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Gluing construction of compact complex surfaces with trivial canonical bundle

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Abstract. We obtain a new construction of compact complex surfaces with trivial canonical bundle. In our construction we glue together two compact complex surfaces with an anticanonical divisor under suitable conditions. Then we show that the resulting compact manifold admits a complex structure with trivial canonical bundle by solving an elliptic partial differential equation. We generalize this result to cases where we have other than two components to glue together. With this generalization, we construct examples of complex tori, Kodaira surfaces and K3 surfaces. Lastly we deal with the smoothing problem of a normal crossing complex surface X with at most double curves. We prove that we still have a family of smoothings of X in a weak sense even when X is not Kählerian or $H^1(X, \mathscr{O}_X) \neq 0$, in which cases the smoothability result of Friedman [**Fr**] is not applicable.

1. Introduction.

Let X be a manifold of dimension n and suppose X contains a compact submanifold X_0 of dimension n with boundary $S = \partial X_0$, such that $X \setminus X_0$ is diffeomorphic to a cylinder $S \times \boldsymbol{R}_+ = \{(p,t) \mid p \in S, 0 < t < \infty\}$. Then we call X a *cylindrical manifold*, and t a *cylindrical parameter* of X. The gluing of cylindrical manifolds is a useful method for constructing compact Riemannian manifolds with a special metric in differential geometry. It was first successful in constructing compact 4-dimensional Riemannian manifolds with an anti-self-dual metric by Floer [**Fl**] and Taubes [**T**], which constructions were later improved by Kovalev and Singer [**KS**]. The method is also used in constructing compact 7-dimensional Riemannian manifolds with holonomy G_2 [**J**], [**K**]. The purpose of this paper is to obtain a new construction of compact complex surfaces with trivial canonical bundle using the gluing method, and the main result is described as follows.

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Theorem 1.1. *Let X be a compact complex surface with a smooth irreducible anticanonical divisor D, and X' another compact complex surface with a smooth irreducible anticanonical divisor D'. Suppose there exists an isomorphism f from D to D' and the holomorphic normal bundles $N_{D/X}$ and $N_{D'/X'}$ are dual to each other via f, i.e., $N_{D/X} \otimes f^* N_{D'/X'} \cong \mathscr{O}_D$. Then there exist tubular neighborhoods W_1, W_2 of D in X with $\overline{W_1} \subset W_2$, tubular neighborhoods W_1', W_2' of D' in X' with $\overline{W_1'} \subset W_2'$, and a diffeomorphism h from $W_2 \setminus \overline{W_1}$ to $W_2' \setminus \overline{W_1'}$ such that the following is true. Via the identification of $W_2 \setminus \overline{W_1}$ with $W_2' \setminus \overline{W_1'}$ by h, we can glue $X \setminus \overline{W_1}$ and $X' \setminus \overline{W_1'}$ together to obtain a compact manifold M. Then the manifold M admits a complex structure with trivial canonical bundle.*

Thus if we are given two compact complex surfaces X and X' as in Theorem 1.1, then we obtain a compact complex surface with trivial canonical bundle from $X \setminus D$ and $X' \setminus D'$. In Theorem 1.1 we don't assume X and X' to be Kählerian. Nor do we assume $N_{D/X}$ and $N_{D'/X'}$ to be trivial, and we need a weaker assumption that the two bundles are dual to each other. We also note that the resulting manifold M is a complex manifold and not a manifold with a special Riemannian metric, which is different from other gluing constructions. In our construction, the complex structures on the regions of $X \setminus D$ and $X' \setminus D'$ to glue together are only close to each other, but not exactly the same. Thus it is not obvious whether the manifold M obtained from $X \setminus D$ and $X' \setminus D'$ is again a complex manifold.

It is also interesting to see that if $D = D'$ then Theorem 1.1 is regarded as giving a kind of smoothing of a surface $X_0 = X \cup X'$ with a normal crossing at D. Indeed, we shall construct a family of smoothings of X_0 in a weak sense in Section 5.3. This result can be compared with the result of Friedman [**Fr**] (see also [**KN**]) that a d-semistable K3 surface has a smoothing. In our case 'd-semistability' means the duality between the normal bundles $N_{D/X}$ and $N_{D/X'}$. Although Friedman's result is powerful and extensive, it needs the assumptions that X_0 is Kählerian and $H^1(X_0, \mathscr{O}_{X_0})$ vanishes, so that it does not cover the smoothability of degenerations of normal crossing complex tori and Kodaira surfaces obtained from our result.

A real $2m$-dimensional manifold admits a complex structure with trivial canonical bundle if and only if it admits a special differential form called an *$SL(m, \boldsymbol{C})$-structure* ψ with $\mathrm{d}\psi = 0$. Other examples of manifolds whose geometric structures are characterized by special d-closed differential forms include Riemannian manifolds with special holonomy [**J**], symplectic manifolds, holomorphic symplectic manifolds, and so on.

Our method is based on the gluing of cylindrical manifolds with an asymptotically $SL(2, \boldsymbol{C})$-structure and analysis as used in constructing compact 8-dimensional Riemannian manifolds with holonomy Spin(7) [**J**].

This paper is organized as follows. In Section 2 we shall introduce the notion of $SL(2,\boldsymbol{C})$- and $SU(2)$-structures on real manifolds of dimension 4. These structures are special cases of $SL(m,\boldsymbol{C})$- and $SU(m)$-structures (torsion-free $SU(m)$-structures are often referred to as *Calabi-Yau structures*) defined on oriented real manifolds of dimension $2m$. (See [**G**] for reference.)

In Section 3 we shall explain the gluing procedure of constructing M in Theorem 1.1. We see that if X, X', D and D' are as in Theorem 1.1, then $X \setminus D$ (resp. $X' \setminus D'$) is a cylindrical manifold with a cylindrical end $S \times \{0 < t < \infty\}$ (resp. $S' \times \{0 < t' < \infty\}$). Since $N_{D/X}$ and $N_{D'/X'}$ are dual to each other, S is diffeomorphic to S'. We set $X_T = (X \setminus D) \setminus (S \times \{t \geq T+1\})$ and $X'_T = (X' \setminus D') \setminus (S' \times \{t' \geq T+1\})$, and define a compact manifold M_T by gluing X_T and X'_T together along the regions $S \times \{T-1 < t < T+1\}$ and $S' \times \{T-1 < t' < T+1\}$. To prove that M admits a complex structure with trivial canonical bundle, we first construct on M_T an approximating holomorphic volume form ψ_T, i.e., an $SL(2,\boldsymbol{C})$-structure with $\mathrm{d}\psi_T$ sufficiently small for large T. We note that X has a meromorphic volume form ψ_0 with a single pole along D, which is asymptotic to a cylindrical d-closed $SL(2,\boldsymbol{C})$-structure on $S \times \{0 < t < \infty\}$. Similarly X' has a meromorphic volume form ψ'_0 with a single pole along D'. Thus we can glue ψ_0 and ψ'_0 together using cut-off functions to obtain an approximating holomorphic volume form ψ_T on M_T. To estimate $\mathrm{d}\psi_T$, we introduce a Hermitian form κ_T such that (ψ_T, κ_T) forms an $SU(2)$-structure on M_T. Then we show that $\mathrm{d}\psi_T$ decays exponentially as $T \to \infty$ with respect to the Riemannian metric associated with (ψ_T, κ_T).

Then in Section 4 we shall find a d-closed $SL(2,\boldsymbol{C})$-structure near ψ_T for sufficiently large T to complete the proof of Theorem 1.1. To do this, we use the analysis developed by Joyce to solve a nonlinear elliptic partial differential equation with respect to ψ_T, which is analogous to the one in [**J**], Chapter 12. The Hermitian form κ_T plays an auxiliary but crucial rôle in solving the partial differential equation, which is different from the cases of G_2- and Spin(7)-structures.

In the last section we shall prove a multiple gluing theorem (Theorem 5.2), which is a generalization of Theorem 1.1 to cases where we have $L(\geq 1)$ surfaces with 2ℓ divisors to glue together. We construct some examples using Theorem 1.1 and Theorem 5.2. According to the classification theory, compact complex surfaces with trivial canonical bundle are divided into 2-dimensional complex tori, Kodaira surfaces and K3 surfaces [**BPV**]. Among these surfaces, complex tori and K3 surfaces are Kählerian and Kodaira surfaces are non-Kählerian. Our gluing examples include all classes of compact complex surfaces with trivial canonical bundle. As another application, we shall treat the smoothing problem of compact complex surfaces with normal crossings. We shall construct a family of smoothings in a weak sense of simple normal crossing complex surfaces with at most double curves (Theorem 5.5), and compare the result with that of Friedman [**Fr**]. Although we can prove an analogous smoothability result for normal crossing complex surfaces with triple points, we will deal with and give a proof for that case in a sequel [**D**] to this paper because some additional care is needed.

2. $\boldsymbol{SL(2, C)}$- and $\boldsymbol{SU(2)}$-structures.

Definition 2.1. Let V be an oriented vector space of dimension 4. Then $\psi_0 \in \wedge^2 V^* \otimes \boldsymbol{C}$ is an $SL(2, \boldsymbol{C})$-*structure* on V if ψ_0 satisfies

$$\psi_0 \wedge \overline{\psi}_0 > 0, \quad \psi_0 \wedge \psi_0 = 0.$$

An $SL(2, \boldsymbol{C})$-structure ψ_0 on V gives complex subspaces

$$V^{0,1} = \{\zeta \in V \otimes \boldsymbol{C} \mid \iota_\zeta \psi_0 = 0\}, \quad V^{1,0} = \overline{V^{1,0}},$$

where ι_ζ is the interior multiplication by ζ. Then we have the decomposition

$$V \otimes \boldsymbol{C} = V^{1,0} \oplus V^{0,1}. \tag{2.1}$$

The decomposition (2.1) defines a complex structure I_{ψ_0} on V such that ψ_0 is a complex differential form of type $(2,0)$ with respect to I_{ψ_0}.

Let $\mathscr{A}_{SL(2,\boldsymbol{C})}(V)$ be the set of $SL(2, \boldsymbol{C})$-structures on V. Then $\mathscr{A}_{SL(2,\boldsymbol{C})}(V)$ is an orbit space under the action of the orientation-preserving general linear group $GL_+(V)$. Since each $\psi \in \mathscr{A}_{SL(2,\boldsymbol{C})}(V)$ has isotropy group $SL(2, \boldsymbol{C})$, the orbit $\mathscr{A}_{SL(2,\boldsymbol{C})}(V)$ is isomorphic to the homogeneous space $GL_+(V)/SL(2, \boldsymbol{C})$.

Definition 2.2. Let M be an oriented manifold of dimension 4. Then $\psi \in C^\infty(\wedge^2 T^* M \otimes \boldsymbol{C})$ is an $SL(2, \boldsymbol{C})$-*structure* on M if ψ satisfies

$$\psi \wedge \overline{\psi} > 0, \quad \psi \wedge \psi = 0.$$

We define $\mathscr{A}_{SL(2,\boldsymbol{C})}(M)$ to be the fibre bundle which has fibre $\mathscr{A}_{SL(2,\boldsymbol{C})}(T_x M)$ over $x \in M$. Then an $SL(2, \boldsymbol{C})$-structure can be regarded as a smooth section of $\mathscr{A}_{SL(2,\boldsymbol{C})}(M)$.

Since an $SL(2, \boldsymbol{C})$-structure ψ on M induces an $SL(2, \boldsymbol{C})$-structure on each tangent space, ψ defines an almost complex structure I_ψ on M such that ψ is a $(2,0)$-form with respect to I_ψ.

LEMMA 2.3 (Grauert, Goto [**G**]). *Let ψ be an $SL(2, \boldsymbol{C})$-structure on an oriented 4-manifold M. If ψ is* d*-closed, then I_ψ is an integrable complex structure on M with trivial canonical bundle and ψ is a holomorphic volume form on M with respect to I_ψ.*

PROOF. Let η be any $(1,0)$-form on ψ. Then we have

$$\psi \wedge \eta = 0$$

since $\psi \in C^\infty(\wedge^{2,0}T^*M)$. Taking the exterior derivative and using $\mathrm{d}\psi = 0$, we obtain

$$\psi \wedge \mathrm{d}\eta = 0,$$

so that we have

$$\mathrm{d}C^\infty(\wedge^{1,0}T^*M) \subset C^\infty(\wedge^{2,0}T^*M) \oplus C^\infty(\wedge^{1,1}T^*M).$$

Hence it follows from Newlander-Nirenberg Theorem that I_ψ is an integrable complex structure on M. $\square$

Conversely, if X is a complex surface with trivial canonical bundle, a holomorphic volume form ψ on X defines a d-closed $SL(2, \boldsymbol{C})$-structure. Hence we have the following characterization of complex surfaces with trivial canonical bundle.

