J. Math. Soc. Japan
Vol. 61, No. 3 (2009) pp. 885–919
doi: 10.2969/jmsj/06130885

Finite rank product theorems for Toeplitz operators on the half-space

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(Received May 9, 2008)
(Revised July 29, 2008)

Abstract. On the harmonic Bergman space of the half space in $\boldsymbol{R}^n$, we show that if the product of two or more Toeplitz operators with harmonic symbols that have certain boundary smoothness has finite rank, then one of the symbols must be identically 0. Our methods require the number of factors in the product to depend on the dimension n.

1. Introduction.

For a fixed positive integer $n > 1$, let $H = \boldsymbol{R}^{n-1} \times \boldsymbol{R}_+$ be the upper half-space where $\boldsymbol{R}_+$ denotes the set of all positive real numbers. We will sometimes write point $z \in H$ as $z = (z', z_n)$ where $z' \in \boldsymbol{R}^{n-1}$ and $z_n > 0$. Let V be the volume measure on H. Throughout the paper we write $dw = dV(w)$ for simplicity. For $1 \le p \le \infty$, we let $L^p = L^p(H, V)$.

The *harmonic Bergman space* b^2 is the space of all complex-valued harmonic functions f on H satisfying

$$||f||_2 := \left\{ \int_H |f|^2 \, dV \right\}^{1/2} < \infty.$$

The space b^2 is a closed subspace of L^2 and thus is a Hilbert space. Basic structures of harmonic Bergman spaces b^2 are studied in [**9**]. For more general Banach space of harmonic functions on the half-space, see [**10**], [**11**] and references therein.

It is easily seen that each point evaluation is a bounded linear functional on b^2. Thus, to each $z \in H$, there corresponds a unique function $R_z = R(z, \cdot)$ in b^2 which has the reproducing property:

2000 *Mathematics Subject Classification.* Primary 47B35; Secondary 46E30.
Key Words and Phrases. Toeplitz operator, zero product, harmonic Bergman space, half-space.
The first two authors were supported by KRF-2007-313-C00025 and the last author was supported by Hanshin University Research Grant.

$$f(z) = \int_H f \overline{R}_z \, dV, \quad z \in H \tag{1.1}$$

for all $f \in b^2$. The kernel $R(z, w)$ is called the harmonic Bergman kernel and its explicit formula is well known:

$$R(z, w) = \frac{4}{n\sigma_n} \frac{n(z_n + w_n)^2 - |z - \overline{w}|^2}{|z - \overline{w}|^{n+2}}, \quad z, w \in H \tag{1.2}$$

where σ_n is the volume measure of the unit ball of $\boldsymbol{R}^n$ and $\overline{w} = (w', -w_n)$. Note that $R(z, w)$ is real and thus the complex conjugation in (1.1) can be removed. See [**1**] for details and related facts.

Let R be the Hilbert space orthogonal projection from L^2 onto b^2. Then the reproducing property (1.1) leads us to the integral realization of the projection R as follows:

$$R\psi(z) = \int_H \psi(w) R(z, w) \, dw, \quad z \in H \tag{1.3}$$

for functions $\psi \in L^2$. For a function $u \in L^\infty$, the *Toeplitz operator* T_u with *symbol* u is defined by

$$T_u f = R(uf)$$

for $f \in b^2$. Note that T_u is clearly bounded on b^2.

Recently, the first two authors [**3**] investigated the problem of characterizing zero products of several Toeplitz operators on the holomorphic Bergman space of the ball in $\mathbf{C}^n$. With harmonic symbols having some boundary regularity, their results assert that a product of Toeplitz operators can be the zero operator only in the trivial case, namely, only when one of the factor is the zero operator. Analogous polydisk versions are also proved in [**4**]. Quite recently, those zero product results have been generalized to finite rank product results in [**5**]. For some time, this "zero product problem", or more generally the "finite rank product problem", has been studied in various situations. See [**3**], [**4**] and references therein for the history of the zero product problem.

Working on higher dimensional balls or polydisks, the authors of [**3**], [**4**] devised a new scheme completely different from the earlier ones that were restricted to the one dimensional case. To be short, that new scheme is to decompose Toeplitz operators into a sum of the "major" and "error" parts, and utilize suitable test functions. Obviously, such a new scheme has another advantage that it may work on more general settings. In this paper we extend that scheme in two directions; the *harmonic* Bergman space on the *unbounded* domain H. With major adjustments to fit to those new settings, we investigate the finite rank product problem and obtain similar results.

In what follows we let h^∞ be the class of all bounded harmonic functions on H. Also, we let $\overline{H} = H \cup \partial H$ where $\partial H = \boldsymbol{R}^{n-1} \times \{0\}$ denotes the boundary of H, not including ∞. The following is one of our main results.

THEOREM 1.1. *Let $u_1, u_2 \in h^\infty \cap C(H \cup W)$ for some nonempty relatively open set $W \subset \partial H$. If $T_{u_1} T_{u_2}$ has finite rank, then either $u_1 = 0$ or $u_2 = 0$.*

In the case where symbols have Lipschitz continuous extensions to the boundary, our method applies to multiple products. Recall that the Lipschitz class on a domain $X \subset \boldsymbol{R}^n$ of order $\epsilon \in (0,1]$, denoted by $\Lambda_\epsilon(X)$, is the class of all complex functions f on X such that $|f(z) - f(w)| = \mathscr{O}(|z-w|^\epsilon)$ for $z, w \in X$. Note that Lipschitz functions on X necessarily extend to Lipschitz functions on $\overline{X}$ of the same order. In what follows we let $\Lambda_\epsilon = \Lambda_\epsilon(H)$. Also, given $\epsilon \in (0,1]$ and $\zeta \in \partial H$, we say $f \in \Lambda_\epsilon(\zeta)$ if f is a Lipschitz function of order ϵ in some neighborhood of ζ.

For Lipschitz symbols, our result is as follows.

THEOREM 1.2. *Let $u_1, \dots, u_{n+1} \in \Lambda_\epsilon \cap h^\infty$ for some ϵ. If $T_{u_1} \cdots T_{u_{n+1}}$ has finite rank, then $u_j = 0$ for some j.*

We also have the following local version, but with a loss of a factor.

THEOREM 1.3. *Let $u_1, \dots, u_n \in \Lambda_\epsilon(\zeta) \cap h^\infty$ for some ϵ and $\zeta \in \partial H$. If $T_{u_1} \cdots T_{u_n}$ has finite rank, then $u_j = 0$ for some j.*

REMARK.

(1) The number of factors in our results comes from the methods we use and may not be critical.

(2) Note that the identity operator is also a Toeplitz operator (with constant symbol 1). Thus, if a zero (or finite rank) product theorem holds for a certain number of factors, it also holds for any smaller number of factors.

(3) The *unboundedness* of H indeed causes a trouble with our method. Theorems 1.2 and 1.3 are harmonic analogues of results in [**5**]. Analogous zero product theorems, with one more factor, are also proved in [**5**]. However, for harmonic analogues of those zero product theorems, our method does not work. The difficulty is caused by the fact the estimate in Lemma 2.2 below always

diverges for $c = 0$. This is in contrast to the case of bounded domains where the corresponding growth rate is usually logarithmic.

(4) Dealing with Toeplitz products with harmonic symbols in the present paper, we do not mean that a single Toeplitz operator with harmonic symbol has been well studied. In fact even the characterization for a compact Toeplitz operator with harmonic symbol does not seem to appear yet in the literature. In the last section we included a proof that if a Toeplitz operator with harmonic symbol is compact, then it is trivial.

In Section 2, we collect technical estimates to be used later. In Section 3, we prove preliminary results concerning the mapping properties of R on Lipschitz spaces, certain boundary behavior of Berezin transform and some basic properties of test functions. In Section 4, we prove our main theorems. Finally, in Section 5, we remark that the zero operator is the only compact Toeplitz operator with harmonic symbol.

CONSTANTS. In the rest of the paper we use the same letter C, often depending on the allowed parameters, to denote various positive constants which may change at each occurrence. For nonnegative quantities X and Y, we often write $X \lesssim Y$ or $Y \gtrsim X$ if X is dominated by Y times some *inessential* positive constant. Also, we write $X \approx Y$ if $X \lesssim Y \lesssim X$.

2. Auxiliary estimates.

As is mentioned in the Introduction, the main idea of our proofs is to follow the schemes of [**3**], [**4**]. That is, we decompose each factor into major and error parts and employ suitable test functions. Thanks to the fact that H is a product domain, major parts are quite simple to deal with; see Lemmas 3.4 and 3.5. However, caused by the fact that H is an unbounded domain, error parts require substantially complicated and technical estimates. All those estimates are collected in this section.

It is clear from (1.2) that there is a constant $C = C(n)$ such that

$$|R(z,w)| \le \frac{C}{|z - \overline{w}|^n}$$

for $z, w \in H$. We will frequently and tacitly use this basic inequality for the rest of the paper. Suggested by this inequality, we need to estimate integrals introduced below.

Given c and s real, let

$$\Phi_{c,s}(z,w) = \frac{1 + |\log z_n|^s + |\log w_n|^s + |\log|z - \overline{w}||^s}{|z - \overline{w}|^{n+c}}$$

for $z, w \in H$ and define corresponding integrals $I_{c,s}(z,w)$ by

$$I_{c,s}(z,w) = \int_H \frac{\Phi_{c,s}(\zeta, w)}{|\zeta - \overline{z}|^n} \, d\zeta.$$

Estimates of these integrals will take care of error terms in repeated Toeplitz integrals which arise in the course of our proofs.

