J. Math. Soc. Japan
Vol. 61, No. 3 (2009) pp. 651–660
doi: 10.2969/jmsj/06130651

Examples of the Hurwitz transform

Dedicated to Professor Jonas Kubilius on his eightieth birthday

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(Received June 25, 2007)

Abstract. Espinosa and Moll [**2**], [**3**] studied "the Hurwitz transform" meaning an integral over [0, 1] of a Fourier series multiplied by the Hurwitz zeta function $\zeta(z,u)$, and obtained numerous results for those which arise from the Hurwitz formula. Ito's recent result [**4**] turns out to be one of the special cases of Espinosa and Moll's theorem. However, they did not give rigorous treatment of the relevant improper integrals.

In this note we shall appeal to a deeper result of Mikolás [**9**] concerning the integral of the product of two Hurwitz zeta functions and derive all important results of Espinosa and Moll. More importantly, we shall record the hidden and often overlooked fact that some novel-looking results are often the result of "duplicate use of the functional equation", which ends up with a disguised form of the original, as in the case of Johnson's formula [**5**]. Typically, Example 9.1 ((1.12) below) is the result of a triplicate use because it depends not only on our Theorem 1, which is the result of a duplicate use, but also on (1.3), the functional equation itself.

1. Introduction and the polylogarithm case.

In his recent paper [**4**], T. Ito has obtained an integral formula for the values of the Riemann zeta-function at positive odd integral arguments > 1. It turns out that it is Example 5.2 of Espinosa and Moll [**2**], and that the latter is a corollary to an anteceding deeper result of Mikolás [**9**]. However, in Ito [**4**] as in Mikolás [**9**], rigorous arguments are given ensuring the (uniform) convergence of the relevant Fourier series, while in [**2**] this important aspect seems to be untouched. This may be perceived e.g. by the passage [**2**, p.184, l.63] "the Hurwitz transform (1.27) defines an analytic function of z as long as the defining integral converges". Also in their main result, Theorem 2.2, unlike in Mikolás', it is not checked whether the integral $\int_0^1 f(z,u)\zeta(z,u)du$ exists or not, and one is not sure in which circumstance one can apply their Theorem 2.2. There are some cases where one can apply their

2000 *Mathematics Subject Classification.* Primary 11M35, 33B15.
Key Words and Phrases. Hurwitz transform, Hurwitz zeta function, Lerch zeta function, Hurwitz formula, Lerch formula, Clausen function, Mikolás' result.

theorem without hesitation, e.g. their Example 9.1 (see our Corollary 1 below), where the relevant Fourier series for $l_s(u)$ is absolutely convergent for $\sigma = \mathrm{Re}\ s > 1$. However, in other cases one needs some analyticity conditions, like the ones in Lemma 1 of Mikolás, on the function $f(z, u)$ to ensure that it can be Hurwitz-transformed.

Since Mikolás' Theorem 3 has a sound basis and has a wider region of convergence than Espinosa-Moll, we shall base our starting point on Mikolás' Theorem 3 and deduce almost all results of Espinosa-Moll, thereby deducing also Ito's result.

We use the following standard notation.

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \qquad \sigma = \mathrm{Re}\ s > 1,\ x > 0$$

indicates the Hurwitz zeta-function;

$$\zeta(s) = \zeta(s, 1) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \sigma > 1$$

is the Riemann zeta-function;

$$l_s(x) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s}, \qquad \sigma > 1\ , x \in \boldsymbol{R}$$

is the polylogarithm function or the Lerch zeta-function;

$$B_m(x) = \sum_{k=0}^{m} \binom{m}{k} B_k x^{m-k}$$

is the m-th Bernoulli polynomial with $B_k = B_k(0)$ the k-th Bernoulli number.

Then Ito's result reads

Proposition 1 ([**4**, Theorem]=[**2**, Example 5.2]). *For $m \in \boldsymbol{N}$*

$$\int_0^1 (\log \sin \pi u) B_m(u) du = \begin{cases} (-1)^{\frac{m}{2}} (2\pi)^{-m} m! \zeta(m+1), & m\ \ even, \\ 0, & m\ \ odd. \end{cases} \tag{1.1}$$

This in turn is a special case of

PROPOSITION 2 ([**2**, Example 5.1]). *For* $z \le 0$

$$\int_0^1 (\log \sin \pi u)\zeta(z,u)du = -\frac{\Gamma(1-z)}{(2\pi)^{1-z}}\sin\frac{\pi z}{2}\zeta(2-z)$$
$$= -\frac{\zeta(z)\zeta(2-z)}{2\zeta(1-z)}. \tag{1.2}$$

Here the second equality (valid for $z \ne 1$) follows from the first by appealing to the asymmetric form of the functional equation for the Riemann zeta-function

$$\zeta(1-s) = \frac{\zeta(s)(2\pi)^{1-s}}{2\Gamma(1-s)\sin\frac{\pi s}{2}}. \tag{1.3}$$

Our main result is the following Theorem 1 which contains most of the results of [**2**].