PROPOSITION 2.4. *Let M be an oriented* 4*-manifold. Then M admits a complex structure with trivial canonical bundle if and only if M admits a* d*-closed $SL(2, \boldsymbol{C})$-structure.*

Similarly d-closed $SL(m, \boldsymbol{C})$-structures characterize complex structures with trivial canonical bundle on an oriented $2m$-manifold. A d-closed $SL(m, \boldsymbol{C})$-structure will be often referred to as a *holomorphic volume form*.

DEFINITION 2.5. Let V be an oriented vector space of dimension 4. Then $(\psi_0, \kappa_0) \in (\wedge^2 V^* \otimes \boldsymbol{C}) \oplus \wedge^2 V^*$ is an $SU(2)$-*structure* on V if (ψ_0, κ_0) satisfies the following conditions:

(i) ψ_0 is an $SL(2, \boldsymbol{C})$-structure on V,
(ii) $\psi_0 \wedge \kappa_0 = 0$,
(iii) an inner product $g_{(\psi_0,\kappa_0)}$ on V defined by $g_{(\psi_0,\kappa_0)}(I_{\psi_0}\cdot, \cdot) = \kappa_0(\cdot,\cdot)$ is positive definite, and

(iv) $2\,\kappa_0^2 = \psi_0 \wedge \overline{\psi}_0$.

Conditions (ii) and (iii) imply that κ_0 is a (1,1)-form associated with the Hermitian inner product $g_{(\psi_0,\kappa_0)}$ on V. Let $\mathscr{A}_{SU(2)}(V)$ be the set of $SU(2)$-structures on the oriented vector space V. Then $\mathscr{A}_{SU(2)}(V)$ is an orbit space under the action of $GL_+(V)$, which is isomorphic to $GL_+(V)/SU(2)$.

For $(\psi_0, \kappa_0) \in \mathscr{A}_{SU(2)}(V)$, we have the orthogonal decomposition with respect to $g_{(\psi_0,\kappa_0)}$

$$\wedge^2 V^* = \wedge^2_+ \oplus \wedge^2_-,$$

where $\wedge^2_+$ and $\wedge^2_-$ are the set of self-dual and anti-self-dual 2-forms respectively. Then $\wedge^2_+$ is spanned by $\{\mathrm{Re}\,\psi_0, \mathrm{Im}\,\psi_0, \kappa_0\}$, where $\mathrm{Re}\,\psi_0$ and $\mathrm{Im}\,\psi_0$ are the real and imaginary part of ψ_0, and $\wedge^2_-$ coincides with the set of primitive real (1,1)-forms with respect to κ_0.

We also have the orthogonal decomposition

$$\begin{aligned}(\wedge^2 V^* \otimes \boldsymbol{C}) \oplus \wedge^2 V^* &\cong T_{(\psi_0,\kappa_0)}((\wedge^2 V^* \otimes \boldsymbol{C}) \oplus \wedge^2 V^*) \\ &= T_{(\psi_0,\kappa_0)}\mathscr{A}_{SU(2)}(V) \oplus T^{\perp}_{(\psi_0,\kappa_0)}\mathscr{A}_{SU(2)}(V),\end{aligned}$$

where $T^{\perp}_{(\psi_0,\kappa_0)}\mathscr{A}_{SU(2)}(V)$ is the orthogonal complement to $T_{(\psi_0,\kappa_0)}\mathscr{A}_{SU(2)}(V)$ with respect to $g_{(\psi_0,\kappa_0)}$. The next lemma is crucial in solving the partial differential equation in the proof of Theorem 1.1.

Lemma 2.6. *The tangent space of $\mathscr{A}_{SU(2)}$ at (ψ_0, κ_0) contains anti-self-dual subspaces*:

$$(\wedge^2_- \otimes \boldsymbol{C}) \oplus \wedge^2_- \subset T_{(\psi_0,\kappa_0)}\mathscr{A}_{SU(2)}(V).$$

Proof. The tangent space $T_{(\psi_0,\kappa_0)}\mathscr{A}_{SU(2)}(V)$ is given by

$$\{(a \cdot \psi_0, a \cdot \kappa_0) \in (\wedge^2 V^* \otimes \boldsymbol{C}) \oplus \wedge^2 V^* \mid a \in gl(V)\},$$

where $gl(V)$ acts on $\wedge^2 V^*$ via the differential representation. We have the decomposition

$$gl(V) = so(V) \oplus \mathrm{S}_0(V) \oplus \boldsymbol{R}\,\mathrm{id}_V,$$

where $\mathrm{S}_0(V)$ is the space of symmetric traceless endomorphisms of V with respect to $g_{(\psi_0,\kappa_0)}$. Then one can show easily that $\mathrm{S}_0(V) \cong \mathrm{Hom}(\wedge^2_+, \wedge^2_-)$, so that $\mathrm{S}_0(V)$

generates $(\wedge^2_- \otimes \boldsymbol{C}) \oplus \wedge^2_-$ because $\{\operatorname{Re}\psi_0, \operatorname{Im}\psi_0, \kappa_0\}$ spans $\wedge^2_+$. $\square$

DEFINITION 2.7. Let V be an oriented vector space of dimension 4. We define a neighborhood of $\mathscr{A}_{SU(2)}(V)$ in $(\wedge^2 V^* \otimes \boldsymbol{C}) \oplus \wedge^2 V^*$ by

$$\begin{aligned}\mathscr{T}_{SU(2)}(V) = \{(\psi_0 + \alpha, \kappa_0 + \beta) \mid (\psi_0, \kappa_0) \in \mathscr{A}_{SU(2)}(V),\ \text{and}\\ (\alpha, \beta) \in T^{\perp}_{(\psi_0,\kappa_0)}\mathscr{A}_{SU(2)}(V) \text{ with } |(\alpha,\beta)|_{g_{(\psi_0,\kappa_0)}} < \rho\},\end{aligned}$$

where ρ is a positive constant and $|(\alpha,\beta)|_{g_{(\psi_0,\kappa_0)}} = |\alpha|_{g_{(\psi_0,\kappa_0)}} + |\beta|_{g_{(\psi_0,\kappa_0)}}$.

LEMMA 2.8. *There exists a positive constant ρ_* such that if $\rho < \rho_*$ then any $(\psi', \kappa') \in \mathscr{T}_{SU(2)}(V)$ can be uniquely written as $(\psi_0 + \alpha, \kappa_0 + \beta)$, where $(\psi_0, \kappa_0) \in \mathscr{A}_{SU(2)}(V)$, $(\alpha, \beta) \in T^{\perp}_{(\psi_0,\kappa_0)}\mathscr{A}_{SU(2)}(V)$.*

Lemma 2.8 implies that for $\rho < \rho_*$ the projection $\Theta : \mathscr{T}_{SU(2)}(V) \to \mathscr{A}_{SU(2)}(V)$ is well-defined.

DEFINITION 2.9. Let M be an oriented 4-manifold. Then $(\psi, \kappa) \in C^\infty(\wedge^2 T^*M \otimes \boldsymbol{C}) \oplus C^\infty(\wedge^2 T^*M)$ is an $SU(2)$-*structure* on M if the restriction $(\psi|_x, \kappa|_x)$ is an $SU(2)$-structure on T_xM for all $x \in M$.

Define $\mathscr{A}_{SU(2)}(M)$ to be the fibre bundle whose fibre over $x \in M$ is $\mathscr{A}_{SU(2)}(T_xM)$. Then an $SU(2)$-structure can be regarded as a smooth section of $\mathscr{A}_{SU(2)}(M)$.

If ψ and κ are both d-closed, then $X = (M, I_\psi, \kappa)$ is a Kähler surface with trivial canonical bundle. Moreover, the Ricci curvature of the Kähler metric g vanishes by condition (iv) of Definition 2.5.

DEFINITION 2.10. Let M be an oriented 4-manifold. Choose $\rho < \rho_*$ so that the projection Θ is well-defined. We define $\mathscr{T}_{SU(2)}(M)$ to be the fibre bundle whose fibre over $x \in M$ is $\mathscr{T}_{SU(2)}(T_xM)$, and denote by Θ the projection from $\mathscr{T}_{SU(2)}(M)$ to $\mathscr{A}_{SU(2)}(M)$.

Let (ψ, κ) be an $SU(2)$-structure on M. If $(\alpha, \beta) \in C^\infty(\wedge^2 T^*M \otimes \boldsymbol{C}) \oplus C^\infty(\wedge^2 T^*M)$ satisfies $\|(\alpha,\beta)\|_{C^0} < \rho$, then $(\psi + \alpha, \kappa + \beta) \in C^\infty(\mathscr{T}_{SU(2)}(M))$ and we have the Taylor expansion

$$\Theta(\psi + \alpha, \kappa + \beta) = (\psi, \kappa) + \pi_{(\psi,\kappa)}(\alpha, \beta) + F_{(\psi,\kappa)}(\alpha, \beta),$$

where $\pi_{(\psi,\kappa)} : \wedge^2 T^*M \otimes \boldsymbol{C} \oplus \wedge^2 T^*M \to T_{(\psi,\kappa)}\mathscr{A}_{SU(2)}(TM)$ is the orthogonal projection and $F_{(\psi,\kappa)}$ is the higher order term with respect to (α, β).

Remark 2.11. We can generalize the notion of $SL(2,\boldsymbol{C})$- and $SU(2)$-structures to higher dimensions and define the projection $\Theta : \mathscr{T}_{SU(m)}(M) \to \mathscr{A}_{SU(m)}(M)$ on an oriented $2m$-manifold M, where $\mathscr{A}_{SU(m)}(M)$ is the set of $SU(m)$-structures on M and $\mathscr{T}_{SU(m)}(M)$ is a neighborhood of $\mathscr{A}_{SU(m)}(M)$.

Lemma 2.12. *Let ρ be a constant as in Definition 2.7. There exist positive constants C_1 and C_2 such that for any $SU(2)$-structure (ψ,κ) and for any $(\alpha,\beta),(\alpha',\beta') \in C^\infty(\wedge^2 T^*M \otimes \boldsymbol{C}) \oplus C^\infty(\wedge^2 T^*M)$ with $\|(\alpha,\beta)\|_{C^0}, \|(\alpha',\beta')\|_{C^0} < \rho/2$, we have the following point-wise estimates on $F = F_{(\psi,\kappa)}$ with respect to $g_{(\psi,\kappa)}$:*

$$|F(\alpha,\beta) - F(\alpha',\beta')| \le C_1 |(\alpha-\alpha',\beta-\beta')|(|(\alpha,\beta)| + |(\alpha',\beta')|), \tag{2.2}$$

$$\begin{aligned}
&|\nabla F(\alpha,\beta) - \nabla F(\alpha',\beta')| \\
&\quad \le C_2\{(|\mathrm{d}\psi| + |\mathrm{d}\kappa|)|(\alpha-\alpha',\beta-\beta')|(|(\alpha,\beta)| + |(\alpha',\beta')|) \\
&\qquad + |(\nabla(\alpha-\alpha'),\nabla(\beta-\beta'))|(|(\alpha,\beta)| + |(\alpha',\beta')|) \\
&\qquad + |(\alpha-\alpha',\beta-\beta')|(|(\nabla\alpha,\nabla\beta)| + |(\nabla\alpha',\nabla\beta')|)\},
\end{aligned} \tag{2.3}$$

where ∇ is the Levi-Civita connection of $g_{(\psi,\kappa)}$.

The proof is essencially the same as in [**J**], Proposition 10.5.9, so we will omit it.

3. The construction of a compact 4-manifold M_T with an approximating holomorphic volume form ψ_T.

In this section we construct a compact manifold M_T from $X \setminus D$ and $X' \setminus D'$ under the assumptions of Theorem 1.1. Then we define an $SU(2)$-structure (ψ_T,κ_T) on M_T and obtain some estimates on (ψ_T,κ_T). Since it is possible to construct M_T and (ψ_T,κ_T) in arbitrary dimension, we leave the dimension m of X and X' undetermined for the most part of this section.

3.1. Compact complex manifolds with an anticanonical divisor.

First we suppose that X is a compact complex manifold of dimension m, and D is a smooth irreducible anticanonical divisor on X.

