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Automorphism groups of q-trigonal planar Klein surfaces and maximal surfaces

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Abstract. A compact Klein surface $X = \mathscr{D}/\Gamma$, where $\mathscr{D}$ denotes the hyperbolic plane and Γ is a surface NEC group, is said to be q-trigonal if it admits an automorphism φ of order 3 such that the quotient $X/<\varphi>$ has algebraic genus q.

In this paper we obtain for each q the automorphism groups of q-trigonal planar Klein surfaces, that is surfaces of topological genus 0 with $k \geq 3$ boundary components. We also study the surfaces in this family, which have an automorphism group of maximal order (maximal surfaces). It will be done from an algebraic and geometrical point of view.

1. Introduction.

A *Klein surface* X is a compact surface equipped with a dianalytic structure, determined by an atlas in which the transition functions can be analytic or antianalytic [**1**]. These surfaces are a generalization of Riemann surfaces and they can have boundary or can be non-orientable surfaces. Klein surfaces are uniformized by non-Euclidean crystallographic groups in a similar way as Riemann surfaces are uniformized by Fuchsian groups. So X is the quotient $\mathscr{D}/\Gamma$ where $\mathscr{D}$ is the hyperbolic plane and Γ is a surface non-Euclidean crystallographic (NEC in short) group. NEC groups are discrete groups of isometries of $\mathscr{D}$, including orientation reversing transformations. For the relation between Klein surfaces and NEC groups see the monograph [**5**].

A Klein surface X is said to be *q-trigonal* if it admits an automorphism ϕ of order three such that $X/<\phi>$ where the algebraic genus is q. If $q = 0$ the surface is called cyclic trigonal. These surfaces were studied and their automorphism groups are found in [**4**]. For $q \neq 0$, q-trigonal surfaces were studied in [**7**]. Let us denote by $\mathscr{K}^{\pm}_{g,k}$ the family of Klein surfaces of topological genus g, k boundary

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components, orientable (+) or not (−). For each family, the q-trigonality was characterized by means of NEC groups in [**7**] and the values of q for which there exist q-trigonal surfaces were also calculated.

A Klein surface X is said to be *planar* if it is a surface of topological genus 0 with $k \geq 3$ boundary components. The automorphism groups of planar Klein surfaces were found in [**3**] and they are $\boldsymbol{Z}_n$, $\boldsymbol{Z}_n \times \boldsymbol{Z}_2$, $\boldsymbol{D}_n$, $\boldsymbol{D}_n \times \boldsymbol{Z}_2$, A_4, $A_4 \times \boldsymbol{Z}_2$, $S_4 \times \boldsymbol{Z}_2$, A_5 and $A_5 \times \boldsymbol{Z}_2$.

In this paper we study the automorphism groups of q-trigonal planar Klein surfaces. For each family $\mathscr{K}^+_{0,k}$ of planar surfaces there is a unique admissible value q [**7**, Corollary 15]. Namely, $q = [k] + \overline{k} - 1$, where $[k]$ and $\overline{k}$ denote the integer part and the remainder mod(3) of k respectively. For each type of group G above described, we characterize the planar Klein surfaces which admit G as its automorphism group. It is done in Section 3. In Section 4 we study these surfaces by means of right-angled fundamental regions.

In the next Section we give necessary preliminaries about NEC groups and Klein surfaces.

