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The $(\mathfrak{g}, K)$-module structures of principal series of $SU(2,2)$

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Abstract. We explicitly describe the $(\mathfrak{g}_{\boldsymbol{C}}, K)$-module structures of the principal series representations of $SU(2,2)$ associated with a maximal parabolic subgroup.

Introduction.

The purpose of this paper is to describe explicitly the $(\mathfrak{g}, K)$-module structure of the principal series representations of $SU(2,2)$, parabolically induced with respect to the minimal parabolic subgroup P_{min}.

This is motivated by the problem of the determination of the precise formulas for various spherical models of the standard representations. Among others we are interested in the Whittaker models (Bayarmagnai [**1**], Hayata [**3**], Ishii [**5**], Miyazaki-Oda [**10**]). Our basic concern is in arithmetic of automorphic forms. However, in our case, we should also recall that the group $SU(2,2)$, which is locally isomorphic to the conformal group $SO(4,2)$, plays a very important role in physics. Our method of proof is similar to that of a recent paper of Oda [**12**], which describes the $(\mathfrak{g}, K)$-module structure of standard representations of $Sp(2,\boldsymbol{R})$. Namely we utilize the concept of simple K-modules with marking, to overcome the problem of multiplicities in K-types.

Our main results are Theorem 3.5 and Theorem 3.6 which are shortly explained below. The template of the formulas is the following:

$$\mathscr{C}_{[\pm,\pm;\pm]}\boldsymbol{S}^{(m)} = \boldsymbol{S}^{(m')}\Gamma_{[\pm,\pm;\pm]}.$$

Here $\boldsymbol{S}^{(m)}$ is the matrix consisting of elementary functions in the representation identified with a closed subspace of $L^2(K)$, $\mathscr{C}_{[\pm,\pm;\pm]}$ is a matrix with entries either in $\mathfrak{p}^+$ or in $\mathfrak{p}^-$, and $\Gamma_{[\pm,\pm;\pm]}$ is a constant matrix whose entries consists of linear forms in the parameters of the representation. The last is called a matrix of

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intertwining constants.

Let us recall the Casimir equation for the Casimir operator $\mathscr{C}$:

$$\mathscr{C}v = \gamma(\mathscr{C})v,$$

where γ is the infinitesimal character and v is a differential vector. Our formula is a "covariant" analogue of this. The details of each symbol is explained in the text.

This paper is arranged as follows. In the Section 1, we establish our notation and define the class of the principal series representations of $SU(2,2)$ corresponding to the minimal parabolic subgroup P_{min}. The marked basis for each K-isotypic component in the principal series representation is introduced in terms of the elementary functions in the Section 2. We begin Section 3 by computing the Clebsch-Gordan coefficients of finite dimensional representations of K (Propositions 3.1 and 3.2). Then we shall determine our main result concerning the $\mathfrak{g}_{\boldsymbol{C}}$-module (Theorems 3.5 and 3.6), and finally give some examples.

We want to refer to the former results on $(\mathfrak{g}, K)$-module structures: Klimyk-Gruber [**6**], [**7**], Molchanov [**11**], Thieleker [**13**], Howe [**4**], and Lee-Loke [**8**]. Their interests are mainly to study the composition series of $(\mathfrak{g}, K)$-modules for degenerate principal series representations which are K-multiplicity free, except for [**4**] and [**8**].

The method of [**4**] for $GL(3, \boldsymbol{R})$ is to find nice elements in the enveloping algebra $U(\mathfrak{g})$ to generate the K-types in a principal series representation, hence it is different from our results. The paper [**8**] is most similar to ours, but this also considers the composition series of degenerate principal series.

The result of the papers of Yamashita [**15**], [**16**] also gives some structure of the composition series of the principal series representations of $SU(2,2)$ by direct determination of intertwining operators.

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1. Preliminaries.

1.1. The group $SU(2,2)$.

In this paper, the group $SU(2,2)$ is the special unitary group of signature $(+2,-2)$ associated to the Hermitian form $\langle\ ,\ \rangle$ defined on $\boldsymbol{C}^4$ by

$$\langle z, w\rangle = \bar{z}_1 w_1 + \bar{z}_2 w_2 - \bar{z}_3 w_3 - \bar{z}_4 w_4$$

for $z = (z_1, z_2, z_3, z_4)$ and $w = (w_1, w_2, w_3, w_4)$. In terms of matrices, the group consists of all matrices $g \in SL_4(\boldsymbol{C})$ that satisfy the following identity:

$$ {}^t\bar{g} I_{2,2} g = I_{2,2}, $$

where $I_{2,2} = \mathrm{diag}(1, 1, -1, -1)$. It is a standard fact that $G = SU(2,2)$ is a quasi-split real semisimple group of real rank two.

Let θ be a Cartan involution given by

$$ \theta(g) = {}^t\bar{g}^{-1}, \ \ g \in G. $$

Then the fixed point set $K = G^\theta$ is the standard maximal compact subgroup $S(U(2) \times U(2))$ of G. The group $K = S(U(2) \times U(2))$ can be represented by matrices

$$ \begin{pmatrix} k_1 & \\ & k_2 \end{pmatrix} \in G, $$

where $k_1, k_2 \in U(2)$ and $\det(k_1 k_2) = 1$.

The Lie algebra $\mathfrak{g}$ of G is the set of matrices $X \in M_4(\boldsymbol{C})$ such that ${}^t\bar{X} I_{2,2} + I_{2,2} X = 0$ and $tr(X) = 0$. We let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and -1 eigen-spaces of the differential of θ, respectively. Then we have

$$ \mathfrak{k} = \left\{ \begin{pmatrix} X_1 & \\ & X_3 \end{pmatrix} \in \mathfrak{sl}_4(\boldsymbol{C}) : X_1, X_3 \in \mathfrak{u}(2) \right\}, $$

and

$$ \mathfrak{p} = \left\{ \begin{pmatrix} & X_2 \\ {}^t\bar{X}_2 & \end{pmatrix} \in M_4(\boldsymbol{C}) : X_2 \in M_2(\boldsymbol{C}) \right\}. $$

For $x \in M_2(\boldsymbol{C})$ we set

$$ p_+(x) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \text{ and } p_-(x) = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}. $$

Let $H_i = p_+(e_{ii}) + p_-(e_{ii})$ $(i = 1, 2)$, where e_{ij} the matrix unit of $M_2(\boldsymbol{R})$ with 1 in the (i,j)-entry and zero elsewhere. Then the space $\mathfrak{a}$ spanned by H_1, H_2 over $\boldsymbol{R}$ is a maximally abelian subalgebra of $\mathfrak{p}$. Let $\{\lambda_1, \lambda_2\}$ be a basis of the dual space $\mathfrak{a}^*$ such that $\lambda_i(H_j) = \delta_{ij}$. Then the restricted root system for $\boldsymbol{\Phi}(\mathfrak{g}, \mathfrak{a})$ is of type C_2, namely

$$\boldsymbol{\Phi}(\mathfrak{g}, \mathfrak{a}) = \{\pm\lambda_1 \pm \lambda_2, \pm 2\lambda_1, \pm 2\lambda_2\}.$$

Choose $\lambda_1 - \lambda_2$ and $2\lambda_2$ as simple roots of $\boldsymbol{\Phi}(\mathfrak{g}, \mathfrak{a})$. Denote by E_{ij} the matrix units in $M_4(\boldsymbol{C})$ for $0 \le i, j \le 4$. Then the corresponding root spaces of dimension two and one are given by

$$\mathfrak{g}_{\lambda_1 - \lambda_2} = \boldsymbol{R} \cdot E_1 \oplus \boldsymbol{R} \cdot E_2 \text{ and } \mathfrak{g}_{2\lambda_2} = \boldsymbol{R} \cdot E_0,$$

where $E_0 = \kappa^{-1} E_{24} \kappa$, $E_1 = \kappa^{-1}(E_{12} - E_{43})\kappa$ and $E_2 = \kappa^{-1}(iE_{12} + iE_{43})\kappa$. Here

$$\kappa = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -i & 0 & i & 0 \\ 0 & -i & 0 & i \end{pmatrix}$$

with $i = \sqrt{-1}$.

We put $A = \exp(\mathfrak{a})$, $M = Z_A(K)$, and choose a minimal parabolic subgroup P_{min} with Langlands decomposition $P_{min} = MAN$.

Here N is the maximal unipotent subgroup of G and an element $n = n(n_0, n_1, n_2, n_3) \in N$ takes the form:

$$\kappa^{-1} \begin{pmatrix} 1 & n_0 & & \\ & 1 & & \\ & & 1 & \\ & & -\bar{n}_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & n_1 & n_2 \\ & 1 & \bar{n}_2 & n_3 \\ & & 1 & \\ & & & 1 \end{pmatrix} \kappa$$

for n_1, $n_3 \in \boldsymbol{R}$, n_0, $n_2 \in \boldsymbol{C}$.

1.2. The $\boldsymbol{K}$-modules.

Let (τ, V_τ) be an irreducible representation of K. The fact is that the dimension of V_τ is finite and τ is unitary. To clarify K-action on V_τ it is enough to consider the $\mathfrak{k}_{\boldsymbol{C}}$-action on that vector space.