Let $\{U_\alpha\}$ be an open covering of X and define $V_\alpha = U_\alpha \cap D$, so that $\{V_\alpha\}$ is an open covering of D. Then there exist collections $z_\alpha = (z_\alpha^1, \dots, z_\alpha^{m-1})$ of holomorphic functions on U_α such that (z_α, w_α) are local coordinates and $V_\alpha = \{w_\alpha = 0\}$. The coordinate tranformation of X is given by

$$\begin{aligned}
z_\alpha &= \phi_{\alpha\beta}(z_\beta, w_\beta), \\
w_\alpha &= f_{\alpha\beta}(z_\beta, w_\beta) w_\beta,
\end{aligned}$$

where $\phi_{\alpha\beta}, f_{\alpha\beta}$ are nonvanishing holomorphic functions on $U_\alpha \cap U_\beta$.

The canonical bundle K_X is given by transition functions

$$h_{\alpha\beta}(z_\beta, w_\beta) = \frac{\mathrm{d}w_\beta \wedge \mathrm{d}z_\beta^1 \wedge \cdots \wedge \mathrm{d}z_\beta^{m-1}}{\mathrm{d}w_\alpha \wedge \mathrm{d}z_\alpha^1 \wedge \cdots \wedge \mathrm{d}z_\alpha^{m-1}} \tag{3.1}$$

on $U_\alpha \cap U_\beta$, and the line bundle $[D]$ is given by transition functions $f_{\alpha\beta}(z_\beta, w_\beta) = w_\alpha / w_\beta$ on $U_\alpha \cap U_\beta$. Since D is an anticanonical divisor on X, we can choose the local coordinates (z_α, w_α) so that

$$f_{\alpha\beta}(z_\beta, w_\beta) h_{\alpha\beta}(z_\beta, w_\beta) = 1. \tag{3.2}$$

Therefore the local holomorphic volume forms

$$\Omega_\alpha = \frac{\mathrm{d}w_\alpha}{w_\alpha} \wedge \mathrm{d}z_\alpha^1 \wedge \cdots \wedge \mathrm{d}z_\alpha^{m-1}$$

together yield a holomorphic volume form Ω on $X \setminus D$.

3.2. The holomorphic normal bundle.

Next we consider the holomorphic normal bundle $N = N_{D/X}$ to D in X. Let $\pi : N \to D$ be the projection. We identify the zero section of N with D. Let x_α be the restriction of z_α to $V_\alpha = U_\alpha \cap D$. Then $\{(V_\alpha, x_\alpha)\}$ is a local coordinate system on D. The coordinate transformation of the normal bundle $N = [D]|_D$ is given by

$$x_\alpha = \psi_{\alpha\beta}(x_\beta),$$
$$y_\alpha = g_{\alpha\beta}(x_\beta) y_\beta,$$

where we set

$$\psi_{\alpha\beta}(x_\beta) = \phi_{\alpha\beta}(x_\beta, 0),$$
$$g_{\alpha\beta}(x_\beta) = f_{\alpha\beta}(x_\beta, 0),$$

and (x_α, y_α) are local coordinates of N on $\pi^{-1}(V_\alpha) \simeq V_\alpha \times \boldsymbol{C}$. Thus restricting equation (3.1) to $V_\alpha \cap V_\beta$, we have

$$h_{\alpha\beta}(x_\beta, 0) = g_{\alpha\beta}(x_\beta)^{-1} \frac{\mathrm{d}x_\beta^1 \wedge \cdots \wedge \mathrm{d}x_\beta^{m-1}}{\mathrm{d}x_\alpha^1 \wedge \cdots \wedge \mathrm{d}x_\alpha^{m-1}} \tag{3.3}$$

on $V_\alpha \cap V_\beta$. Restricting (3.2) to V_α and putting (3.3), we have

$$\frac{\mathrm{d}x_\beta^1 \wedge \cdots \wedge \mathrm{d}x_\beta^{m-1}}{\mathrm{d}x_\alpha^1 \wedge \cdots \wedge \mathrm{d}x_\alpha^{m-1}} = 1$$

on $V_\alpha \cap V_\beta$. Therefore the local holomorphic volume forms

$$\Omega_{D,\alpha} = \mathrm{d}x_\alpha^1 \wedge \cdots \wedge \mathrm{d}x_\alpha^{m-1}$$

on V_α together yield a holomorphic volume form Ω_D on D, so that the canonical bundle K_D of D is trivial, which also follows from the adjunction formula, $K_D = (K_X \otimes [D])|_D \cong \mathscr{O}_D$.

The holomorphic volume form Ω_D obtained from Ω regarded as a meromorphic volume form on X with a single pole along D is called the *Poincaré residue* of Ω, which is independent of the choice of local coordinates representing Ω (see [**GH**], pp. 147–148). We note that if Ω and Ω' are two meromorphic volume forms on X with a single pole along D, then they differ by a nonzero multiplicative constant.

Let $\|\cdot\|$ be a Hermitian metric on the normal bundle N and define a cylindrical parameter t on $N \setminus D$ by

$$t(s) = -t_0 - \log \|s\|^2 \quad \text{for } s \in N \setminus D, \tag{3.4}$$

where t_0 is a constant. The following result is immediate from the tubular neighborhood theorem.

Proposition 3.1. *There exists a constant t_0 and a diffeomorphism Φ from a neighborhood V of the zero section of N containing $t^{-1}((0,\infty))$ to a tubular neighborhood U of D in X such that Φ can be locally represented as*

$$\begin{aligned} z_\alpha &= x_\alpha + O(|y_\alpha|^2) = x_\alpha + O(e^{-t}), \\ w_\alpha &= y_\alpha + O(|y_\alpha|^2) = y_\alpha + O(e^{-t}) \end{aligned} \tag{3.5}$$

by shrinking $\{U_\alpha\}$ if necessary.

Proposition 3.1 implies that the complex structure on V in $N_{D/X}$ approaches the complex structure on U in X exponentially as $t \to \infty$.

We consider local holomorphic volume forms $\Omega_{0,\alpha}$ on $\pi^{-1}(V_\alpha) \setminus D$ defined by

$$\Omega_{0,\alpha} = \frac{\mathrm{d}y_\alpha}{y_\alpha} \wedge \pi^*\Omega_D.$$

Since on $\pi^{-1}(V_\alpha \cap V_\beta) \setminus D$ we have

$$\Omega_{0,\alpha} - \Omega_{0,\beta} = \mathrm{d} \log g_{\alpha\beta} \wedge \pi^* \Omega_D = \partial \log g_{\alpha\beta} \wedge \pi^* \Omega_D = 0,$$

$\Omega_{0,\alpha}$ together yield a holomorphic volume form Ω_0 on $N \setminus D$.

Let ω_D be a Hermitian form on D normalized so that

$$\omega_D^{m-1} = c_{m-1} \Omega_D \wedge \overline{\Omega}_D, \tag{3.6}$$

where c_k are constants defined by $c_k = (\sqrt{-1})^{k^2} 2^{-k} k!$. We define ω_0 on $N \setminus D$ by

$$\omega_0 = \frac{\sqrt{-1}}{2} \partial t \wedge \overline{\partial} t + \pi^* \omega_D. \tag{3.7}$$

LEMMA 3.2. *The pair (Ω_0, ω_0) of the complex m-form and real 2-form on $N \setminus D$ satisfies*

$$\omega_0^m = c_m \Omega_0 \wedge \overline{\Omega}_0, \tag{3.8}$$

and the metric g_0 associated with (Ω_0, ω_0) is cylindrical and positive definite. In particular, if $m = 2$, then (Ω_0, ω_0) is an $SU(2)$-structure on $N \setminus D$.

PROOF. Equation (3.8) follows easily from (3.6) and (3.7).

Next we will have a local expression for the metric g_0 associated with (Ω_0, ω_0). The Hermitian metric $\|\cdot\|$ on N is locally represented as

$$\|(x_\alpha, y_\alpha)\|^2 = e^{\phi_\alpha(x_\alpha)} |y_\alpha|^2 \quad \text{for } (x_\alpha, y_\alpha) \in \pi^{-1}(V_\alpha),$$

where ϕ_α are real-valued functions on V_α, satisfying $\phi_\alpha(x_\alpha) = \phi_\beta(x_\beta) - \log |g_{\alpha\beta}(x_\beta)|^2$ on $V_\alpha \cap V_\beta$. Then we have

$$t(x_\alpha, y_\alpha) = -\phi_\alpha(x_\alpha) - \log |y_\alpha|^2 \quad \text{for } (x_\alpha, y_\alpha) \in \pi^{-1}(V_\alpha).$$

If we set

$$r_\alpha + \sqrt{-1} \theta_\alpha = -\log y_\alpha,$$

then we can check easily that $(x_\alpha, t, \theta_\alpha)$ are local coordinates of $N \setminus D$ on

$\pi^{-1}(V_\alpha)\setminus D$. Thus a direct computation shows that $g_0(\cdot,\cdot)=\omega_0(\cdot, I_{\Omega_0}\cdot)$ is expressed as

$$\begin{aligned} g_0 &= \left(\mathrm{d}r_\alpha - \frac{1}{2}\,\mathrm{d}\phi_\alpha\right)^2 + \left(\mathrm{d}\theta_\alpha + \frac{1}{2}\,\mathrm{d}^c\phi_\alpha\right)^2 + \pi^* g_D \\ &= \frac{1}{4}\,\mathrm{d}t^2 + \left(\mathrm{d}\theta_\alpha + \frac{1}{2}\,\mathrm{d}^c\phi_\alpha\right)^2 + \pi^* g_D, \end{aligned} \tag{3.9}$$

where g_D is the metric associated with (Ω_D, ω_D), and $\mathrm{d}^c = \sqrt{-1}(\partial - \overline{\partial})$. (In effect, g_0 is given by $(\partial t\otimes\overline{\partial}t + \overline{\partial}t\otimes\partial t)/2 + \pi^* g_D$.) □

Define an S^1-bundle $p : S \to D$ in terms of coordinate transformation by

$$\begin{aligned} x_\alpha &= \psi_{\alpha\beta}(x_\beta), \\ \theta_\alpha &= \theta_\beta - \arg g_{\alpha\beta}(x_\beta), \end{aligned}$$

where each $(x_\alpha, \theta_\alpha)$, $\theta_\alpha \in \boldsymbol{R}/2\pi\boldsymbol{Z}$ is a local coordinate on $p^{-1}(V_\alpha)\simeq V_\alpha\times S^1$. Then $N\setminus D$ is the Riemannian product $(S, g_S)\times\left(\boldsymbol{R}, \frac{1}{4}\,\mathrm{d}t^2\right)$, where $g_S = (\mathrm{d}\theta_\alpha + \frac{1}{2}\,\mathrm{d}^c\phi_\alpha)^2 + p^* g_D$ on $p^{-1}(V_\alpha)$.

Lemma 3.3. *For any integer $k\geq 0$, we have*

$$\begin{aligned} \left|\nabla^k_{g_0}(\mathrm{d}z_\alpha - \mathrm{d}x_\alpha)\right|_{g_0} &= O(e^{-t}), \\ \left|\nabla^k_{g_0}\left(\frac{\mathrm{d}w_\alpha}{w_\alpha} - \frac{\mathrm{d}y_\alpha}{y_\alpha}\right)\right|_{g_0} &= O(e^{-t/2}). \end{aligned} \tag{3.10}$$

Proof. We obtain $|\mathrm{d}t|_{g_0} = 2$, $|\mathrm{d}\theta_\alpha|_{g_0} = O(1)$ and $|\mathrm{d}x_\alpha|_{g_0} = O(1)$ from the explicit representation (3.9) of g_0. Note that the remainders $O(e^{-t})$ in the Taylor expansions (3.5) are of the form

$$A(x_\alpha, \overline{x}_\alpha, y_\alpha, \overline{y}_\alpha)\, y_\alpha^2 + B(x_\alpha, \overline{x}_\alpha, y_\alpha, \overline{y}_\alpha)\, y_\alpha\overline{y}_\alpha + C(x_\alpha, \overline{x}_\alpha, y_\alpha, \overline{y}_\alpha)\, \overline{y}_\alpha^2, \tag{3.11}$$

where A, B and C are C^∞ functions. If we rewrite equation (3.11) as $R(x_\alpha, \overline{x}_\alpha, t, \theta_\alpha)\, e^{-t}$, then R is a C^∞ function for $t\neq\infty$ with bounded derivatives, so that $R\, e^{-t}$ extends smoothly to $t=\infty$. Thus differentiating (3.5) gives

$$\left|\mathrm{d}z_\alpha - \mathrm{d}x_\alpha\right|_{g_0} = \left|e^{-t}(\mathrm{d}R + R\,\mathrm{d}t)\right|_{g_0} = O(e^{-t}).$$

On the other hand, it follows from (3.5) and (3.11) that

$$\begin{aligned}\frac{\mathrm{d}w_\alpha}{w_\alpha} - \frac{\mathrm{d}y_\alpha}{y_\alpha} &= \mathrm{d}\log\frac{w_\alpha}{y_\alpha} \\ &= \mathrm{d}\log\left(1 + A\,y_\alpha + B\,\overline{y}_\alpha + C\,\frac{\overline{y}_\alpha^2}{y_\alpha}\right) \\ &= \mathrm{d}\log(1 + R'(x_\alpha, \overline{x}_\alpha, t, \theta_\alpha)\,e^{-t/2}),\end{aligned}$$

where R' is a C^∞ function for $t \neq \infty$ with bounded derivatives, $R'e^{-t/2}$ extending smoothly to $t = \infty$. Consequently we have

$$\left|\frac{\mathrm{d}w_\alpha}{w_\alpha} - \frac{\mathrm{d}y_\alpha}{y_\alpha}\right|_{g_0} = \left|\frac{e^{-t/2}\big(\mathrm{d}R' - \frac{1}{2}R'\mathrm{d}t\big)}{1 + R'e^{-t/2}}\right|_{g_0} = O(e^{-t/2}).$$

Thus we establish (3.10) for $k = 0$.