2. Preliminaries.

An NEC group Γ is a discrete subgroup of isometries of the hyperbolic plane $\mathscr{D}$ (including orientation reversing isometries) with compact quotient $\mathscr{D}/\Gamma$ [**13**]. The *signature* of Γ is the following symbol and it determines its algebraic structure [**9**]:

$$\sigma(\Gamma) : (g; \pm; [m_1, \dots, m_r], \{(n_{1,1}, \dots, n_{1,s_1}), \dots, (n_{k,1}, \dots, n_{k,s_k})\}), \tag{1}$$

where $g, k \geq 0$, $m_i, n_{i,j} \geq 2$ and every number is an integer. The quotient $\mathscr{D}/\Gamma$ is an orbifold of topological genus g with k boundary components. The brackets $(n_{i,1}, \dots, n_{i,s_i})$ are called *cycle-periods* and the numbers m_i and $n_{i,j}$ are called *proper periods* and *link periods*, respectively. If $r = 0$ and $k = 0$ or $s_i = 0$, we write in each respective case [−], {−}, (−), and it happens that the quotient $\mathscr{D}/\Gamma$ is a surface. Also we write m_i^t, $n_{i,j}^t$ or $(-)^t$ when a period or an empty cycle-period is repeated t times.

The *algebraic genus* of Γ is the number $p = \eta g + k - 1$ where $\eta = 2$ or 1 according to the sign in σ be '+' or '−'. It is also called the algebraic genus of the quotient $\mathscr{D}/\Gamma$. The area of Γ is the area of any fundamental region of Γ. It is denoted by $|\Gamma|$ and it satisfies

$$|\Gamma| = 2\pi\left(\eta g + k - 2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1 - \frac{1}{n_{i,j}})\right).$$

An NEC group Γ with signature given by (1) there exists if and only if $|\Gamma| > 0$.

Let Γ be an NEC group with signature as (1). Γ is generated by $\{x_i\}_{i=1,\dots,r}$ elliptic transformations, $\{e_i\}_{i=1,\dots,k}$ hyperbolic transformations, $\{c_{i,j}\}_{\substack{i=1,\dots,k \\ j=0,1,\dots,s_i}}$ reflections and $\{a_i, b_i\}_{i=1,\dots,g}$ hyperbolic transformations (if sign is '+') or $\{d_i\}_{i=1,\dots,g}$ glide reflections (if sign is '−'). The generators satisfy the following relations:

$$\begin{aligned}
&x_i^{m_i} = 1 && i = 1, \dots, r, \\
&c_{i,j-1}^2 = c_{i,j}^2 = (c_{i,j-1}c_{i,j})^{n_{i,j}} = 1 && i = 1, \dots, k,\ j = 1, \dots, s_i, \\
&e_i^{-1}c_{i,0}e_ic_{i,s_i} = 1 && i = 1, \dots, k, \\
&\prod_{i=1}^{r} x_i \prod_{i=1}^{k} e_i \prod_{i=1}^{g}[a_ib_i] = 1 && \text{if sign '+'},
\end{aligned}$$

or

$$\prod_{i=1}^{r} x_i \prod_{i=1}^{k} e_i \prod_{i=1}^{g} d_i^2 = 1 \qquad \text{if sign '−'},$$

where $[a_ib_i]$ denotes the commutator $a_ib_ia_i^{-1}b_i^{-1}$.

An NEC group with sign '+' in the signature and $k = 0$, (hence $g \geq 2$) is a Fuchsian group. An NEC group which is not a Fuchsian group is called a *proper* NEC group. The subgroup of all orientation preserving elements of a proper NEC group Γ is called *the canonical Fuchsian group* of Γ and denoted by Γ^+.

NEC groups uniformize Klein surfaces in the same way as Fuchsian groups uniformize Riemann surfaces. The Uniformization Theorem, obtained in [**12**], asserts: Let X be a Klein surface of topological genus g with k boundary components, there exists an NEC group Γ with signature

$$(g; \pm; [-], \{(-)^k\}), \tag{2}$$

such that $X = \mathscr{D}/\Gamma$. In that case Γ is said a *surface* NEC group and we say that Γ uniformizes the surface X.

NEC groups permit also to characterize the automorphism groups of Klein surfaces. A group G is a group of automorphisms of X with order N, if and only if there exists an NEC group Λ with $\Gamma \lhd_N \Lambda$ such that $G = \Lambda/\Gamma$ [**11**]. The automorphism group of X, denoted by $Aut(X)$, is the quotient $N_{\mathscr{G}}(\Gamma)/\Gamma$ where the group $N_{\mathscr{G}}(\Gamma)$ denotes the normalizer of Γ in the group $\mathscr{G}$ of isometries of $\mathscr{D}$.