We introduce some auxiliary integrals depending on parameters $s \geq 0$ and c real. First, let

$$J_{c,s}(z) = \int_H \frac{|\log w_n|^s + |\log|w - \overline{z}||^s}{|w - \overline{z}|^{n+c}} \, dw$$

for $z \in H$. Given $a > 0$, we decompose the integral $J_{c,s}(z)$ into two pieces

$$J_{c,s}(z) = U_{c,s}(z,a) + L_{c,s}(z,a)$$

where $U_{c,s}(z,a)$ and $L_{c,s}(z,a)$ are integrals defined by

$$U_{c,s}(z,a) = \int_{H \setminus B_a(\overline{z})} \frac{|\log w_n|^s + |\log|w - \overline{z}||^s}{|w - \overline{z}|^{n+c}} \, dw$$
$$L_{c,s}(z,a) = \int_{H \cap B_a(\overline{z})} \frac{|\log w_n|^s + |\log|w - \overline{z}||^s}{|w - \overline{z}|^{n+c}} \, dw.$$

Here, $B_a(\overline{z})$ denotes the Euclidean ball in $\boldsymbol{R}^n$ with center at $\overline{z}$ and radius $a > 0$. Note that $H \cap B_a(\overline{z}) = \emptyset$ if $a \leq z_n$. So, $U_{c,s}(z,a) = J_{c,s}(z)$ and $L_{c,s}(z,a) = 0$ for $a \leq z_n$. We will often use the notation

$$L^s(t) = |\log t|^s, \quad t > 0$$

for simplicity. We begin with the following lemma.

LEMMA 2.1. *Let $c > 0$ and $s \geq 0$. Then the following estimates hold for $0 < \epsilon < 1/2$ and $a > 0$:*

$$\int_\epsilon^1 \frac{|\log r|^s}{r^{1+c}}\,dr \approx \epsilon^{-c}|\log \epsilon|^s; \tag{2.1}$$

$$\int_0^\epsilon \frac{|\log r|^s}{r^{1-c}}\,dr \approx \epsilon^{c}|\log \epsilon|^s; \tag{2.2}$$

$$\int_a^\infty \frac{|\log r|^s}{r^{1+c}}\,dr \approx a^{-c}(1+|\log a|^s). \tag{2.3}$$

The constants suppressed above are independent of ϵ and a.

PROOF. For a proof of (2.1) and (2.2), see [**3**, Lemma 3.4]. To see (2.3), we consider three cases $a < 1/2$, $1/2 \le a \le 2$ and $a > 2$ separately. First, we have by (2.2)

$$\int_a^\infty \frac{|\log r|^s}{r^{1+c}}\,dr \approx a^{-c}|\log a|^s \approx a^{-c}(1+|\log a|^s)$$

for $a > 2$. Next, we have

$$\int_a^\infty \frac{|\log r|^s}{r^{1+c}}\,dr \approx 1 \approx a^{-c} \approx a^{-c}(1+|\log a|^s).$$

for $1/2 \le a \le 2$. Finally, we have by (2.1)

$$\int_a^\infty \frac{|\log r|^s}{r^{1+c}}\,dr \approx 1 + \int_a^1 \frac{|\log r|^s}{r^{1+c}}\,dr \approx 1 + a^{-c}|\log a|^s \approx a^{-c}[1+|\log a|^s]$$

for $a < 1/2$. The proof is complete. □

For the integrals $J_{c,s}(z)$, we have the following estimate.

LEMMA 2.2. *Let $s \ge 0$ and c be real. Then the following estimates hold for $z \in H$:*

$$J_{c,s}(z) \approx \begin{cases} z_n^{-c}(1+|\log z_n|^s) & \text{for } c > 0 \\ \infty & \text{for } c \le 0. \end{cases}$$

The constant suppressed above is independent of z.

PROOF. Let $z \in H$. We may assume $z = (0', z_n)$. Note

$$|w-\overline{z}| \le z_n + w_n + |w'| \le 2|w-\overline{z}| \tag{2.4}$$

for all $z, w \in H$. Thus we have

$$J_1 := \int_H \frac{L^s(w_n)}{|w-\overline{z}|^{n+c}}\,dw \approx \int_0^\infty \int_{\boldsymbol{R}^{n-1}} \frac{L^s(w_n)}{(w_n+z_n+|w'|)^{n+c}}\,dw'\,dw_n.$$

Thus, integration in polar coordinates yields

$$J_1 \approx \left\{\int_0^\infty \frac{L^s(t)}{(t+z_n)^{1+c}}\,dt\right\}\left\{\int_0^\infty \frac{r^{n-2}}{(1+r)^{n+c}}\,dr\right\}. \tag{2.5}$$

Note that the first integral of the above diverges for $c \le 0$. Thus we have $J_{c,s}(z) \gtrsim J_1 = \infty$ for $c \le 0$.

Assume $c > 0$ for the rest of the proof. Since the second integral of (2.5) is finite, we have by (2.3)

$$\begin{aligned} J_1 &\approx \int_0^{z_n} \frac{L^s(t)}{(t+z_n)^{1+c}}\,dt + \int_{z_n}^\infty \frac{L^s(t)}{(t+z_n)^{1+c}}\,dt \\ &\approx z_n^{-1-c}\int_0^{z_n} L^s(t)\,dt + \int_{z_n}^\infty \frac{L^s(t)}{t^{1+c}}\,dt \\ &\approx z_n^{-c}[1+L^s(z_n)]. \end{aligned}$$

Note that this implies the lower estimate of $J_{c,s}(z)$. Also, since

$$J_2 := \int_H \frac{L^s(|w-\overline{z}|)}{|w-\overline{z}|^{n+c}}\,dw \le \int_{\boldsymbol{R}^n\setminus B_{z_n}(0)} \frac{L^s(|x|)}{|x|^{n+c}}\,dx \approx \int_{z_n}^\infty \frac{L^s(r)}{r^{1+c}}\,dr,$$

we have by (2.3)

$$J_2 \lesssim z_n^{-c}[1+L^s(z_n)].$$

Now, combining the estimates of J_1 and J_2, we have the upper estimate of $J_{c,s}(z)$. The proof is complete. □

Next, we have the following estimate for the integrals $U_{c,s}(z,a)$.

LEMMA 2.3. *Let $s \ge 0$ and c be real. Then the following estimates hold for*

$z \in H$ and $a > 0$:

$$U_{c,s}(z,a) \approx \begin{cases} \dfrac{1 + |\log(z_n + a)|^s}{(a + z_n)^c} & \text{if } c > 0 \\ \infty & \text{if } c \le 0. \end{cases}$$

The constants suppressed above are independent of z and a.

PROOF. Let $z \in H$ and $a > 0$. In case $a < z_n$ the lemma goes back to Lemma 2.2, because $U_{c,s}(z,a) = J_{c,s}(z)$. So, assume $a \ge z_n$ for the rest of the proof. Also, we may assume $z = (0', z_n)$. We first prove the lower estimate. Since the set $H \setminus B_a(\overline{z})$ contains all points w with $w_n \ge a$ and $|w'| \ge a$, we have by (2.4)

$$\begin{aligned} U_{c,s}(z,a) &\gtrsim \int_a^\infty \int_{|w'| \ge a} \frac{L^s(w_n)}{(w_n + z_n + |w'|)^{n+c}}\, dw'\, dw_n \\ &\approx \int_a^\infty \int_a^\infty \frac{L^s(t)}{(t + z_n + r)^{n+c}} r^{n-2}\, dr\, dt \\ &= \int_a^\infty \frac{L^s(t)}{(t+z_n)^{1+c}} \int_{a/(t+z_n)}^\infty \frac{r^{n-2}}{(1+r)^{n+c}}\, dr\, dt. \end{aligned}$$

Since $a/(t + z_n) < 1$ for $t \ge a$, the inner integral of the above is bigger than some positive constant. Thus we have

$$U_{c,s}(z,a) \gtrsim \int_a^\infty \frac{L^s(t)}{(t+z_n)^{1+c}}\, dt.$$

For $c \le 0$, this integral diverges and thus $U_{c,s}(z,a) = \infty$. For $c > 0$, we have by (2.3)

$$\int_a^\infty \frac{L^s(t)}{(t+z_n)^{1+c}}\, dt \approx \int_a^\infty \frac{L^s(t)}{t^{1+c}}\, dt \approx a^{-c}[1 + L^s(a)]$$

and thus

$$U_{c,s}(z,a) \gtrsim a^{-c}[1 + L^s(a)],$$

which is equivalent to the desired lower estimate; recall $z_n \le a$.

We now assume $c > 0$ and prove the upper estimate. We will show

$$U_{c,s}(z,a) \lesssim a^{-c}[1 + L^s(a)], \tag{2.6}$$

which is again equivalent to the desired upper estimate. By (2.4) we have

$$\begin{aligned}
U_{c,s}(z,a) &\lesssim \iint_{w_n+|w'|\geq a-z_n} \frac{L^s(w_n) + L^s(w_n + z_n + |w'|)}{(w_n + z_n + |w'|)^{n+c}}\, dw'\, dw_n \\
&\lesssim \iint_{t+r\geq a-z_n} \frac{L^s(t) + L^s(t + z_n + r)}{(t + z_n + r)^{2+c}}\, dr\, dt \\
&= \int_{a-z_n}^{\infty}\int_0^{\infty} + \int_0^{a-z_n}\int_{a-z_n-t}^{\infty} \\
&:= U_1 + U_2.
\end{aligned}$$

We first consider the integral U_1. By (2.3) we have

$$\begin{aligned}
U_1 &\lesssim \int_{a-z_n}^{\infty} \frac{1 + L^s(t) + L^s(t + z_n)}{(t + z_n)^{1+c}}\, dt \\
&\approx a^{-c}[1 + L^s(a)] + \int_{a-z_n}^{\infty} \frac{L^s(t)}{(t + z_n)^{1+c}}\, dt.
\end{aligned}$$

Since $t + z_n \approx t$ for $t \geq a$ and $t + z_n \approx a$ for $a - z_n \leq t \leq a$, we have by (2.3)

$$\begin{aligned}
\int_{a-z_n}^{\infty} \frac{L^s(t)}{(t + z_n)^{1+c}}\, dt &= \int_a^{\infty} \frac{L^s(t)}{(t + z_n)^{1+c}}\, dt + \int_{a-z_n}^{a} \frac{L^s(t)}{(t + z_n)^{1+c}}\, dt \\
&\lesssim \int_a^{\infty} \frac{L^s(t)}{t^{1+c}}\, dt + a^{-1-c}\int_0^a L^s(t)\, dt \\
&\approx a^{-c}[1 + L^s(a)]
\end{aligned}$$

and thus conclude that U_1 is dominated by $a^{-c}[1 + L^s(a)]$. For the integral U_2, it is easily seen that

$$U_2 \lesssim a^{-1-c}\int_0^a L^s(t)\, dt + a\int_a^{\infty} \frac{L^s(r)}{r^{2+c}}\, dr.$$

This, together with (2.3), implies that U_2 is also dominated by $a^{-c}[1 + L^s(a)]$. Thus we obtain (2.6), as required. The proof is complete. □

We now turn to the estimate of integrals $L_{c,s}(z,a)$. For $c > 0$, the trivial inequality $L_{c,s}(z,a) \le J_{c,s}(z)$ will be enough for our purpose. For $c \le 0$, we need the following estimate.