With respect to the coordinate system $\{(x_\alpha, t, \theta_\alpha)\}$, the components of g_0 are independent of t, and so are the components of ∇_{g_0}. This proves (3.10) for $k \geq 1$. □

3.3. An approximating holomorphic volume form and a Hermitian form on $X \setminus D$.

We extend $(\Phi^{-1})^* t$ to a smooth function on $X \setminus D$ which is nonpositive on $X \setminus U$. By abuse of notation we denote the function again by t. We also consider (Ω_0, ω_0) as an $SU(m)$-structure defined on $U \setminus D \subset X \setminus D$ via the diffeomorphism Φ.

Let $\rho : \boldsymbol{R} \to [0,1]$ denote the cut-off function

$$\rho(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ 0 & \text{if } x \geq 1, \end{cases}$$

and define $\rho_T : \boldsymbol{R} \to [0,1]$ by

$$\rho_T(x) = \rho(x - T + 1) = \begin{cases} 1 & \text{if } x \leq T - 1, \\ 0 & \text{if } x \geq T. \end{cases}$$

Proposition 3.4. *There exist a complex 1-form ξ on the region $\{t > 0\} \subset X \setminus D$ and positive constants $C'_{1,k-1}$ for $k \geq 0$ such that*

$$\Omega - \Omega_0 = \mathrm{d}\xi, \quad |\nabla_{g_0}^k \xi|_{g_0} \leq C'_{1,k-1} e^{-t/2}.$$

For simplicity, if a differential form α satisfies $|\alpha|_{g_0} \leq Ce^{-t/2}$ for a constant C, we will often write as $\alpha = O(e^{-t/2})$,

Proof. Let $\{f_t\}$ be the 1-parameter family of diffeomorphisms generated by the vector field $\{\partial/\partial t\}$ on $\{t > 0\}$. Then for $(p,s) \in \{t > 0\} \simeq S \times (0,\infty)$, $f_t(p,s) = (p, s+t)$. Let $\alpha = \Omega - \Omega_0$. Then α is of order $O(e^{-t/2})$ with all derivatives by Lemma 3.3. Thus we obtain

$$\begin{aligned}\alpha(p,t) &= \int_\infty^t \frac{\mathrm{d}}{\mathrm{d}s}(f_{s-t}^*\alpha)(p,t)\mathrm{d}s \\ &= \int_\infty^t (f_{s-t}^*\mathscr{L}_{\partial/\partial s}\alpha)(p,t)\mathrm{d}s \\ &= \mathrm{d}\int_\infty^t (f_{s-t}^*\iota_{\partial/\partial s}\alpha)(p,t)\mathrm{d}s,\end{aligned}$$

where the last equality holds because both $(f_{s-t}^*\mathscr{L}_{\partial/\partial s}\alpha)(p,t)$ and $(f_{s-t}^*\iota_{\partial/\partial s}\alpha)(p,t)$ are continuous forms of order $O(e^{-s/2})$ and integrable on $S \times (0,\infty)$. Letting ξ be the integral of the right-hand side, we have $\Omega - \Omega_0 = \mathrm{d}\xi$. Moreover $\nabla_{g_0}^k \xi = O(e^{-t/2})$ for $k \geq 0$ since $(f_{s-t}^*\iota_{\partial/\partial s}\alpha)(p,t)$ is of order $O(e^{-s/2})$ with all derivatives. □

We define a d-closed complex m-form Ω_T on $X \setminus D$ by

$$\Omega_T = \begin{cases} \Omega - \mathrm{d}(1-\rho_{T-1})\xi & \text{on } \{t \leq T-1\}, \\ \Omega_0 + \mathrm{d}\rho_{T-1}\xi & \text{on } \{t \geq T-2\}. \end{cases}$$

On $\{T-2 < t < T-1\}$ we have

$$\Omega_T - \Omega_0 = \mathrm{d}\rho_{T-1}\xi = O(e^{-T/2}), \tag{3.12}$$

so that Ω_T is an approximating holomorphic volume form for large T.

Next we will define a Hermitian form ω on $X \setminus D$ such that the associated metric g is a Hermitian metric asymptotic to the cylindrical metric g_0. Let ω_1 be a Hermitian form on $\{t \leq 1\}$ normalized so that

$$\omega_1^m = c_m \Omega \wedge \overline{\Omega},$$

and g_1 the metric associated with (Ω, ω_1). Let $\pi_{I_\Omega}^{1,1}\omega_0$ be the $(1,1)$-part of ω_0 with respect to the complex structure I_Ω defined by Ω, i.e., the standard complex structure on $X \setminus D$. Then we see that $\pi^{1,1}_{I_\Omega}$ is a smooth operator with $|\omega_0 - \pi^{1,1}_{I_\Omega}\omega_0|_{g_0} = O(e^{-t/2})$. There exists a positive function λ_0 on $\{t \geq 0\}$ such that

$$(\lambda_0 \pi^{1,1}_{I_\Omega}\omega_0)^m = c_m \Omega \wedge \overline{\Omega}$$

and $\lambda_0 = 1 + O(e^{-t/2})$. We define a 2-form on $\{t \geq 0\}$ by

$$\omega_2 = \lambda_0 \pi^{1,1}_{I_\Omega}\omega_0.$$

Since $\omega_2 - \omega_0 = O(e^{-t/2})$, for sufficiently large t_0 in (3.4) the metric g_2 associated with (Ω, ω_2) becomes positive definite, so that ω_2 becomes a Hermitian form on $\{t \geq 0\}$. Then we glue g_1 and g_2 together along $\{0 \leq t \leq 1\}$ to obtain a metric $\widehat{g}$ on $X \setminus D$, which is Hermitian with respect to I_Ω:

$$\widehat{g} = \rho_1 g_1 + (1 - \rho_1) g_2. \tag{3.13}$$

The Hermitian form $\widehat{\omega}$ associated with $(I_\Omega, \widehat{g})$ satisfies

$$(\widehat{\lambda}\widehat{\omega})^m = c_m \Omega \wedge \overline{\Omega}$$

for some positive function $\widehat{\lambda}$. Note that $\widehat{\lambda} = 1$ on $(X \setminus D) \setminus \{0 < t < 1\}$ by construction. Then the Hermitian metric g on $X \setminus D$ associated with $\omega = \widehat{\lambda}\widehat{\omega}$ is asymptotic to the cylindrical metric g_0.

Finally we glue together $\omega = \widehat{\lambda}\widehat{\omega}$ and ω_0 along $\{T - 2 \leq t \leq T - 1\}$ to define an approximating Hermitian form ω_T:

$$\omega_T = \rho_{T-1}\omega + (1 - \rho_{T-1})\omega_0. \tag{3.14}$$

On $\{t \leq T - 2, T - 1 \leq t\}$ ω_T is a Hermitian form with respect to I_{Ω_T}, and

$$\omega_T^m = c_m \Omega_T \wedge \overline{\Omega}_T. \tag{3.15}$$

On the other hand, for $T - 2 < t < T - 1$,

$$\omega_T - \omega_0 = \rho_{T-1}(\omega - \omega_0) = O(e^{-T/2}), \tag{3.16}$$

so that ω_T is an approximating Hermitian form. Thus by (3.12) and (3.16), (Ω_T, ω_T) is a smooth section of $\mathscr{T}_{SU(m)}(X \setminus D)$ for sufficiently large T.

3.4. The construction of M_T and (ψ_T, κ_T).

Let X' be another compact complex manifold of dimension m with a smooth irreducible anticanonical divisor D', and N' the holomorphic normal bundle $N_{D'/X'}$ of D' in X'. We suppose there exists an isomorphism $f : D \to D'$, and N, N' are dual line bundles via f, i.e., $N \otimes f^*N' \cong \mathscr{O}_D$.

Let $V'_\alpha = f(V_\alpha)$ and $x'_\alpha = (f^{-1})^* x_\alpha$. Then $\{V'_\alpha\}$ is an open covering of D' and $\{(V'_\alpha, x'_\alpha)\}$ is a local coordinate system on D'. We choose a covering $\{U'_\alpha\}$ of X' so that $V'_\alpha = U'_\alpha \cap D'$. Since the holomorphic normal bundle $N' = [D']|_{D'}$ is isomorphic to $(f^{-1})^* N^{-1}$, one can show that by choosing sufficiently small U'_α, there exist local coordinates (z'_α, w'_α) of X' on U'_α such that w'_α are local defining functions of D' on V'_α, and that (z'_α, w'_α) satisfy

$$
\begin{aligned}
z'_\alpha|_{V'_\alpha} &= x'_\alpha, \\
w'_\alpha / w'_\beta|_{V'_\alpha \cap V'_\beta} &= g'_{\alpha\beta}(x'_\beta),
\end{aligned}
\tag{3.17}
$$

where $g'_{\alpha\beta} = (f^{-1})^* g^{-1}_{\alpha\beta}$.

Since D' is an anticanonical divisor on X', there exist holomorphic functions f'_α on U'_α such that the local holomorphic volume forms

$$
\Omega'_\alpha = -f'_\alpha(z'_\alpha, w'_\alpha)^{-1} \frac{\mathrm{d}w'_\alpha}{w'_\alpha} \wedge \mathrm{d}{z'_\alpha}^1 \wedge \cdots \wedge \mathrm{d}{z'_\alpha}^{m-1}
$$

on $U'_\alpha \setminus D'$ together yield a holomorphic volume form Ω' on $X' \setminus D'$. The local holomorphic volume forms

$$
{f'}^{-1}_\alpha \mathrm{d}{z'_\alpha}^1 \wedge \cdots \wedge \mathrm{d}{z'_\alpha}^{m-1}|_{V'_\alpha} = {g'}^{-1}_\alpha \mathrm{d}{x'_\alpha}^1 \wedge \cdots \wedge \mathrm{d}{x'_\alpha}^{m-1},
$$

where $g'_\alpha(x'_\alpha) = f'_\alpha(x'_\alpha, 0)$, together yield a holomorphic volume form on D', which must be a constant multiple of $\Omega_{D'} = (f^{-1})^* \Omega_D = \mathrm{d}{x'_\alpha}^1 \wedge \cdots \wedge \mathrm{d}{x'_\alpha}^{m-1}$. We multiply all f'_α by this constant so that $g'_\alpha = 1$. Since ${f'}^{-1}_\alpha|_{V'_\alpha} = 1$, $({f'}^{-1}_\alpha - 1)/w'_\alpha$ is a nonvanishing holomorphic function on U'_α. We may assume that each U'_α is convex. If we redefine w'_α to be

$$
w'_\alpha \exp\left(\int_0^{w'_\alpha} \frac{{f'}^{-1}_\alpha - 1}{w'_\alpha} \mathrm{d}w'_\alpha \right),
$$

then (z'_α, w'_α) still defines a local coordinate on U'_α satisfying (3.17), and w'_α is a local defining function of D'. Moreover, $\Omega'_\alpha = \Omega'|_{V'_\alpha}$ is expressed as

$$\Omega'_\alpha = -\frac{\mathrm{d}w'_\alpha}{w'_\alpha} \wedge \mathrm{d}{z'_\alpha}^1 \wedge \cdots \wedge \mathrm{d}{z'_\alpha}^{m-1}.$$