Note that the group $\tilde{K} = SU(2) \times SU(2) \times \boldsymbol{C}^{(1)}$ is a twofold covering of K with a projection given by

$$pr(g_1, g_2; u) = \mathrm{diag}(ug_1, u^{-1}g_2),$$

where $g_1, g_2 \in SU(2)$ and $u \in \boldsymbol{C}^{(1)}$. The kernel of this homomorphism is

$$\mathrm{Ker}(pr) = \{\pm(1_2, 1_2; 1)\}.$$

Let (sym^m, V_m) be the m-th symmetric tensor representation of the group $SU(2)$. Then the unitary dual of K can be parameterized by the set

$$\hat{K} = \{(\tau_{[m_1,m_2;\, l]}, V_{m_1m_2}) \mid m_1, m_2 \in \boldsymbol{N} \cup 0,\ l \in \boldsymbol{Z},\ m_1 + m_2 + l \in 2\boldsymbol{Z}\}.$$

Here $V_{m_1m_2}$ is the outer tensor product of the spaces V_{m_1} and V_{m_2}, and if $g_1, g_2 \in SU(2)$ and $u \in \boldsymbol{C}^{(1)}$, then the action is

$$\tau_{[m_1,m_2;\, l]}(g_1, g_2; u) = \mathrm{sym}^{m_1}(g_1) \otimes \mathrm{sym}^{m_2}(g_2) \otimes u^l.$$

We fix now a basis for $\mathfrak{k}_{\boldsymbol{C}} = \mathrm{Lie}(K)_{\boldsymbol{C}}$:

$$h^1 = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}, \quad h^2 = \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}, \quad I_{2,2} = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix},$$
$$e^1_\pm = \begin{pmatrix} e_\pm & 0 \\ 0 & 0 \end{pmatrix}, \quad e^2_\pm = \begin{pmatrix} 0 & 0 \\ 0 & e_\pm \end{pmatrix},$$

where $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then we have the following proposition:

Lemma 1.1. *Let* $\{f_i\}_{0 \le i \le m_j}$ *be a basis of* V_{m_j} *as* $SU(2)$*-module for* $j = 0, 1$. *For a given* K*-module* $(\tau_{[m_1,m_2;l]}, V_{m_1m_2})$ *the set*

$$\{f_{pq} : f_{pq} = f_p \otimes f_q,\ 0 \le p \le m_1, 0 \le q \le m_2\}$$

forms a basis of $V_{m_1m_2}$ *as* K*-module and the infinitesimal action corresponding to* K *on* $V_{m_1m_2}$ *is expressed by*

$$\begin{aligned} &h^1(f_{pq}) = (2p - m_1)f_{pq}, && h^2(f_{pq}) = (2q - m_2)f_{pq}, \\ &e^1_+(f_{pq}) = (m_1 - p)f_{p+1,q}, && e^2_+(f_{pq}) = (m_2 - q)f_{p,q+1}, \\ &e^1_-(f_{pq}) = pf_{p-1,q}, && e^2_-(f_{pq}) = qf_{p,q-1}, \\ &I_{2,2}f_{pq} = lf_{pq}. && \end{aligned}$$

For a simple K-module τ, we can normalize the one dimensional space of K-homomorphisms of τ onto itself by the following definition.

Definition 1.1. A simple K-module τ equipped with a canonical basis is called a marked simple K-module or a simple K-module with marking.

1.3. Iwasawa decomposition.

The set $\{E_{i,j+2}, E_{i+2,j} \mid i,j = 1,2\}$ forms a basis of the 8-dimensional vector space $\mathfrak{p}_{\boldsymbol{C}}$ and one has

$$E_{i,j+2} = p_+(e_{ij}) \quad \text{and} \quad E_{i+2,j} = p_-(e_{ij}),$$

where $i, j = 1, 2$.

Lemma 1.2. *Put*

$$\begin{aligned} E_{2\lambda_1} &= \kappa^{-1}E_{13}\kappa,\ E^1_{\lambda_1+\lambda_2} = \kappa^{-1}E_{14}\kappa,\ E^1_{\lambda_1-\lambda_2} = \kappa^{-1}E_{43}\kappa, \\ E_{2\lambda_2} &= \kappa^{-1}E_{24}\kappa,\ E^2_{\lambda_1+\lambda_2} = \kappa^{-1}E_{23}\kappa,\ E^2_{\lambda_1-\lambda_2} = \kappa^{-1}E_{12}\kappa. \end{aligned}$$

Then we have

$$p_{\pm}(e_{ii}) = \frac{1}{2}(\mp 2\sqrt{-1}E_{2\lambda_i} + H_i \pm \frac{1}{2}(I_{2,2} - \epsilon(i)(h^1 - h^2))),$$

$$p_{\pm}(e_{ij}) = \frac{1}{2}(-\epsilon(i)E^j_{\lambda_1-\lambda_2} \mp \sqrt{-1}E^i_{\lambda_1+\lambda_2}) - \epsilon(j)\begin{cases} e^j_{\epsilon(j)}, & \textit{if } (+) \\ e^i_{-\epsilon(i)}, & \textit{if } (-) \end{cases}$$

where $\epsilon(i) := \mathrm{sign}(-1)^i$ $(i \neq j,\ i,j \in \{1,2\})$.

Proof. We can show this by direct computation. □

1.4. The adjoint representation.

Now we consider the adjoint representation Ad of K on the complexification $\mathfrak{p}_{\boldsymbol{C}}$ of $\mathfrak{p}$. It splits into two K-irreducible components, namely, the holomorphic part $\mathfrak{p}_+$ generated by the set of matrix units $\{E_{ij} \mid i = 1,2, j = 3,4\}$ and the antiholomorphic part $\mathfrak{p}_-$ generated by the set $\{E_{ij} \mid i = 3,4, j = 1,2\}$ over $\boldsymbol{C}$. Moreover, we have:

Lemma 1.3 (cf. [**3**, 3.10]). *The linear maps from* $\mathfrak{p}_+$ *and* $\mathfrak{p}_-$ *to* V_{11} *given by*

$$(E_{23}, E_{13}, E_{24}, E_{14}) \to (f_{00}, f_{10}, -f_{01}, -f_{11})$$

and

$$(E_{41}, E_{31}, E_{42}, E_{32}) \to (f_{00}, f_{01}, -f_{10}, -f_{11}),$$

respectively, induce the K-isomorphisms

$$(Ad, \mathfrak{p}_+) \cong (\tau_{[1,1;2]}, V_{11}) \text{ and } (Ad, \mathfrak{p}_-) \cong (\tau_{[1,1;-2]}, V_{11}).$$

1.5. Principal series representations.

In this paper we will be dealing with principal series representations which are parabolically induced with respect to the minimal parabolic subgroup. We take a moment here to review basic definition to that of $SU(2,2)$.

Let P_{min} be a minimal parabolic subgroup of G with Langlands decomposition $P_{min} = MAN$ with $M = Z_K(A)$. In particularly, any element of M can be represented by a matrix

$$[\exp(i\theta)]\gamma^j \text{ for some } \theta \in \boldsymbol{R} \text{ and } j = 0,\ 1,$$

where $\gamma = \mathrm{diag}(1, -1, 1, -1) \in G$ and

$$[\exp(i\theta)] = \mathrm{diag}(\exp(i\theta), \exp(-i\theta), \exp(i\theta), \exp(-i\theta)).$$

Let χ be a character of the group $\{\pm 1\}$. For an integer s, we define a unitary character of M by

$$\sigma_{\chi,s}([\exp(i\theta)]\gamma^j) = \chi(-1)^j \exp(is\theta).$$

Given a complex valued real linear form $\mu = \lambda_1\mu_1 + \lambda_2\mu_2$ on $\mathfrak{a}$, define a character e^μ of A by

$$e^\mu(a) = \exp(\mu_1 a_1 + \mu_2 a_2),$$

for $a = \exp(a_1 H_1 + a_2 H_2) \in A$. We extend it to a character of AN so that the restriction to N is trivial. Define an admissible character of P_{min} by tensoring these characters. Then one has the induced representation

$$\pi = \mathrm{Ind}_{P_{min}}^G(\sigma_{\chi,s} \otimes e^{\mu+\rho} \otimes 1_N)$$

and call it *the principal series representation* of G. Here ρ denotes the half sum of the positive roots of $\boldsymbol{\Phi}(\mathfrak{g}, \mathfrak{a})$. Now look at the compact realization of π. Then representation space H_π of π can be realized on the Hilbert space

$$L^2_{\sigma_{\chi,s}}(K) = \{f \in L^2(K) \mid \\ f(mk) = \sigma_{\chi,s}(m)f(k) \text{ for } m \in M,\ k \in K,\ a.e.\}$$

with G-action defined by

$$(\pi(g)f)(x) = e^{\mu+\rho}(a(xg))f(k(xg)),\ x \in K, g \in G,$$

where $xg = n(xg)a(xg)m(xg)k(xg)$ is the Iwasawa decomposition of the element xg.