LEMMA 2.4. *Given $c \le 0$ and $s \ge 0$, there is a constant $C = C(c,s)$ such that*

$$L_{c,s}(z,a) \le C \times \begin{cases} 1 + |\log z_n|^{s+1} + |\log a|^{s+1} & \text{if } c = 0 \\ a^{-c}(1 + |\log z_n|^s + |\log a|^s) & \text{if } c < 0 \end{cases}$$

for $z \in H$ and $a > z_n$.

PROOF. Let $c \le 0$ and $s \ge 0$. Let $z \in H$ and $a > z_n$. We may assume $z = (0', z_n)$. Writing

$$\begin{aligned} L_{c,s}(z,a) &= \int_{H\cap B_a(\overline{z})} \frac{L^s(w_n)}{|w-\overline{z}|^{n+c}}\, dw + \int_{H\cap B_a(\overline{z})} \frac{L^s(|w-\overline{z}|)}{|w-\overline{z}|^{n+c}}\, dw \\ &:= L_1 + L_2, \end{aligned}$$

we will show that both integrals L_1 and L_2 satisfy the desired estimates.

The estimate for the second integral L_2 is simpler. Since $H \cap B_a(\overline{z}) \subset B_a(\overline{z}) \setminus B_{z_n}(\overline{z})$, we have by integration in polar coordinates and (2.1)

$$L_2 \lesssim \int_{z_n}^{a} \frac{L^s(r)}{r^{1+c}}\, dr \tag{2.7}$$

$$\lesssim z_n^{-c} \int_1^{a/z_n} \frac{L^s(r) + L^s(z_n)}{r^{1+c}}\, dr \tag{2.8}$$

$$\approx \begin{cases} \log(a/z_n)[L^s(a/z_n) + L^s(z_n)] & \text{if } c = 0 \\ a^{-c}[L^s(a/z_n) + L^s(z_n)] & \text{if } c < 0 \end{cases}$$

$$\lesssim \begin{cases} L^{s+1}(a) + L^{s+1}(z_n) & \text{if } c = 0 \\ a^{-c}[L^s(a) + L^s(z_n)] & \text{if } c < 0. \end{cases} \tag{2.9}$$

Next, we estimate L_1. Since $z_n + w_n \le |w - \overline{z}|$ and $|w'| \le |w - \overline{z}|$, we have by (2.4)

$$
\begin{aligned}
L_1 &\lesssim \int_0^{a-z_n} \int_{|w'|<a} \frac{L^s(w_n)}{(w_n + z_n + |w'|)^{n+c}} \, dw' \, dw_n \\
&\approx \int_0^{a-z_n} \int_0^a \frac{L^s(t)}{(t + z_n + r)^{n+c}} r^{n-2} \, dr \, dt \\
&= z_n^{-c} \int_0^{a/z_n - 1} \frac{L^s(tz_n)}{(1+t)^{1+c}} \int_0^{a/z_n(1+t)} \frac{r^{n-2}}{(1+r)^{n+c}} \, dr \, dt. \qquad (2.10)
\end{aligned}
$$

Thus, for $c < -1$, we see from (2.3) that

$$
\begin{aligned}
L_1 &\lesssim z_n^{-c} \int_0^{a/z_n - 1} L^s(tz_n)(1+t)^{-1-c} \int_0^{a/z_n(1+t)} r^{-2-c} \, dr \, dt \\
&\lesssim z_n a^{-1-c} \int_0^{a/z_n} L^s(t) + L^s(z_n) \, dt \\
&\approx a^{-c}[1 + L^s(a/z_n) + L^s(z_n)] \\
&\lesssim a^{-c}[1 + L^s(z_n) + L^s(a)].
\end{aligned}
$$

Note that we have

$$
L_1 \lesssim z_n^{-c} \int_0^{a/z_n - 1} \frac{L^s(tz_n)}{(1+t)^{1+c}} \left\{ 1 + \int_1^{a/z_n(1+t)} \frac{dr}{r^{2+c}} \right\} dt \qquad (2.11)
$$

for $-1 \leq c \leq 0$. If $c = 0$, then from (2.11) and (2.9) that

$$
L_1 \lesssim 1 + \int_1^{a/z_n} \frac{L^s(t) + L^s(z_n)}{t} \, dt \lesssim 1 + L^{s+1}(z_n) + L^{s+1}(a).
$$

Similarly, if $-1 < c < 0$, then we obtain

$$
L_1 \lesssim z_n^{-c} \int_0^{a/z_n} \frac{L^s(t) + L^s(z_n)}{t^{1+c}} \, dt \lesssim a^{-c}[1 + L^s(z_n) + L^s(a)].
$$

If $c = -1$, we see from (2.11) and (2.3) that

$$\begin{aligned} L_1 &\lesssim z_n \int_0^{a/z_n} L^s(tz_n)[1+\log(a/z_n)+\log(1+t)]\,dt \\ &\lesssim z_n[1+\log(a/z_n)] \int_0^{a/z_n} L^s(t)+L^s(z_n)\,dt \\ &\lesssim a[1+L^s(z_n)+L^s(a)], \end{aligned}$$

which completes the proof. □

REMARK. Recall $L_{c,s}(z,a)=0$ for $a \le z_n$. Thus the estimate in Lemma 2.4 is far from being sharp as $a/z_n \to 1$. However, as long as the behavior $L_{c,s}(z,a)$ as $a/z_n \to 1$ is concerned, one may get a better upper bound as follows:

$$L_{c,s}(z,a) \lesssim z_n^{-c}(a/z_n-1)^{(n+1)/2}[1+L^s(a-z_n)+L^s(z_n)]. \tag{2.12}$$

In order to see this, assume $z_n < a \le 2z_n$. Then, since $|w-\overline{z}| \approx z_n$ for $|w-\overline{z}|<a$ and $a+t+z_n \approx z_n$ for $0 \le t \le a-z_n$, we have

$$\begin{aligned} L_1 &\approx z_n^{-n-c} \int_0^{a-z_n} L^s(t)[a^2-(t+z_n)^2]^{(n-1)/2}\,dt \\ &\approx z_n^{-n-c} z_n^{(n-1)/2} \int_0^{a-z_n} L^s(t)(a-t-z_n)^{(n-1)/2}\,dt \\ &= z_n^{-c}(a/z_n-1)^{(n+1)/2} \int_0^1 L^s(t(a-z_n))(1-t)^{(n-1)/2}\,dt \\ &\lesssim z_n^{-c}(a/z_n-1)^{(n+1)/2}[1+L^s(a-z_n)]. \end{aligned}$$

Similarly, we have

$$L_2 \lesssim z_n^{-c}(a/z_n-1)^{(n+1)/2}[1+L^s(z_n)].$$

Now, combining these estimates, we obtain (2.12). Note that the above argument works even for $c>0$. Thus (2.12) is also valid for $c>0$.

We are now ready to prove the following estimate.

PROPOSITION 2.5. *Given $c>-n$ and $s \ge 0$, there exists a constant $C = C(c,s)$ such that*

$$I_{c,s}(z,w) \le C \times \begin{cases} w_n^{-c}\,\Phi_{0,s+1}(z,w) & \text{if } c > 0 \\ \Phi_{c,s+1}(z,w) & \text{if } c \le 0 \end{cases}$$

for $z, w \in H$.

PROOF. Let $c > -n$, $s \ge 0$ and fix $z, w \in H$. Decompose H into three pieces E_1, E_2 and E_3 given by

$$\begin{aligned} E_1 &= \{\zeta \in H : 2|z - \overline{w}| \le |\zeta - \overline{w}|\} \\ E_2 &= \{\zeta \in H : |z - \overline{w}|/2 \le |\zeta - \overline{w}| < 2|z - \overline{w}|\} \\ E_3 &= \{\zeta \in H : |\zeta - \overline{w}| < |z - \overline{w}|/2\} \end{aligned}$$

and consider corresponding integrals

$$I_j := \int_{E_j} \frac{1 + L^s(w_n) + L^s(\zeta_n) + L^s(|\zeta - \overline{w}|)}{|\zeta - \overline{z}|^n |\zeta - \overline{w}|^{n+c}}\, d\zeta$$

for $j = 1, 2, 3$.