We consider the holomorphic normal bundle $\pi' : N' = N_{D'/X'} \to D'$. Let y'_α be fibre coordinates of ${\pi'}^{-1}(V'_\alpha) \simeq V'_\alpha \times \boldsymbol{C}$, satisfying

$$y'_\alpha = g'_{\alpha\beta}(x'_\beta) y'_\beta. \tag{3.18}$$

Let (x_α, y_α) be local coordinates of $\pi^{-1}(V_\alpha) \simeq V_\alpha \times \boldsymbol{C}$ as in Section 3.2. We define an isomorphism $h_T : N \setminus D \to N' \setminus D'$ locally by

$$\begin{array}{ccl} \pi^{-1}(V_\alpha) & \to & {\pi'}^{-1}(V'_\alpha) \\ \Psi & & \Psi \\ (x_\alpha, y_\alpha) & \mapsto & (x'_\alpha, y'_\alpha) = (x_\alpha, e^{-T}/y_\alpha). \end{array} \tag{3.19}$$

This mapping is well-defined since $N \otimes f^* N' \cong \mathscr{O}_D$. We introduce a cylindrical parameter t' on $N' \setminus D'$ by

$$t' = -t \circ h_0^{-1}.$$

Then we can define a pair (Ω'_0, ω'_0) of a holomorphic volume form and a Hermitian form on $N' \setminus D'$ by

$$\begin{aligned} \Omega'_0 &= -\frac{\mathrm{d}y'_\alpha}{y'_\alpha} \wedge {\pi'}^* \Omega_{D'}, \\ \omega'_0 &= \frac{\sqrt{-1}}{2} \partial t' \wedge \overline{\partial} t' + {\pi'}^* \omega_{D'}, \end{aligned}$$

which satisfies

$${\omega'_0}^m = c_m \Omega'_0 \wedge \overline{\Omega'}_0$$

and induces a cylindrical metric g'_0, where $\omega_{D'} = (f^{-1})^* \omega_D$. Then again by Proposition 3.1, there exists a diffeomorphism Φ' from a neighborhood V' of the zero section of N' containing ${t'}^{-1}((0, \infty))$ to a tubular neighborhood U' of D' in X' such that

$$\begin{aligned} z'_\alpha &= x'_\alpha + O(|y'_\alpha|^2) = x'_\alpha + O(e^{-t'}), \\ w'_\alpha &= y'_\alpha + O(|y'_\alpha|^2) = y'_\alpha + O(e^{-t'}). \end{aligned}$$

Parallel to the argument in Section 3.3, via the identification of V' with U' by Φ', we can also define (Ω'_T, ω'_T) such that

$$\begin{aligned}
\Omega'_T &= \begin{cases} \Omega' & \text{if } t' \le T-2, \\ \Omega'_0 & \text{if } t' \ge T-1, \end{cases} \\
\omega'_T &= \omega'_0 \quad \text{if } t' \ge T-1, \\
{\omega'_T}^m &= c_m \Omega'_T \wedge \overline{\Omega'}_T \quad \text{if } t' \le T-2 \text{ or } t' \ge T-1
\end{aligned} \tag{3.20}$$

and

$$\Omega'_T - \Omega'_0 = O(e^{-T/2}), \quad \omega'_T - \omega'_0 = O(e^{-T/2}) \quad \text{if } T-2 < t' < T-1.$$

We define a subset X_T of $X \setminus D$ and a subset X'_T of $X' \setminus D'$ by

$$X_T = \{t < T+1\} \subset X \setminus D, \quad X'_T = \{t' < T+1\} \subset X' \setminus D'.$$

Then we glue X_T and X'_T together along $\{T-1 < t < T+1\} \subset X \setminus D$ and $\{T-1 < t' < T+1\} \subset X' \setminus D'$ by the mapping h_T, and define a compact manifold M_T. Since we have

$$h_T^* \Omega'_0 = \Omega_0, \quad h_T^* \omega'_0 = \omega_0,$$

we can also glue (Ω_T, ω_T) and (Ω'_T, ω'_T) together and define $(\widetilde{\Omega}_T, \widetilde{\omega}_T)$ on M_T, with $\mathrm{d}\widetilde{\Omega}_T = 0$. We also define a cylindrical parameter τ on M_T with centre $t = t' = T$ by

$$\tau = \begin{cases} t - T & \text{on } X_T, \\ T - t' & \text{on } X'_T. \end{cases}$$

There exists a constant T_* depending on ρ such that for all $T > T_*$ we have $(\widetilde{\Omega}_T, \widetilde{\omega}_T) \in C^\infty(\mathscr{T}_{SU(m)}(M_T))$. Hence for T with $T > T_*$, we can define an $SU(m)$-structure (ψ_T, κ_T) on M_T by

$$(\psi_T, \kappa_T) = \Theta(\widetilde{\Omega}_T, \widetilde{\omega}_T).$$

Let $\phi_T = \widetilde{\Omega}_T - \psi_T$. Then $\mathrm{d}\psi_T + \mathrm{d}\phi_T = 0$. It follows from (3.15) and (3.20) that

$$(\psi_T, \kappa_T) = (\widetilde{\Omega}_T, \widetilde{\omega}_T), \quad \mathrm{d}\psi_T = 0, \quad \phi_T = 0 \quad \text{if } |\tau| \le 1 \text{ or } |\tau| \ge 2.$$

REMARK 3.5. For $T \in (0, \infty)$ and $\theta \in \boldsymbol{R}/2\pi\boldsymbol{Z}$, we can also use the gluing map $h_{T,\theta}$ locally defined by

$$\begin{array}{ccc} \pi^{-1}(V_\alpha) & \to & \pi'^{-1}(V'_\alpha) \\ \cup & & \cup \\ (x_\alpha, y_\alpha) & \mapsto & (x'_\alpha, y'_\alpha) = (x_\alpha, e^{-T-\sqrt{-1}\theta}/y_\alpha). \end{array} \tag{3.21}$$

instead of h_T, and construct a compact 4-manifold $M_{T,\theta} = X_T \cup_{h_{T,\theta}} X'_T$. Thus we can parametrize a family $\{M_{T,\theta}\}$ of compact 4-manifolds by a complex variable $\zeta = e^{-T-\sqrt{-1}\theta}$.

3.5. The main estimates.

Now we will derive the following estimates.

PROPOSITION 3.6. *Let ρ be a constant as in Definition 2.7. Then for all T with $T > T_* = T_*(\rho)$ there exist positive constants C_3, C_4, and C_5 independent of T such that with respect to the metric g_T on M_T associated with (ψ_T, κ_T), we have the following estimates*:

$$\|\phi_T\|_{L^p} \le C_3\, e^{-T/2}, \tag{3.22}$$
$$\|\mathrm{d}\phi_T\|_{L^p} \le C_4\, e^{-T/2}, \tag{3.23}$$
$$\|\mathrm{d}\kappa_T\|_{C^0} \le C_5. \tag{3.24}$$

PROOF. It is sufficient to obtain the estimation on X_T. We will use D_1, D_2, etc. as constants.

We expand $\Theta(\Omega_0 + \alpha, \omega_0 + \beta)$ for an $SU(2)$-structure (Ω_0, ω_0) with respect to (α, β) with $\|(\alpha, \beta)\|_{C^0} < \rho$ as

$$\Theta_1(\Omega_0 + \alpha, \omega_0 + \beta) = \Omega_0 + p_1(\alpha) + q_1(\beta) + F_1(\alpha, \beta), \tag{3.25}$$
$$\Theta_2(\Omega_0 + \alpha, \omega_0 + \beta) = \omega_0 + p_2(\alpha) + q_2(\beta) + F_2(\alpha, \beta), \tag{3.26}$$

where Θ_i are the projection of Θ to the i-th component, $p_i(\alpha), q_i(\beta)$ the linear terms, and $F_i(\alpha, \beta)$ the higher order terms for $i = 1, 2$. Set

$$\alpha = \Omega - \Omega_0 = \mathrm{d}\xi, \quad \beta = \omega - \omega_0,$$

and

$$\alpha_T = \Omega_T - \Omega_0 = \mathrm{d}\rho_{T-1}\xi = \rho_{T-1}\alpha + \mathrm{d}\rho_{T-1} \wedge \xi,$$
$$\beta_T = \omega_T - \omega_0 = \rho_{T-1}\beta.$$

Then ϕ_T and $\mathrm{d}\phi_T$ are expressed as

$$\begin{aligned}\phi_T &= \Omega_T - \psi_T \\ &= \Omega_0 + \alpha_T - \Theta(\Omega_0 + \alpha_T, \omega_0 + \beta_T) \\ &= (\alpha_T - p_1(\alpha_T)) - q_1(\beta_T) - F_1(\alpha_T, \beta_T)\end{aligned}$$

and

$$\mathrm{d}\phi_T = -\mathrm{d}\psi_T = -\mathrm{d}p_1(\alpha_T) - \mathrm{d}q_1(\alpha_T) - \mathrm{d}F_1(\alpha_T, \beta_T).$$

LEMMA 3.7. *There exist constants $C'_{1,k}, C'_{2,k}, C'_{3,k}$, and $C'_{4,k}$ for $k \geq 0$ such that for $t \geq 1$, we have*

$$|\nabla^k_{g_0}(\Omega - \Omega_0)|_{g_0} \leq C'_{1,k}\, e^{-t/2}, \quad |\nabla^k_{g_0}\Omega_0|_{g_0} \leq C'_{2,k},$$
$$|\nabla^k_{g_0}(\omega - \omega_0)|_{g_0} \leq C'_{3,k}\, e^{-t/2}, \quad |\nabla^k_{g_0}\omega_0|_{g_0} \leq C'_{4,k}.$$

PROOF OF LEMMA 3.7. The first inequality follows immediately from Proposition 3.4. The estimation for $\omega - \omega_0$ is similar. The inequalities for Ω_0 and ω_0 follow from the t-invariance of ∇_{g_0}, Ω_0 and ω_0. □

From the estimation $|\mathrm{d}\rho_{T-1}|_{g_0} = O(1)$ and the t-independence of the components of ∇_{g_0}, we also have $|\nabla_{g_0}\mathrm{d}\rho_{T-1}|_{g_0} = O(1)$. Consequently there exist positive constants D_1, D_2, D_3 and D_4 such that

$$|\alpha_T|_{g_0} \leq D_1\, e^{-T/2}, \quad |\nabla_{g_0}\alpha_T|_{g_0} \leq D_2\, e^{-T/2},$$
$$|\beta_T|_{g_0} \leq D_3\, e^{-T/2}, \quad |\nabla_{g_0}\beta_T|_{g_0} \leq D_4\, e^{-T/2}.$$

Thus it follows from (2.2) that

$$|\phi_T|_{g_0} \leq |\alpha_T|_{g_0} + |\beta_T|_{g_0} + C_1|(\alpha_T, \beta_T)|^2_{g_0} \leq D_5\, e^{-T/2}.$$

Next we consider

$$|\mathrm{d}\phi_T|_{g_0} \leq |\nabla_{g_0}p_1(\alpha_T)|_{g_0} + |\nabla_{g_0}q_1(\alpha_T)|_{g_0} + |\nabla_{g_0}F_1(\alpha_T, \beta_T)|_{g_0}.$$

The terms on the right-hand side are estimated as

$$\begin{aligned}
|\nabla_{g_0} p_1(\alpha_T)|_{g_0} &\leq |\nabla_{g_0} \alpha_T|_{g_0} + D_1 |\alpha_T|_{g_0} (|\mathrm{d}\Omega_0|_{g_0} + |\mathrm{d}\omega_0|_{g_0}) \\
&\leq |\nabla_{g_0} \alpha_T|_{g_0} + D_1 |\alpha_T|_{g_0} |\nabla_{g_0} \omega_0|_{g_0} \\
&\leq D_6 \, e^{-T/2}, \\
|\nabla_{g_0} q_1(\beta_T)|_{g_0} &\leq |\nabla_{g_0} \beta_T|_{g_0} + D_7 |\beta_T|_{g_0} (|\mathrm{d}\Omega_0|_{g_0} + |\mathrm{d}\omega_0|_{g_0}) \\
&\leq D_8 \, e^{-T/2}, \\
|\nabla_{g_0} F_1(\alpha_T, \beta_T)|_{g_0} &\leq C_2 \{ (|\mathrm{d}\Omega_0|_{g_0} + |\mathrm{d}\omega_0|_{g_0}) |(\alpha_T, \beta_T)|_{g_0}^2 \\
&\quad + 2 |(\alpha_T, \beta_T)|_{g_0} |\nabla_{g_0} (\alpha_T, \beta_T)|_{g_0} \} \\
&\leq D_9 \, e^{-T} \leq D_9 \, e^{-T/2},
\end{aligned}$$

where we used $\mathrm{d}\Omega_0 = 0$ and (2.3). Hence we have

$$|\mathrm{d}\phi_T|_{g_0} \leq D_{10} e^{-T/2}.$$

Now there exists positive constants ϵ_1, ϵ_2 such that for $t \geq 1$,

$$\begin{gathered}
|\cdot|_{g_T} \leq (1 + \epsilon_1) |\cdot|_{g_0}, \\
\psi_T \wedge \overline{\psi}_T \leq (1 + \epsilon_2) \Omega_0 \wedge \overline{\Omega}_0.
\end{gathered}$$

Thus we have

$$\begin{aligned}
\|\phi_T\|_{L^p(X_T)} &= \left\{ \int_{t=T-2}^{T-1} |\phi_T|_{g_T}^p \psi_T \wedge \overline{\psi}_T \right\}^{1/p} \\
&\leq D_5 (1 + \epsilon_1) \left\{ (1 + \epsilon_2) \int_{t=T-2}^{T-1} \Omega_0 \wedge \overline{\Omega}_0 \right\}^{1/p} e^{-T/2} \\
&\leq D_{11} e^{-T/2},
\end{aligned}$$

and similarly

$$\|\mathrm{d}\phi_T\|_{L^p(X_T)} \leq D_{12} e^{-T/2},$$

so that we obtain (3.22) and (3.23).