2. The structure of K-types of the principal series representation.

In this section we express the K-isotypic components of H_π in terms of the elementary functions obtained from the tautological representation of $SU(2)$. Combining it with Lemma 1.1, the K-module structures on H_π^K is described explicitly.

2.1. Elementary functions in $L^2(K)$.

We begin this subsection with the parametrization of the unitary dual of $SU(2)$. Let $S(x)$ ($x \in SU(2)$) be a square matrix function associated to $SU(2)$ given by

$$S(x) = \begin{pmatrix} s_1(x) & s_2(x) \\ -\bar{s}_2(x) & \bar{s}_1(x) \end{pmatrix}, \quad \text{with } \det(S(x)) = 1.$$

Then we have $S(xy) = S(x)S(y)$ and $s_i(-x) = -s_i(x)$ for $i = 1, 2$. Consider $S(x)$ as a linear transformation from (X, Y) to (X', Y'), *i.e.*,

$$(X', Y') = (X, Y)\begin{pmatrix} s_1(x) & s_2(x) \\ -\bar{s}_2(x) & \bar{s}_1(x) \end{pmatrix},$$

where X, Y are independent variables. For each positive integer $n \geq 2$, there is a linear transformation

$$\mathrm{Sym}^{(n)}(S(x)) = \begin{bmatrix} s^{(n)}_{nn}(x) & \cdots & s^{(n)}_{n0}(x) \\ \vdots & \ddots & \vdots \\ s^{(n)}_{0n}(x) & \cdots & s^{(n)}_{00}(x) \end{bmatrix} = \{s^{(n)}_{ij}(x)\}_{0 \leq i,j \leq n}$$

between the homogeneous forms of (X, Y) and (X', Y') of degree n via

$$((X')^n, (X')^{n-1}Y', \ldots, (Y')^n) = (X^n, X^{n-1}Y, \ldots, Y^n) \cdot \mathrm{Sym}^{(n)}(S(x)).$$

First recall the following well-known observation without proof.

LEMMA 2.1. *The $n+1$ entries of each i-th row vector of $\mathrm{Sym}^{(n)}(S(x))$ make a canonical basis of the irreducible right $SU(2)$-representation of dimension $n+1$ in $L^2(SU(2))$. In particular, we have*

1. $\mathrm{Sym}^{(n)}(S(xy)) = \mathrm{Sym}^{(n)}(S(x))\mathrm{Sym}^{(n)}(S(y))$, $x, y \in SU(2)$,
2. $\mathrm{Sym}^{(n)}(S(x)) = \mathrm{diag}_{0 \le i \le n}(e^{\sqrt{-1}t(n-2i)})$ *if* $x = \mathrm{diag}(e^{\sqrt{-1}t}, e^{-\sqrt{-1}t})$

with $t \in \boldsymbol{R}$.

2.2. Elementary functions in $L^2(\tilde{K})$.

Fix positive integers m_1, m_2 and an integer l. Put $m = [m_1, m_2; l]$. For each quadruple $(i, j, p, q) \in \boldsymbol{Z}_+^4$ such that $i, p \le m_1$ and $j, q \le m_2$, we define a $\boldsymbol{C}$-valued function on $\tilde{K}$ by

$$S_{ij,pq}(g_1, g_2, u) = s_{ip}^{(m_1)}(g_1) s_{jq}^{(m_2)}(g_2) u^l,$$

where $g_1, g_2 \in SU(2)$ and $u \in \boldsymbol{C}^{(1)}$. For a fixed pair (i, j), a space $W_{ij}^{(m)}$ generated by

$$\{S_{ij,pq} \mid 0 \le p \le m_1, 0 \le q \le m_2\}$$

is a $\tilde{K}$-module with the action τ_m defined by

$$\tau_m(g_1, g_2; u) S_{ij,pq}(x, y; v) = S_{ij,pq}(xg_1, yg_2; vu)$$

for g_1, g_2, $x, y \in SU(2)$ and u, $v \in \boldsymbol{C}^{(1)}$. Note that for each pair (i, j), we have that $(\tau_m, W_{00}^{(m)}) \cong (\tau_m, W_{ij}^{(m)})$ and the τ_m-isotypic component in the right $\tilde{K}$-module $L^2(\tilde{K})$ is just the sum of all spaces $W_{ij}^{(m)}$, where $0 \le i \le m_1, 0 \le j \le m_2$.

2.3. K-isotypic components of the principal series representations.

For $x \in SU(2)$, Lemma 2.1 implies that

$$\mathrm{Sym}^{(n)}(S(-x)) = (-1)^n \mathrm{Sym}^{(n)}(S(x)),$$

hence $S_{ij,pq}(k) = S_{ij,pq}(-(1_2, 1_2; 1)k)$ for $k \in \tilde{K}$ when $m_1 + m_2 + l \in 2\boldsymbol{Z}$. Therefore in this case the functions $S_{ij,pq}(k)$ are well defined on K *i.e.*, we may say that

$$\hat{K} = \{(\tau_m, W_{00}^{(m)}) \mid m = [m_1, m_2; l], m_1 + m_2 + l \in 2\boldsymbol{Z}\}.$$

Note also that Lemma 2.1 shows $S_{ij,pq}(k) = \delta_{ij,pq}$ at the point $k = 1_4$. This property will be used several times later.

Set $\sigma = \sigma_{\chi,s}$. Since $L^2_\sigma(K) \subset L^2(K)$, as a right unitary representation of K, it has an irreducible decomposition of $K \times K$-bimodules

$$L^2_\sigma(K) \cong \hat{\oplus}_{\tau \in \hat{K}} \{(\tau^* \mid_M)[\sigma^{-1}] \otimes \tau\}$$

by the Peter-Weyl theorem. Here $(\tau^* \mid M)[\sigma^{-1}]$ is the σ^{-1}-isotypic component in $\tau^* \mid_M$. Hence one can explicitly describe the K-isotypic components of the principal series representation π.

LEMMA 2.2 (cf. [**3**, 3.6]). *Assume* $m_1 + m_2 \geq |s|$ *and* $l \equiv 2m_2 + s + 1 - \chi(-1) \pmod 4$. *Then the* τ_m*-isotypic component* $H_\pi(\tau_m)$ *in the principal series representation* π *is isomorphic to*

$$\oplus_\gamma W_\gamma^{(m)} \text{ with } \gamma = (t, (m_1 + m_2 + s)/2 - t)),$$

where t *runs over integers satisfying,*

$$\begin{cases} 0 \leq t \leq (m_1 + m_2 + s)/2, & if\ s < \min(m_1 - m_2, m_2 - m_1) \\ (m_1 - m_2 + s)/2 \leq t \leq m_1, & if\ s \geq \max(m_2 - m_1, m_1 - m_2) \end{cases}$$

and when $\min(m_1 - m_2, m_2 - m_1) \leq s < \max(m_1 - m_2, m_2 - m_1)$

$$\begin{cases} 0 \leq t \leq m_1, & if\ m_1 < m_2 \\ (m_1 - m_2 + s)/2 \leq t \leq (m_1 + m_2 + s)/2, & if\ m_1 > m_2. \end{cases}$$

Extending the notion given in Definition 1.1 slightly, we can define a set of markings for each isotypic component of $L^2(K)$.

DEFINITION 2.1. Let (τ_m, V_m) be an irreducible representation of K with parametrization $m = [m_1, m_2; l]$. For each possible pair (i, j), the marking on the simple K-module $(\tau_m, W_{ij}^{(m)})$ specified by the basis

$$\{S_{ij,pq}(k) \in L^2_\sigma(K) \mid 0 \leq p \leq m_1, 0 \leq q \leq m_2\}$$

is called the marking by elementary functions.

CONVENTIONS. Fix π and a marked simple K-module τ_m in $\pi|_K$ with parametrization $m = [m_1, m_2; l]$. Denote by $I(\pi, \tau_m)$ the set of all γ such that $\gamma = (t, (m_1 + m_2 + s)/2 - t))$ as in Lemma 2.2 and $W_\gamma^{(m)}$ occurs in $\pi|_K$. Then the multiplicity $m(\pi, \tau_m)$ of τ_m in $\pi|_K$ is the cardinality of the finite set $I(\pi, \tau_m)$.

When $\gamma \in I(\pi, \tau_m)$, there is a K-isomorphism from V_m onto $W_\gamma^{(m)}$ by sending the set of marked basis onto the set of marked elementary functions and hence denote this K-isomorphism by $[\gamma]$.

3. $(\mathfrak{g}, \boldsymbol{K})$-module structures.

In this section we investigate the action $\mathfrak{g} = \mathrm{Lie}(G)$ (or $\mathfrak{g}_{\boldsymbol{C}} = \mathfrak{g} \otimes \boldsymbol{C}$) on the subspace $H_{\pi,K}$ of the K-finite vectors in the representation space H_π. Because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, it suffices to investigate the action of $\mathfrak{p}$ or $\mathfrak{p}_{\boldsymbol{C}}$.

3.1. Clebsch-Gordan coefficients.

We recall that the adjoint representation of K on $\mathfrak{p}_{\boldsymbol{C}}$ splits into two irreducible components, namely the holomorphic part $\mathfrak{p}_+$ and the antiholomorphic part $\mathfrak{p}_-$. Let (τ_m, V_m) be an irreducible representation of K with parametrization $(m = [m_1, m_2; l])$.

