we see that the case (3) is impossible. Hence we have two possibilities:

(i) C is irreducible or

(ii) C consists of a line and a curve of degree 9.

If C has a line component, this line must be defined by $\{y = 0\}$. In fact, this case is given by the normal forms of f_2, f_5:

$$f_2(x,y) = a_{02}\, y^2 + (a_{11}\, x + 1)\, y - k^2\, x^2,$$
$$f_5(x,y) = (t + a_{02}\, b_{04})\, y^5 + \phi_4(x)\, y^4 + \phi_3(x)\, y^3 + \phi_2(x)\, y^2 + \phi_1(x)\, y - k^5\, x^5$$

where $\phi_1, \phi_2, \phi_3, \phi_4$ take the forms:

$$\phi_4(x) = (a_{02}\, b_{13} - a_{02}^2\, b_{12} + a_{11}\, b_{04})x + b_{04},$$
$$\phi_3(x) = (b_{13}\, a_{11} - k^2\, b_{04} - 2b_{12}\, a_{02}\, a_{11} + b_{22}\, a_{02})\, x^2 + b_{13}\, x,$$
$$\phi_2(x) = (a_{02}\, k^3 + k^2\, a_{02}\, b_{12} - k^2\, b_{13} - b_{12} a_{11}^2 + b_{22}\, a_{11})\, x^3 + b_{22}\, x^2 + b_{12}\, x,$$
$$\phi_1(x) = (a_{11}\, k^3 + b_{12}\, k^2\, a_{11} - b_{22}\, k^2)\, x^4 + (k^3 - k^2\, b_{12})\, x^3.$$

The branched sequence (b), (c) in Section 3.2 are obtained by degenerating (C_5, O), fixing the tangent cone of (C_5, O) and keeping irreducibility of C.

CASE $(C, O) \sim B_{20,5}$: This is the last singularity in the main sequence. We will show that C can not be irreducible in this case. As $\mu(B_{20,5}) = 76$, the number of irreducible components r of C must be at least 5 by the generalized Plücker's formula. On the other hand, the singularity $B_{20,5}$ consists of 5 smooth local components. Any two components intersects with intersection multiplicity 4. Thus each local component corresponds to a global component and its degree must be 2, namely a conic.

4. Calculation of $\Delta_C(t)$ I: Non-degenerate case.

We divide the calculation of the Alexander polynomial $\Delta_C(t)$ in two cases, according to (C, O) being non-degenerate or not in the sense of Newton boundary ([**12**]). In this section, we treat the first case.

4.1. Characterization of the adjunction ideal for non-degenerate singularities.

In general, the computation of the ideal $\mathscr{J}_{O,k,d}$ requires an explicit computation of the resolution of the singularity (C, O). However for the case of non-degenerate singularities, the ideal $\mathscr{J}_{O,k,d}$ can be obtained combinatorially by a toric modification. Let (u, v) be a local coordinate system centered at O such that (C, O) is defined by a function germ $f(u, v)$ and the Newton boundary $\Gamma(f; u, v)$ is non-degenerate. Let $Q_1, \dots, Q_s$ be the primitive weight vectors which correspond to the faces $\Delta_1, \dots, \Delta_s$ of $\Gamma(f; u, v)$. Let $\pi : \tilde{U} \to U$ be the canonical toric modification and let $\hat{E}(Q_i)$ be the exceptional divisor corresponding to Q_i. Recall that the order of zeros of the canonical two form $\pi^*(du \wedge dv)$ along the divisor $\hat{E}(Q_i)$ is simply given by $|Q_i| - 1$ where $|Q_i| = p + q$ for a weight vector $Q_i = {}^t(p_i, q_i)$ (see [**12**]). For a function germ $g(u, v)$, let $m(g, Q_i)$ be the multiplicity of the pull-back $(\pi^* g)$ on $\hat{E}(Q_i)$. Then

LEMMA 2 ([**4**], [**13**]). *A function germ $g \in \mathscr{O}_O$ is contained in the ideal $\mathscr{J}_{O,k,d}$ if and only if g satisfies following condition*:

$$m(g, Q_i) \geq \left[\frac{k}{d} m(f, Q_i)\right] - |Q_i| + 1, \quad i = 1, \dots, s.$$

The ideal $\mathscr{J}_{O,k,d}$ is generated by the monomials satisfying the above conditions.

We consider the following integers for each singular point $P \in \Sigma(C)$:

$$\rho_k(P) := \dim V_k(P),\ \tilde{\rho}(k) := \sum_{P \in \Sigma(C)} \rho_k(P) - \dim O(k-3),\ \iota_k(P) := \min_{g \in \mathscr{J}_{P,k,d}} I(g, f; P),$$

where $V_k(P) = \mathscr{O}_P / \mathscr{J}_{P,k,d}$. Then the multiplicity ℓ_k in the formula (1) of Loeser-Vaquié is given as

$$\ell_k = \dim \operatorname{Coker} \bar{\sigma}_k = \tilde{\rho}(k) + \dim \operatorname{Ker} \bar{\sigma}_k.$$

where $\bar{\sigma}_k$ is defined in Section 2.1. We consider the integer $\sum_{P \in \Sigma(C)} \iota_k(P)$. The following is essential due to [**4**].

PROPOSITION 1. *If $\sum_{P \in \Sigma(C)} \iota_k(P) > d(k-3)$, then*
(a) *C is irreducible and $\bar{\sigma}_k$ is injective and $\ell_k = \tilde{\rho}(k)$ or*
(b) *C is reducible.*

Proof. Suppose $0 \neq g \in \operatorname{Ker} \bar{\sigma}_k \subset O(k-3)$. Then by Bézout theorem, we have

$$d(k-3) \geq I(G,C) \geq \sum_{P \in \Sigma(C)} I(G,C;P) \geq \sum_{P \in \Sigma(C)} \iota_k(P) > d(k-3)$$

where $G = \{g = 0\}$. This is an obvious contradiction unless $g \mid f$. Thus this implies either f is irreducible and $\bar{\sigma}_k$ is injective or f is reducible (and $g \mid f$). □

4.2. The singularities $B_{15,2} \circ B_{10,3}$ and $B_{25,4}$.

Now we consider the following two non-degenerate singularities $B_{15,2} \circ B_{10,3}$, and $B_{25,4}$ which appear as the last singularities of the respective degenerations with C being irreducible. We assume that we have chosen local analytic coordinates (u,v) so that

$$B_{15,2} \circ B_{10,3}: \quad f(u,v) = u^{25} + u^{10}v^2 + v^5 + \text{(higher terms)},$$
$$B_{25,4}: \quad f(u,v) = u^{25} + v^4 + \text{(higher terms)}.$$

The local data are given by the following tables.