We now estimate the integrals introduced above. First, using the inequalities

$$|\zeta - \overline{z}| \ge |\zeta - \overline{w}| - |\overline{z} - \overline{w}| \ge |\zeta - \overline{w}| - |z - \overline{w}| \ge |\zeta - \overline{w}|/2$$

valid for $\zeta \in E_1$, we have

$$\begin{aligned} I_1 &\lesssim [1 + L^s(w_n)] \int_{E_1} \frac{d\zeta}{|\zeta - \overline{w}|^{2n+c}} + \int_{E_1} \frac{L^s(\zeta_n) + L^s(|\zeta - \overline{w}|)}{|\zeta - \overline{w}|^{2n+c}}\, d\zeta \\ &= U_{n+c,s}(w, a_1) + [1 + L^s(w_n)] U_{n+c,0}(w, a_1) \end{aligned}$$

where $a_1 = 2|z - \overline{w}|$. Since $n + c > 0$, we conclude

$$I_1 \lesssim \Phi_{c,s}(z,w) \lesssim \Phi_{c,s+1}(z,w) \tag{2.13}$$

by Lemma 2.3. Next, using the inequalities

$$|\zeta - \overline{z}| \le |\zeta - \overline{w}| + |\overline{z} - \overline{w}| \le |\zeta - \overline{w}| + |z - \overline{w}| < 3|z - \overline{w}|$$

and $|z - \overline{w}| \approx |\zeta - \overline{w}|$ valid for $\zeta \in E_2$, we obtain

$$
\begin{aligned}
I_2 &\lesssim \frac{1 + L^s(w_n) + L^s(|z-\overline{w}|)}{|z-\overline{w}|^{n+c}} \int_{H\cap B_{a_2}(\overline{z})} \frac{d\zeta}{|\zeta-\overline{z}|^n} \\
&\quad + \frac{1}{|z-\overline{w}|^{n+c}} \int_{H\cap B_{a_2}(\overline{z})} \frac{L^s(\zeta_n)}{|\zeta-\overline{z}|^n}\, d\zeta \\
&\le \frac{[1 + L^s(w_n) + L^s(|z-\overline{w}|)]L_{0,0}(z,a_2) + L_{0,s}(z,a_2)}{|z-\overline{w}|^{n+c}}
\end{aligned}
$$

where $a_2 = 3|z-\overline{w}|$ and thus conclude

$$
I_2 \lesssim \Phi_{c,s+1}(z,w) \tag{2.14}
$$

by Lemma 2.4. Finally, using the inequalities

$$
|\zeta-\overline{z}| \ge |\zeta - z| \ge |z-\overline{w}| - |\zeta-\overline{w}| \ge |z-\overline{w}|/2
$$

valid for $\zeta \in E_3$, we obtain

$$
\begin{aligned}
I_3 &\lesssim \frac{1 + L^s(w_n)}{|z-\overline{w}|^n} \int_{H\cap B_{a_3}(\overline{w})} \frac{d\zeta}{|\zeta-\overline{w}|^{n+c}} \\
&\quad + \frac{1}{|z-\overline{w}|^n} \int_{H\cap B_{a_3}(\overline{w})} \frac{L^s(\zeta_n) + L^s(|\zeta-\overline{w}|)}{|\zeta-\overline{w}|^{n+c}}\, d\zeta \\
&= \frac{[1 + L^s(w_n)]L_{c,0}(w,a_3) + L_{c,s}(w,a_3)}{|z-\overline{w}|^n}
\end{aligned}
$$

where $a_3 = |z-\overline{w}|/2$. Accordingly, for $c > 0$, we have by Lemma 2.2

$$
I_3 \lesssim \frac{w_n^{-c}[1 + L^s(w_n)]}{|z-\overline{w}|^n} \lesssim w_n^{-c}\Phi_{0,s+1}(z,w). \tag{2.15}
$$

Meanwhile, for $c = 0$, we have by Lemma 2.4

$$
I_3 \lesssim \frac{1 + L^{s+1}(w_n) + L^{s+1}(|z-\overline{w}|)}{|z-\overline{w}|^n} \le \Phi_{0,s+1}(z,w). \tag{2.16}
$$

Also, for $-n < c < 0$, we have by Lemma 2.4

$$
I_3 \lesssim \frac{|z-\overline{w}|^{-c}[1 + L^s(w_n) + L^s(|z-\overline{w}|)]}{|z-\overline{w}|^n} \lesssim \Phi_{c,s+1}(z,w). \tag{2.17}
$$

Now, since $\Phi_{c,s+1}(z,w) \le w_n^{-c}\Phi_{0,s+1}(z,w)$ for $c>0$, we conclude the proposition by (2.13)–(2.17). The proof is complete. □

3. Some preliminary results.

In this section we prove some preliminary results: (i) the mapping properties of R on (local) Lipschitz spaces, (ii) the boundary continuous extension property of Berezin transform and (iii) some basic properties of test functions. Some of these results may be of independent interest.

3.1. Lipschitz spaces.

In the holomorphic case it is well known on bounded strictly pseudoconvex domains that Lipschitz spaces (of non-integer order) are invariant under Bergman projections. This is a consequence of a theorem due to Ahern and Schneider [**2**]. The harmonic analogue is noticed by Kang and Koo [**8**] on bounded smooth domains. We need to establish the analogous result, as well as its local version, on our *unbounded* domain H.

Let D_j be the differentiation with respect to the j-th component, i.e., $D_jf(w) = \partial f/\partial w_j(w)$. If both variables z and w are present, we will let D_{z_j} or D_{w_j} etc, in place of D_j, to specify the variable of differentiation. For the derivatives of $R(z,w)$ we have the following size estimate for a given multi-index α:

$$|D_z^\alpha R(z,w)| \le \frac{C}{|z-\overline{w}|^{n+|\alpha|}} \tag{3.1}$$

for some constant $C = C(\alpha)$; see [**9**].

The proof of the following lemma is parallel to the well known argument and thus omitted; see [**12**, Lemma 6.4.8].

LEMMA 3.1. *If a function $f \in C^1(H)$ satisfies*

$$|\nabla f(z)| \le z_n^{\epsilon-1}, \quad z \in H$$

for some $\epsilon \in (0,1)$, then $f \in \Lambda_\epsilon$.

Let b^p be the subspace consisting of all harmonic functions in L^p. The case $p=2$ of the following theorem will be used later. The general case $1<p<\infty$ is included, because it may be of some independent interest.

THEOREM 3.2. *Let $0<\epsilon<1$, $1<p<\infty$ and $\zeta \in \partial H$. Then*

$$R[L^p \cap \Lambda_\epsilon] \subset b^p \cap \Lambda_\epsilon \tag{3.2}$$

and

$$R[L^p \cap \Lambda_\epsilon(\zeta)] \subset b^p \cap [\Lambda_\epsilon + \mathscr{H}(\zeta)] \tag{3.3}$$

where $\mathscr{H}(\zeta)$ *denotes the class of all functions harmonic on some open set containing* $H \cup \{\zeta\}$.

The proof of (3.2) below is also parallel for most part to the well known argument. The only difference is that we need the b^1-cancelation property ([**9**, Theorem 2.2]): If $f \in b^1$, then

$$\int_{\boldsymbol{R}^{n-1}} f(x,t)\,dx = 0 \tag{3.4}$$

for every $t > 0$.

PROOF. We first show (3.2). Let $f \in L^p \cap \Lambda_\epsilon$. It is well known that $R : L^p \to b^p$ is a bounded projection; see [**9**, Theorem 3.2]. Thus we only need to prove $Rf \in \Lambda_\epsilon$. Put $F = Rf$ and let $z \in H$. Differentiating under the integral sign, we have

$$D_j F(z) = \int_H f(w) D_{z_j} R(z,w)\,dw$$

for each j. Note that each function $D_{z_j} R(z,\cdot)$ is integrable on H by (3.1). Hence the b^1-cancelation property (3.4) yields

$$D_{z_j} F(z) = \int_H f(w) D_{z_j} R(z,w)\,dw = \int_H (f(w) - f(z)) D_{z_j} R(z,w)\,dw$$

for each j. Since $f \in \Lambda_\epsilon$, this, together with (3.1), implies

$$|\nabla F(z)| \lesssim \int_H \frac{dw}{|z - \overline{w}|^{n+1-\epsilon}} \approx z_n^{\epsilon-1};$$

the last equivalence comes from Lemma 2.2. So, we conclude (3.2) by Lemma 3.1.

We now show (3.3). Let $f \in L^p \cap \Lambda_\epsilon(\zeta)$. Then $f \in \Lambda_\epsilon(\overline{U})$ for some bounded neighborhood U of ζ. Choose a neighborhood U_1 of ζ such that $\overline{U_1} \subset U$. Pick a smooth cut-off function ψ on $\boldsymbol{R}^n$ with $0 \le \psi \le 1$ such that $\psi = 1$ on $\overline{U_1}$ and $\psi = 0$ on $\boldsymbol{R}^n \setminus U$. We certainly have $f\psi \in \Lambda_\epsilon(\overline{H} \setminus U)$. Also, we have $f\psi \in \Lambda_\epsilon(\overline{U})$, because ψ is smooth. Let $z \in \overline{H} \cap U$ and $w \in \overline{H} \setminus U$. Pick a point $a \in \partial U$ that stays closest to z. Since $\psi(a) = 0$ and $\psi \in \Lambda_\epsilon(\overline{U})$, we have

$$|\psi(z)| = |\psi(z) - \psi(a)| \le C_1 |z-a|^\epsilon \le C_1 |z-w|^\epsilon$$

for some constant $C_1 > 0$ independent of z and w. Since $|f|$ is bounded on $\overline{U}$, say by C_2, we obatin

$$|f(z)\psi(z) - f(w)\psi(w)| = |f(z)\psi(z)| \le C_1 C_2 |z-w|^\epsilon.$$

Thus $f\psi \in \Lambda_\epsilon$ and $R[f\psi] \in \Lambda_\epsilon$ by (3.2). Meanwhile, since

$$R[f - f\psi](z) = \int_{H \setminus U_1} f(w)(1-\psi(w)) R(z,w)\, dw,$$

we see that $R[f - f\psi]$ extends to a harmonic function across $U_1 \cap \partial H$. So, we have (3.3). The proof is complete. $\square$

REMARK. It is known that the kernel $R(z,w)$ also reproduces b^1-functions. So, the proof of (3.2) shows the following (even with $p = 1$): For $\epsilon \in (0,1)$, $1 \le p < \infty$ and $f \in b^p$, we have $f \in \Lambda_\epsilon$ if and only if $|\nabla f(z)| = \mathcal{O}(z_n^{\epsilon-1})$. The same characterization extends to $p = \infty$, namely, for functions $f \in h^\infty$. This latter assertion can be verified by means of the modified reproducing formula ([**9**], Lemma 5.6]) for harmonic Bloch functions.

3.2. Berezin transform.

Recall that the *Berezin transform* $\widetilde{T}$ of a bounded linear operator T on b^2 is defined by

$$\widetilde{T}(z) = \langle T r_z, r_z \rangle, \quad z \in H$$

where $r_z = R_z \|R_z\|_2^{-1}$ is the normalized kernel. Clearly, these Berezin transforms are continuous on H. Moreover, Berezin transforms of Toeplitz products has the continuous extension property up to the boundary, when the inducing symbols are continuous at a boundary point as in the next theorem.

THEOREM 3.3. *Suppose that functions $u_1, \ldots, u_N \in L^\infty$ are continuous at $\zeta \in \partial H$. Let $T = T_{u_1} \cdots T_{u_N}$. Then $\widetilde{T}$ continuously extends to $H \cup \{\zeta\}$ and*

$\widetilde{T}(\zeta) = (u_1 \cdots u_N)(\zeta)$.