$\mathfrak{p}_+$-side. By the well known Clebsch-Gordan theorem and Lemma 1.3, the irreducible components in the K-module $\mathfrak{p}_+ \otimes_{\boldsymbol{C}} \tau_m$ are precisely the K-representations

$$\{\ \tau_{[m_1+e_1, m_2+e_2; l+2]} \mid e_1, e_2 \in \{\pm 1\}\},$$

and we will denote these by $\tau_{[\pm,\pm;+]}$ or $\tau_{[e_1,e_2;+]}$ respectively.

For a fixed pair (e_1, e_2), $e_j \in \{\pm 1\}$ with $j = 1, 2$, we define c_t^j by

$$c_t^j = \frac{t}{m_j + 1} \ (0 \leq t \leq m_j + e_j).$$

When $\tau_{[e_1,e_2;+]}$ is non zero, we now express the canonical basis vectors of $\tau_{[e_1,e_2;+]}$ in terms of the basis vectors of $\mathfrak{p}_+ \otimes_{\boldsymbol{C}} \tau_m$ induced from those of $\mathfrak{p}_+$ and τ. In this case, denote by $I_{[\pm,\pm;+]}$ a generator of the vector space $\mathrm{Hom}_K(\tau_{[e_1,e_2;+]}, \mathfrak{p}_+ \otimes_{\boldsymbol{C}} \tau_m)$, which is unique up to constant multiple. More precisely, we have

Proposition 3.1. *The image of the* (p,q)*-th canonical basis vector* f'_{pq} *of* $\tau_{[e_1,e_2;+]}$ *under the* K*-homomorphism* $I_{[e_1,e_2;+]}$ *is given by*

i. *If* $(e_1,e_2)=(-1,-1)$ *then*

$$E_{23}\otimes f_{p+1q+1}-E_{13}\otimes f_{pq+1}+E_{24}\otimes f_{p+1q}-E_{14}\otimes f_{pq},$$

ii. *If* $(e_1,e_2)=(+1,-1)$ *then*

$$(1-c^1_p)(E_{23}\otimes f_{pq+1}+E_{24}\otimes f_{pq})+c^1_p(E_{13}\otimes f_{p-1q+1}+E_{14}\otimes f_{p-1q}),$$

iii. *If* $(e_1,e_2)=(-1,+1)$ *then*

$$(1-c^2_q)(E_{13}\otimes f_{pq}-E_{23}\otimes f_{p+1q})+c^2_q(E_{24}\otimes f_{p+1q-1}-E_{14}\otimes f_{pq-1}),$$

iv. *If* $(e_1,e_2)=(+1,+1)$ *then*

$$\begin{aligned}&-(1-c^2_q)((1-c^1_p)E_{23}\otimes f_{pq}+c^1_pE_{13}\otimes f_{p-1q})\\&+c^2_q((1-c^1_p)E_{24}\otimes f_{pq-1}+c^1_pE_{14}\otimes f_{p-1q-1})\end{aligned}$$

where $0\le p\le m_1+e_1$ *and* $0\le q\le m_2+e_2$*, respectively.*

Proof. Denote by u_{pq} the element in $\mathfrak{p}_+\otimes_{\boldsymbol{C}}\tau_m$ defined in our Proposition. To prove $I_{[e_1,e_2;+]}(f'_{pq})=u_{pq}$, it is enough to show that the correspondence $f_{pq}\to u_{pq}$ is a K-module homomorphism by utilizing the infinitesimal representation of K. Hence we only consider the first case as an example. We now claim that the weight of the vector $u_{m_1-1m_2-1}$ given by

$$E_{23}\otimes f_{m_1m_2}-E_{13}\otimes f_{m_1-1m_2}+E_{24}\otimes f_{m_1m_2-1}-E_{14}\otimes f_{m_1-1m_2-1}$$

is the same as that of $f_{m_1-1m_2-1}$ in $\tau_{[-,-;+]}$. Note that the algebra generated by h^1,h^2 and $I_{2,2}$ form a Cartan subalgebra. Moreover, it is clear that $I_{2,2}\cdot u_{m_1-1m_2-1}=(l+2)u_{m_1-1m_2-1}$. By Lemma 1.1 and Lemma 1.3, it follows that

$$\begin{aligned}h^1\cdot E_{14}\otimes f_{m_1-1m_2-1}&=(1+2(m_1-1)-m_1)E_{14}\otimes f_{m_1-1m_2-1},\\h^1\cdot E_{13}\otimes f_{m_1-1m_2}&=(1+2(m_1-1)-m_1)E_{13}\otimes f_{m_1-1m_2},\\h^1\cdot E_{24}\otimes f_{m_1m_2-1}&=(-1+2m_1-m_1)E_{24}\otimes f_{m_1m_2-1},\\h^1\cdot E_{23}\otimes f_{m_1m_2}&=(m_1+1-2)E_{23}\otimes f_{m_1m_2}.\end{aligned}$$

Hence the eigenvalue of u_{m_1-1,m_2-1} under h^1 is just $m_1 - 1$. Similarly, one can check that the eigenvalue via h^2 is equal to $m_2 - 1$. The next claim is

$$u_{p-1,q} = \frac{e_-^1 \cdot u_{p,q}}{p}$$

for all possible values of (p, q). By using Lemma 1.1 and Lemma 1.3 again, we obtain that

$$\begin{aligned}
e_-^1 \cdot E_{23} \otimes f_{p+1q+1} &= (p+1)E_{23} \otimes f_{pq+1}, \\
e_-^1 \cdot E_{13} \otimes f_{pq+1} &= E_{23} \otimes f_{pq+1} + pE_{13} \otimes f_{p-1q+1}, \\
e_-^1 \cdot E_{24} \otimes f_{p+1q} &= (p+1)E_{24} \otimes f_{pq}, \\
e_-^1 \cdot E_{14} \otimes f_{pq} &= E_{24} \otimes f_{pq} + p \cdot E_{14} \otimes f_{p-1q}.
\end{aligned}$$

Hence the claim follows from the above. Similarly, for all possible indices (p, q), we can show that $u_{pq-1} = e_-^2 \cdot u_{pq}/q$. Therefore the natural correspondence $f_{pq} \to u_{pq}$ gives a non zero K-isomorphism. □

$\mathfrak{p}_-$-side. Since $(Ad, \mathfrak{p}_-) \cong \tau_{[1,1;-2]}$, the tensor product $\mathfrak{p}_- \otimes_{\boldsymbol{C}} \tau_m$ has four irreducible K-components:

$$\{\tau_{[m_1+e_1,m_2+e_2;l-2]} \mid e_1, e_2 \in \{\pm 1\}\}$$

and we will denote these by $\tau_{[e_1,e_2;-]}$ respectively. Let $I_{[e_1,e_2;-]}$ be a generator of the vector space $\mathrm{Hom}_K(\tau_{[e_1,e_2;-]}, \mathfrak{p}_- \otimes_{\boldsymbol{C}} \tau_m)$ when $\tau_{[e_1,e_2;-]}$ is non zero. Similar to the previous Proposition, we have the following:

Proposition 3.2. *The image of the* (p, q)*-th canonical basis vector* f'_{pq} *of* $\tau_{[e_1,e_2;-]}$ *under the* K*-homomorphism* $I_{[e_1,e_2;-]}$ *is given by*

i. *If* $(e_1, e_2) = (-1, -1)$ *then*

$$E_{41} \otimes f_{p+1q+1} + E_{42} \otimes f_{pq+1} - E_{31} \otimes f_{p+1q} - E_{32} \otimes f_{pq},$$

ii. *If* $(e_1, e_2) = (+1, -1)$ *then*

$$(1 - c_p^1)(E_{31} \otimes f_{pq} - E_{41} \otimes f_{pq+1}) + c_p^1(E_{42} \otimes f_{p-1q+1} - E_{32} \otimes f_{p-1q}),$$

iii. *If* $(e_1, e_2) = (-1, +1)$ *then*

$$(1 - c_q^2)(E_{42} \otimes f_{pq} + E_{41} \otimes f_{p+1q}) + c_q^2(E_{31} \otimes f_{p+1q-1} + E_{32} \otimes f_{pq-1}),$$

iv. *If* $(e_1, e_2) = (+1, +1)$ *then*

$$\begin{aligned} &- (1 - c_q^2)((1 - c_p^1)E_{41} \otimes f_{pq} - c_p^1 E_{42} \otimes f_{p-1q}) \\ &- c_q^2((1 - c_p^1)E_{31} \otimes f_{pq-1} - c_p^1 E_{32} \otimes f_{p-1q-1}), \end{aligned}$$

where $0 \leq p \leq m_1 + e_1$ *and* $0 \leq q \leq m_2 + e_2$*, respectively.*

Proof. The proof is quite similar to that of Proposition 3.1. □

3.2. Matrix form of the Clebsch-Gordan decompositions.
For the further convenience, it is useful to describe the K-isomorphisms $I_{[e_1,e_2;\pm]}$ described in Propositions 3.1 and 3.2 in terms of the canonical basis of V_m.