$B_{15,2} \circ B_{10,3}$:

k	$\mathscr{I}_{O,k,10}$	$\rho_k(O)$	$\iota_k(O)$
3	$\langle u, v \rangle$	1	5
4	$\langle u^3, v \rangle$	3	15
5	$\langle u^5, uv, v^2 \rangle$	6	23
6	$\langle u^7, u^3v, v^2 \rangle$	10	33
7	$\langle u^{10}, u^5v, uv^2, v^3 \rangle$	16	43
8	$\langle u^{12}, u^6v, u^3v^2, v^3 \rangle$	21	52
9	$\langle u^{15}, u^8v, u^5v^2, uv^3, v^4 \rangle$	29	63

$B_{25,4}$:

k	$\mathscr{I}_{O,k,10}$	$\rho_k(O)$	$\iota_k(O)$
3	$\langle u, v \rangle$	1	4
4	$\langle u^3, v \rangle$	3	12
5	$\langle u^6, v \rangle$	6	24
6	$\langle u^8, u^2v, v^2 \rangle$	10	32
7	$\langle u^{11}, u^5v, v^2 \rangle$	16	44
8	$\langle u^{13}, u^7v, uv^2, v^3 \rangle$	21	52
9	$\langle u^{16}, u^{10}v, u^3v^2, v^3 \rangle$	29	62

CASE $(C,O) \sim B_{15,2} \circ B_{10,3}$ and $B_{25,4}$: In this case, we have the inequalities $\iota_k(O) > 10(k-3)$ for all $k = 3, \dots, 9$ by the local data. Hence $\bar{\sigma}_k$ is injective for all k by Proposition 1 and we obtain the property ($\sharp$):

$$\ell_k = \tilde{\rho}(k) = \begin{cases} 1 & k = 7, 9, \\ 0 & k \neq 7, 9. \end{cases}$$

Therefore $\Delta_C(t) = \Delta_{5,2}(t) = t^4 - t^3 + t^2 - t + 1$.

4.3. Exceptional case: $(C, O) \sim B_{20,5}$.

In this section, we consider the last singularity $B_{20,5}$ which takes place for reducible C. Recall that C is a union of five conics. We assume that we have chosen local coordinates (u, v) so that (C, O) is defined by

$$B_{20,5}: \; f(u, v) = u^{20} + v^5 + (\text{higher terms}),$$

where we ignore the coefficients of the monomials and other monomials corresponding to other integral points on the Newton boundary.

$B_{20,5}$:

k	$\mathscr{I}_{O,k,10}$	$\rho_k(O)$	$\iota_k(O)$
3	$\langle u^2, v \rangle$	2	10
4	$\langle u^4, v \rangle$	4	20
5	$\langle u^6, u^2v, v^2 \rangle$	8	30
6	$\langle u^8, u^4v, v^2 \rangle$	12	40
7	$\langle u^{10}, u^6v, u^2v^2, v^3 \rangle$	18	50
8	$\langle u^{12}, u^8v, u^4v^2, v^3 \rangle$	24	60
9	$\langle u^{14}, u^{10}v, u^6v^2, u^2v^3, v^4 \rangle$	32	70

Again we have the inequalities $\iota_k(O) - 10(k-3) > 0$ for all $k = 3, \dots, 9$. We claim that $\bar{\sigma}_k$ *is injective for all* k. In fact, assuming $0 \neq g \in \operatorname{Ker} \bar{\sigma}_k$, we have $g \mid f$ by the proof of Proposition 1 and this means g is a union of conics which are components of f. Consider the factorization $f = h_1 h_2 h_3 h_4 h_5$ where $\{h_i = 0\}$ is a smooth conic component of C. Then we may assume that

$$f \stackrel{\sigma}{\longmapsto} u^{20} + v^5 + (\text{higher terms}), \quad h_i \stackrel{\sigma}{\longmapsto} u^4 + \zeta^i v + (\text{higher terms}), \; i = 1, \dots, 5$$

where $\zeta = \exp(\pi i/5)$. Thus suppose that $g = h_{i_1} \cdots h_{i_j}$. Then $2j \leq k-3$ or $j \leq [\frac{k-3}{2}]$ and $\sigma_k(g)$ must contain v^j with a non-zero coefficient. This implies that $j \leq$

$0, 0, 1, 1, 2, 2, 3$ for $k = 3, 4, \ldots, 9$ respectively. On the other hand, $v^j \in \mathscr{J}_{O,k,10}$ implies from the table of $B_{20,5}$ that $j \geq 1, 1, 2, 2, 3, 3, 4$ for $k = 3, \ldots, 9$ respectively. This gives an obvious contradiction. Hence we have

$$\ell_k = \tilde{\rho}(k) = \begin{cases} 1 & k = 3, 4, \\ 2 & k = 5, 6, \\ 3 & k = 7, 8, \\ 4 & k = 9. \end{cases}$$

Therefore by the formula (1) in Lemma 1 we obtain the equality:

$$\Delta_C(t) = (t-1)^4(t+1)^4(t^4 - t^3 + t^2 - t + 1)^4(t^4 + t^3 + t^2 + t + 1)^3.$$

REMARK 3. This case can be also computed using the observation of the fundamental groups of maximal contact conics is a free product of $\boldsymbol{Z}_2$ and a free group of rank 4.

5. Calculation of $\boldsymbol{\Delta_C(t)}$, II: Degenerate cases.

Next we calculate the Alexander polynomial of following two degenerate singularities:

- $(B_{9,2}^2)^{B_{5,2}} \circ B_{2,1}$: this is the last singularity of the sequence of (b-i) or (c-i).
- $B_{29,2} \circ B_{2,1} \circ (B_{2,1}^2)^{B_{5,2}}$: this is the last singularity of the sequence of (b-ii) or (c-ii).

5.1. Characterization of the adjunction ideal for degenerate cases.

For degenerate singularities, we proceed several toric modifications to obtain their resolutions. Consider an embedded resolution of $(C, O) \subset (\boldsymbol{C}^2, O)$, $\pi : \tilde{U} \to U$ where U is an open neighborhood of O and let $E_1, \ldots, E_s$ be the exceptional divisors. We put the ideal $\bar{\mathscr{J}}_{O,k,d}$ of $\mathscr{O}_O$

$$\bar{\mathscr{J}}_{O,k,d} := \langle M \in \mathscr{O}_O \mid M\text{: monomial, } (\pi^* M) \geq \sum_i ([km_i/d] - k_i)E_i \rangle, \ 1 \leq k \leq d-1.$$

In general, $\bar{\mathscr{J}}_{O,k,d} \subset \mathscr{J}_{O,k,d}$ and $\bar{\mathscr{J}}_{O,k,d} = \mathscr{J}_{O,k,d}$ if (C, O) is non-degenerate from Lemma 2. If (C, O) is degenerate singularity, there exist several other (non-monomial) polynomials h_i, $i = 1, \ldots, r$ such that $h_i \in \mathscr{J}_{O,k,d} \setminus \bar{\mathscr{J}}_{O,k,d}$ and

$$\mathscr{J}_{O,k,d} = \langle M, h_i \mid M \in \bar{\mathscr{J}}_{O,k,d}, \ i = 1, \ldots, r \rangle.$$

5.1.1 Formulation of the multiplicities.

We recall how the multiplicities of the pull-back of a function after toric modifications along the exceptional divisors can be computed.