This theorem is the harmonic analogue of [**3**, Proposition 2.1] (for the holomorphic Bergman spaces of the ball in $\boldsymbol{C}^n$) whose proof utilizes automorphisms. One may also use the maps ϕ_z introduced in the proof of Lemma 3.4 to modify the proof of [**3**, Proposition 2.1]. Here, we provide below a different proof that may work on more general settings. Also, since $r_z \to 0$ uniformly on compact sets as $z \to \infty$, the theorem remains true for $\zeta = \infty$ by an easy modification.

PROOF. We first prove

$$\lim_{z \to \zeta} \|T_u r_z\|_2 = 0 \tag{3.5}$$

for any function $u \in L^\infty$ that continuously extends to ζ with $u(\zeta) = 0$. Let $z \in H$. Note $\|T_u r_z\|_2 = \|R(ur_z)\|_2 \le \|ur_z\|_2$. Thus, for $\epsilon > 0$ small, we have

$$\begin{aligned} \|T_u r_z\|_2^2 &\le \int_H |u(w)|^2 |r_z(w)|^2 \, dw \\ &= \int_{|w-\zeta|<\epsilon} + \int_{|w-\zeta|\ge\epsilon} |u(w)|^2 |r_z(w)|^2 \, dw \\ &\le \sup_{|w-\zeta|<\epsilon} |u(w)|^2 + \|u\|_\infty^2 \int_{|w-\zeta|\ge\epsilon} |r_z(w)|^2 \, dw. \end{aligned} \tag{3.6}$$

Note $|R(z,w)| \lesssim |z - \overline{w}|^{-n}$ by (1.2). Meanwhile, since

$$|z - \overline{w}| \ge |w - \zeta| - |z - \zeta|, \quad w \in H,$$

we have by Lemma 2.2

$$\begin{aligned} \int_{|w-\zeta|\ge\epsilon} |r_z(w)|^2 \, dw &\lesssim R(z,z)^{-1} (\epsilon - |z-\zeta|)^{1-n} \int_H \frac{dw}{|z-\overline{w}|^{n+1}} \\ &\approx z_n^{n-1} (\epsilon - |z-\zeta|)^{1-n} \end{aligned}$$

for z sufficiently close ζ. Note that the last expression above converges to 0 as $z \to \zeta$. Hence, taking the limit $z \to \zeta$ with ϵ fixed and then taking the limit $\epsilon \to 0$ in (3.6), we conclude (3.5), as desired.

Now, put $c_j = u_j(\zeta)$. Then an inductive argument yields

$$T = T_{c_1} \cdots T_{c_N} + \sum_{j=1}^{N} T_{u_1} \cdots T_{u_{j-1}} T_{u_j - c_j} T_{c_{j+1}} \cdots T_{c_N}.$$

Note $T_c = cI$ for constants c where I is the identity operator on b^2. Also, note $T_{u_j - c_j} r_z \to 0$ in b^2 as $z \to \zeta$ by (3.5) for each j. Thus we see from the above that $(T - c_1 \cdots c_N I) r_z \to 0$ in b^2 and thus $\widetilde{T}(z) \to c_1 \cdots c_N$ as $z \to \zeta$. This completes the proof. □

3.3. Test functions.

We introduce our test functions and prove some basic properties. We fix the reference point

$$\boldsymbol{e} := (0', 1) \in H$$

for the rest of the paper. Also, we use the notation $\mathscr{D} = D_n$ to emphasize the normal differentiation. Our test functions will be the functions λ_t^k defined by

$$\lambda_t^k = \mathscr{D}^k R_{t\boldsymbol{e}}$$

for integers $k \geq 0$ and $t > 0$.

In the next two lemmas we observe information on how test functions grow along the diagonal and on how major part of a given Toeplitz operator acts on test functions. It is clear from (3.1) that there is a constant $C = C(n, k)$ such that

$$|\mathscr{D}^k R_z(w)| \leq \frac{C}{|z - \overline{w}|^{n+k}} \tag{3.7}$$

for $z, w \in H$. Moreover, the upper bound on the right side turns out to be precise along the diagonal, as in the next lemma.

Lemma 3.4. *Given an integer $k \geq 1$, there is a constant $c_k = c_k(n) > 0$ such that*

$$\mathscr{D}^k R_z(z) = (-1)^k c_k z_n^{-n-k}$$

for $z \in H$.

Proof. Let $k \geq 1$ be an integer and $z \in H$. A straightforward calculation yields

$$R_z(w) = z_n^{-n} R_{\boldsymbol{e}}(\phi_z(w)) \tag{3.8}$$

for $w \in H$ where $\phi_z(w) = z_n^{-1}(w' - z', w_n)$. Applying $\mathscr{D}^k$ to both sides of the above, we obtain

$$\mathscr{D}^k R_z(w) = z_n^{-n-k} \mathscr{D}^k R_{\boldsymbol{e}}(\phi_z(w))$$

for $w \in H$. Thus, evaluating at $w = z$, we have $\mathscr{D}^k R_z(z) = z_n^{-n-k} \mathscr{D}^k R_{\boldsymbol{e}}(\boldsymbol{e})$. In order to compute $\mathscr{D}^k R_{\boldsymbol{e}}(\boldsymbol{e})$, note that we have

$$R_{\boldsymbol{e}}(w) = \frac{4}{n\sigma_n} g(|w'|, 1 + w_n)$$

by (1.2) where

$$g(s,t) = \frac{(n-1)t^2 - s^2}{(s^2 + t^2)^{(n+2)/2}}.$$

Thus, using $g(0,t) = (n-1)t^{-n}$ and real-analyticity, we have

$$\mathscr{D}^k R_{\boldsymbol{e}}(\boldsymbol{e}) = \frac{4}{n\sigma_n} \left[\frac{d^k}{dt^k} g(0,t) \right]_{t=2} = (-1)^k \frac{4}{n\sigma_n 2^{n+k}} \frac{(n+k-1)!}{(n-2)!},$$

as required. The proof is complete. □

LEMMA 3.5. *The identity*

$$T_{w_n} \mathscr{D}^k R_z = -\frac{1}{2} \mathscr{D}^{k-1} R_z$$

holds for integers $k \geq 1$ and $z \in H$.

PROOF. We first recall a reproducing formula. Among many other reproducing formulas obtained in [**9**], we recall that the kernel $R(z,w)$ has the following generalized reproducing property ([**9**, Lemma 4.6]):

$$u(z) = \frac{(-2)^m}{m!} \int_H w_n^m \mathscr{D}^m u(w) R(z,w)\, dw$$

for integers $m \geq 0$ and functions $u \in b^2$.

Let $k \geq 1$ be an integer and fix $z \in H$. Note that $\mathscr{D}^{k-1} R_z \in b^2$ by (3.7). Thus the lemma follows from the above generalized reproducing property (with $m = 1$ and $u = \mathscr{D}^{k-1} R_z$). □

4. Proofs of main Theorems.

In this section, we prove our main results. We introduce some notation. First, in conjunction with Proposition 2.5, we let Log^s, $s \geq 0$, denote the class of all harmonic functions f on H such that

$$\|f\|_{Log^s} := \sup_{w \in H} \frac{|f(w)|}{\Phi_{0,s}(w, \boldsymbol{e})} < \infty.$$

Note $T_u Log^s \subset Log^{s+1}$ for each $s \geq 0$ and $u \in L^\infty$ by Proposition 2.5. Also, given $1 < p < \infty$ and $s \geq 0$, the estimate

$$\int_H |\Phi_{0,s}(w, \boldsymbol{e})|^p \, dw \approx I_{np-n,sp}(\boldsymbol{e}, \boldsymbol{e})$$

yields $Log^s \subset L^p$ by Proposition 2.5. Next, we let $\mathscr{D}$ be the class of all harmonic functions on H such that

$$\sup_{w \in H} |u(w)|(1 + |w|^n) < \infty.$$

Note $R_z \in \mathscr{D}$ for each $z \in H$. Also, note $Log^0 = \mathscr{D}$. Finally, given nontrivial functions $f, g \in b^2$, we let $f \otimes g$ denote the rank-one operator on b^2 defined by

$$(f \otimes g)h = \langle h, g \rangle f$$

for $h \in b^2$.

The following is the key lemma for our results.

Lemma 4.1. *Let $u_1, \ldots, u_m \in L^\infty$. Let $\{f_j\}_{j=1}^N$ and $\{g_j\}_{j=1}^N$ be linearly independent collections of functions in b^2. Assume*

$$T_{u_1} T_{u_2} \cdots T_{u_m} = \sum_{j=1}^N f_j \otimes g_j$$

on b^2. Then $f_j, g_j \in Log^m$ for all j. If, in addition, $u_1, \dots, u_m \in \Lambda_\epsilon(\zeta)$ for some $\epsilon \in (0,1)$ and $\zeta \in \partial H$, then $f_j, g_j \in \Lambda_\epsilon(\zeta)$ for all j.

PROOF. We first show $f_j \in Log^m$ for all j. Put $T = T_{u_1} T_{u_2} \cdots T_{u_m}$. Given $z \in H$, we have $T_{u_m} R_z \in T_{u_m} \mathscr{D} = T_{u_m} Log^0 \subset Log^1$ by the remarks above. Repeating similar arguments, we have $TR_z \in Log^m$. Note $TR_z = \sum_{j=1}^N \overline{g_j(z)} f_j$ by (1.1). Thus, for arbitrary points $z^1, \dots, z^N$ in H, we have

$$\begin{pmatrix} TR_{z^1} \\ \vdots \\ TR_{z^N} \end{pmatrix} = \begin{pmatrix} \overline{g_1(z^1)} & \dots & \overline{g_N(z^1)} \\ \vdots & & \vdots \\ \overline{g_1(z^N)} & \dots & \overline{g_N(z^N)} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}.$$

By [**5**, Lemma 2.4] we can pick some points $z^1, \dots, z^N \in H$ such that the $N \times N$ matrix in the above displayed equation is invertible. Now, since functions TR_{z^j} all belong to Log^m, we conclude that $f_j \in Log^m$ for all j, as desired.