THEOREM 1. *Let* $x = \mathrm{Re}\ z$. *Then in the region* $\max\{0, 1-\sigma\} + \max\{0, x\} < 1$, *we have*

$$\int_0^1 l_s(u)\zeta(z,u)du = (2\pi)^{z-1}e^{\frac{\pi i}{2}(1-z)}\Gamma(1-z)\zeta(1+s-z). \tag{1.4}$$

LEMMA 1 (Mikolás [**9**, Satz 3]). *In the region* $D : \max\{0, \sigma\} + \max\{0, x\} < 1$, *we have*

$$\int_0^1 \zeta(s,u)\zeta(z,u)du$$
$$= 2(2\pi)^{s+z-2}\Gamma(1-s)\Gamma(1-z)\cos\frac{\pi}{2}(s-z)\zeta(2-s-z) \tag{1.5}$$

and

$$\int_0^1 \zeta(s,u)\zeta(z,1-u)du$$
$$= -2(2\pi)^{s+z-2}\Gamma(1-s)\Gamma(1-z)\cos\frac{\pi}{2}(s+z)\zeta(2-s-z). \tag{1.6}$$

We also appeal to the functional equation for the Hurwitz zeta-function (the Hurwitz formula)

$$l_s(x) = i\,\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left(e^{-\frac{\pi i s}{2}}\zeta(1-s,x) - e^{\frac{\pi i s}{2}}\zeta(1-s,1-x)\right), \qquad 0 < x < 1. \tag{1.7}$$

Proof of Theorem 1. Substituting (1.7) in the integral in (1.4), we obtain

$$\begin{aligned} S &:= \int_0^1 l_s(u)\zeta(z,u)du \\ &= i\,\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\left(e^{-\frac{\pi i s}{2}}\int_0^1 \zeta(1-s,u)\zeta(z,u)du \right.\\ &\qquad\qquad \left. - e^{\frac{\pi i s}{2}}\int_0^1 \zeta(1-s,1-u)\zeta(z,u)du\right). \end{aligned}$$

Hence substituting (1.5) and (1.6), we deduce that

$$\begin{aligned} S =& i\,\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\,2(2\pi)^{-1-s+z}\Gamma(s)\Gamma(1-z)\zeta(1+s-z) \\ &\times\left(e^{-\frac{\pi i s}{2}}\cos\frac{\pi}{2}(1-s-z) + e^{\frac{\pi i s}{2}}\cos\frac{\pi}{2}(1+s-z)\right). \end{aligned}$$

Now the factor on the right of the above equality is seen to be

$$e^{-\frac{\pi i s}{2}}\sin\pi z.$$

Hence, using the reciprocal relation

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}, \tag{1.8}$$

we find that

$$S = i(2\pi)^{z-1}\Gamma(1-z)e^{-\frac{\pi i z}{2}}\zeta(1+s-z),$$

which leads to (1.4). This completes the proof. □

We recall the well-known formula [**11**, Proposition 3.2]:

$$l_k(x) = \frac{(2\pi i)^{k-1}}{k!}\,(A_k(x) - \pi i B_k(x)), \qquad 0 < x < 1,\ k \in \boldsymbol{N}, \tag{1.9}$$

where

$$A_k(x) = k \int_0^x A_{k-1}(t)dt + \begin{cases} \dfrac{(-1)^{\frac{k-1}{2}}k!}{(2\pi)^{k-1}}\zeta(k), & k \text{ odd}, \\ 0, & k \text{ even}. \end{cases}$$

We also recall the Clausen functions

$$Cl_{2n}(2\pi x) = \sum_{k=1}^{\infty} \frac{\sin 2\pi kx}{k^{2n}} = (-1)^{n-1} \frac{(2\pi)^{2n-1}}{(2n)!} A_{2n}(x)$$

and

$$Cl_{2n+1}(2\pi x) = \sum_{k=1}^{\infty} \frac{\sin 2\pi kx}{k^{2n+1}} = (-1)^{n} \frac{(2\pi)^{2n}}{(2n+1)!} A_{2n+1}(x).$$

Using these we may deduce

Corollary 1 ([**2**, Example 9.1]). *For $x = \operatorname{Re} z < 1$, we have*

$$\int_0^1 \frac{(2\pi i)^{k-1}}{k!}(A_k(u) - \pi i B_k(u))\zeta(z,u)du$$
$$= (2\pi)^{z-1} e^{\frac{\pi i}{2}(1-z)}\Gamma(1-z)\zeta(1-z+k). \tag{1.10}$$

The real and imaginary parts of (1.10) *read*

$$\frac{(2\pi)^{k-1}}{k!}\int_0^1 A_k(u)\zeta(z,u)du$$
$$= (2\pi)^{z-1}\cos\frac{\pi}{2}(2-z-k)\Gamma(1-z)\zeta(1-z+k) \tag{1.11}$$

and

$$\int_0^1 B_k(u)\zeta(z,u)du = (-1)^{k+1}\frac{k!\zeta(z-k)}{(1-z)_k}, \tag{1.12}$$

respectively, where $(z)_k = z(z+1)\cdot\dots\cdot(z+k-1)$ signifies the Pochhammer symbol.