Let $D = \{g = 0\}$ be a plane curve and let $P \in D$ be a singular point. Suppose that its Newton boundary $\Gamma(g; u, v)$ consists of m-faces $\Delta_1, \dots, \Delta_m$ where (u, v) is a local coordinates centered at P. Then the face function of g with respect to a face Δ_i takes the form:

$$g_{\Delta_i}(u, v) = c\, u^{w_i} v^{t_i} \prod_{j=1}^{k_i} (v^{a_i} - \gamma_{i,j} u^{b_i})^{\nu_{i,j}}, \quad c \neq 0$$

where $P_i = {}^t(a_i, b_i)$ is the weight vector corresponding to Δ_i. Let $\{E_0, P_1, \dots, P_m, E_2\}$ be the vertices of the dual Newton diagram $\Gamma^*(g; u, v)$ where $E_1 = {}^t(1, 0)$ and $E_2 = {}^t(0, 1)$. Let $\pi_1 : X_1 \to \boldsymbol{C}^2$ be the toric modification associated with $\{\Sigma_1^*, (u, v), P\}$ where $\Sigma_1^* = \{E_1, Q_1, \dots, Q_{m'}, E_2\}$ is the canonical regular simplicial cone subdivision of $\{E_1, P_1, \dots, P_m, E_2\}$ ([**12**]). Then we can write the divisor $(\pi_1^* g)$ as

$$(\pi_1^* g) = \tilde{D} + \sum_{s=1}^{m'} m(g, Q_s) \hat{E}(Q_s)$$

where $\tilde{D}$ is the strict transform of D and $\hat{E}(Q_j)$ is the exceptional divisor corresponding to the vertex Q_j. We assume that $P_i = Q_{\nu_i}$ for $i = 1, \dots, m$. Then the exceptional divisors $\hat{E}(Q_{\nu_i}) = \hat{E}(P_i)$ intersects with the strict transform $\tilde{D}$. We take the toric coordinates $(\boldsymbol{C}^2_{\sigma_{\nu_i}}, (u_i, v_i))$ where $\sigma_{\nu_i} = \mathrm{Cone}\,(Q_{\nu_i}, Q_{\nu_i+1})$ so that $\{u_i = 0\}$ defines $\hat{E}(Q_{\nu_i}) \cap \boldsymbol{C}^2_{\sigma_{\nu_i}}$. Then $\tilde{D}$ and the total transform $\pi_1^* D$ are defined in this coordinate as

$$\begin{aligned} \tilde{D} : &\quad \tilde{g}(u_i, v_i) = c_i\, (v_i - \gamma_{i,j})^{\nu_{i,j}} + R(u_i, v_i) = 0, \quad c_i \neq 0 \\ \pi_1^* D : &\quad \pi^* g(u_i, v_i) = u_i^{d(P_i; g)} v_i^{d(Q_{\nu_i+1}; g)}\, \tilde{g}(u_i, v_i) \end{aligned}$$

where $R \equiv 0$ modulo (u_i). Thus $\xi_{i,j} := (0, \gamma_{i,j})$ is the intersection points of $\tilde{D}$ and $\hat{E}(Q_{\nu_i})$ for $j = 1, \dots, k_i$. We take an admissible translated coordinates (u_i, v_i') with $v_i' = v_i - \gamma_{i,j} + h(u_i)$ in an open neighbourhood of $\xi_{i,j}$ where h is a suitable polynomial with $h(0) = 0$. Suppose that $(\tilde{D}, \xi_{i,j})$ has a non-degenerate singularity with respect to the coordinates (u_i, v_i') and suppose that the Newton boundary has a unique face $\Delta_{i,j}$ for $j = 1, \dots, k_i$. (For our purpose, this case is enough to be considered.) Let $S_{i,j} = {}^t(s_{i,j}, t_{i,j})$ be the primitive dual vector which corresponds to the face $\Delta_{i,j}$ and assume the germ $(\tilde{D}, \xi_{i,j})$ is equivalent to the Brieskorn singularity $B_{c_{i,j},d_{i,j}}$ with $t_{i,j}c_{i,j} = s_{i,j}d_{i,j}$. This means the dual Newton diagram $\Gamma^*(\tilde{g}; u_i, v_i')$ is given by $\{E_1, S_{i,j}, E_2\}$.

We take the canonical regular subdivision $\Sigma_{i,j}^*$ of $\Gamma^*(\tilde{g}; u_i, v_i')$. Put

$$\Sigma_{i,j}^* = \{T_{i,j,0}, T_{i,j,1}, \dots, T_{i,j,m_j}, T_{i,j,m_j+1}\},\ T_{i,j,0} = E_1, T_{i,j,m_j+1} = E_2.$$

We may assume $S_{i,j} = T_{i,j,k_0}$ for some $k_0 \in \{1, \dots, m_j\}$. At each point $\xi_{i,j}$, we take the toric modification $\pi_{ij} : X_{ij} \to X_1$ with respect to $\{\Sigma_{i,j}^*, (u_i, v_i'), \xi_{i,j}\}$. These modifications are compatible with each other and let $\pi_2 : X_2 \to X_1$ be the composition of these modifications for every i, j so that the exceptional divisors of π_2 are bijectively corresponding to the vertices of $\Sigma_{i,j}^*$, $i = 1, \dots, m$, $j = 1, \dots, k_i$. What is necessary to be checked are the multiplicities of $\pi^* g$ and $\pi^*(du \wedge dv)$ along the exceptional divisors $\hat{E}(T_{i,j,k})$ where $\pi : X_2 \to \boldsymbol{C}^2$ is the composition of $\pi_2 : X_2 \to X_1$ and $\pi_1 : X_1 \to \boldsymbol{C}^2$. Then we can write:

$$(\pi^* g) = \tilde{D} + \sum_{s=1}^{m'} m(g, Q_s)\hat{E}(Q_s) + \sum_{i=1}^{m}\sum_{j=1}^{k_i}\sum_{k=1}^{m_j} m(g, T_{i,j,k})\hat{E}(T_{i,j,k}).$$

$$(\pi^* K) = \sum_{s=1}^{m'} k(Q_s)\hat{E}(Q_s) + \sum_{i=1}^{m}\sum_{j=1}^{k_i}\sum_{k=1}^{m_j} k(T_{i,j,k})\hat{E}(T_{i,j,k})$$

where $K = du \wedge dv$ is the canonical two form in the base space.

Lemma 3. *Under the above situations, the multiplicities are given as follows. Put* $T_{i,j,k} = {}^t(\varepsilon_{i,j,k}, \eta_{i,j,k})$.

(1) *The multiplicities* $m(g, P_i)$, $m(g, T_{i,j,k})$ *of* $\pi^* g$ *along the divisors* $\hat{E}(P_i)$ *and* $\hat{E}(T_{i,j,k})$ *are given by*

$$m(g, P_i) = d(P_i, g), \quad m(g, T_{i,j,k}) = \varepsilon_{i,j,k} m(g, P_i) + d(T_{i,j,k}, \tilde{g}).$$

(2) *The multiplicities* $k(Q_s)$, $k(T_{i,j,k})$ *of the pull-back of the canonical two form* $K = du \wedge dv$ *along the divisors* $\hat{E}(Q_s)$ *and* $\hat{E}(T_{i,j,k})$ *are given by*

$$k(Q_s) = |Q_s| - 1, \quad k(T_{i,j,k}) = |T_{i,j,k}| - 1 + \varepsilon_{i,j,k} k(P_i)$$

where $|{}^t(a, b)| = a + b$.

The proof follows easily from Theorem 3.8 and Proposition 7.2, Chapter III of [**12**].

5.2. Generalization of Lemma 2.

LEMMA 4. *Under the above assumptions, a germ $\varphi \in \mathscr{O}_P$ is contained in the ideal $\mathscr{J}_{P,k,d}$ if and only if φ satisfies*:

(1) $m(\varphi, P_i) \geq [\frac{k}{d} m(g, P_i)] - k(P_i)$ *for* $i = 1, \ldots, m$, *and*
(2) $m(\varphi, S_{i,j}) \geq [\frac{k}{d} m(g, S_{i,j})] - k(S_{i,j})$ *for* $j = 1, \ldots, k_i$.

Note that there are no conditions on other exceptional divisors $\hat{E}(T_{i,j,k})$.