Let $\epsilon \in (0,1)$ and $\zeta \in \partial H$. Then, given $u \in L^\infty \cap \Lambda_\epsilon(\zeta)$, we have by Theorem 3.2

$$T_u[b^2 \cap \Lambda_\epsilon(\zeta)] \subset R[L^2 \cap \Lambda_\epsilon(\zeta)] \subset b^2 \cap [\Lambda_\epsilon + \mathscr{H}(\zeta)] \subset b^2 \cap \Lambda_\epsilon(\zeta),$$

because $\mathscr{H}(\zeta) \subset \Lambda_\epsilon(\zeta)$ and functions in Λ_ϵ are easily seen to have extensions in $\Lambda_\epsilon(\boldsymbol{R}^n)$. Thus, if $u_1, \dots, u_m \in L^\infty \cap \Lambda_\epsilon(\zeta)$, then

$$T[b^2 \cap \Lambda_\epsilon(\zeta)] \subset b^2 \cap \Lambda_\epsilon(\zeta)$$

and, in particular, $TR_{z^j} \in \Lambda_\epsilon(\zeta)$ for each j. Thus we conclude $f_j \in \Lambda_\epsilon(\zeta)$ for all j.

Since $T_u^* = T_{\overline{u}}$ and $(f \otimes g)^* = g \otimes f$ in general where the superscript * denotes the Hilbert space adjoint operator, we have

$$T^* = T_{\overline{u}_m} \cdots T_{\overline{u}_1} = \sum_{j=1}^N g_j \otimes f_j.$$

Now, what we've proved above implies the assertions on functions g_j. The proof is complete. □

Using Lemma 4.1, we obtain the following growth estimates for finite rank operators, when applied to test functions.

Lemma 4.2. *Let $u_1, \dots, u_m \in L^\infty$ and put $T = T_{u_1} T_{u_2} \cdots T_{u_m}$. Let $1 < p < \infty$ and $k \geq 1$ be an integer. If T has finite rank, then there exists a constant $C = C(p, k, T)$ such that*

$$|T\lambda_t^k(t\boldsymbol{e})| \leq C \frac{1 + |\log t|^m}{t^{k+n(1-1/p)}}$$

for all $0 < t < 1$. If, in addition, $u_1, \dots, u_m \in \Lambda_\epsilon(0)$ for some $\epsilon \in (0, 1)$, then there exists a constant $C = C(k, T)$ such that

$$|T\lambda_t^k(t\boldsymbol{e})| \leq \frac{C}{t^{k-\epsilon}} \tag{4.1}$$

for $0 < t < 1$.

Proof. Assume T has finite rank, say N. Then there exists linearly independent collections $\{f_j\}_{j=1}^N$ and $\{g_j\}_{j=1}^N$ of functions in b^2 such that $T = \sum_{j=1}^N f_j \otimes g_j$ and thus

$$T\lambda_t^k(t\boldsymbol{e}) = \sum_{j=1}^N \langle \lambda_t^k, g_j \rangle f_j(t\boldsymbol{e})$$

for $t > 0$. Note that functions f_j, g_j all belong to Log^m by Lemma 4.1. Let q be the conjugate exponent of p. Recall $Log^m \subset L^q$ and note

$$\int_H |\lambda_t^k(w)|^p \, dw \lesssim \int_H \frac{dw}{|t\boldsymbol{e} - \overline{w}|^{p(n+k)}} \approx t^{n-p(n+k)}$$

for $t > 0$ by (3.7) and Lemma 2.2. Hence, applying Hölder's inequality and denoting the L^q-norm by $\| \, \|_q$, we obtain

$$\begin{aligned} |T\lambda_t^k(t\boldsymbol{e})| &\lesssim \frac{\Phi_{0,m}(t\boldsymbol{e}, \boldsymbol{e})}{t^{k+n(1-1/p)}} \sum_{j=1}^N \|g_j\|_q \|f_j\|_{Log^m} \\ &\lesssim \frac{1 + |\log t|^m}{t^{k+n(1-1/p)}} \sum_{j=1}^N \|g_j\|_q \|f_j\|_{Log^m} \end{aligned} \tag{4.2}$$

for $0 < t < 1$. This completes the proof of the first part of the lemma.

Now, assume further $u_1, \dots, u_m \in \Lambda_\epsilon(0)$ for some $\epsilon \in (0, 1)$ and show (4.1).

Note that functions f_j, g_j all belong to $\Lambda_\epsilon(0)$ by Lemma 4.1. Thus

$$|T\lambda_t^k(t\boldsymbol{e})| \le M \sum_{j=1}^{N} |\langle \lambda_t^k, g_j\rangle|$$

where $M = \sup_{0<t<1,j} |f_j(t\boldsymbol{e})| < \infty$. So, in order to complete the proof of (4.1), it is sufficient to show that, given a function $g \in L^2 \cap \Lambda_\epsilon(0)$, there is a constant $C = C(k, g)$ such that

$$|\langle \lambda_t^k, g\rangle| \le \frac{C}{t^{k-\epsilon}} \tag{4.3}$$

for $0 < t < 1$. Assume $g \in \Lambda_\epsilon(U)$ where U is some neighborhood of 0. Let $0 < t < 1$. Note $\lambda_t^k \in b^1$ by (3.7). Thus we have

$$\langle \lambda_t^k, g\rangle = \langle \lambda_t^k, g - g(0)\rangle$$

by the b^1-cancelation property (3.4). Meanwhile, we have

$$|\langle \lambda_t^k, g - g(0)\rangle| \le \int_H \frac{|g(w) - g(0)|}{|t\boldsymbol{e} - \overline{w}|^{n+k}}\, dw = \int_{H\cap U} + \int_{H\setminus U} \frac{|g(w) - g(0)|}{|t\boldsymbol{e} - \overline{w}|^{n+k}}\, dw.$$

Using the Lipschitz condition at 0, we have by Lemma 2.2

$$\int_{H\cap U} \frac{|g(w) - g(0)|}{|t\boldsymbol{e} - \overline{w}|^{n+k}}\, dw \lesssim \int_H \frac{dw}{|t\boldsymbol{e} - \overline{w}|^{n+k-\epsilon}} \approx \frac{1}{t^{k-\epsilon}}.$$

Also, we have by Hölder's inequality

$$\begin{aligned} \int_{H\setminus U} \frac{|g(w) - g(0)|}{|t\boldsymbol{e} - \overline{w}|^{n+k}}\, dw \lesssim \|g\|_2 \left\{ \int_{H\setminus U} \frac{dw}{|t\boldsymbol{e} - \overline{w}|^{2(n+k)}} \right\}^{1/2} \\ + |g(0)| \int_{H\setminus U} \frac{dw}{|t\boldsymbol{e} - \overline{w}|^{n+k}}. \end{aligned}$$

Note that the integrals on the right side of the above are bounded uniformly in t by Lemma 2.3. Now, combining these estimates, we have (4.3) and thus conclude (4.1). The proof is complete. □

We also need a uniqueness result for harmonic functions as in the next lemma. This lemma is proved on the ball in [**3**, Proposition 4.1] and we omit the proof which is much simpler on the half-space. Note that $\mathscr{D}u$ always exists by the reflection principle (see [**1**, Theorem 4.12]) in the hypothesis of the next lemma.

LEMMA 4.3. *Suppose that u is a function harmonic on H and continuous on $H \cup W$ for some nonempty relatively open set $W \subset \partial H$. If both u and $\mathscr{D}u$ vanish on W, then $u = 0$ on H.*

We are now ready to prove our main results. Our proof of Theorem 1.1 will depend on Theorem 1.3, which in turn depends on the proof Theorem 1.2. So, we first prove Theorem 1.2.

PROOF OF THEOREM 1.2. Put $T = T_{u_1} \cdots T_{u_{n+1}}$ and assume T has finite rank. Since T has finite rank (and thus is compact) and $r_z \to 0$ weakly in b^2 as $z \to \partial H$ (see [**6**, Lemma 5.2]), we have $\widetilde{T}(z) \to 0$ as $z \to \partial H$. It follows from Theorem 3.3 that $u_1 \cdots u_{n+1} = 0$ on ∂H.

If $u_1 = 0$ on H, there is nothing to do. So, assume that u_1 is not identically 0. Since a bounded harmonic function is recovered by the Poisson integral of its boundary values, u_1 vanish nowhere (by continuity) on some nonempty relatively open subset of ∂H, say W_1. If u_2 is not identically 0 on W_1, we can find a smaller relatively nonempty open set $W_2 \subset W_1$ on which u_2 also vanishes nowhere. Continuing this process, we see that there exists a nonempty relatively open set $W \subset \partial H$ such that

$$\begin{aligned} \text{either} \quad & u_j(\zeta) \neq 0, \quad \zeta \in W \\ \text{or} \quad & u_j = 0 \quad \text{on } W \end{aligned}$$

holds for each j. Note u_1 vanishes nowhere on W. Also, note $u_{j_0} = 0$ on W for some j_0, because $u_1 \cdots u_{n+1} = 0$ on ∂H. If, in addition, $\mathscr{D}u_{j_0} = 0$ on (some nonempty relatively open subset of) W, we have $u_{j_0} = 0$ by Lemma 4.3. So, given j, assume that u_j and $\mathscr{D}u_j$ do not simultaneously vanish on any nonempty relatively open subset of W. Thus there exists some nonempty relatively open subset of W, still denoted by W, such that

$$\begin{aligned} \text{either} \quad & u_j(\zeta) \neq 0, \quad \zeta \in W \\ \text{or} \quad & u_j(\zeta) = 0,\ \mathscr{D}u_j(\zeta) \neq 0, \quad \zeta \in W \end{aligned} \tag{4.4}$$

for each $j = 1, \ldots, n+1$. We may assume $0 \in W$ without loss of generality. This will lead us to a contradiction.