To the set of all canonical basis $\{ f_{pq} \mid 0 \leq p \leq m_1, 0 \leq q \leq m_2\}$ of the simple K-module V_m, we associate a row vector of size $(m_1 + 1)(m_2 + 1)$ with entries f_{pq} given by

$$\boldsymbol{F}_\tau = (f_{00}, f_{01}, \ldots, f_{0m_2}, f_{10}, f_{11}, \ldots, f_{m_1,m_2-1}, f_{m_1m_2}).$$

$\mathfrak{p}_+$-side. Define a matrix $\mathscr{C}_{[-,-;+]} = \{C_{ij}\}$ of size $(m_1m_2) \times (m_1 + 1)(m_2 + 1)$ with entries consisting of elements in $\mathfrak{p}_+$ by

$$\begin{aligned} C_{m_2p+q+1,(m_2+1)p+q+1} &= -E_{14}, \\ C_{m_2p+q+1,(m_2+1)p+q+2} &= -E_{13}, \\ C_{m_2p+q+1,(m_2+1)(p+1)+q+1} &= E_{24}, \\ C_{m_2p+q+1,(m_2+1)(p+1)+q+2} &= E_{23}, \end{aligned}$$

for each $0 \leq p \leq m_1 - 1$ and $0 \leq q \leq m_2 - 1$, but all other entries are 0.

Define a matrix $\mathscr{C}_{[+,-;+]} = \{C_{ij}\}$ of size $(m_1 + 2)m_2 \times (m_1 + 1)(m_2 + 1)$ with entries consisting of elements in $\mathfrak{p}_+$ by

$$\begin{aligned} C_{m_2p+q+1,(m_2+1)p+q+1} &= (1 - c_p^1)E_{24}, \\ C_{m_2p+q+1,(m_2+1)p+q+2} &= (1 - c_p^1)E_{23}, \\ C_{m_2p+q+1,(m_2+1)(p-1)+q+1} &= c_p^1 E_{14}, \\ C_{m_2p+q+1,(m_2+1)(p-1)+q+2} &= c_p^1 E_{13}, \end{aligned}$$

for $0 \leq p \leq m_1+1$ and $0 \leq q \leq m_2-1$, but all other entries are 0.

Define a matrix $\mathscr{C}_{[-,+;+]} = \{C_{ij}\}$ of size $m_1(m_2+2) \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}_+$ by

$$\begin{aligned} C_{(m_2+2)p+q+1,(m_2+1)p+q+1} &= (1-c_q^2)E_{13}, \\ C_{(m_2+2)p+q+1,(m_2+1)p+q} &= -c_q^2 E_{14}, \\ C_{(m_2+2)p+q+1,(m_2+1)(p+1)+q+1} &= -(1-c_q^2)E_{23}, \\ C_{(m_2+2)p+q+1,(m_2+1)(p+1)+q} &= c_q^2 E_{24}, \end{aligned}$$

for $0 \leq p \leq m_1+1$ and $0 \leq q \leq m_2-1$, but all other entries are 0.

Define a matrix $\mathscr{C}_{[+,+;+]} = \{C_{ij}\}$ of size $(m_1+2)(m_2+2) \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}_+$ by

$$\begin{aligned} C_{(m_2+2)p+q+1,(m_2+1)p+q+1} &= -(1-c_p^1)(1-c_q^2)E_{23}, \\ C_{(m_2+2)p+q+1,(m_2+1)p+q} &= (1-c_p^1)c_q^2 E_{24}, \\ C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q+1} &= -c_p^1(1-c_q^2)E_{13}, \\ C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q} &= c_p^1 c_q^2 E_{14}, \end{aligned}$$

for each $0 \leq p \leq m_1+1$ and $0 \leq q \leq m_2+1$, but all other entries are 0. Then Proposition 3.1 reads as the following proposition.

PROPOSITION 3.3. *Let* $\mathscr{C}_{[e_1,e_2;+]}, \boldsymbol{F}_\tau$ *be as above. Then for each pair* e_1, e_2 *the simple* K*-module* $V_{[e_1,e_2;+]}$ *is generated by the entries of the matrix* $\mathscr{C}_{[e_1,e_2;+]}{}^{\boldsymbol{t}}\boldsymbol{F}_\tau$*. Moreover, these entries make a set of canonical basis.*

PROOF. Note that for the (i,j)-th entry of $\mathscr{C}_{[e_1,e_2;+]}$, the index i indicates the i-th coordinate in $\boldsymbol{F}_{[e_1,e_2;+]}$ and the index j indicates the j-th coordinate in $\boldsymbol{F}_\tau$. The i-th coordinate in $\boldsymbol{F}_{[e_1,e_2;+]}$ is uniquely expressed as

$$i = (m_2+1+e_2)p+q+1$$

for some pair (p,q) so that $0 \leq p \leq m_1+e_1$ and $0 \leq q \leq m_2+e_2$. Hence it is just the (p,q)-th canonical basis vector in $\tau_{[e_1,e_2;+]}$ by definition of $\mathscr{C}_{[e_1,e_2;+]}$. Similarly, the j-th coordinate in $\boldsymbol{F}_\tau$ corresponds to the (p,q)-th basis vector in τ. Thus the proposition follows from Proposition 3.1. □

$\mathfrak{p}_-$-side. Define a matrix $\mathscr{C}_{[-,-;-]} = \{C_{ij}\}$ of size $m_1 m_2 \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}_-$ by

$$\begin{aligned}
C_{m_2p+q+1,(m_2+1)p+q+1} &= -E_{32},\\
C_{m_2p+q+1,(m_2+1)p+q+2} &= E_{42},\\
C_{m_2p+q+1,(m_2+1)(p+1)+q+1} &= -E_{31},\\
C_{m_2p+q+1,(m_2+1)(p+1)+q+2} &= E_{41},
\end{aligned}$$

for $0 \le i \le m_1 - 1$ and $0 \le q \le m_2 - 1$, but all other entries are 0.

Define a matrix $\mathscr{C}_{[+,-;-]} = \{C_{ij}\}$ of size $(m_1+2)m_2 \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}_-$ by

$$\begin{aligned}
C_{m_2p+q+1,(m_2+1)p+q+1} &= (1-c_p^1)E_{31},\\
C_{m_2p+q+1,(m_2+1)p+q+2} &= -(1-c_p^1)E_{41},\\
C_{m_2p+q+1,(m_2+1)(p-1)+q+1} &= -c_p^1E_{32},\\
C_{m_2p+q+1,(m_2+1)(p-1)+q+2} &= c_p^1E_{42},
\end{aligned}$$

for $0 \le p \le m_1 + 1$ and $0 \le q \le m_2 - 1$, but all other entries are 0.

Define a matrix $\mathscr{C}_{[-,+;-]} = \{C_{ij}\}$ of size $m_1(m_2+2) \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}_-$ by

$$\begin{aligned}
C_{(m_2+2)p+q+1,(m_2+1)p+q+1} &= (1-c_q^2)E_{42},\\
C_{(m_2+2)p+q+1,(m_2+1)p+q} &= c_q^2E_{32},\\
C_{(m_2+2)p+q+1,(m_2+1)(p+1)+q+1} &= (1-c_q^2)E_{41},\\
C_{(m_2+2)p+q+1,(m_2+1)(p+1)+q} &= c_q^2E_{31},
\end{aligned}$$

for $0 \le p \le m_1 - 1$ and $0 \le q \le m_2 + 1$, but all other entries are 0.

Define a matrix $\mathscr{C}_{[+,+;-]} = \{C_{ij}\}$ of size $(m_1+2)(m_2+2) \times (m_1+1)(m_2+1)$ with entries consisting of elements in $\mathfrak{p}_-$ by

$$\begin{aligned}
C_{(m_2+2)p+q+1,(m_2+1)p+q+1} &= -(1-c_p^1)(1-c_q^2)E_{41},\\
C_{(m_2+2)p+q+1,(m_2+1)p+q} &= -(1-c_p^1)c_q^2E_{31},\\
C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q+1} &= c_p^1(1-c_q^2)E_{42},\\
C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q} &= c_p^1c_q^2E_{32},
\end{aligned}$$

for each $0 \le p \le m_1 + 1$ and $0 \le q \le m_2 + 1$, but all other entries are 0. Then Proposition 3.2 reads as the following proposition.

PROPOSITION 3.4. *Let* $\mathscr{C}_{[e_1,e_2;-]}, \boldsymbol{F}_\tau$ *be as above. Then for each pair* e_1, e_2 *the simple* K*-module* $V_{[e_1,e_2;-]}$ *is generated by the entries of the matrix* $\mathscr{C}_{[e_1,e_2;-]}{}^{\boldsymbol{t}}\boldsymbol{F}_\tau$.

Moreover, these entries make a set of canonical basis.

PROOF. The proof is similar to that of Proposition 3.3. □

3.3. The Dirac-Schmid operators.

In this subsection we discuss the main result of this paper, that is, to compute the matrix forms of intertwining constants explicitly.