PROOF. The proof is almost parallel to that of Lemma 2 of [**13**]. Assume that φ satisfies the conditions (1) and (2). It is enough to show that

$$(2 - bis) \quad m(\varphi, T_{i,j,k}) \geq \left[\frac{k}{d} m(g, T_{i,j,k})\right] - k(T_{i,j,k}), \quad j = 1, \ldots, k_i, \ k = 1, \ldots, m_j.$$

Note that the condition (2) is equivalent to
(2)' $m(\varphi, S_{i,j}) > \frac{k}{d} m(g, S_{i,j}) - (|S_{i,j}| + s_{i,j} k(P_i))$ for $j = 1, \ldots, k_i$.
First we observe that $m(g, T_{i,j,0}) = m(g, P_i)$ and $m(g, T_{i,j,m_j+1}) = 0$. Take $T_{i,j,k}$ for $k < k_0$ for example. We can write $T_{i,j,k} = \alpha_k S_{i,j} + \beta_k T_{i,j,0}$ for some positive rational numbers α_k, β_k. Note that

$$|T_{i,j,k}| = \alpha_k |S_{i,j}| + \beta_k |T_{i,j,0}| = \alpha_k |S_{i,j}| + \beta_k,$$
$$m(g, T_{i,j,k}) = \alpha_k m(g, S_{i,j}) + \beta_k m(g, T_{i,j,0}),$$

Here the second equality follows as $\Delta(S_{i,j}, \pi_1^* g) \cap \Delta(T_{i,j,0}, \pi_1^* g) \neq \emptyset$ by the admissibility of the canonical subdivision $\Sigma_{i,j}^*$. Thus we have

$$\begin{aligned} m(\varphi, T_{i,j,k}) &\geq \alpha_k m(\varphi, S_{i,j}) + \beta_k m(\varphi, T_{i,j,0}) \\ &> \alpha_k \left(\frac{k}{d} m(g, S_{i,j}) - (|S_{i,j}| + s_{i,j} k(P_i))\right) + \beta_k \left(\frac{k}{d} m(g, T_{i,j,0}) - (1 + k(P_i))\right) \\ &= \frac{k}{d} m(g, T_{i,j,k}) - (|T_{i,j,k}| + \varepsilon_{i,j,k} k(P_i)) \end{aligned}$$

as $\varepsilon_{i,j,k} = \alpha_k s_{i,j} + \beta_k$ by the equality $T_{i,j,k} = \alpha_k S_{i,j} + \beta_k T_{i,j,0}$. This inequality is equivalent:

$$m(\varphi, T_{i,j,k}) \geq \left[\frac{k}{d} m(g, T_{i,j,k})\right] - k(T_{i,j,k}).$$

For $T_{i,j,k}$ with $k > k_0$, the argument is similar. Hence we have $\varphi \in \mathscr{J}_{P,k,d}$. □

Now we consider the ideal $\mathscr{J}_{P,k,d}$ in more detail. Take $\varphi \in \mathscr{O}_P$. We compute the multiplicity of φ along the divisors $\hat{E}(P_i)$ and $\hat{E}(S_{i,j})$. We divide our consideration into the two cases:

(1) φ is a monomial,

(2) φ is a polynomial (non-monomial).

First we see the case (1) and we put $\varphi(u,v) = u^\alpha v^\beta$. As $\pi_1^*\varphi$ is also a monomial in u_i, v_i, we can check easily following

$$m(\varphi, P_i) = d(P_i, \varphi) = a_i\alpha + b_i\beta, \quad m(\varphi, S_{i,j}) = s_{i,j} m(\varphi, P_i).$$

Next we consider the case (2). We can write $\varphi(u,v) = \varphi_{P_i}(u,v) + R(u,v)$ where $R(u,v)$ consist of monomials of degree strictly greater than $d(P_i, \varphi)$. If $\Delta(\varphi, P_i)$ is zero dimensional, then the multiplicities $m(\varphi, P_i)$ and $m(\varphi, S_{i,j})$ are equal to that of the monomial $\varphi_{P_i}(u,v)$. If $\Delta(\varphi, P_i)$ is one dimensional, then the face function $\varphi_{P_i}(u,v)$ can be written by

$$\varphi_{P_i}(u,v) = c_i\, u^\alpha v^\beta \prod_{j=1}^{\kappa_i} (v^{a_i} - \delta_{i,j} u^{b_i})^{\mu_{i,j}}, \quad c_i,\ \delta_{i,j} \neq 0.$$

Then the multiplicities $m(\varphi, P_i)$ is given by

$$m(\varphi, P_i) = a_i\alpha + b_i\beta + a_i b_i \sum_{j=1}^{\kappa_i} \mu_{i,j}.$$

In the admissible translated coordinates (u_i, v_i'), the function $\pi_1^*\varphi$ is written by

$$\begin{aligned} \pi_1^*\varphi(u_i, v_i') &= c_i u_i^{m(\varphi, P_i)} \tilde{\varphi}(u_i, v_i'), \\ \tilde{\varphi}(u_i, v_i') &= \prod_{j=1}^{\kappa_i} (v_i' + (\gamma_{i,j} - \delta_{i,j}) - h(u_i))^{\mu_{i,j}} + \tilde{R}(u_i, v_i') \end{aligned}$$

where $\tilde{R}(u_i, v_i') \equiv 0 \bmod (u_i)$. Thus we obtain

$$m(\varphi, S_{i,j}) = \begin{cases} s_{i,j} m(\varphi, P_i) & \text{if } \delta_{i,j} \neq \gamma_{i,j} \text{ for all } j, \\ s_{i,j} m(\varphi, P_i) + d(S_{i,j}, \tilde{\varphi}) & \text{if } \delta_{i,j} = \gamma_{i,j} \text{ for some } j. \end{cases}$$

Note that the multiplicity $d(S_{i,j}, \tilde{\varphi})$ depends on the form h, R and $S_{i,j}$.

5.3. The case of $(B^2_{9,2})^{B_{5,2}} \circ B_{2,1}$.

By the local classification in [**6**], this singularity $(B^2_{9,2})^{B_{5,2}} \circ B_{2,1}$ appears when the associated curves C_2 and C_5 satisfy following conditions:

(1) C_2 is irreducible and $I(C_2, C_5; O) = 10$.

(2) The multiplicity of (C_5, O) is 3 and the tangent cone of C_5 consists of a line L_1 with multiplicity 2 and a single line L_2.

(3) The conic C_2 is tangent to the line L_1 at O.

Suppose that C_2 and C_5 satisfy the above conditions. Then we may assume that the defining polynomials of C_2 and C_5 are the following forms:

$$
\begin{aligned}
f_2(x,y) &= y + a_{20}x^2 + a_{11}xy + a_{02}y^2,\ a_{20} \neq 0,\\
f_5(x,y) &= b_{05}y^5 + ((a_{02}^2 b_{12} + a_{11}b_{04})x + b_{04})y^4 + ((2b_{12}a_{02}a_{11} + a_{20}b_{04})x^2 + 2a_{02}b_{12}x)y^3\\
&\quad + ((2a_{20}a_{02}b_{12} + b_{12}a_{11}^2)x^3 + 2b_{12}a_{11}x^2 + b_{12}x)y^2\\
&\quad + (2a_{11}b_{12}a_{20}x^4 + 2b_{12}a_{20}x^3)y + a_{20}^2 b_{12}x^5
\end{aligned}
$$

where $b_{12} \neq 0$ and $a_{20} + b_{12}^2 \neq 0$ in general. If $a_{20} + b_{12}^2 = 0$, (C, O) has the same type of singularity but C is not irreducible and has a line component which is defined by $\{y = 0\}$. Now we take a local coordinates (u, v) of the following type so that

$$
\begin{gathered}
x = u, \quad y = v + \varphi(u), \quad \varphi(u) = -a_{20}u^2 + \cdots,\ a_{20} \neq 0,\\
\begin{aligned}
f_2(u, v + \varphi(u)) &= v + c_1u^5 + \cdots,\\
f_5(u, v + \varphi(u)) &= b_{04}v^4 + b_{12}uv^2 + c_2u^{10} + (\text{higher terms}),\ b_{12}, c_2 \neq 0,\\
f(u, v + \varphi(u)) &= v^5 + u^2(b_{12}v^2 + c_2u^9)^2 + (\text{higher terms}).
\end{aligned}
\end{gathered}
$$