Using these characterizations based on NEC groups, the automorphism groups of planar Klein surfaces were calculated in [**3**, Theorem 3.1]. The results are shown in the following

THEOREM 1 ([**3**, Theorem 3.1]). *Let X be a planar Klein surface with $k \geq 3$ boundary components. Then the group $Aut(X)$ is one of the following, and each group is achieved for a given k if and only if the corresponding conditions are satisfied*

$\boldsymbol{Z}_N$	*If N is odd and N divides strictly k, $k-4$ or $k-2$; or* *If N and k are even and N divides strictly k or $k-2$; or* *If N even, k odd and N divides strictly $2k$, $2k-4$ or $k-1$*
$\boldsymbol{Z}_N \times \boldsymbol{Z}_2$ *$N \neq 2$ even*	*If N divides k and $N \neq k$, $\frac{k}{2}$* *or* *N divides $k-2$ and $N \neq k-2$, $\frac{k-2}{2}$*
D_N	*If N is odd, N divides k, $k-4$ or $k-2$ and $n \neq k$; or* *If N and k are even and N divides k or $k-2$ and $n \neq k$; or* *If N even, k odd and N divides $2k$, $2k-4$ or $k-1$*
$D_N \times \boldsymbol{Z}_2$	*N even and N divides k or $k-2$*
A_4	$k = 4a + 6b + 12c$ *where* $a = 0, 1, 2;\ b = 0, 1;\ c = 1, 2, 3, \ldots$
$A_4 \times \boldsymbol{Z}_2$	$k = 4a + 6b + 12c$ *where* $a = 0, 2;\ b = 0, 1;\ c = 1, 2, 3, \ldots$
S_4	$k = 4a + 6b + 12c + 24d$ *where* $a = 0, 1, 2;\ b = 0, 1;\ c = 0, 1;$ $d = 0, 1, 2, \ldots;\ a + b + c + d > 0$ *and if* $b = 1$ *then* $a + c + d > 0$
$S_4 \times \boldsymbol{Z}_2$	$k = 4a + 6b + 12c + 24d$ *where* $a = 0, 1;\ b = 0, 1;\ c = 0, 1;$ $d = 0, 1, 2, \ldots;\ a + b + c + d > 0$
A_5	$k = 12a + 20b + 30c + 60d$ *where* $a = 0, 1;\ b = 0, 1;\ c = 0, 1;$ $d = 1, 2, 3, \ldots$
$A_5 \times \boldsymbol{Z}_2$	$k = 12a + 20b + 30c + 60d$ *where* $a = 0, 1;\ b = 0, 1;\ c = 0, 1;$ $d = 0, 1, 2, \ldots; a + b + c + d > 0$

Let $X = \mathscr{D}/\Gamma$ be a *q-trigonal* Klein surface, as we defined in the Introduction. Let ϕ be an order three automorphism such that the quotient $X/ < \phi >$ has algebraic genus q. In terms of NEC groups the existence of such automorphism ϕ is equivalent to the existence of an NEC group Γ^* such that $\Gamma \lhd_3 \Gamma^*$. This group Γ^* is called *a q-trigonal group* of X and uniformizes the quotient $X/ < \phi >$.