Proof. Only (1.12) needs a proof. Formula (1.10) gives in the first instance

$$\frac{\pi(2\pi)^{k-1}}{k!}\int_0^1 B_k(u)\zeta(z,u)du$$
$$= (-1)^{k-1}(2\pi)^{z-1}\sin\frac{\pi}{2}(z-k)\Gamma(1-z)\zeta(1-z+k).$$

Noting that

$$\Gamma(1-z) = \frac{\Gamma(1+k-z)}{(1-z)(2-z)\cdot\cdots\cdot(k-z)} = \frac{\Gamma(1+k-z)}{(1-z)_k},$$

we obtain

$$\frac{2\pi(2\pi)^{k-1}}{k!}\int_0^1 B_k(u)\zeta(z,u)du$$
$$=(-1)^{k-1}(2\pi)^{z-1}\sin\frac{\pi}{2}(z-k)\,\frac{2\Gamma(1-(z-k))}{(1-z)_k}\,\zeta(1-(z-k))$$
$$=\frac{(-1)^{k-1}(2\pi)^k\zeta(z-k)}{(1-z)_k}$$

by (1.3), and this lead to (1.12). □

Remark 1.

(i) We note that (1.12) is Example 3.5 of [**2**] in which $z \le 0$. In the form of (1.12), one cannot clearly see that it is a result of applying the functional equations (1.3) and (1.7). More importantly, by [**8**], [**1**], it is known that the functional equations are equivalent in a loose sense, i.e. they are equivalent if we take some known formulas for granted. Thus we may think of (1.12) as the result of a duplicate use of the functional equation—a double reflection (with respect to $\sigma = 1/2$), which makes things return back to the nearly original form (for another example, see [**5**]).

(ii) On the other hand, (1.11) leads to [**2**, Example 9.2]. Indeed

$$(-1)^{n-1}\int_0^1 Cl_{2n}(2\pi n)\zeta(z,u)du$$
$$= (2\pi)^{z-1}\cos\frac{\pi}{2}(2-2n-z)\Gamma(1-z)\zeta(1-z+2n)$$
$$= (-1)^{n-1}(2\pi)^{z-1}\cos\frac{\pi}{2}z\,\Gamma(1-z)\zeta(1-z+2n)$$

which is [**2**, (9.7)]. Similarly ($n \geq 0$),

$$(-1)^n \int_0^1 Cl_{2n+1}(2\pi n)\zeta(z,u)du$$
$$= (-1)^n (2\pi)^{z-1} \sin\frac{\pi}{2} z\,\Gamma(1-z)\zeta(2-z+2n) \tag{1.13}$$

leads to [**2**, (9.8)]. The special case $n = 0$ of (1.13) leads to the first equality in (1.2).

(iii) We conclude that in the same way as in (1.12), Proposition 1 is a result of three times reflections.

Corollary 2 ([**2**, Theorem 3.7]). *For* $n \in \boldsymbol{N}$

$$\int_0^1 u^n \zeta(z,u)du = n! \sum_{j=1}^{n} (-1)^{j+1} \frac{\zeta(z-j)}{(1-z)_j (n-j+1)!}. \tag{1.14}$$

Proof. Although Espinosa and Moll use the induction, we show that this is an immediate consequence of (1.12) and the known identity

$$u^n = \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} B_j(u). \tag{1.15}$$

Indeed, substituting (1.15), the left-hand side of (1.14) becomes

$$\frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} \int_0^1 B_j(u)\zeta(z,u)du,$$

whence by (1.12) the result follows. For generalizations, see [**2**, Example 2.2] and [**6**]. □

The exponential function case was treated earlier by Mikolás [**10**], and we are left with the case of derivatives of $\zeta(s,u)$, which we shall study in Section 2.

2. The derivative case.

Mikolás [**9**] proved that the product $\zeta(s,u)\zeta(z,u)$ is absolutely integrable over $(0,1)$, whence we may conclude that the left-hand side of (1.5) is an analytic function in the region D. Hence we may differentiate with respect to s under the integral sign to deduce [**2**, Proposition 7.1] rigorously.