$\mathfrak{p}_+$-side. Note that the homomorphisms $[\gamma]$ with $\gamma \in I(\pi, \tau_m)$ defined in the Section 2 form a basis of the vector space $\mathrm{Hom}_K(\tau_m, H_\pi(\tau_m))$ and hence we fix this basis for each τ_m in π. Take an element $i \in \mathrm{Hom}_K(\tau_m, H_\pi(\tau_m))$, then the $(\mathfrak{g}, K)$-module property of H_π^K gives us the canonical surjective K-homomorphism

$$\mathfrak{p}_+ \otimes_{\boldsymbol{C}} \tau_m \to \mathfrak{p}_+ \operatorname{Im}(\tau_m).$$

For the K-module $\tau_{[e_1,e_2;+]}$, by composing this K-homomorphism with the injection $\tau_{[e_1,e_2;+]} \subset \mathfrak{p}_+ \otimes_{\boldsymbol{C}} \tau_m$, we obtain a $\boldsymbol{C}$-linear map ϕ

$$\phi : \mathrm{Hom}_K(\tau_m, H_\pi(\tau_m)) \to \mathrm{Hom}_K(\tau_{[e_1,e_2;+]}, H_\pi(\tau_{[e_1,e_2;+]})),$$

which is determining the action of $\mathfrak{p}_+$ on H_π^K.

Our goal is to determine the matrix representation $\Gamma_{[e_1,e_2;+]}$ of ϕ *i.e.*, to find a matrix $\Gamma_{[e_1,e_2;+]}$ such that

$$\phi\left(\sum_{\gamma \in I(\pi,\tau_m)} [\gamma]\right) = \left(\sum_{\gamma' \in I(\pi,\tau_{m'})} [\gamma']\right) \times \Gamma_{[e_1,e_2;+]},$$

where $m' = [e_1, e_2; +]$. Therefore we have to compute the image (under ϕ) of the K-isomorphism $[\gamma] : \tau_m \to W_\gamma^{(m)}$ for each $\gamma \in I(\pi, \tau_m)$, that is, to express the K-homomorphism ϕ_γ in the commutative diagram

$$\begin{array}{ccc} \tau_{[e_1,e_2;+]} & \longrightarrow & \mathfrak{p}_+ \otimes_{\boldsymbol{C}} \tau_m \\ & \overset{\phi_\gamma}{\searrow} & \downarrow [\gamma] \\ & & \mathfrak{p}_+ W_\gamma^{(m)} \longrightarrow H_\pi(\tau_{[e_1,e_2;+]}) \end{array}$$

Diagram 1.

in terms of the fixed basis $[\gamma']$ with $\gamma' \in I(\pi, \tau_{[e_1,e_2;+]})$.

Set $\nu = (m_1 + m_2 + s)/2$. For each τ_m, we regard the vector space $\mathrm{Hom}_K(\tau_m, H_\pi(\tau_m))$ as a subspace of the $\nu + 1$-dimensional vector space

$\mathrm{Hom}_K(\tau_m, \oplus_\gamma W_\gamma^{(m)})$ with γ running over all nonnegative integer pairs (t_1, t_2) such that $t_1 + t_2 = \nu$ and hence define $\Gamma_{[e_1,e_2;+]}$ as a matrix of size $(\nu + 1 + (e_1 + e_2)/2) \times (\nu + 1)$.

REMARK 3.1. For fixed e_1, e_2, we remark that $\Gamma_{[e_1,e_2;\pm]}$ is a matrix of size $I(\pi, \tau_{[e_1,e_2;\pm]}) \times I(\pi, \tau)$ but is represented here as an embedded one inside of a matrix of size$(\nu + 1 + (e_1 + e_2)/2) \times (\nu + 1)$. Note that the explicit formula of $m(\pi, \tau_{[e_1,e_2;\pm]})$ seems to be involved.

Fix a K-module τ_m with $m = [m_1, m_2; l]$. Set $r = (s + l)/2$ and $m' = [m_1 + e_1, m_2 + e_2; l + 2]$. In the following list, we use the coefficients c_p^1 and c_q^2 defined in Subsection 3.1.

1. Define a matrix $\Gamma_{[-,-;+]} = \{a_{ij}\}_{0\le i\le \nu-1, 0\le j\le \nu}$ of size $\nu \times (\nu + 1)$ so that its all non zero entries are given by

$$\begin{aligned} a_{t-1,t} &= a_t \quad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t - 1, \nu - t) \in I(\pi, \tau_{m'}), \\ a_{t,t} &= b_t \quad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t, \nu - t - 1) \in I(\pi, \tau_{m'}), \end{aligned}$$

where

$$\begin{aligned} a_t &= \frac{1}{2}(\mu_2 + 1 + m_1 + r - 2t), \\ b_t &= -\frac{1}{2}(\mu_1 - 1 - m_2 + r - 2t) \end{aligned}$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

2. Define a matrix $\Gamma_{[+,+;+]} = \{a_{ij}\}_{0\le i\le \nu+1, 0\le j\le \nu}$ of size $(\nu + 2) \times (\nu + 1)$ so that its all non zero entries are given by

$$\begin{aligned} a_{t,t} &= a_t \quad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t, \nu - t + 1) \in I(\pi, \tau_{m'}), \\ a_{t+1,t} &= b_t \quad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t + 1, \nu - t) \in I(\pi, \tau_{m'}), \end{aligned}$$

where

$$\begin{aligned} a_t &= \frac{1}{2}(\mu_2 + 1 + m_1 + r - 2t)(1 - c_t^1)c_{\nu-t+1}^2, \\ b_t &= -\frac{1}{2}(\mu_1 + 3 + 2m_1 + m_2 + r - 2t)c_{t+1}^1(1 - c_{\nu-t}^2) \end{aligned}$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

3. Define a square matrix $\Gamma_{[-,+;+]} = \{a_{ij}\}_{0 \le i \le \nu, 0 \le j \le \nu}$ of size $(\nu + 1) \times (\nu + 1)$ so that its all non zero entries are given by

$$\begin{aligned} a_{t-1,t} &= a_t \quad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t-1, \nu - t + 1) \in I(\pi, \tau_{m'}) \\ a_{t,t} &= b_t \quad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t, \nu - t) \in I(\pi, \tau_{m'}), \end{aligned}$$

where

$$\begin{aligned} a_t &= \frac{1}{2}(\mu_2 + 1 + m_1 + r - 2t) c^2_{\nu - t + 1}, \\ b_t &= \frac{1}{2}(\mu_1 + 1 + m_2 + r - 2t)(1 - c^2_{\nu - t}) \end{aligned}$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

4. Define a square matrix $\Gamma_{[+,-;+]} = \{a_{ij}\}_{0 \le i \le \nu, 0 \le j \le \nu}$ of size $(\nu + 1) \times (\nu + 1)$ so that its all non zero entries are given by

$$\begin{aligned} a_{t,t} &= a_t \quad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t, \nu - t) \in I(\pi, \tau_{m'}), \\ a_{t+1,t} &= b_t \quad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t+1, \nu - t - 1) \in I(\pi, \tau_{m'}), \end{aligned}$$

where

$$\begin{aligned} a_t &= \frac{1}{2}(\mu_2 + 1 + m_1 + r - 2t)(1 - c^1_t), \\ b_t &= \frac{1}{2}(\mu_1 + 1 + 2m_1 - m_2 + r - 2t) c^1_{t+1} \end{aligned}$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

Our main result is these constructions of $\Gamma_{[e_1,e_2;+]}$. In the following, we show that these matrices are the desired ones.

THEOREM 3.5. *Let* (e_1, e_2) *be a pair so that* $e_1, e_2 \in \{\pm 1\}$. *Then the matrix* $\Gamma_{[e_1,e_2;+]}$ *defined above is the* $\boldsymbol{C}$*-linear homomorphism between the vector spaces* $\mathrm{Hom}_K(\tau_m, H_\pi(\tau_m))$ *and* $\mathrm{Hom}_K(\tau_{[e_1,e_2;+]}, H_\pi(\tau_{[e_1,e_2;+]}))$.