As it was said before the q-trigonal surfaces were characterized in terms of NEC groups in [**7**]. Now we particularize these results to the case of planar surfaces. We start with the following

LEMMA 1 ([**7**, Corollary 15]). *The family $\mathscr{K}^+_{0,k}$ (planar surfaces) contains q-trigonal surfaces for every $k \geq 3$, where q is unique and equal to $(k-3)/3, (k-1)/3$, or $(k+1)/3$ according to $k \equiv 0$, 1 or 2 mod(3), respectively.*

Let us write the number of boundary components k as $k = 3[k] + \overline{k}$, where $\overline{k}$ is the integer remainder of k modulo 3. We have the immediate

COROLLARY 1. *Let X be a q-trigonal planar Klein surface with k boundary components. Then $q = [k] + \overline{k} - 1$.*

The q-trigonal groups are given in the following

PROPOSITION 1. *Let X be a planar q-trigonal Klein surface with k boundary components. Then, every q-trigonal group of X has the following signature*

$$(0, +, [3^{2-\overline{k}}], \{(-)^{[k]+\overline{k}}\}).$$

PROOF. From [**7**, Proposition 11] we have that a q-trigonal group Γ^* of X has one of the following signatures

$$(g^*, \pm, [3^{\frac{p+2-3q}{2}}], \{(-)^{q+1-\eta g^*}\}),$$

for each g^*, $0 \leq g^* \leq \dfrac{3q-k+3}{3\eta}$, where $sign(\Gamma) = sign(\Gamma^*)$ and if $sign(\Gamma) =$ “ $-$ ” then g^* is even. In our case $p = k-1$ and $q = [k] + \overline{k} - 1$ and so the number of proper periods is $2 - \overline{k}$. We now deal with the number of cycle-periods. We have

$$0 \leq g^* \leq \frac{3q-k+3}{3\eta} = \frac{\overline{k}}{3\eta} < 1,$$

and hence $g^* = 0$ and $sign(\Gamma^*) =$ “ $+$ ”. So that the number of cycle-periods is $[k] + \overline{k}$. □

The signature of the q-trigonal group tell us about the geometrical properties of the action of a q-trigonal automorphism: it leaves $2 - \bar{k}$ fixed points of order 3 and $q+1=[k]+\overline{k}$ invariant curves.

3. The automorphism group.

In this Section we study which of the groups shown in Theorem 1 can be realized as the automorphism group of a q-trigonal planar Klein surface. We divide our study according to q be odd or even. The answer depends on the integer remainder $k\,\mathrm{mod}(6)$.

Let us write $q=2t+1$ or $q=2t$. As a consequence of Lemma 1, for each t there are only three values of k such that there exist planar Klein surfaces with k boundary components which are q-trigonal surfaces. They are three consecutive even (respectively, odd) numbers as it is shown in the following

COROLLARY 2. *Let $q=2t+1$ or $q=2t$. Then there exist q-trigonal planar Klein surfaces with k boundary components if and only if $k=6t+2, 6t+4$ or $6(t+1)$ in the first case and $k=6t-1, 6t+1$ or $6t+3$ in the second one.*

PROOF. It follows from $q=[k]+\overline{k}-1$ and $k=3[k]+\overline{k}$. □

Now we run over the groups in Theorem 1 having in mind that the order N of the groups in that Theorem is $N=3n$ in our case. These groups can be cyclic $\boldsymbol{Z}_{3n}$, dihedral $\boldsymbol{D}_{3n}$, alternating groups A_4 and A_5, the symmetric group S_4, and the direct products $\boldsymbol{Z}_{3n}\times\boldsymbol{Z}_2$, $\boldsymbol{D}_{3n}\times\boldsymbol{Z}_2, A_4\times\boldsymbol{Z}_2$, $A_5\times\boldsymbol{Z}_2$, $S_4\times\boldsymbol{Z}_2$. Let us denote by $D(x)$ the set of divisors of the number x and by $D^*(x)$ the set of divisors different to x. We separate the cases in which the automorphism group involves cyclic and dihedral groups from the cases with alternating and symmetric groups.