Finally, we have

$$\|\mathrm{d}\kappa_T\|_{C^0(X_T)} = \max\left\{\sup_{t\leq 1}|\mathrm{d}\omega|_g, (1+\epsilon_1)C'_{4,1}\right\} \leq D_{13},$$

where g is the metric associated with (Ω, ω). This completes the proof of Proposition 3.6. $\square$

4. Proof of Theorem 1.1.

In this section we will see that the $SL(m, \boldsymbol{C})$-structure ψ_T constructed in the last section can be deformed into a d-closed $SL(m, \boldsymbol{C})$-structure for $m = 2$, although the deformation is not always possible for $m > 2$.

Theorem 1.1. *Let X be a compact complex surface with a smooth irreducible anticanonical divisor D, and X' another compact complex surface with a smooth irreducible anticanonical divisor D'. Suppose there exists an isomorphism f from D to D' and the holomorphic normal bundles $N_{D/X}$ and $N_{D'/X'}$ are dual to each other via f, i.e., $N_{D/X} \otimes f^* N_{D'/X'} \cong \mathscr{O}_D$. Then there exist tubular neighborhoods W_1, W_2 of D in X with $\overline{W}_1 \subset W_2$, tubular neighborhoods W'_1, W'_2 of D' in X' with $\overline{W'}_1 \subset W'_2$, and a diffeomorphism h from $W_2 \setminus \overline{W}_1$ to $W'_2 \setminus \overline{W'}_1$ such that the following is true. Via the identification of $W_2 \setminus \overline{W}_1$ with $W'_2 \setminus \overline{W'}_1$ by h, we can glue $X \setminus \overline{W}_1$ and $X' \setminus \overline{W'}_1$ together to obtain a compact manifold M. Then the manifold M admits a complex structure with trivial canonical bundle.*

To prove the above theorem, we will find a smooth 2-form η_T on M_T with $\|\eta_T\|_{C^0} < \rho$ satisfying the equation

$$\mathrm{d}\Theta_1(\psi_T + \eta_T, \kappa_T) = 0 \tag{4.1}$$

for sufficiently large T. Using the Taylor expansion as in (3.25) and setting $F(\alpha) = -F_1(\alpha, 0)$, we have from equation (4.1)

$$\mathrm{d}p_1(\eta_T) = \mathrm{d}\phi_T + \mathrm{d}F(\eta_T). \tag{4.2}$$

Let $\wedge^2_- T^* M_T$ be the bundle of anti-self-dual 2-forms on M_T with respect to the Riemannian metric g_T associated with (ψ_T, κ_T). For $\eta_T \in C^\infty(\wedge^2_- T^* M_T)$, equation (4.2) is equivalent to the equation

$$\mathrm{d}\eta_T = \mathrm{d}\phi_T + \mathrm{d}F(\eta_T) \tag{4.3}$$

by Lemma 2.6. The proof of Theorem 1.1 is based on the following two theorems.

THEOREM 4.1. *Let μ, ν, and ϵ be positive constants, and suppose (M, g) is a complete Riemannian 4-manifold, whose injectivity radius $\delta(g)$ and Riemann curvature $R(g)$ satisfy $\delta(g) \geq \mu\epsilon$ and $\|R(g)\| \leq \nu\epsilon^{-2}$. Then there exist $C_6, C_7 > 0$ depending only on μ and ν, such that if $\chi \in L^8_1(\wedge^2_- T^*M \otimes \boldsymbol{C}) \cap L^2(\wedge^2_- T^*M \otimes \boldsymbol{C})$ then*

$$\|\nabla\chi\|_{L^8} \leq C_6(\|\mathrm{d}\chi\|_{L^8} + \epsilon^{-5/2}\|\chi\|_{L^2}),$$
$$\|\chi\|_{C^0} \leq C_7(\epsilon^{1/2}\|\nabla\chi\|_{L^8} + \epsilon^{-2}\|\chi\|_{L^2}).$$

THEOREM 4.2. *Let λ, C_6, and C_7 be positive constants. Then there exist a positive constant ϵ_* such that whenever $0 < \epsilon < \epsilon_*$, the following is true.*

Let M be a compact 4-manifold, (ψ, κ) an $SU(2)$-structure on M, and g the metric associated with (ψ, κ). Suppose that ϕ is a smooth complex 2-form on M with $\mathrm{d}\psi + \mathrm{d}\phi = 0$, and

(i) $\|\phi\|_{L^2} \leq \lambda\,\epsilon^3, \|\mathrm{d}\phi\|_{L^8} \leq \lambda\,\epsilon$, *and* $\|\mathrm{d}\kappa\|_{L^8} \leq \lambda\,\epsilon^{-1/2}$,
(ii) *if* $\chi \in L^8_1(\wedge^2_- T^*M \otimes \boldsymbol{C})$ *then* $\|\nabla\chi\|_{L^8} \leq C_6(\|\mathrm{d}\chi\|_{L^8} + \epsilon^{-5/2}\|\chi\|_{L^2})$,
(iii) *if* $\chi \in L^8_1(\wedge^2_- T^*M \otimes \boldsymbol{C})$ *then* $\|\chi\|_{C^0} \leq C_7(\epsilon^{1/2}\|\nabla\chi\|_{L^8} + \epsilon^{-2}\|\chi\|_{L^2})$.

Let ρ be as in Definition 2.7. Then there exists $\eta \in C^\infty(\wedge^2_- T^*M \otimes \boldsymbol{C})$ with $\|\eta\|_{C^0} < \rho$ such that $\mathrm{d}\Theta_1(\psi + \eta, \kappa) = 0$.

Theorem 4.1 is a geometric result similar to Theorems G1 and S1 in [**J**], and the proof is almost the same, so we will omit it. Theorem 4.2 will be proved later.

PROOF OF THEOREM 1.1. We define $W_1, W_2 \subset X$ to be $W_1 = \{T+1 < t\} \cup D$, $W_2 = \{T-1 < t\} \cup D$ and W'_1, W'_2 to be $W'_1 = \{T+1 < t'\} \cup D'$, $W'_2 = \{T-1 < t'\} \cup D'$ respectively. We also define $h : W_2 \setminus \overline{W}_1 \to W'_2 \setminus \overline{W'}_1$ to be the h_T defined in (3.19), Section 3.4. Then $M = M_T$.

Since M_T is cylindrical, the injectivity radius and the Riemann curvature of M_T are uniformly bounded with respect to T. Thus Theorem 4.1 holds and conditions (ii) and (iii) of Theorem 4.2 follow automatically from Theorem 4.1. Now we see that condition (i) is also satisfied for $M = M_T, (\psi, \kappa) = (\psi_T, \kappa_T)$, and $\phi = \phi_T$ for sufficiently large T. We choose γ so that $0 < \gamma < 1/6$, and set $\epsilon = e^{-\gamma T}$. Then by Proposition 3.6, we have for $T > T_*(\rho)$

$$\|\phi\|_{L^2} \leq C_3\, e^{-T/2} \leq C_3\, e^{-3\gamma T} = C_3\, \epsilon^3,$$
$$\|\mathrm{d}\phi\|_{L^8} \leq C_4\, e^{-T/2} \leq C_4\, e^{-\gamma T} = C_4\, \epsilon.$$

To estimate $\|\mathrm{d}\kappa\|_{L^8}$, we note that there exists a positive constant C_8 such that

$$\mathrm{Vol}_{g_T}(M_T) \leq C_8 T,$$

where $\mathrm{Vol}_{g_T}(M_T)$ is the volume of M_T with respect to the metric g_T. Thus we obtain

$$\begin{aligned}\|\mathrm{d}\kappa\|_{L^8} &\leq \|\mathrm{d}\kappa\|_{C^0}\mathrm{Vol}_{g_T}(M_T)^{1/8}\\ &\leq C_5(C_8T)^{1/8} = C_5(-C_8\gamma^{-1}\log\epsilon)^{1/8}\\ &\leq C_5(C_8\gamma^{-1})^{1/8}\epsilon^{-1/2}.\end{aligned}$$

Let $\lambda = \max\{C_3, C_4, C_5(C_8\gamma^{-1})^{1/8}\}$. Then we see that condition (i) is satisfied.

Therefore by Theorem 4.2, for all $T > \max\{T_*(\rho), -\gamma^{-1}\log\epsilon_*\}$ there exists a smooth 2-form η_T on M_T with $\|\eta_T\|_{C^0} < \rho$ such that $\mathrm{d}\Theta_1(\psi_T + \eta_T, \kappa_T) = 0$. Hence $\Theta_1(\psi_T + \eta_T, \kappa_T)$ is a d-closed $SL(2, \boldsymbol{C})$-structure on M_T, which induces on M_T a complex structure with trivial canonical bundle. This completes the proof of Theorem 1.1. □

The rest of this section is devoted to the proof of Theorem 4.2.

Proof of Theorem 4.2. We begin with the following result.

Proposition 4.3. *There exists positive constants ϵ_*, C_9 and K depending only on λ, C_6, C_7 such that if $0 < \epsilon < \epsilon_*$ then there exists a sequence $\{\eta_j\}$ in $C^\infty(\wedge^2_- T^*M \otimes \boldsymbol{C})$ with $\eta_0 = 0$ satisfying for each $j > 0$ the equation*

$$\mathrm{d}\eta_j = \mathrm{d}\phi + \mathrm{d}F(\eta_{j-1}) \tag{4.4}$$

and the inequalities

$$\begin{array}{ll} \text{(a) } \|\eta_j\|_{L^2} \leq 4\lambda\,\epsilon^3, & \text{(d) } \|\eta_j - \eta_{j-1}\|_{L^2} \leq 4\lambda 2^{-j}\epsilon^3,\\ \text{(b) } \|\nabla\eta_j\|_{L^8} \leq C_9\,\epsilon^{1/2}, & \text{(e) } \|\nabla(\eta_j - \eta_{j-1})\|_{L^8} \leq C_9 2^{-j}\epsilon^{1/2},\\ \text{(c) } \|\eta_j\|_{C^0} \leq K\epsilon < \rho/2, & \text{(f) } \|\eta_j - \eta_{j-1}\|_{C^0} \leq K2^{-j}\epsilon. \end{array}$$

Proof. The proof is by induction on j, and will follow from the following two lemmas.

Lemma 4.4. *Suppose by induction that $\eta_0, \ldots, \eta_k$ exist and satisfy* (4.4) *and parts* (a), (c) *and* (d) *of Proposition 4.3 for $j \leq k$. Then there exists a unique $\eta_{k+1} \in C^\infty(\wedge^2_- T^*M \otimes \boldsymbol{C})$ satisfying* (4.4) *and parts* (a), (d) *for $j = k+1$, and such that $\eta_{k+1} - \phi - F(\eta_k)$ is L^2-orthogonal to $\mathscr{H}^2_-$.*

PROOF. According to Hodge theory, there exists a unique $\eta_{k+1} \in C^\infty(\wedge^2_- T^*M)$ satisfying equation (4.4) such that $\eta_{k+1} - \phi - F(\eta_k)$ is L^2-orthogonal to $\mathscr{H}^2_-$. We shall prove that η_{k+1} satisfies (d) for $k = 0$ and $k > 0$ separately. Part (a) follows immediately from (d).