PROOF. We only consider the case $(e_1, e_2) = (-1, -1)$, because the remaining cases are proved similarly. Set $m' = [m_1 - 1, m_2 - 1; l + 2]$ and fix a basis vector $[\gamma]$. From the K-equivariant property of ϕ_γ induced from $[\gamma]$ in the Diagram 1, the image of a fixed basis element $f_{pq}^{(m')}$ in $V_{m'}$ can be expressed as

$$\phi_\gamma(f_{pq}^{(m')}) = \sum_{\gamma' \in I(\pi, m')} c_{\gamma'} S_{\gamma', pq}^{(m')}(x).$$

Note that we omit the index (m) of basis vectors for only τ_m *i.e.*, write f_{pq} instead of $f_{pq}^{(m)}$. Consider the above expression at $x = 1_4$, by using $S_{\gamma,pq}(1_4) = \delta_{\gamma,pq}$, we then get

$$\phi_\gamma(f_{pq}^{(m')})(1_4) = c_{\gamma'}, \text{ if } \gamma' = (p, q).$$

On the other hand, the commutativity of the Diagram 1 and Proposition 3.1 imply that $\phi_\gamma(f_{pq}^{(m')})$ is equal to

$$E_{23} S_{\gamma,p+1q+1}(k) - E_{13} S_{\gamma,pq+1}(k) + E_{24} S_{\gamma,p+1q}(k) - E_{14} S_{\gamma,pq}(k).$$

Note that $XS_{\gamma,pq}(k)\mid_{k=1_4} = 0$ for any $X \in \mathfrak{n}$. By considering the Iwasawa decomposition of E_{ij} $(i = 1, 2, j = 3, 4)$ given in Lemma 1.2, one can calculate that

$$\begin{aligned}
(E_{13} S_{\gamma,pq})(1_4) &= \frac{1}{2}\left(H_1 + \frac{1}{2}(I_{2,2} + h^1 - h^2)\right) S_{\gamma,pq}(k)\mid_{k=1_4} \\
&= \frac{1}{4}(2\mu_1 + 6 + l + (2p - m_1) - (2q - m_2)) S_{\gamma,pq}(1_4), \\
(E_{24} S_{\gamma,pq})(1_4) &= \frac{1}{2}\left((H_2 + \frac{1}{2}(I_{2,2} - h^1 + h^2)\right) S_{\gamma,pq}(k)\mid_{k=1_4} \\
&= \frac{1}{4}(2\mu_2 + 2 + l - (2p - m_1) + (2q - m_2)) S_{\gamma,pq}(1_4), \\
(E_{14} S_{\gamma,pq})(1_4) &= -e_+^2 S_{\gamma,pq}(k)\mid_{k=1_4} = (q - m_2) S_{\gamma,pq+1}(1_4), \\
(E_{23} S_{\gamma,pq})(1_4) &= e_-^1 S_{\gamma,pq}(k)\mid_{k=1_4} = p S_{\gamma,p-1q}(1_4).
\end{aligned}$$

Combining these observations, we obtain that $\phi_\gamma(f_{pq}^{(m')})(1_4)$ is equal to

$$\begin{aligned}
&\frac{1}{2}\left(\mu_2 + q - p + \frac{m_1 - m_2 + l}{2}\right) S_{\gamma,p+1q}(1_4) + S_{\gamma,pq+1}(1_4) \\
&\quad \times \left(-\frac{1}{2}\left(\mu_1 + 2 + p - q + \frac{m_2 - m_1 + l}{2}\right) + p + 1 - (q - m_2)\right).
\end{aligned}$$

Using $S_{\gamma,pq}(1_4) = \delta_{\gamma,pq}$ again, one has

$$\gamma' \text{ is equal to } \gamma - (1,0) \text{ or } \gamma - (0,1)$$

and hence the corresponding coefficients $c_{\gamma'}$ are just

$$c_{\gamma'} = \frac{1}{2}\left[\mu_2 + 1 + m_1 + \frac{s+l}{2} - 2t\right]$$

or

$$c_{\gamma'} = -\frac{1}{2}\left[\mu_1 - 1 - m_2 + \frac{l+s}{2} - 2t\right],$$

respectively when $\gamma = (t, \nu - t) \in I(\pi, \tau)$. It shows the coincidence of $\Gamma_{[-,-;+]}$ with ϕ. □

$\mathfrak{p}_-$-side. By the same computation as the case $\mathfrak{p}_+$-side we obtain similar results for the matrix form of the $\boldsymbol{C}$-linear map

$$\Gamma_{[e_1,e_2;-]} : \mathrm{Hom}_K(\tau_m, H_\pi(\tau_m)) \to \mathrm{Hom}_K(\tau_{[e_1,e_2;-]}, H_\pi(\tau_{[e_1,e_2;-]})).$$

1. Define a matrix $\Gamma_{[-,-;-]} = \{a_{ij}\}_{0\le i\le \nu-1, 0\le j\le \nu}$ of size $\nu \times (\nu+1)$ so that its all non zero entries are given by

$$\begin{aligned} a_{t,t} &= a_t && \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t, \nu - t - 1) \in I(\pi, \tau_{m'}),\\ a_{t-1,t} &= b_t && \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t - 1, \nu - t) \in I(\pi, \tau_{m'}),\end{aligned}$$

   where

$$\begin{aligned} a_t &= \frac{1}{2}(\mu_2 + 1 - m_1 - r + 2t),\\ b_t &= -\frac{1}{2}(\mu_1 - 1 - 2m_1 - m_2 - r + 2t)\end{aligned}$$

   for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

2. Define a matrix $\Gamma_{[+,+;-]} = \{a_{ij}\}_{0\le i\le \nu+1, 0\le j\le \nu}$ of size $(\nu+2) \times (\nu+1)$ so that its all non zero entries are given by

$$
\begin{aligned}
a_{t+1,t} &= a_t \quad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t+1, \nu - t) \in I(\pi, \tau_{m'}),\\
a_{t,t} &= b_t \quad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t, \nu - t + 1) \in I(\pi, \tau_{m'}),
\end{aligned}
$$

where

$$
\begin{aligned}
a_t &= \frac{1}{2}(\mu_2 + 1 - m_1 - r + 2t)c^1_{t+1}(1 - c^2_{\nu-t}),\\
b_t &= -\frac{1}{2}(\mu_1 + 3 + m_2 - r + 2t)(1 - c^1_t)c^2_{\nu-t+1}
\end{aligned}
$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

3. Define a square matrix $\Gamma_{[-,+;-]} = \{a_{ij}\}_{0\le i\le \nu, 0\le j\le \nu}$ of size $(\nu+1)\times(\nu+1)$ so that its all non zero entries are given by

$$
\begin{aligned}
a_{t,t} &= a_t \quad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t, \nu - t) \in I(\pi, \tau_{m'}),\\
a_{t-1,t} &= b_t \quad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t-1, \nu - t + 1) \in I(\pi, \tau_{m'}),
\end{aligned}
$$

where

$$
\begin{aligned}
a_t &= \frac{1}{2}(\mu_2 + 1 - m_1 - r + 2t)(1 - c^2_{\nu-t}),\\
b_t &= \frac{1}{2}(\mu_1 + 1 - 2m_1 + m_2 - r + 2t)c^2_{\nu-t+1}
\end{aligned}
$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

4. Define a square matrix $\Gamma_{[+,-;-]} = \{a_{ij}\}_{0\le i\le \nu, 0\le j\le \nu}$ of size $(\nu+1)\times(\nu+1)$ so that its all non zero entries are given by

$$
\begin{aligned}
a_{t+1,t} &= a_t \quad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t+1, \nu - t - 1) \in I(\pi, \tau_{m'}),\\
a_{t,t} &= b_t \quad \text{if } (t, \nu - t) \in I(\pi, \tau_m),\ (t, \nu - t) \in I(\pi, \tau_{m'}),
\end{aligned}
$$

where

$$
\begin{aligned}
a_t &= \frac{1}{2}(\mu_2 + 1 - m_1 - r + 2t)c^1_{t+1},\\
b_t &= \frac{1}{2}(\mu_1 + 1 - m_2 - r + 2t)(1 - c^1_t)
\end{aligned}
$$

for $\gamma = (t, \nu - t) \in I(\pi, \tau)$.

Thus we have the following results similar to that of $\mathfrak{p}_+$-side.

THEOREM 3.6. *Let (e_1, e_2) be a pair so that $e_1, e_2 \in \{\pm 1\}$. Then the matrix $\Gamma_{[e_1,e_2;-]}$ defined above is the $\boldsymbol{C}$-linear homomorphism between the vector spaces $\mathrm{Hom}_K(\tau_m, H_\pi(\tau_m))$ and $\mathrm{Hom}_K(\tau_{[e_1,e_2;-]}, H_\pi(\tau_{[e_1,e_2;-]}))$.*

PROOF. Set $m' = [m_1 + e_1, m_2 + e_2; l - 2]$ and fix a basis vector $[\gamma]$. From the K-equivariant property of ϕ_γ induced from $[\gamma]$ in the Diagram 1, the image of a fixed basis element $f_{pq}^{(m')}$ in $V_{m'}$ can be expressed as

$$\phi_\gamma(f_{pq}^{(m')}) = \sum_{\gamma' \in I(\pi, m')} c_{\gamma'} S_{\gamma',pq}^{(m')}(x).$$

On the other hand, the commutativity of the Diagram 1 and Proposition 3.2 imply that $\phi_\gamma(f_{pq}^{(m')})$ is equal to

$$E_{41} S_{\gamma,p+1q+1}(k) + E_{42} S_{\gamma,pq+1}(k) - E_{31} S_{\gamma,p+1q}(k) - E_{32} S_{\gamma,pq}(k).$$