THEOREM 2. *Let $q=2t+1$, where t is a natural number. Then for each n satisfying the conditions in the following table there exist q-trigonal planar Klein surfaces X with k boundary components such that $Aut(X)=G$.*

Group	n	$k=6t+2$	$k=6t+4$	$k=6(t+1)$
$\boldsymbol{Z}_{3n}$	*odd* *even*	$n\in D(t)$ $n\in D^*(2t)$	$n\in D^*(2t+1)$ $---$	$n\in D^*(2t+2)$ $n\in D^*(2t+2)$
$\boldsymbol{Z}_{3n}\times\boldsymbol{Z}_2$	*even*	$n\in D^*(2t)$ $n\neq t$	$---$	$n\in D^*(2t+2)$ $n\neq t+1$
$\boldsymbol{D}_{3n}$	*odd* *even*	$n\in D(2t)$ $n\in D(2t)$	$n\in D(2t+1)$ $---$	$n\in D^*(2t+2)$ $n\in D^*(2t+2)$
$\boldsymbol{D}_{3n}\times\boldsymbol{Z}_2$	*even*	$n\in D(2t)$	$---$	$n\in D(2t+2)$

PROOF. Let X be a q-trigonal planar surface X where $q = 2t+1$. From Corollary 5 we have that the number k of boundary components of X must be equal to $6t+2$, $6t+4$ or $6t+6$. By Theorem 1 the automorphism group $Aut(X)$ is a cyclic group of odd order $3n$ if and only if

$$3n \text{ is a proper divisor of } k,\ k-1 \text{ or } k-2 \tag{3}$$

If $k = 6t+2$ then (3) is equivalent to $3n$ divides properly $6t$. Since n is odd we have n divides t.

If $k = 6t+4$ then (3) is satisfied if and only if $3n$ divides properly $6t+3$, so $n \in D^*(2t+1)$.

In the case $k = 6t+6$ the condition (3) is equivalent to n divides $t+1$.

The necessary and sufficient condition to be $Aut(X)$ a cyclic group of even order $3n$, that is if n is even, is

$$3n \text{ divides properly } k \text{ or } k-2. \tag{4}$$

So k can be $6t+2$ or $6(t+1)$. In the first case (4) is equivalent to n divides properly $2t$, in the second one n must be a proper divisor of $2t+2$.

The remaining cases can be obtained in the same way using Theorem 1 and Corollary 5. □

THEOREM 3. *Let $q = 2t+1$, where t is a natural number. Then there exist q-trigonal planar Klein surfaces X with k boundary components such that $Aut(X) = G$ for each t satisfying the conditions in the following table*

Group	$k = 6t+2$		$k = 6t+4$	$k = 6(t+1)$	
A_4	$t \geq 3$		$t \geq 2$	$t \geq 1$	
$A_4 \times \boldsymbol{Z}_2$	$t \geq 3$		$---$	$t \geq 1$	
S_4	$t \geq 1$		$t \geq 0$	$t \geq 1$	
$S_4 \times \boldsymbol{Z}_2$	$t \geq 1$		$---$	$t \geq 0$	
A_5	$t = 10l$ $t = 10l+3$ $t = 10l+5$ $t = 10l+8$	$t \geq 1$	$---$	$t = 10l-1$ $t = 10l+1$ $t = 10l+4$ $t = 10l+6$	$l \geq 1$
$A_5 \times \boldsymbol{Z}_2$	$t = 10l$ $t = 10l+3$ $t = 10l+5$ $t = 10l+8$	$t \geq 1$ $t \geq 0$ $t \geq 0$ $t \geq 0$	$---$	$t = 10l-1$ $t = 10l+1$ $t = 10l+4$ $t = 10l+6$	$l \geq 1$ $l \geq 0$ $l \geq 0$ $l \geq 0$

PROOF. Let X be a q-trigonal planar surface X where $q = 2t+1$. From

Corollary 5 we have that the number k of boundary components of X must be equal to $6t+2$, $6t+4$ or $6t+6$. By Theorem 1 the automorphism group of X is the alternating group A_5 if and only if

$$k = 12a + 20b + 30c + 60d, \text{ with } a = 0,1,\ b = 0,1,\ c = 0,1,\ d = 1,2,3,\dots$$