We introduce more notation. In the rest of the proof we let $t \in (0,1)$ be arbitrary and $z \in H$ represent an arbitrary point, unless otherwise specified. Recall that $\mathscr{D}u_j(0) \neq 0$ by (4.4), in case $u_j(0) = 0$. Let $d_j = 1$ if $u_j(0) = 0$, and $d_j = 0$ otherwise. Note $d_1 = 0$. Now, define the major part m_j of u_j by

$$m_j(z) = \begin{cases} u_j(0) & \text{if } d_j = 0 \\ \mathscr{D}u_j(0) z_n & \text{if } d_j = 1 \end{cases}$$

and put $e_j = u_j - m_j$ for each j. Note that we have

$$e_j(z) = \mathscr{O}\left(|z|^{\epsilon + d_j}\right) \tag{4.5}$$

for each j. To see this we only need to consider z near 0, because u_j is bounded. Thus (4.5) is a consequence of the Lipschitz hypothesis if $d_j = 0$. In case $d_j = 1$, since $u_j = 0$ on W, u_j is harmonic and thus smooth across W by the reflection principle. Also, note $\frac{\partial u_j}{\partial z_i}(0) = 0$ for $i = 1, \ldots, n-1$. Thus Taylor's theorem yields $e_j(z) = \mathscr{O}(|z|^2)$, which implies (4.5). Similarly, we have

$$u_j(z) = \mathscr{O}\left(|z|^{d_j}\right) \tag{4.6}$$

for each j.

We introduce further notation. Put $M = T_{m_1} \cdots T_{m_{n+1}}$ and $R = T_{u_1} \cdots T_{u_{n+1}} - M$. Then one may verify by an inductive argument

$$R = \sum_{j=1}^{n+1} R_j$$

where

$$R_j = T_{u_1} \cdots T_{u_{j-1}} T_{e_j} T_{m_{j+1}} \cdots T_{m_{n+1}}$$

for each j. Note $M = T - R$. Fix $k > n+2$. We will estimate the same expression $M\lambda_t^k = T\lambda_t^k - R\lambda_t^k$ along the vertical ray emanating from the origin in two different ways and reach a contradiction.

Put

$$d = d_1 + \cdots + d_{n+1}$$

and

$$p_j = d_1 + \cdots + d_j, \quad q_j = d_j + \cdots + d_{n+1}$$

for each j. Note $d \geq 1$, because $(u_1 \ldots u_{n+1})(0) = 0$. Also, recall $d_1 = 0$.

We first estimate $M\lambda_t^k(t\boldsymbol{e})$. Let

$$c_j = \begin{cases} u_j(0) & \text{if } u_j(0) \neq 0 \\ \mathscr{D}u_j(0) & \text{if } u_j(0) = 0. \end{cases}$$

Then we have $M = cT_{w_n}^d$ where $c = c_1 \cdots c_{n+1} \neq 0$. It follows from Lemma 3.5 that $T_{w_n}^d \lambda_t^k = (-1/2)^d \lambda_t^{k-d}$ and thus we have

$$|M\lambda_t^k(t\boldsymbol{e})| \approx t^{-n-k+d} \tag{4.7}$$

by Lemma 3.4.

Next, we estimate $R\lambda_t^k(t\boldsymbol{e})$. So, fix j and consider R_j. Since $T_{m_{j+1}} \cdots T_{m_{n+1}} = c_{j+1} \cdots c_{n+1} T_{w_n}^{q_{j+1}}$, we have

$$T_{m_{j+1}} \cdots T_{m_{n+1}} \lambda_t^k = c_{j+1} \cdots c_{n+1} \left(-\frac{1}{2}\right)^{q_{j+1}} \lambda_t^{k-q_{j+1}}$$

by Lemma 3.5. Thus we have by (4.5) and Proposition 2.5

$$\begin{aligned} \left|T_{e_j}\left(T_{m_{j+1}} \cdots T_{m_{n+1}}\right)\lambda_t^k(z)\right| &\lesssim \int_H \frac{|e_j(w)||\lambda_t^{k-q_{j+1}}(w)|}{|z-\overline{w}|^n}\, dw \\ &\lesssim \int_H \frac{dw}{|t\boldsymbol{e}-\overline{w}|^{n+k-q_{j+1}-d_j-\epsilon}|z-\overline{w}|^n} \\ &\lesssim t^{-(k-q_j-\epsilon)}\Phi_{0,1}(t\boldsymbol{e}, z). \end{aligned} \tag{4.8}$$

Now, a similar argument using (4.6) and Proposition 2.5 yields

$$\begin{aligned} \left|T_{u_{j-1}}T_{e_j}T_{m_{j+1}} \cdots T_{m_{n+1}}\lambda_t^k(z)\right| &\lesssim t^{-(k-q_j-\epsilon)} \int_H \frac{|w|^{d_{j-1}}\Phi_{0,1}(t\boldsymbol{e}, w)}{|z-\overline{w}|^n}\, dw \\ &\lesssim t^{-(k-q_j-\epsilon)}\Phi_{-d_{j-1},2}(t\boldsymbol{e}, z) \end{aligned}$$

for all $j \geq 2$. Note that

$$p_{j-1} \le j-2 < n, \quad j = 2, \dots, n+1, \tag{4.9}$$

because $d_1 = 0$; *it is this step which requires the restriction on the number of factors in the product.* Hence, by the same argument repeatedly using (4.9) and Proposition 2.5, we eventually obtain

$$\left|R_j \lambda_t^k(z)\right| \lesssim t^{-(k-q_j-\epsilon)} \Phi_{-p_{j-1},j}(t\boldsymbol{e}, z)$$

for each $j \ge 2$. This also holds for $j = 1$ by (4.8) if we set $p_0 = 0$. So, evaluating at $z = t\boldsymbol{e}$, we therefore have

$$\left|R_j \lambda_t^k(t\boldsymbol{e})\right| \lesssim \frac{\left(1 + |\log t|^j\right)}{t^{n+k-d-\epsilon}} \lesssim \frac{\left(1 + |\log t|^{n+1}\right)}{t^{n+k-d-\epsilon}};$$

this holds for arbitrary j. Consequently, we obtain

$$\left|R \lambda_t^k(t\boldsymbol{e})\right| \lesssim \frac{\left(1 + |\log t|^{n+1}\right)}{t^{n+k-d-\epsilon}}. \tag{4.10}$$

Finally, we have by Lemma 4.2

$$|T \lambda_t^k(t\boldsymbol{e})| \lesssim \frac{1}{t^{k-\epsilon}}. \tag{4.11}$$

Now, we have by (4.7), (4.10) and (4.11)

$$1 = \frac{|T\lambda_t^k(t\boldsymbol{e}) - R\lambda_t^k(t\boldsymbol{e})|}{|M\lambda_t^k(t\boldsymbol{e})|} \lesssim t^{\epsilon}(1 + |\log t|^{n+1}) + t^{\epsilon+n-d}$$

for $0 < t < 1$. The constant suppressed above is independent of t. Recall $d \le n$. Thus, upon taking the limit $t \to 0$, we reach a contradiction. The proof is complete. □

Next, we prove Theorem 1.3. The proof is almost the same as that of Theorem 1.2. We only indicate what the difference is.

PROOF OF THEOREM 1.3. Let $u_{n+1} = 1$ and follow the proof of Theorem 1.2. In the course of the proof of Theorem 1.2, we were able to assume $d_1 = 0$ under the global Lipschitz hypothesis, because the location of the boundary set W is of no significance. However, we cannot assume $d_1 = 0$ in general under the present local Lipschitz hypothesis. This causes loss of a factor. The rest of the proof is unchanged. $\square$

Finally, we prove Theorem 1.1.

PROOF OF THEOREM 1.1. Put $T = T_{u_1}T_{u_2}$ and assume T has finite rank. As in the proof of Theorem 1.2, we have $u_1u_2 = 0$ on W. There are two cases to consider:

(i) Both u_1 and u_2 vanish everywhere on W;
(ii) Either u_1 or u_2 vanish nowhere on W (shrinking W if necessary).

Assume $0 \in W$ without loss of generality. The case (i) is contained in Theorem 1.3, because $u_1, u_2 \in \Lambda_1(0)$ by (4.6). So, assume (ii) holds. We may assume that u_1 vanishes nowhere on W; otherwise consider the adjoint operator $(T_{u_1}T_{u_2})^* = T_{\overline{u}_2}T_{\overline{u}_1}$. We now have $u_2 = 0$ on W. If $\mathscr{D}u_2 = 0$ on W, we are done by Lemma 4.3. Thus we may further assume $\mathscr{D}u_2$ vanishes nowhere on W. This will lead us to a contradiction.

Let $c_1 = u_1(0) \neq 0$ and $c_2 = \mathscr{D}u_2(0) \neq 0$. Let $e_1 = u_1 - c_1$ and $e_2 = u_2 - c_2z_n$. Then we have

$$T = T_{c_1+e_1}T_{c_2w_n+e_2} = c_1c_2T_{w_n} + c_2T_{e_1}T_{w_n} + T_{u_1}T_{e_2}$$

and thus

$$c_1c_2T_{w_n} = T - c_2T_{e_1}T_{w_n} - T_{u_1}T_{e_2}. \tag{4.12}$$

Now we apply each side of the above to the same test functions λ_t^k with an integer $k > 2$ and derive a contradiction, as in the proof of Theorem 1.2. In the rest of the proof we let $t \in (0, 1)$ be arbitrary and $z \in H$ represent an arbitrary point, unless otherwise specified.

First, we have by Lemmas 3.5 and 3.4

$$2|T_{w_n}\lambda_t^k(t\boldsymbol{e})| = |\lambda_t^{k-1}(t\boldsymbol{e})| \approx t^{-n-k+1}. \tag{4.13}$$

Next, note that $e_2(w) = \mathscr{O}(|w|^2)$; see (4.5) in the proof of Theorem 1.2. Thus we have

$$\begin{aligned}|T_{e_2}\lambda_t^k(z)| &\le \int_H |e_2(w)||\lambda_t^k(w)||R(z,w)|\,dw\\ &\lesssim \int_H \frac{dw}{|t\boldsymbol{e}-\overline{w}|^{n+k-2}|z-\overline{w}|^n}\\ &\lesssim t^{-k+2}\Phi_{0,1}(t\boldsymbol{e},z)\end{aligned}$$

by Proposition 2.5 and therefore

$$\left|T_{u_1}T_{e_2}\lambda_t^k(t\boldsymbol{e})\right| \lesssim \|u_1\|_\infty t^{-k+2}I_{0,1}(t\boldsymbol{e},t\boldsymbol{e}) \lesssim t^{-n-k+2}(1+|\log t|^2) \tag{4.14}$$

again by Proposition 2.5.