Combining the fact $XS_{\gamma,pq}(k)\,|_{k=1_4} = 0$ for any $X \in \mathfrak{n}$ and the Iwasawa decomposition of E_{ji} $(i = 1, 2, j = 3, 4)$ given in Lemma 1.2, one can also calculate that

$$\begin{aligned}
(E_{31} S_{\gamma,pq})(1_4) &= \frac{1}{2}\left(H_1 - \frac{1}{2}(I_{2,2} + h^1 - h^2)\right) S_{\gamma,pq}(k)\,|_{k=1_4} \\
&= \frac{1}{4}(2\mu_1 + 6 - l - (2p - m_1) + (2q - m_2)) S_{\gamma,pq}(1_4), \\
(E_{42} S_{\gamma,pq})(1_4) &= \frac{1}{2}\left(H_2 - \frac{1}{2}(I_{2,2} - h^1 + h^2)\right) S_{\gamma,pq}(k)\,|_{k=1_4} \\
&= \frac{1}{4}(2\mu_2 + 2 - l + (2p - m_1) - (2q - m_2)) S_{\gamma,pq}(1_4), \\
(E_{32} S_{\gamma,pq})(1_4) &= -e_+^1 S_{\gamma,pq}(k)\,|_{k=1_4} = (p - m_1) S_{\gamma,p+1q}(1_4), \\
(E_{41} S_{\gamma,pq})(1_4) &= e_-^2 S_{\gamma,pq}(k)\,|_{k=1_4} = (q + a_2) S_{\gamma,pq-1}(1_4).
\end{aligned}$$

It follows that $\phi_\gamma(f_{pq}^{(m')})(1_4)$ is equal to

$$\begin{aligned}
&\left(-\frac{1}{2}\left(\mu_1 + q - p - \frac{m_2 - m_1 + l}{2}\right) + q + m_1 - p\right) S_{\gamma,p+1q}(1_4) \\
&\quad - \frac{1}{2}\left(\mu_2 + p - q - \frac{m_1 - m_2 + l}{2}\right) S_{\gamma,pq+1}(1_4).
\end{aligned}$$

As seen in the previous theorem

$$\gamma' \text{ is equal to } \gamma - (0,1) \text{ or } \gamma - (1,0)$$

and hence the corresponding coefficients $c_{\gamma'}$ are just

$$c_{\gamma'} = \frac{1}{2}\,[\mu_2 + 1 - m_1 - r + 2t]$$

or

$$c_{\gamma'} = -\frac{1}{2}\,[\mu_1 - 1 - 2m_1 - m_2 - r + 2t],$$

respectively when $\gamma = (t, \nu - t) \in I(\pi, \tau)$. It shows the coincidence of $\Gamma_{[e_1,e_2;-]}$ with ϕ. □

3.4. Matrix representations.

We now describe the relations between the matrices $\mathscr{C}_{[e_1,e_2;\pm]}$ and $\Gamma_{[e_1,e_2;\pm]}$ in terms of the marked elementary basis functions in the K-isotypic component of π. Fix τ_m with $m = [m_1, m_2; l]$. For a pair (i,j) such that $i + j = \nu$ and $i, j \in \boldsymbol{Z}_+$, we define a row matrix $\boldsymbol{F}^{(m)}_{(i,j)}$ of size $1 \times (m_1+1)(m_2+1)$ with entries in the set of all marked elementary functions of $W^{(m)}_{ij}$ introduced in Definition 2.1 as follows

$$\boldsymbol{F}^{(m)}_{\gamma} = (S_{\gamma,00}, S_{\gamma,01}, \dots, S_{\gamma,0m_2}, S_{\gamma,10}, S_{\gamma,11}, \dots, S_{\gamma,m_1(m_2-1)}, S_{\gamma,m_1m_2})$$

with $\gamma = (i,j)$. To the K-isotypic component of τ_m in π we associate a matrix $\boldsymbol{S}^{(m)}$ of size $(m_1+1)(m_2+1) \times (\nu+1)$ such that the non zero columns are those ${}^t\boldsymbol{F}^{(m)}_{\gamma}$ with entries in the K-isotypic component $H_\pi(\tau_m)$, that is,

$$\boldsymbol{S}^{(m)} = [{}^t\boldsymbol{F}^{(m)}_{(0,\nu)}, \dots, {}^t\boldsymbol{F}^{(m)}_{(\nu,0)}],$$

where the symbol ^{t} is the transpose and $\boldsymbol{F}^{(m)}_{\gamma} = 0$ when $\gamma \notin I(\pi, \tau_m)$.

Now we are in a position to state the main result which includes all results in this paper.

Theorem 3.7. *Let $\tau_{[e_1,e_2;\pm]}$ be a simple K-submodule of the K-module $\mathfrak{p}_\pm \otimes_{\boldsymbol{C}} \tau_m$ for a given simple K-module τ_m and the K-module* $(\mathrm{Ad}, \mathfrak{p}_\pm)$. *Then we have that*

$$\mathscr{C}_{[e_1,e_2;\pm]}\boldsymbol{S}^{(m)} = \boldsymbol{S}^{([e_1,e_2;\pm])}\Gamma_{[e_1,e_2;\pm]},$$

where the product of the entries of matrices of the left hand side is the differential *operation.*

3.5. Examples of contiguous relations and their composites.

Here are some examples of contiguous relations along some certain K-types in a fixed principal series π. We refer the reader to [**12**] for further reference and contiguous relations.

Let $\tau = \tau_{[m_1,m_2;l]}$ be a K-submodule of $\pi = \mathrm{Ind}_P^G(\sigma_{\chi,s} \otimes e^{\mu+\rho} \otimes 1_N)$. Then Lemma 2.2 implies that $[\pi\,|_K \colon \tau] = 1$ if $|s| = m_1 + m_2$ and $l \equiv 2m_2 + s + 1 - \chi(-1) \pmod 4$. Hence, in this case, we may assume that the size of the matrices $\Gamma_{[+,-;\pm]}, \Gamma_{[+,-;\pm]}$ are just 1×1 *i.e.*, they are constants and $\Gamma_{[+,+;\pm]}$ is of size 2×1, because the other entries are zero. Although there is no $\Gamma_{[-,-;\pm]}$, since $\tau_{[-,-;\pm]}$ does not occur in π.

Note that $H_\pi(\tau) \cong W^{(m)}_{(m_1,m_2)}$ if $s \geq 0$ and $H_\pi(\tau) \cong W^{(m)}_{(0,0)}$ if $s \leq 0$. Put

$$\nu_1 = \frac{l + m_1 - m_2}{2} \text{ and } \nu_2 = \frac{l + m_2 - m_1}{2}.$$

FORMULA 3.8. *Assume $s \geq 0$. Then we have*

$$\begin{aligned}
\mathscr{C}_{[+,-;+]}\,{}^{\boldsymbol{t}}\boldsymbol{F}^{\tau}_{(m_1,m_2)} &= \frac{1}{2}(\mu_1 + 1 + \nu_1)\,{}^{\boldsymbol{t}}\boldsymbol{F}^{\tau_{[+,-;+]}}_{(+,-)}, \\
\mathscr{C}_{[-,+;+]}\,{}^{\boldsymbol{t}}\boldsymbol{F}^{\tau}_{(m_1,m_2)} &= \frac{1}{2}(\mu_2 + 1 + \nu_2)\,{}^{\boldsymbol{t}}\boldsymbol{F}^{\tau_{[-,+;+]}}_{(-,+)}, \\
\mathscr{C}_{[+,-;-]}\,{}^{\boldsymbol{t}}\boldsymbol{F}^{\tau}_{(m_1,m_2)} &= \frac{1}{2}(\mu_2 + 1 - \nu_2)\,{}^{\boldsymbol{t}}\boldsymbol{F}^{\tau_{[+,-;-]}}_{(+,-)}, \\
\mathscr{C}_{[-,+;-]}\,{}^{\boldsymbol{t}}\boldsymbol{F}^{\tau}_{(m_1,m_2)} &= \frac{1}{2}(\mu_1 + 1 - \nu_1)\,{}^{\boldsymbol{t}}\boldsymbol{F}^{\tau_{[-,+;-]}}_{(-,+)}.
\end{aligned}$$

Here the symbols $(\pm,\pm)$ mean $(m_1 \pm 1, m_2 \pm 1)$, respectively.

FORMULA 3.9. *Assume $s \leq 0$ and set $n = (0,0)$. Then we have*

$$\begin{aligned}
\mathscr{C}_{[+,-;+]}\,{}^{\boldsymbol{t}}\boldsymbol{F}^{\tau}_{n} &= \frac{1}{2}(\mu_2 + 1 + \nu_1)\,{}^{\boldsymbol{t}}\boldsymbol{F}^{\tau_{[+,-;+]}}_{n}, \\
\mathscr{C}_{[-,+;+]}\,{}^{\boldsymbol{t}}\boldsymbol{F}^{\tau}_{n} &= \frac{1}{2}(\mu_1 + 1 + \nu_2)\,{}^{\boldsymbol{t}}\boldsymbol{F}^{\tau_{[-,+;+]}}_{n}, \\
\mathscr{C}_{[+,-;-]}\,{}^{\boldsymbol{t}}\boldsymbol{F}^{\tau}_{n} &= \frac{1}{2}(\mu_1 + 1 - \nu_2)\,{}^{\boldsymbol{t}}\boldsymbol{F}^{\tau_{[+,-;-]}}_{n}, \\
\mathscr{C}_{[-,+;-]}\,{}^{\boldsymbol{t}}\boldsymbol{F}^{\tau}_{n} &= \frac{1}{2}(\mu_2 + 1 - \nu_1)\,{}^{\boldsymbol{t}}\boldsymbol{F}^{\tau_{[-,+;-]}}_{n}.
\end{aligned}$$

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