This condition is equivalent to

$$k \geq 60 \text{ and } k \bmod 60 \in \{0, 2, 12, 20, 30, 32, 42, 50\} \tag{5}$$

Suppose $k = 6t+2$, then $t \geq 10$. So we can write $t = 10l + r$, $l \geq 1$ and $r \in \{0, 1, \dots, 9\}$. With this notation $k = 60l + 6r + 2$ and (5) is satisfied if and only if $r \in \{0, 3, 5, 8\}$.

If $k = 6t+4$, as in previous case $t = 10l + r$, $l \geq 1$ and $r \in \{1, 2, \dots, 9\}$ and $k = 60l + 6r + 4$, but for all r condition (5) is not satisfied.

Finally, in the case $k = 6t+6$ we have $t \geq 9$, so we may write $t = 10l + r$, $l \geq 1$ and $r \in \{-1, 0, 1, 2, \dots, 9\}$. Then $k = 60l + 6r + 6$ and the remainder of $k \bmod 60$ is one in (5) if and only if $r \in \{-1, 1, 4, 6, 9\}$.

The remaining cases are obtained in a similar way by checking conditions in Theorem 1. □

THEOREM 4. *Let $q = 2t$, where t is a natural number. Then for each n satisfying the conditions in the following table there exist q-trigonal planar Klein surfaces X with k boundary components such that $Aut(X) = G$.*

Group	n	$k = 6t-1$	$k = 6t+1$	$k = 6t+3$
$\boldsymbol{Z}_{3n}$	*odd*	$n \in D^*(2t-1)$	$n \in D(t)$	$n \in D^*(2t+1)$
	even	$n \in D^*(4t-2)$	$n \in D^*(2t)$	$n \in D^*(4t+2)$
$\boldsymbol{D}_{3n}$	*odd*	$n \in D(2t-1)$	$n \in D(t)$	$n \in D^*(2t+1)$
	even	$n \in D(4t-2)$	$n \in D(2t)$	$n \in D(4t+2)$

PROOF. Let us observe that if q is even then k is odd, so the only possible automorphism groups are $\boldsymbol{Z}_{3n}$ or $\boldsymbol{D}_{3n}$. The order of these groups is computed, in the three cases: $\bar{k} = 1, 3$ or 5, in the same way as it was done in previous theorems. □

4. Maximal symmetry.

A classical problem related with automorphism groups of surfaces is the study of those groups with maximal order. May proved in [**10**] that the maximal order of an automorphism group G, of a bordered planar Klein surface X with k

boundary components is $12(k-2)$. It is said these surfaces have *maximal symmetry* and the groups G are called *M^*-groups*.

THEOREM 5. *Let X be a q-trigonal planar Klein surface with k boundary components. If X has maximal symmetry then*

(i) $k=3,\ q=0$ *and* $Aut(X) \simeq D_6$ *or*
(ii) $k=4,\ q=1$ *and* $Aut(X) \simeq S_4$ *or*
(iii) $k=12,\ q=3$ *and* $Aut(X) \simeq A_5 \times \boldsymbol{Z}_2$.

PROOF. First of all, let us observe that if X has maximal symmetry, then the order of an M^*-group G is $12(k-2)$. We proceed following the cases appearing in previous theorems depending on the number of boundary components of the surface.

CASE 1: if $k=6t+1$ then by Theorem 4 the greatest cardinal is attained for the group $D_{3n},\ n=2t$ and $t=(k-1)/3$. But this is not an M^*-group because its order is $2(k-1)$.

CASE 2: if $k=6t+2$ the upper bound is $72t$. The greatest cardinal obtained from Theorem 2 is for the group $D_{3n} \times \boldsymbol{Z}_2$, where $n=2t$. But $|D_{3n} \times \boldsymbol{Z}_2| = 24t < 72t$. On the other hand, from Theorem 3 the candidate is $A_5 \times \boldsymbol{Z}_2$ under the condition $t \geq 3$; but in this case $|A_5 \times \boldsymbol{Z}_2| = 120 < 72t$.