Next, we estimate $T_{e_1}T_{w_n}\lambda_t^k(t\boldsymbol{e})$. Given $\epsilon>0$, let

$$\omega(\epsilon)=\sup_{|w|<\epsilon}|e_1(w)|$$

be the modulus of continuity of u_1 at 0. Note $-2T_{e_1}T_{w_n}\lambda_t^k = T_{e_1}\lambda_t^{k-1}$ by Lemma 3.5. Meanwhile, we have by (3.7)

$$\begin{aligned}\left|T_{e_1}\lambda_t^{k-1}(t\boldsymbol{e})\right| &\lesssim \int_H \frac{|e_1(w)|}{|t\boldsymbol{e}-\overline{w}|^{(n+k-1)+n}}\,dw\\ &= \int_{|w|<\epsilon}+\int_{|w|\ge\epsilon}\frac{|e_1(w)|}{|t\boldsymbol{e}-\overline{w}|^{2n+k-1}}\,dw\\ &:= I_1+I_2.\end{aligned}$$

By Lemma 2.2 we have

$$I_1 \lesssim t^{-n-k+1}\omega(\epsilon)$$

for $t>0$. Note that $|t\boldsymbol{e}-\overline{w}|\ge|w|-t\ge\epsilon-t$ for $|w|\ge\epsilon$. Thus, for $0<t<\epsilon$, by Lemma 2.2 again we have

$$I_2 \lesssim \frac{\|u_1\|_\infty}{(\epsilon-t)^n}\int_H \frac{dw}{|t\boldsymbol{e}-\overline{w}|^{n+k-1}} \approx \frac{\|u_1\|_\infty}{(\epsilon-t)^n t^{k-1}}.$$

Combining the estimates of I_1 and I_2 together, we have

$$\left|T_{e_1}T_{w_n}\lambda_t^k(t\boldsymbol{e})\right| \lesssim \frac{\omega(\epsilon)}{t^{n+k-1}} + \frac{1}{(\epsilon - t)^n t^{k-1}} \tag{4.15}$$

for $0 < t < \epsilon$.

Also, we have by Lemma 4.2

$$|T\lambda_t^k(t\boldsymbol{e})| \lesssim \frac{1 + |\log t|^2}{t^{k+n(1-1/p)}}. \tag{4.16}$$

where p is chosen so that $1 < p < n$.

Now, setting $M = c_1c_2T_{w_n}$ and $R = c_2T_{e_1}T_{w_n} + T_{u_1}T_{e_2}$ and $\delta = \min\{1, n/p - 1\} > 0$, we obtain from (4.13)–(4.16) that

$$1 = \frac{|T\lambda_t^k(t\boldsymbol{e}) - R\lambda_t^k(t\boldsymbol{e})|}{|M\lambda_t^k(t\boldsymbol{e})|} \lesssim t^\delta(1 + |\log t|^2) + \omega(\epsilon) + t^n(\epsilon - t)^{-n}$$

for $0 < t < \epsilon$. The constant suppressed above is independent of t and ϵ. So, first taking the limit $t \to 0$ with $\epsilon > 0$ fixed and then taking the limit $\epsilon \to 0$, we have

$$1 \lesssim \omega(\epsilon) \to 0$$

by continuity of u_1 at 0. Thus we have a contradiction as desired, completing the proof. □

5. Remarks.

Although we have studied Toeplitz products in the present paper, not much is known for Toeplitz operators (with general symbols). For example, even for bounded symbols, characterization for most basic operator theoretic property such as compactness is not known. Only positive compact Toeplitz operators are characterized in [**6**]. Even for bounded harmonic symbols, it seems not easy to see that compactness implies the symbol being zero; the analogue for holomorphic Bergman space on the disk is an easy consequence of the fact that bounded harmonic functions are fixed by the holomorphic Berezin transform. Motivated by these observations, we provide here a proof of the fact that the zero operator is the only compact Toeplitz operator with harmonic symbol on b^2.

To begin with, we recall the pseudohyperbolic distance $\rho(z, w)$ between two points z and w in H:

$$\rho(z,w) = \frac{|z-w|}{|z-\overline{w}|}.$$

This pseudohyperbolic distance is horizontal translation invariant and dilation invariant. In particular, we have

$$\rho(z,w) = \rho(\phi_a(z), \phi_a(w)) \tag{5.1}$$

for $a, z, w \in H$ where ϕ_a denotes the mapping introduced in the proof of Lemma 3.4. For $z \in H$ and $0 < \delta < 1$, let $E_\delta(z)$ denote the pseudohyperbolic ball centered at z with radius δ. It is known that

$$\frac{1-\delta}{1+\delta} < \frac{|z-\overline{a}|}{|w-\overline{a}|} < \frac{1+\delta}{1-\delta} \tag{5.2}$$

whenever $w \in E_\delta(z)$ and $a \in H$; see [**7**, Lemma 3.3].

We now recall the notion of nontangential limits. Given $\alpha > 1$ and $\zeta \in \partial H$, let $\Gamma_\alpha(\zeta)$ be the nontangential approach region with vertex ζ consisting of all points $z \in H$ such that

$$|z-\zeta| < \alpha z_n.$$

Give an function u on H, we say that u has a nontangential limit at $\zeta \in \partial H$, denoted by $u^*(\zeta)$, if

$$\lim_{z\to\zeta, z\in\Gamma_\alpha(\zeta)} u(z) = u^*(\zeta)$$

for each $\alpha > 1$. It then turns out that the following nontangential version of Theorem 3.3 holds.

THEOREM 5.1. *Suppose that functions $u_1, \dots, u_N \in L^\infty$ have nontangential limits at $\zeta \in \partial H$. Let $T = T_{u_1} \cdots T_{u_N}$. Then $\widetilde{T}$ has a nontangential limit at ζ with $(\widetilde{T})^*(\zeta) = (u_1 \cdots u_N)^*(\zeta)$.*

PROOF. By the proof of Theorem 3.3, it is sufficient to prove

$$\lim_{z\to\zeta, z\in\Gamma_\alpha(\zeta)} \|T_u r_z\|_2 = 0, \quad \alpha > 1 \tag{5.3}$$

for any function $u \in L^\infty$ that has a nontangential limit 0 at ζ. We fix $\alpha > 1$ and assume $z \in \Gamma_\alpha(\zeta)$ for the rest of the proof.

Note $\|T_u r_z\|_2 = \|R(ur_z)\|_2 \le \|ur_z\|_2$. Thus, given $0 < \delta < 1$, we have

$$\begin{aligned}\|T_u r_z\|_2^2 &\le \int_{E_\delta(z)} + \int_{H\setminus E_\delta(z)} |u(w)|^2 |r_z(w)|^2\, dw \\ &:= I_1 + I_2.\end{aligned} \tag{5.4}$$

We first consider the first term I_1. Let $w \in E_\delta(z)$. Note that

$$1 - \delta^2 < 1 - \rho^2(z,w) = \frac{4z_n w_n}{|z - \overline{w}|^2} < \frac{4w_n}{z_n} < \frac{4\alpha w_n}{|z - \zeta|}$$

where we used the assumption that $z \in \Gamma_\alpha(\zeta)$ for the last inequality. Also, note

$$|w - \overline{z}| < 2\,\frac{1+\delta}{1-\delta}\, w_n$$

by (5.2). Thus, if $w \in E_\delta(z)$, then

$$|w - \zeta| < |w - \overline{z}| + |z - \zeta| < 2\left(\frac{1+\delta}{1-\delta} + \frac{2\alpha}{1-\delta^2}\right) w_n.$$

This means that $\cup_{z\in\Gamma_\alpha(\zeta)} E_\delta(z)$ is contained in some fixed nontangential approach region with vertex ζ, say $\Gamma_\beta(\zeta)$, depending on α and δ. Consequently, we have

$$I_1 \le \sup_{w\in\Gamma_\beta(\zeta)} |u(w)|^2.$$

Since $u^*(\zeta) = 0$, this yields $I_1 \to 0$ as $z \to \zeta$ (within $\Gamma_\alpha(\zeta)$) for each fixed δ.

We now consider the second term I_2. Note $\phi_z E_\delta(z) = E_\delta(\boldsymbol{e})$ by (5.1). Also, note $R(z,z)^{-1} z_n^{-n} = c_n$ where $c_n = n\sigma_n 2^{n-2}(n-1)^{-1}$ by (1.2). Thus, using (3.8) and making a change of variables, we obtain

$$\begin{aligned}I_2 &= c_n \int_{H\setminus E_\delta(\boldsymbol{e})} |u \circ \phi_z^{-1}(w)|^2 |R_{\boldsymbol{e}}(w)|^2\, dw \\ &\le c_n \|u\|_\infty^2 \int_{H\setminus E_\delta(\boldsymbol{e})} |R_{\boldsymbol{e}}(w)|^2\, dw.\end{aligned}$$

This shows that I_2 is dominated by a quantity which is independent of z and converges to 0 as $\delta \to 1$ by the dominated convergence theorem. Thus, taking the limit $z \to \zeta$ with δ fixed and then taking the limit $\delta \to 1$ in (5.4), we conclude (5.3), as required. This completes the proof. □

As a consequence, we obtain the following.

COROLLARY 5.2. *Let $u \in h^\infty$. If T_u is compact, then $u = 0$.*

PROOF. Assume T_u is compact. By compactness we have $\widetilde{T_u}(z) \to 0$ as $z \to \partial H$, as in the proof of Theorem 1.2. On the other hand, being a bounded harmonic function, u has nontangential limits $u^*(\zeta)$ at almost all points $\zeta \in \partial H$; see [**1**, Theorem 7.28]. Thus we have $u^* = 0$ by Theorem 5.1. Now, since u is recovered by the Poisson integral of $u^* = 0$, we conclude $u = 0$. □

In case $u \geq 0$ this corollary is already known and is a consequence of the maximum principle and the Carleson measure characterization (see [**6**, Theorem 5.3]) in terms of averaging functions.

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