Then the Newton boundary $\Gamma(f; u, v)$ consists of two faces Δ_i $(i = 1, 2)$ so that the respective face functions are given by

$$
f_{\Delta_1}(u,v) = v^4(v + b_{12}^2u^2), \quad f_{\Delta_2}(u,v) = u^2(b_{12}v^2 + c_2u^9)^2.
$$

Note that $f(u, v)$ is degenerate on Δ_2. We take the canonical toric modification $\pi_1 : X_1 \to \boldsymbol{C}^2$ with respect to $\{\Sigma_1^*, (u,v), O\}$ where Σ_1^* is the canonical regular simplicial cone subdivision with vertices $\{E_1, Q_1, \ldots, Q_6, E_2\}$ where

$$
Q_1 = \begin{pmatrix}1\\1\end{pmatrix},\ Q_2 = \begin{pmatrix}1\\2\end{pmatrix},\ Q_3 = \begin{pmatrix}1\\3\end{pmatrix},\ Q_4 = \begin{pmatrix}1\\4\end{pmatrix},\ Q_5 = \begin{pmatrix}2\\9\end{pmatrix},\ Q_6 = \begin{pmatrix}1\\5\end{pmatrix}
$$

and the weight vectors Q_2 and Q_5 correspond to the faces Δ_1 and Δ_2 respectively. Then we can write the divisor $(\pi_1^* f)$ as

$$(\pi_1^* f) = \tilde{C} + \sum_{i=1}^{6} m(f, Q_i)\hat{E}(Q_i),$$

where $\tilde{C}$ is the strict transform of C and intersects only with the exceptional divisors $\hat{E}(Q_2)$ and $\hat{E}(Q_5)$. We can see that $\tilde{C}$ is smooth and intersects transversely at $\tilde{C} \cap \hat{E}(Q_2)$ but $\tilde{C}$ has the singularity at the intersection $\tilde{C} \cap \hat{E}(Q_5)$. Put $\xi = \tilde{C} \cap \hat{E}(Q_5)$. In the toric coordinates (u_1, v_1) of $\boldsymbol{C}_\tau^2$ with $\tau = \mathrm{Cone}\,(Q_5, Q_6)$ (see [**12**] for the notations), $\xi = (0, -c_2/b_{12})$. To see the singularity $(\tilde{C}, \xi)$, we take the admissible translated toric coordinates (u_1, v_1') with $v_1' = v_1 + c_2/b_{12} + h(u_1)$ where h take the form $h(u_1) = q_1 u_1 + q_2 u_1^2$. Then we can see that $\pi_1^* f(u_1, v_1') = c\, u_1^{40}({v_1'}^2 + \beta u_1^5 + \text{(higher terms)})$ and $(\tilde{C}, \xi) \sim B_{5,2}$. Now we take the second toric modification $\pi_2 : X_2 \to X_1$ with respect to $\{\Sigma_2^*, (u_1, v_1'), \xi\}$ where Σ_2^* is the canonical regular simplicial cone subdivision with vertices $\{E_1, T_1, \dots, T_4, E_2\}$ where

$$T_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ T_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ T_3 = \begin{pmatrix} 2 \\ 5 \end{pmatrix},\ T_4 = \begin{pmatrix} 1 \\ 3 \end{pmatrix},$$

and the weight vector T_3 corresponds to the unique face of $\pi_1^* f(u_1, v_1')$. Note also the exceptional divisor which corresponds to E_1 is nothing but the exceptional divisor $\hat{E}(Q_5)$ in the previous modification π_1. Then we have

$$\begin{aligned}(\pi^* f) = {} & 5\hat{E}(Q_1) + 10\hat{E}(Q_2) + 14\hat{E}(Q_3) + 18\hat{E}(Q_4) + 40\hat{E}(Q_5) + 20\hat{E}(Q_6) \\ & + 42\hat{E}(T_1) + 44\hat{E}(T_2) + 90\hat{E}(T_3) + 45\hat{E}(T_4)\end{aligned}$$

$$\begin{aligned}(\pi^* K) = {} & \hat{E}(Q_1) + 2\hat{E}(Q_2) + 3\hat{E}(Q_3) + 4\hat{E}(Q_4) + 10\hat{E}(Q_5) + 5\hat{E}(Q_6) \\ & + 11\hat{E}(T_1) + 12\hat{E}(T_2) + 26\hat{E}(T_3) + 13\hat{E}(T_4)\end{aligned}$$

and we consider two polynomials $h_2(u, v)$ and $r_2(u, v)$ which are defined by $h_2(u, v) = b_{12} v^2 + c_2 u^9$ and $r_2(u, v) = h_2(u, v) - (q_1 b_{12}^2 / c_2) u^5 v$. Then we can see by a direct computation

$$\begin{aligned}
&\pi_1^* h_2(u_1, v_1') = u^{18}(d_3 v_1' + d_4 u_1 + \text{(higher terms)}),\\
&\pi_1^* r_2(u_1, v_1') = u^{18}(d_3' v_1' + d_4' u_1^2 + \text{(higher terms)}),\\
&m(h_2, Q_2) = 4, \quad m(h_2, Q_5) = 18, \quad m(h_2, T_3) = 38,\\
&m(r_2, Q_2) = 4, \quad m(r_2, Q_5) = 18, \quad m(h_2, T_3) = 40.
\end{aligned}$$

ASSERTION 1. *The adjunction ideals* $\mathscr{J}_{O,k,10}$ *are given by*

$$\mathscr{J}_{O,3,10} = \langle u, v\rangle,\ \mathscr{J}_{O,4,10} = \langle u^3, v\rangle,\ \mathscr{J}_{O,5,10} = \langle u^5, uv, v^2\rangle,\ \mathscr{J}_{O,6,10} = \langle u^7, u^3v, v^2\rangle,$$
$$\mathscr{J}_{O,7,10} = \langle u^{10}, u^5v, uv^2, v^3\rangle,\ \mathscr{J}_{O,8,10} = \langle u^{12}, u^7v, u^3v^2, v^3, h_2^{(2,0)}\rangle,$$
$$\mathscr{J}_{O,9,10} = \langle u^{14}, u^{10}v, u^5v^2, uv^3, v^4, r_2^{(4,0)}\rangle$$

where $h_2^{(2,0)}(u,v) := u^2h_2(u,v)$ *and* $r_2^{(4,0)}(u,v) := u^4r_2(u,v)$.

The proof follows from Lemma 3 and Lemma 4 and by an easy computation.

Thus we have $\rho_8(O) = 21$, $\rho_9(O) = 29$ and

$$\tilde\rho(k) = \begin{cases} 1 & k = 7, 9, \\ 0 & k \neq 7, 9. \end{cases},\quad \iota_k(O) > 10(k-3),\quad 3 \le k \le 9.$$

ASSERTION 2. *The map* $\bar\sigma_k$ *is injective for all* $k = 3, \ldots, 9$.

PROOF. Recall that C can be either irreducible or reducible in this case. As $\iota_k(O) > 10(k-3)$, if C is irreducible, then the assertion follows from Proposition 1.