THEOREM 2 (Mikolás and Espinosa-Moll). *For s, z in the region D,*

$$\begin{aligned}\int_0^1 &\frac{\partial}{\partial s}\zeta(s,u)\zeta(z,u)du \\ &= -\frac{2\Gamma(1-s)\Gamma(1-z)}{(2\pi)^{2-s-z}}\cos\frac{\pi}{2}z\Gamma(s-z)\zeta(2-s-z) \\ &\times\left(\frac{\zeta'}{\zeta}(2-s-z)+\frac{\pi}{2}\tan\frac{\pi}{2}(s-z)-2\log\sqrt{2\pi}+\psi(1-s)\right),\end{aligned} \tag{2.1}$$

where $\psi(z)$ signifies the Euler digamma function $\psi(z)=\frac{\Gamma'}{\Gamma}(z)$.

As pointed out by Espinosa and Moll [**2**, p.89], if we apply Lerch's formula

$$\zeta'(0,u)=\log\frac{\Gamma(u)}{\sqrt{2\pi}}, \tag{2.2}$$

we may deduce the result on the *loggamma* function

COROLLARY 3 ([**2**, Example 6.1]). *For $x=\mathrm{Re}\ z<1$, we have*

$$\begin{aligned}\int_0^1 &\log\frac{\Gamma(u)}{\sqrt{2\pi}}\zeta(z,u)du \\ &= \frac{2\Gamma(1-z)}{(2\pi)^{2-z}}\zeta(2-z)\left(\pi\sin\frac{\pi}{2}z+2\cos\frac{\pi z}{2}\left(A-\frac{\zeta'}{\zeta}(2-z)\right)\right),\end{aligned} \tag{2.3}$$

where $A=2\log\sqrt{2\pi}+\gamma=-2\frac{d}{dz}\left(\zeta(z)\Gamma(1-z)\right)\big|_{z=0}$ and $\gamma=-\psi(1)$ is Euler's constant.

COROLLARY 4. *Correspondingly to Theorem 2 and Corollary 3 we have the formula*

$$\begin{aligned}\int_0^1 &\frac{\partial}{\partial s}\zeta(s,1-u)\zeta(z,u)du \\ &= -\frac{2\Gamma(1-s)\Gamma(1-x)}{(2\pi)^{2-s-x}}\zeta(2-s-x)\cos\frac{\pi}{2}z\Gamma(s+x) \\ &\times\left(\frac{\zeta'}{\zeta}(2-s-x)+\frac{\pi}{2}\tan\frac{\pi}{2}(s+x)-2\log\sqrt{2\pi}+\psi(1-s)\right)\end{aligned} \tag{2.4}$$

and

$$\int_0^1 \log \frac{\Gamma(1-u)}{\sqrt{2\pi}} \zeta(z,u)du$$
$$= \frac{2\Gamma(1-z)}{(2\pi)^{2-z}} \zeta(2-z) \cos\frac{\pi}{2} z\left(\frac{\zeta'}{\zeta}(2-z) + \frac{\pi}{2}\tan\frac{\pi}{2} z - 2\log\sqrt{2\pi} - \psi\right). \quad (2.5)$$

PROOF OF PROPOSITION 2. Summing (2.3) and (2.5), we obtain

$$\int_0^1 \log \frac{\Gamma(u)\Gamma(1-u)}{\sqrt{2\pi}} \zeta(z,u)du = \frac{\Gamma(1-z)}{(2\pi)^{2-z}} \zeta(2-z) 2\pi \sin\frac{\pi z}{2},$$

whence by means of the reciprocal relation (1.8) and the basic formula

$$\int_0^1 \zeta(s,u)du = 0, \quad \sigma > 1, \quad (2.6)$$

(1.2) follows, where the integral on the left is proper for $\sigma \geq 0$ and convergent (Mikolás [**9**, (5.1)]). This proves Proposition 2, and *a fortiori*, Proposition 1, Ito's main result. □

REMARK 2. Mikolás' [**9**, (footnote 21), p.158] remarked that the integral

$$\int_0^1 \zeta(s,u)\zeta(z,u)du$$

may not exist outside of the region D as the (Cauchy mean value of) Riemann integral. In connection with this, Espinosa and Moll's study on the positive value case of z ([**2**, p.184]) may be of some interest. It seems that the function represented by the integral can be continued up to the singularity of the resulting function.

As an example, note that the region of validity of (1.12) extends to $x < 2$ in the case k is an odd integer since the trivial zero of ζ at $z = 1-k$ cancels the zero (at $z = 1$) of the denominator, making the right-hand side analytic over to $x < 2$. But in [**2**, Example 12.3] the region of validity may be only $x < 1$. We hope to return to this "extending the region of validity" problem elsewhere.

ACKNOWLEDGEMENTS. The authors would like to thank Professor Yoshio Tanigawa and Professor Haruo Tsukada for enlightening discussions and many

useful comments on the manuscript.

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