First suppose $k = 0$. Then $\eta_1 - \phi$ is a d-closed 2-form L^2-orthogonal to $\mathscr{H}^2_-$, so that it defines a cohomology class in $H^2_+(M, \boldsymbol{C})$. Thus

$$\|\phi_+\|^2_{L^2} - \|\eta_1 - \phi_-\|^2_{L^2} = \int_M (|\phi_+|^2 - |\eta_1 - \phi_-|^2)\mathrm{vol} = [\eta_1 - \phi] \cup [\eta_1 - \phi] \geq 0,$$

which implies

$$\|\eta_1\|_{L^2} \leq \|\phi_+\|_{L^2} + \|\phi_-\|_{L^2} \leq 2\|\phi\|_{L^2} \leq 2\lambda\,\epsilon^3, \tag{4.5}$$

where we write $\phi = \phi_+ + \phi_-$, $\phi_\pm \in C^\infty(\wedge^2_\pm T^*M)$.

Next suppose $k > 0$. Then $\eta_{k+1} - \eta_k - F(\eta_k) + F(\eta_{k-1})$ is a d-closed 2-form L^2-orthogonal to $\mathscr{H}^2_-$, so that it defines a cohomology class in $H^2_+(M, \boldsymbol{C})$. Thus writing $F(\eta_k) = F(\eta_k)_+ + F(\eta_k)_-$, $F(\eta_k)_\pm \in C^\infty(\wedge^2_\pm T^*M)$, we have similarly as above

$$\|F(\eta_k)_+ - F(\eta_{k-1})_+\|^2_{L^2} - \|\eta_{k+1} - \eta_k - F(\eta_k)_- + F(\eta_{k-1})_-\|^2_{L^2} \geq 0,$$

which implies

$$\|\eta_{k+1} - \eta_k\|_{L^2} \leq 2\|F(\eta_k) - F(\eta_{k-1})\|_{L^2}.$$

Now by equation (2.2) and part (c) for $j = k-1, k$, we have

$$\begin{aligned} 2\|F(\eta_k) - F(\eta_{k-1})\|_{L^2} &\leq 2C_1(\|\eta_k\|_{C^0} + \|\eta_{k-1}\|_{C^0})\|\eta_k - \eta_{k-1}\|_{L^2} \\ &\leq 4C_1 K\,\epsilon\|\eta_k - \eta_{k-1}\|_{L^2}. \end{aligned}$$

Thus by choosing ϵ_* so that $4C_1 K\,\epsilon_* \leq 1/2$, part (d) holds for $j = k+1$. This completes the proof. $\square$

Now we set $C_9 = 6\lambda C_6$ and $K = C_7(C_9 + 4\lambda)$.

LEMMA 4.5. *Parts* (b), (c), (e) *and* (f) *of Proposition* 4.3 *hold for* $j = 1$. *Suppose by induction that* (4.4) *and parts* (a)–(f) *hold for* $j \leq k$, *and part* (d) *and* (4.4) *hold for* $j = k+1$. *Then parts* (b), (c), (e) *and* (f) *hold for* $j = k+1$.

Proof. Again we shall deal with the cases $k=0$ and $k>0$ separately.

First suppose $k=0$. Then applying $\mathrm{d}\eta_1=\mathrm{d}\phi$, conditions (i) and (ii) of Theorem 4.2, and equation (4.5), we have

$$\begin{aligned}\|\nabla\eta_1\|_{L^8}&\leq C_6(\|\mathrm{d}\eta_1\|_{L^8}+\epsilon^{-5/2}\|\eta_1\|_{L^2})\\&\leq C_6(\lambda\,\epsilon+2\lambda\,\epsilon^{1/2})\\&\leq 3\lambda C_6\,\epsilon^{1/2}=\frac{1}{2}\,C_9\,\epsilon^{1/2}.\end{aligned}$$

Thus parts (b) and (e) hold for $k=0$. By the above inequality, equation (4.5) and condition (iii) of Theorem 4.2, we have

$$\|\eta_1\|_{C^0}\leq C_7\left(\frac{1}{2}\,C_9\,\epsilon+2\lambda\,\epsilon\right)=\frac{1}{2}\,K\,\epsilon.$$

Thus by choosing ϵ_* so that $K\,\epsilon_*\leq\rho$, parts (c) and (f) hold for $k=0$.

Next suppose $k>0$. It follows from (2.3) of Lemma 2.12, condition (i) of Theorem 4.2, and parts (b), (c), (e), and (f) for $j=k-1,k$ that

$$\begin{aligned}\|\mathrm{d}(\eta_{k+1}-\eta_k)\|_{L^8}&=\|\mathrm{d}(F(\eta_k)-F(\eta_{k-1}))\|_{L^8}\leq\|\nabla(F(\eta_k)-F(\eta_{k-1}))\|_{L^8}\\&\leq C_2\{(\|\mathrm{d}\phi\|_{L^8}+\|\mathrm{d}\kappa\|_{L^8})\|\eta_k-\eta_{k-1}\|_{C^0}(\|\eta_k\|_{C^0}+\|\eta_{k-1}\|_{C^0})\\&\quad+\|\nabla(\eta_k-\eta_{k-1})\|_{L^8}(\|\eta_k\|_{C^0}+\|\eta_{k-1}\|_{C^0})\\&\quad+\|\eta_k-\eta_{k-1}\|_{C^0}(\|\nabla\eta_k\|_{L^8}+\|\nabla\eta_{k-1}\|_{L^8})\}\\&\leq C_2\{(\lambda\,\epsilon+\lambda\,\epsilon^{-1/2})\cdot K2^{-k}\epsilon\cdot 2K\,\epsilon\\&\quad+C_92^{-k}\epsilon^{1/2}\cdot 2K\,\epsilon+K2^{-k}\epsilon\cdot 2C_9\epsilon^{1/2}\}\\&\leq 2C_2K(2\lambda K+4C_9)2^{-k-1}\epsilon^{3/2}.\end{aligned}$$

Now we choose ϵ_* so that $C_2K(2\lambda K+4C_9)\epsilon_*\leq\lambda$. Then for $0<\epsilon<\epsilon_*$, we have

$$\|\mathrm{d}(\eta_{k+1}-\eta_k)\|_{L^8}\leq 2\lambda 2^{-k-1}\epsilon^{1/2}. \tag{4.6}$$

Using condition (ii) of Theorem 4.2, equation (4.6), and part (d) for $j=k+1$, we have

$$\begin{aligned}\|\nabla(\eta_{k+1}-\eta_k)\|_{L^8}&\leq C_6(\|\mathrm{d}(\eta_{k+1}-\eta_k)\|_{L^8}+\epsilon^{-5/2}\|\eta_{k+1}-\eta_k\|_{L^2})\\&\leq C_6(2\lambda 2^{-k-1}\epsilon^{1/2}+4\lambda 2^{-k-1}\epsilon^{1/2})=C_92^{-k-1}\epsilon^{1/2}.\end{aligned}$$

Thus part (e) holds for $j = k + 1$. Using condition (iii) of Theorem 4.2 we find

$$\|\eta_{k+1} - \eta_k\|_{C^0} \leq C_7(C_9 2^{-k-1}\epsilon + 4\lambda 2^{-k-1}\epsilon) = K2^{k-1}\epsilon.$$

Thus part (f) holds for $j = k + 1$. Parts (b) and (c) follow immediately from parts (e) and (f). □

By the induction steps established in the above two lemmas, we have a sequence $\{\eta_j\}$ in $C^\infty(\wedge^2_- T^*M)$ with $\eta_0 = 0$ satisfying (a)–(f). This completes the proof of Proposition 4.3. □

The sequence $\{\eta_k\}$ converges to some η in $L^8_1(\wedge^2_- T^*M)$. Now it remains to show that this η is smooth. We follow the argument in [**J**, p. 365]. Taking the Hodge star of equation (4.3) and using $*\mathrm{d}\eta = \mathrm{d}^*\eta$ since η is anti-self-dual and $\mathrm{d}^* = - * \mathrm{d} *$, we have

$$(\mathrm{d} + \mathrm{d}^*)\eta = \mathrm{d}\phi + *\mathrm{d}\phi + \mathrm{d}F(\eta) + *\mathrm{d}F(\eta). \tag{4.7}$$

We may write as

$$\mathrm{d}F(\eta) + *\mathrm{d}F(\eta) = G(\eta, \nabla\eta) + H(\eta),$$

where $G(x, y)$ is linear in y. Then $G(x, y)$ and $H(x)$ are smooth functions of x and y, and $G(0, y) = 0$. Thus if we consider a first-order partial differential operator $P(\eta) : L^8_1(V) \to L^8(V)$ with $V = \bigoplus_{i=0}^4 \wedge^i T^*M$ defined by

$$P(\eta)\zeta = (\mathrm{d} + \mathrm{d}^*)\zeta - G(\eta, \nabla\zeta),$$

then $P(0) = \mathrm{d} + \mathrm{d}^*$ is an elliptic operator on $L^8_1(V)$. Since ellipticity is an open condition, we see that $P(\eta)$ is elliptic for $\|\eta\|_{C^0} < 2\epsilon K$ with $\epsilon < \epsilon_*$ by taking ϵ_* smaller if necessary. Now we rewrite equation (4.7) as

$$P(\eta)\eta = \mathrm{d}\phi + *\mathrm{d}\phi + H(\eta). \tag{4.8}$$

By the Sobolev embedding $L^8_1 \hookrightarrow C^{0,1/2}$ in 4 dimensions, we have $\eta \in C^{0,1/2}(\wedge^2_- T^*M) \subset C^{0,1/2}(V)$. Since $\eta \in C^{0,1/2}(V)$ is a solution of equation (4.8) and the coefficients of P belong to $C^{0,1/2}$, we have $\eta \in C^{1,1/2}(V)$ by the elliptic regularity. Similarly if $\eta \in C^{k,1/2}(V)$, then we have $\eta \in C^{k+1,1/2}(V)$. Hence η is smooth by induction. This completes the proof of Theorem 4.2. □

5. Applications.

5.1. A basic example.

Example 5.1. For $d \in \{0,1,2,3\}$, let C, C_1, C_2 be smooth curves of degree $3, 3-d, 3+d$ in $\boldsymbol{C}P^2$ respectively. Let X_d be the blow-up of $\boldsymbol{C}P^2$ at $3(3-d)$ points $C \cap C_1$, X'_d the blow-up of $\boldsymbol{C}P^2$ at $3(3+d)$ points $C \cap C_2$, and D, D' the proper transforms of C in X_d, X'_d respectively. Then D, D' are isomorphic to C, and we can compute as

$$N_{D/X_d} \cong \mathscr{O}_C(d), \quad N_{D'/X'_d} \cong \mathscr{O}_C(-d).$$

Then by Theorem 1.1, we glue $X_d \setminus D$ and $X'_d \setminus D'$ together to obtain a compact complex surface with trivial canonical bundle. We can see easily that the resulting surface is simply connected. Thus it is by definition, a $K3$ *surface*.

5.2. Multiple gluing theorem.

The following theorem is a generalization of Theorem 1.1.

Theorem 5.2. *Let $X_1, \dots, X_L$ be compact complex surfaces, and $D_1, \dots, D_{2\ell}$ be irreducible smooth divisors on the disjoint union $X = \coprod_{a=1}^{L} X_a$ such that $D_i \cap D_j = \emptyset$ for $i \neq j$. Define index sets $I_a = \{i \mid D_i \subset X_a\}$ for $a = 1, \dots, L$. Let $\sum_{i \in I_a} D_i$ be an anticanonical divisor on X_a, Ω_a a meromorphic volume form on X_a with a single pole along $\sum_{i \in I_a} D_i$ and holomorphic elsewhere for $a = 1, \dots, L$, and Ω_{D_i} the Poincaré residue of Ω_a on D_i for $i = 1, \dots, 2\ell$. Suppose there exist isomorphisms $f_i : D_{2i-1} \to D_{2i}$ such that the normal bundle of D_{2i-1} and D_{2i} are dual to each other via f_i and $f_i^* \Omega_{D_{2i}} = -\Omega_{D_{2i-1}}$ for $i = 1, \dots, \ell$. Then we obtain a compact complex surface with trivial canonical bundle by gluing together $X_a \setminus \bigcup_{i \in I_a} D_i$ for $a = 1, \dots, L$.*

The proof of Theorem 5.2 is essentially the same as Theorem 1.1, so we will omit it.