Assume C is not irreducible. We have seen in the previous argument in Section 3.4, C has two irreducible components of respective degree 1 and 9. Namely we can write $C = C_1 \cup C_9$ where $C_1 = \{y = 0\}$. Suppose that there exists a non-zero $g \in \operatorname{Ker}\bar\sigma_k \subset O(k-3)$. As $\iota_k(O) > 10(k-3)$, g divides f by the proof of Proposition 1. This is possible only if $k \ge 4$ and $\deg g = 1$. By the assumption, we have $g = cy$ with $c \neq 0$. As $y = v + \varphi(u)$, we see that g can not be in the ideal $\mathscr{J}_{O,k,10}$ for $k \ge 5$, as $v \notin \mathscr{J}_{O,k,10}$ by Assertion 1. This implies that $\bar\sigma_k$ is injective for $k \neq 4$. Assume $k = 4$. As $a_{20} \neq 0$, $\operatorname{ord}_u \varphi(u) = 2$ and $\mathscr{J}_{O,4,10} = \langle u^3, v\rangle$, again we see that $v + \varphi(u) \notin \mathscr{J}_{O,4,10}$. This is a contradiction for $g \in \operatorname{Ker}\bar\sigma_4$ and the proof is completed. □

Therefore we obtain the property (♯): $\ell_k = 1$ for $k = 7, 9$ and $\ell_k = 0$ otherwise. Thus the reduced Alexander polynomial is given by $\tilde\Delta_C(t) = t^4 - t^3 + t^2 - t + 1$ for the case $(C, O) \sim (B_{9,2}^2)^{B_{5,2}} \circ B_{2,1}$.

5.4. The case of $B_{29,2} \circ B_{2,1} \circ (B_{2,1}^2)^{B_{5,2}}$.

By local classification [**6**], this singularity appears in the case that the associated curves C_2 and C_5 satisfy following conditions:

(1) C_2 is irreducible and $I(C_2, C_5; O) = 10$.

(2) The multiplicity of (C_5, O) is 3 and the tangent cone of C_5 at O consists of a line L_1 with multiplicity 2 and a single line L_2.

(3) The conic C_2 is tangent to the line L_1 at O.
Suppose that C_2 and C_5 satisfy the above conditions. Then we may assume that the defining polynomials of C_2 and C_5 have following forms:

$$\begin{aligned}
f_2(x,y) &= y + a_{20}x^2 + a_{11}xy + a_{02}y^2,\ a_{20} \neq 0 \\
f_5(x,y) &= b_{05}y^5 + a_{02}^2 b_{12}\,xy^4 + 2a_{02}b_{12}x(a_{11}x+1)y^3 \\
&\quad + \left(\frac{1}{27}b_{12}(4a_{02}b_{12}^2 + 54a_{02}a_{20} + 27a_{11}^2)x^3 + 2a_{11}b_{12}x^2 + b_{12}x\right)y^2 \\
&\quad + \left(\frac{2}{27}b_{12}x^3(2b_{12}^2 + 27a_{20})(a_{11}x+1)\right)y + \frac{1}{27}b_{12}a_{20}(27a_{20} + 4b_{12}^2)x^5
\end{aligned}$$

where $b_{12} \neq 0$ and $b_{12}^2 + 9a_{20} \neq 0$ in general. If $b_{12}^2 + 9a_{20} = 0$, C has the line component which is defined by $\{y = 0\}$. Now we take a local coordinates (u, v) of the following type so that

$$\begin{aligned}
&x = u, \quad y = v + \varphi(u), \quad \varphi(u) = -a_{20}u^2 + \cdots,\ a_{20} \neq 0, \\
&f_2(u, v+\varphi(u)) = v + \psi(u) = v + \beta_7 u^7 + (\text{higher terms}), \quad \beta_7 \neq 0, \\
&f_5(u, v+\varphi(u)) = b_{05}\,v^5 + b_{12}uv\left(v + \frac{4}{27}b_{12}^2u^2\right) + c_4\,u^{18} + (\text{higher terms}),\ b_{12} \neq 0, \\
&f(u, v+\varphi(u)) = v^2(v + d_1u^2)(v + d_2u^2)^2 + \beta_7^5 u^{35} + (\text{higher terms}),\ d_1, d_2 \neq 0.
\end{aligned}$$

By an explicit calculation, we have $d_2 = (4/9)\,b_{12}^2$ and $d_2 + a_{20} \neq 0$. (If $d_2 + a_{20} = 0$, f becomes a non-reduced polynomial.) Then the Newton boundary $\Gamma(f; u, v)$ consists of two faces Δ_1 and Δ_2 so that their face functions are given by

$$f_{\Delta_1}(u,v) = v^2(v + d_1u^2)(v + d_2u^2)^2, \quad f_{\Delta_2}(u,v) = u^6(d_1d_2^2v^2 + \beta_7^5u^{29}).$$

Note that $f(u, v)$ is degenerate on Δ_1. We take the canonical toric modification $\pi_1 : X_1 \to \boldsymbol{C}^2$ with respect to $\{\Sigma_1^*, (u,v), O\}$ where Σ_1^* is the canonical regular simplicial cone subdivision with vertices

$$E_1, \quad Q_k = \begin{pmatrix} 1 \\ k \end{pmatrix} (1 \le k \le 14), \quad Q_{15} = \begin{pmatrix} 2 \\ 29 \end{pmatrix}, \quad Q_{16} = \begin{pmatrix} 1 \\ 15 \end{pmatrix}, \quad E_2$$

where Q_2 and Q_{15} are the weight vectors of the faces Δ_1 and Δ_2 respectively. Then the divisor $(\pi_1^* f)$ is given by

$$(\pi_1^* f) = \tilde{C} + \sum_{i=1}^{16} m(f, Q_i)\hat{E}(Q_i),$$

where $\tilde{C}$ is the strict transform of C and intersects only with the exceptional divisors $\hat{E}(Q_2)$ and $\hat{E}(Q_{15})$. We can see that $\tilde{C}$ is smooth at $\tilde{C} \cap \hat{E}(Q_{15})$ and the intersection is transverse. On the other hand, $\tilde{C}$ intersects with $\hat{E}(Q_2)$ at two points $\xi_{1,1}$, $\xi_{1,2}$ where $\xi_{1,1} = (0, -d_1)$, $\xi_{1,2} = (0, -d_2)$ in the toric coordinates (u_1, v_1) of the chart $\boldsymbol{C}_\tau^2$ with $\tau = \text{Cone}\ (Q_2, Q_3)$. Note that $(\tilde{C}, \xi_{1,1})$ is smooth and the intersection with $\hat{E}(Q_2)$ is transverse at $\xi_{1,1} = (0, -d_1)$. On the other hand, $(\tilde{C}, \xi_{1,2})$ has singularity. To see the singularity $(\tilde{C}, \xi_{1,2})$, we take the admissible translated coordinates (u_1, v_1') with $v_1' = v_1 + d_2 + h(u_1)$ where h takes the form $h(u_1) = q_1 u + q_2 u^2$. Then we see that $\pi_1^* f(u_1, v_1') = c\, u_1^{10}({v_1'}^2 + \beta u_1^5 + \text{(higher terms)})$ and $(\tilde{C}, \xi_{1,2}) \sim B_{5,2}$. Now we take the second toric modification $\pi_2 : X_2 \to X_1$ with respect to $\{\Sigma_2^*, (u_1, v_1'), \xi_{1,2}\}$ where Σ_2^* is the canonical regular simplicial cone subdivision with vertices

$$E_1, \quad T_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ T_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ T_3 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \ T_4 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad E_2$$

where the weight vector T_3 corresponds to the unique face of $\Gamma(\pi_1^* f; u_1, v_1')$. Then we have

$$\begin{aligned}(\pi^* f) &= 5\hat{E}(Q_1) + \sum_{i=2}^{14} 2(i+3)\hat{E}(Q_i) + 70\hat{E}(Q_{15}) + 35\hat{E}(Q_{16}) \\ &\quad + 12\hat{E}(T_1) + 14\hat{E}(T_2) + 30\hat{E}(T_3) + 15\hat{E}(T_4). \\ (\pi^* K) &= \hat{E}(Q_1) + \sum_{i=2}^{14} i\hat{E}(Q_i) + 30\hat{E}(Q_{15}) + 15\hat{E}(Q_{16}) \\ &\quad + 3\hat{E}(T_1) + 4\hat{E}(T_2) + 10\hat{E}(T_3) + 5\hat{E}(T_4)\end{aligned}$$

and we consider two polynomials $h_1(u,v)$ and $r_1(u,v)$ which are defined by $h_1(u,v) = v + d_2 u^2$ and $r_1(u,v) = h_1(u,v) - (q_1/d_2)u^3$. Then

$$\begin{aligned} &\pi_1^* h_1(u_1, v_1') = u^2(e_3 v_1' + e_4 u_1 + \text{(higher terms)}), \\ &\pi_1^* r_1(u_1, v_1') = u^2(e_3' v_1' + e_4' u_1^2 + \text{(higher terms)}), \\ &m(h_1, Q_2) = 2, \quad m(h_1, Q_{15}) = 4, \quad m(h_1, T_3) = 6, \\ &m(r_1, Q_2) = 2, \quad m(r_1, Q_{15}) = 4, \quad m(r_1, T_3) = 8. \end{aligned}$$

Assertion 3. *Under the above situation,*
(a) *The ideals* $\mathscr{I}_{O,k,10}$ *are given by*

$$\mathscr{I}_{O,3,10} = \langle u, v\rangle, \quad \mathscr{I}_{O,4,10} = \langle u^2, v\rangle, \quad \mathscr{I}_{O,5,10} = \langle u^3, uv, v^2\rangle, \quad \mathscr{I}_{O,6,10} = \langle u^6, u^2v, v^2\rangle,$$
$$\mathscr{I}_{O,7,10} = \langle u^{10}, u^4v, u^2v^2, v^3, h_1^{(1,1)}\rangle, \quad \mathscr{I}_{O,8,10} = \langle u^{13}, u^5v, u^3v^2, uv^3, v^4, h_1^{(2,1)}, h_1^{(0,2)}\rangle,$$
$$\mathscr{I}_{O,9,10} = \langle u^{17}, u^7v, u^5v^2, u^3v^3, uv^4, v^5, r_1^{(3,1)}, r_1^{(1,2)}, h_1^{(0,3)}\rangle,$$

where $h_1^{(r,s)}(u,v) := u^r v^s h_1(u,v)$ *and* $r_1^{(r,s)}(u,v) := u^r v^s r_1(u,v)$.
(b) *The kernel of* $\bar{\sigma}_k$ *are given by*

$$\operatorname{Ker}\bar{\sigma}_3 = \langle 0\rangle, \quad \operatorname{Ker}\bar{\sigma}_4 = \langle y\rangle, \quad \operatorname{Ker}\bar{\sigma}_5 = \langle y^2, xy\rangle, \quad \operatorname{Ker}\bar{\sigma}_6 = \langle yf_2, y^3\rangle,$$
$$\operatorname{Ker}\bar{\sigma}_7 = \langle y^2 f_2\rangle.$$

Remark 4. *We have also*

$$\operatorname{Ker}\bar{\sigma}_8 = \langle y^3 f_2\rangle,$$
$$\operatorname{Ker}\bar{\sigma}_9 = \langle 3yf_5 - 2b_{12}xyf_2^2 - c_0y^2f_2^2\rangle, \quad \text{with} \quad c_0 = \frac{81a_{11}a_{20}}{4b_{12}(9a_{20} + 4b_{12}^2)}.$$

We do not need this calculation.

Proof. The assertion (a) follows from Lemma 3 and Lemma 4. We consider the assertion (b). By the choice of the local coordinates (u,v), we have relations:

$$x = u, \quad y = v + \varphi(u) = v - a_{20}u^2 + \cdots,$$
$$f_2(u, v + \varphi(u)) = v + \psi(u) = v + \beta_7 u^7 + (\text{higher terms}), \ \beta_7 \neq 0,$$
$$f_5(u, v + \varphi(u)) = b_{05}\, v^5 + b_{12}uv\left(v + \frac{4}{27}b_{12}^2u^2\right) + c_4\, u^{18} + (\text{higher terms}).$$

Put $\varphi(u) = \sum_{j=2}^{\infty} \alpha_j u^j$ with $\alpha_2 = -a_{20}$. We define

$$\operatorname{ord} \mathscr{I}_{O,k,10} := \min\{\operatorname{ord}_{(u,v)} h \mid h \in \mathscr{I}_{O,k,10}\}.$$

Thus for any $g \in \operatorname{Ker}\bar{\sigma}_k$, we have

$$\operatorname{ord}_{(x,y)} g = \operatorname{ord}_{(u,v)}\sigma_k(g) \geq \operatorname{ord} \mathscr{I}_{O,k,10}. \tag{2}$$

□

CASE $k=4$: $\operatorname{Ker}\bar{\sigma}_4 = \langle y\rangle$:

PROOF. The inclusion $\langle y\rangle \subset \operatorname{Ker}\bar{\sigma}_4$ holds by the definition of σ_4. For any $g \in \operatorname{Ker}\bar{\sigma}_4 \subset O(1)$, writing $g(x,y) = c_1 + c_2 x + c_3 y$,

$$\sigma_4(g)(u,v) = c_1 + c_2 u + c_3(v + \varphi(u)) \in \mathscr{I}_{O,4,10} = \langle u^2, v\rangle.$$

Hence we have $c_1 = c_2 = 0$ and $\operatorname{Ker}\bar{\sigma}_4 \subset \langle y\rangle$. □

CASE $k=5$: $\operatorname{Ker}\bar{\sigma}_5 = \langle y^2, xy\rangle$:

PROOF. First we show that $y^2,\ xy \in \operatorname{Ker}\bar{\sigma}_5$. By the definition of σ_5, we have

$$\sigma_5(y^2) = (v+\varphi(u))^2 = v^2 - 2a_{20}u^2 v + a_{20}^2 u^4 + (\text{higher terms}) \in \mathscr{I}_{O,5,10}$$
$$\sigma_5(xy) = u(v+\varphi(u)) = uv - a_{20}u^3 + (\text{higher terms}) \in \mathscr{I}_{O,5,10}$$

as $\mathscr{I}_{O,5,10} = \langle u^3, uv, v^2\rangle$. Next we show that $\operatorname{Ker}\bar{\sigma}_5 \subset \langle y^2, xy\rangle$. Take $g \in \operatorname{Ker}\bar{\sigma}_5 \subset O(2)$. As $\operatorname{ord}\mathscr{I}_{O,5,10} = 2$, we can write $g(x,y) = c_1x^2 + c_2xy + c_3y^2$ by (2) and

$$\sigma_5(g)(u,v) = c_1u^2 + c_2uv + c_3v^2 + (\text{higher terms}) \in \mathscr{I}_{O,5,10} = \langle u^3, uv, v^2\rangle.$$

Hence we have $c_1 = 0$ and $\operatorname{Ker}\bar{\sigma}_5 = \langle y^2, xy\rangle$. □

CASE $k=6$: $\operatorname{Ker}\bar{\sigma}_6 = \langle yf_2, y^3\rangle$:

PROOF. First we show that $yf_2,\ y^3 \in \operatorname{Ker}\bar{\sigma}_6$. By the definition of σ_6, we have

$$\begin{aligned}\sigma_6(yf_2) &= (v+\varphi(u))(v+\psi(u))\\ &= v^2 - a_{20}u^2v - a_{20}\beta_7u^9 + (\text{higher terms}) \in \mathscr{I}_{O,6,10}\end{aligned}$$
$$\sigma_6(y^3) = (v+\varphi(u))^3 = v^3 - 3a_{20}u^2v^2 + 3a_{20}^2u^4v - a_{20}^3u^6 + (\text{higher terms}) \in \mathscr{I}_{O,6,10}$$

as $\mathscr{I}_{O,6,10} = \langle u^6, u^2v, v^2\rangle$.