Example 5.3. Let C be a cubic curve in $\boldsymbol{C}P^2$, and $Y_d = \boldsymbol{P}(\mathscr{O}_{\boldsymbol{C}P^2} \oplus \mathscr{O}_{\boldsymbol{C}P^2}(d))|_C$. Let D_0 and D_∞ be the zero section and the infinity section of Y_d respectively. Then D_0 and D_∞ are naturally isomorphic to C, and $D_0 + D_\infty$ is an anticanonical divisor on Y_d. The normal bundles of D_0 and D_∞ are computed as

$$N_{D_0/Y_d} \cong \mathscr{O}_C(d), \quad N_{D_\infty/Y_d} \cong \mathscr{O}_C(-d).$$

Thus we can glue $Y_d \setminus D_0 \cup D_\infty$ with itself along both ends to obtain a compact complex surface M_d with trivial canonical bundle. One can show that M_d is topologically $S_d \times S^1$, where S_d is the $U(1)$-bundle associated with the complex line bundle $\mathscr{O}_C(d)$, and that the Betti numbers of S_d are given by $b_1(S_0) = b_2(S_0) = 3,\ b_1(S_d) = b_2(S_d) = 2$ for $d \neq 0$. Thus by the classification of compact complex surfaces with trivial canonical bundle [**BPV**], we see that M_0 is a *complex torus* and M_d for $d \neq 0$ is a *Kodaira surface.*

EXAMPLE 5.4. Let C be a cubic curve in $\boldsymbol{C}P^2$, and X_d, X'_d, and Y_d as in Example 5.1 and 5.3. Then for $d \in \{0, 1, 2, 3\}$, we obtain a K3 surface from X_d, X'_d and any number of copies of Y_d.

5.3. Smoothings of normal crossing complex surfaces with at most double curves.

In this section we shall approach the smoothing problem of normal crossing complex surfaces in a differential-geometric way.

Let X be a compact complex analytic surface with irreducible components $X_1, \ldots, X_N$. Then we say that X is *a simple normal crossing complex surface* if X is locally embedded in $\boldsymbol{C}^3$ as $\{(\zeta_1, \zeta_2, \zeta_3) \in \boldsymbol{C}^3 \mid \zeta_1 \cdots \zeta_\ell = 0\}$ for some $\ell \in \{1, 2, 3\}$ and each X_i is smooth (see also [**KN**] for a definition).

We consider a simple normal crossing complex surface $X \cup X'$ with a connected double curve $D = X \cap X'$. We suppose D is an anticanonical divisor on both X and X', and the holomorphic normal bundles $N_{D/X}$ and $N_{D/X'}$ are dual to each other. We adopt almost the same notation as in Section 3. Let $\{(U_\alpha, (z_\alpha, w_\alpha))\}$ be a local coordinate system on X and $\{(U'_\alpha, (z'_\alpha, w'_\alpha))\}$ a local coordinate system on X' as implemented in Section 3, such that $\{(V_\alpha, x_\alpha)\}$ defines a local coordinate system on D, where $V_\alpha = U_\alpha \cap D = U'_\alpha \cap D$ and $x_\alpha = z_\alpha|_{V_\alpha} = z'_\alpha|_{V_\alpha}$. Let (x_α, y_α) be local coordinates on $N = N_{D/X}$ and (x_α, y'_α) local coordinates on $N' = N_{D/X'}$ such that $y_\alpha y'_\alpha = y_\beta y'_\beta$. We also let t be a cylindrical parameter on $N \setminus D$, $\Phi : t^{-1}((0, \infty)) \to X \setminus D$ a diffeomorphism onto image as in Proposition 3.1, and similarly for t', Φ'.

Let $\Delta = \{\zeta = e^{-T-\sqrt{-1}\theta} \in \boldsymbol{C} \mid |\zeta| = e^{-T} < \epsilon\}$ be a domain in $\boldsymbol{C}$ for some $\epsilon > 0$. A family of local smoothings of $N \cup N'$ parametrized by Δ is given by

$$\mathscr{V}_\Delta = \{(x_\alpha, y_\alpha, y'_\alpha) \in N \oplus N' \mid t > 0, t' > 0, \text{ and } y_\alpha y'_\alpha = \zeta \in \Delta\}.$$

Then the projection $\varpi : \mathscr{V}_\Delta \to \Delta$ is given by $\varpi(x_\alpha, y_\alpha, y'_\alpha) = y_\alpha y'_\alpha$, such that $V_\zeta = \varpi^{-1}(\zeta), \zeta \in \Delta^* = \Delta \setminus \{0\}$ is a smoothing of $V_0 = \varpi^{-1}(0) \subset N \cup N'$. We have a diffeomorphism onto image defined by

$$\begin{array}{ccc} \widetilde{\Phi} : \mathscr{V}_\Delta \setminus N' & \to & (X \setminus D) \times \Delta \\ \Psi & & \Psi \\ (x_\alpha, y_\alpha, y'_\alpha) & \mapsto & (\Phi(x_\alpha, y_\alpha), y_\alpha y'_\alpha), \end{array}$$

and similarly a diffeomorphism onto image $\widetilde{\Phi}' : \mathscr{V}_\Delta \setminus N \to (X' \setminus D) \times \Delta$. Then we can glue $\mathscr{V}_\Delta$, $(X \setminus D) \times \Delta$ and $(X' \setminus D) \times \Delta$ together by the gluing maps $\widetilde{\Phi}$ and $\widetilde{\Phi}'$ to obtain a fibration $\mathscr{X}_\Delta$ of $X \cup X'$ and the projection map $\mathscr{X}_\Delta \to \Delta$, which we also denote by ϖ. We can see easily that $(X \setminus D) \times \{\zeta\}$ is glued to $(X' \setminus D) \times \{\zeta\}$ by $h_{T,\theta}$ in equation (3.21), so that $M_\zeta = \varpi^{-1}(\zeta), \zeta \in \Delta^* = \Delta \setminus \{0\}$ is $M_{T,\theta} = X_T \cup_{h_{T,\theta}} X'_T$. Thus each fibre M_ζ of ϖ over $\zeta \in \Delta^*$ is smooth, while the central fibre is $M_0 = \varpi^{-1}(0) = X \cup X'$. Note that at this point we don't know whether each smooth fibre admits a complex structure.

Let I_0 be the complex structure on $X \cup X'$ and Ω, Ω' holomorphic volume forms defining I_0 on $X \setminus D, X' \setminus D$ respectively. Under a diffeomorphism $\mathscr{X} \setminus M_0 \simeq M \times \Delta^*$, we have a family $\{(\psi_\zeta, \kappa_\zeta) \mid \zeta \in \Delta^*\}$ of $SU(2)$-structures on M, where ψ_ζ and κ_ζ are smooth with respect to ζ. Taking ϵ sufficiently small, the resulting family $\{\eta_\zeta \mid \zeta \in \Delta^*\}$ of solutions of equation (4.1) is *continuous* with respect to ζ. Let Ω_ζ be the $SL(2, \boldsymbol{C})$-structure $\Theta_1(\psi_\zeta + \eta_\zeta, \kappa_\zeta)$ on M_ζ. Then we have

$$\|\Omega - \Omega_\zeta\|_{C^0(X_T, g)} \to 0, \quad \|\Omega' - \Omega_\zeta\|_{C^0(X'_T, g')} \to 0 \quad \text{as } \zeta \to 0, \tag{5.1}$$

where g is the asymptotically cylindrical metric on $X \setminus D$ defined in Section 3.4, and similarly for the metric g' on $X' \setminus D$. In this sense we have a family $\{X_\zeta = (M_\zeta, I_\zeta) \mid \zeta \in \Delta\}$ of compact complex surfaces, continuous with respect to ζ outside $D \subset M_0$, where $I_\zeta, \zeta \in \Delta^*$ is the complex structure with trivial canonical bundle induced by the $SL(2, \boldsymbol{C})$-structure Ω_ζ, while the central fibre $M_0 = X \cup X'$ is endowed with the original complex structure I_0. Using Theorem 5.2, we generalize this result as follows.

Theorem 5.5. *Let* $X = \bigcup_{i=1}^N X_i$ *be a simple normal crossing complex surface with at most double curves. Suppose that*

(i) *the holomorphic normal bundles* N_{D_{ij}/X_i} *and* N_{D_{ij}/X_j} *are dual to each other for each double curve* $D_{ij} = X_i \cap X_j$,
(ii) $D_i = \sum_{\ell(\neq i)} D_{i\ell}$ *is an anticanonical divisor on each* X_i, *and*
(iii) *there exist meromorphic volume forms* Ω_i *on* X_i *with a pole along* D_i *such that the Poincaré residue of* Ω_i *on* D_{ij} *is minus the Poincaré residue of* Ω_j *on* D_{ij}.

Then there exist $\epsilon > 0$ *and a surjective mapping* $\varpi : \mathscr{X} \to \Delta = \{\zeta \in \boldsymbol{C} \mid |\zeta| < \epsilon\}$ *such that*

(a) $\mathscr{X}$ *is a smooth* 6*-dimensional manifold and* ϖ *is a smooth mapping,*
(b) $X_0 = \varpi^{-1}(0) = X$,
(c) *for each* $\zeta \in \Delta^*$, $X_\zeta = \varpi^{-1}(\zeta)$ *is a smooth compact complex surface with trivial canonical bundle, and*
(d) *the complex structures on* X_ζ *depend continuously on* ζ *outside the singular locus* $D = \bigcup_{i \neq j} D_{ij}$ *of* X_0 *(in the sense of* (5.1)*), or more precisely, for any point* $p \in \mathscr{X} \setminus D$ *there exist a neighborhood* U *of* p *and a diffeomophism* $U \simeq V \times D$ *with* $D \subset \Delta$*, such that the induced complex structures on* V *depend continuously on* $\zeta \in D$*.*

Lastly we compare this result with that of Friedman in [**Fr**]. Let $X = \bigcup_{i=1}^N X_i$ be a simple normal crossing complex surface. Then X is *d-semistable* if for each $D_{ij} = X_i \cap X_j$ with $i \neq j$ we have

$$N_{ij} \otimes N_{ji} \otimes [T_{ij}] \cong \mathscr{O}_{D_{ij}}, \tag{5.2}$$

where N_{ij} denotes the holomorphic normal bundle N_{D_{ij}/X_i}, and $T_{ij} = \sum_{k \neq i,j} D_{ij} \cap X_k$ a divisor on D_{ij} defined by the triple points (this condition is equivalent to Friedman's original one $\bigotimes_{i=1}^N \mathscr{I}_{X_i}/\mathscr{I}_{X_i}\mathscr{I}_D \cong \mathscr{O}_D$ for the singular locus D on X). We say that X is a *d-semistable K*3 *surface* if X is a d-semistable normal crossing Kähler surface with trivial canonical bundle and $H^1(X, \mathscr{O}_X) = 0$. As is well-known, d-semistable K3 surfaces are classified into Types I–III, which comes from the classification of degenerations of K3 surfaces (see Theorems 5.1, 5.2 and Definition 5.5 in [**Fr**]). Friedman proved that a d-semistable K3 surface has a family of smoothings $\varpi : \mathscr{X} \to \Delta \subset \boldsymbol{C}$ of X with the canonical bundle of $\mathscr{X}$ trivial, where $\mathscr{X}$ is a 3-dimensional complex manifold and ϖ is a holomorphic mapping. If X is a d-semistable K3 surface at most double curves, it is of Type I—a smooth K3 surface, or of Type II—a chain $X_1 \cup \cdots \cup X_N$ of surfaces with X_1, X_N rational, and X_i for $2 \le i \le N-1$ elliptic ruled with the double curves two disjoint sections of the ruling. We note that in either case X satisfies the hypotheses of Theorem 5.5. Thus Theorem 5.5 implies that even when X is not Kählerian or $H^1(X, \mathscr{O}_X) \neq 0$, there still exists a family of smoothings $\varpi : \mathscr{X} \to \Delta$ of X in a weak sense, whose general fibre is a smooth compact complex surface with trivial canonical bundle. This result strongly suggests that X as in Theorem 5.5 admits a family of smoothings in the standard holomorphic sense. We can further generalize Theorem 5.5 to include cases where X is a normal crossing complex surface with triple points (in particular the Type III case), which will be treated in [**D**].

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