Next we show that $\operatorname{Ker}\bar{\sigma}_6 \subset \langle yf_2, y^3\rangle$. Take $g \in \operatorname{Ker}\bar{\sigma}_6 \subset O(3)$. As $\sigma_6(g) \in \mathscr{I}_{O,6,10}$, we can write

$$\sigma_6(g)(u,v) = g(u, v+\varphi(u)) = a_1(u,v)u^6 + a_2(u,v)u^2v + a_3(u,v)v^2$$

where $a_i \in \mathscr{O}_O$ $(i=1,2,3)$. Define $g'(u,v)$ by the above right side polynomial. Then we see that

$$I(y,g;O) = \mathrm{ord}_u g'(u, -\varphi(u)) \geq 4.$$

On the other hand, if y does not divide g, $I(y,g;O) \leq 3$ by Bézout's theorem which is an obvious contradiction. Therefore y divides g. Thus we can write $g(x,y) = yg_2(x,y)$ where $g_2 \in O(2)$. Dividing g_2 by f_2 as a polynomial of x, we can write g_2 as $g_2 = c_0 f_2 + (c_1 + c_2 y)x + c_3 y^2 + c_4 y + c_5$ for some constants $c_0, \ldots, c_5$. As yf_2, $y^3 \in \mathrm{Ker}\,\bar{\sigma}_6$, we need to have $y((c_1 + c_2 y)x + c_4 y + c_5) \in \mathrm{Ker}\,\bar{\sigma}_6$. By a simple computation, we conclude $c_1 = c_2 = c_4 = c_5 = 0$ and

$$g(x,y) = c_0 y f_2(x,y) + c_3 y^3 \in \langle y f_2, y^3 \rangle. \qquad \square$$

CASE $k = 7$: $\mathrm{Ker}\,\bar{\sigma}_7 = \langle y^2 f_2 \rangle$:

PROOF. First we show that $y^2 f_2 \in \mathrm{Ker}\,\bar{\sigma}_7$. By the definition of σ_7, we have

$$\begin{aligned}\sigma_7(y^2 f_2)(u,v) &= (v + \varphi(u))^2 (v + \psi(u)) \\ &= v^3 - 2a_{20} u^2 v^2 + a_{20}^2 u^4 v + a_{20}^2 \beta_7 u^{11} + (\text{higher terms}) \in \bar{\mathscr{I}}_{O,7,10}\end{aligned}$$

as

$$\bar{\mathscr{I}}_{O,7,10} = \langle u^{10}, u^4 v, u^2 v^2, v^3 \rangle, \quad \mathscr{I}_{O,7,10} = \langle u^{10}, u^4 v, u^2 v^2, v^3, h_1^{(1,1)} \rangle$$

where $h_1^{(1,1)}(u,v) := uv(v + d_2 u^2)$. Next we show that $\mathrm{Ker}\,\bar{\sigma}_7 \subset \langle y^2 f_2 \rangle$. Take $g \in \mathrm{Ker}\,\bar{\sigma}_7 \subset O(4)$ and we can write $\sigma_7(g)$ as

$$\sigma_7(g)(u,v) = \sum_{i \geq 0} g_i(u) v^i, \quad \mathrm{ord}_u g_0(u) \geq 10,\ \mathrm{ord}_u g_1(u) \geq 3,\ \mathrm{ord}_u g_2(u) \geq 1.$$

Then we see that $I(g,y;O) = \mathrm{ord}_u\, \sigma_7(g)(u, -\varphi(u)) \geq 5$ and by Bézout's theorem, y divides g. Similarly we can see that we have $I(g, f_2; O) = \mathrm{ord}_u\, \sigma_7(g)(u, -\psi(u)) \geq 10$ and again by Bézout's theorem, we conclude f_2 divides g. Thus we can write $g(x,y) = y f_2 (c_0 + c_1 x + c_2 y)$ for some $c_0, c_1, c_2 \in \boldsymbol{C}$. The assumption g, $y^2 f_2 \in \mathrm{Ker}\,\bar{\sigma}_7$ implies that $g(x,y) - c_2\, y^2 f_2 = (c_0 + c_1 x) y f_2 \in \mathrm{Ker}\,\bar{\sigma}_7$. Thus we have $\sigma_7(c_0 y f_2)(u,0) = -c_0 a_{20} \beta_7 u^9 + \cdots \in \mathscr{I}_{O,7,10}$. Therefore $c_0 = 0$ as $\mathrm{ord}_u \sigma_7(c_0 y f_2)$ $(u,0) \geq 10$. Moreover we have

$$\sigma_7(g) \equiv \sigma_7(c_1 x y f_2) \equiv c_1\, uv(v - a_{20} u^2) \mod \bar{\mathscr{I}}_{O,7,10}.$$

As $d_2 + a_{20} \neq 0$, we see that $uv(v - a_{20}u^2) \notin \mathscr{J}_{O,7,10}$. Hence we have $c_1 = 0$ and we conclude $g(x,y) = c_2\, y^2 f_2$. □

The proof of Assertion 3 is now completed.

Now we are ready to compute the Alexander polynomial for the case $(C,O) \sim B_{29,2} \circ B_{2,1} \circ (B_{2,1}^2)^{B_{5,2}}$. By above assertions, we have $\rho_7(O) = 15$, hence we obtain the property ($\sharp$): $\ell_6 = 0$, $\ell_7 = 1$. This implies also $\ell_8 = 0$, $\ell_9 = 1$ by Remark 2. Therefore we have $\tilde{\Delta}_C(t) = t^4 - t^3 + t^2 - t + 1$ by Lemma 1. Thus the proof of Theorem 1 is completed.

5.5. Linear torus curve.

The singularity $B_{50,2}$ appears also as a linear torus curve of type (5,2):

$$C: \quad f_5(x,y)^2 - y^{10} = 0$$

with $I(f_5, y; O) = 5$ ([**2**]). In this case, C consists of two smooth quintics and the Alexander polynomial is given by following ([**2**]):

$$\Delta_C(t) = \frac{(t^{10} - 1)}{t + 1}.$$

5.6. Proofs of Corollary 1 and Corollary 2.

The assertion of Corollary 1 is an immediate consequence of the Sandwich principle. The assertion of Corollary 2 is a result of [**2**]. In fact, we only need to observe that the equivalence class of such torus curves correspond bijectively to the partitions of 10 by locally intersection numbers of C_2 and C_5. In particular, such a curve degenerates into an irreducible torus curve with a unique singularity $B_{50,2}$ which corresponds to the partition $10 = 10$.

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