CASE 3: if $k=6t+3$ the group with greatest possible cardinal is $G=D_{3n}$ where $n=4t+2$ and $|G|=12(2t+1)$. G is an M^*-group if and only if $12(k-2)=12(6t+1)$, that is $t=0$. So we conclude D_6 is an M^*-group acting on a 0-trigonal planar surface X with $k=3$ boundary components.

CASE 4: if $k=6t+4$ the order of an M^*-group is $12(6t+2)$. From Theorem 2, the greatest order for a dihedral or cyclic automorphism group G is attained when $G=D_{3n}$ and $n=2t+1$. But in these conditions $|G| = 6(2t+1) < 12(6t+2)$. Now, we check Theorem 3. If $t \geq 1$ then the order of an M*-group G must be $|G| \geq 96$ but the possible groups A_4 or S_4 have smaller cardinal. If $t=0$, that is $k=4$, we have S_4 acts as an M*-group on a 1-trigonal planar surface with 4 boundary components.

CASE 5: if $k=6t-1,\ t \geq 1$, then from Theorem 4 the group of greatest order is $G=D_{3n}$ where $n=4t-2$ and $|G|=12(2t-1) < 12(k-2)$. So there is no M*-groups in this case.

CASE 6: if $k=6(t+1)$ the upper bound for the order of an automorphism group is $12(6t+4)$. All groups in Theorem 2 have smaller order, in fact, the greatest one is $D_{3n} \times \boldsymbol{Z}_2$ for $n = 2t+2$ and its cardinal is $12(2t+2) < 12(k-2)$. Now, from Theorem 3, for $t = 1$, we obtain as M*-group $A_5 \times \boldsymbol{Z}_2$ which acts on a 3-trigonal Klein surface with $k = 12$ boundary components. $\square$

4.1. Constructing maximal surfaces.

We devote this Section to construct geometrically maximal surfaces. It will be done by studying the action of the automorphism group on such type of surfaces. To do it we use appropriate fundamental polygons.

It is known (see [**11**]), a group G is an M^*-group which acts on a Klein surface $X = \mathscr{D}/\Gamma$ if and only if $G = \Lambda/\Gamma$ where Λ is an NEC group with signature $(0, +, [-], \{(2,2,2,3)\})$. The epimorphism $\theta : \Lambda \to G$ such that $\ker\theta = \Gamma$ is called the *canonical epimorphism*. The group Λ is generated by four reflections $\{c_1, c_2, c_3, c_4\}$ satisfying the following relations:

$$c_i^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_4)^2 = (c_4c_1)^3 = 1$$

Let us consider as a fundamental polygon for the discontinuous action of Λ on the hyperbolic plane $\mathscr{D}$, the hyperbolic quadrilateral $\mathscr{Q}$ in Figure 1. We call λ_i the side of $\mathscr{Q}$ lying on the axis of the reflection c_i and $l_i = |\lambda_i|$ its length, for $i = 1, \ldots, 4$. This polygon is known as a *Lambert quadrilateral* and the lengths of its sides satisfy the following relations (see [**2**, p. 156])

$$\sinh l_2 \, \sinh l_3 = \cos\frac{\pi}{3}\,,\ \cosh l_1 = \cosh l_3 \sin\frac{\pi}{3}\,,\ \cosh l_4 = \cosh l_2 \sin\frac{\pi}{3}\,. \qquad (6)$$

So, for each $\ell \in (0, \infty)$ there is a unique quadrilateral $\mathscr{Q}_\ell$ such that $l_2 = \ell$.

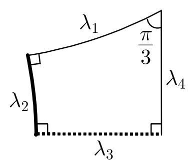

Figure 1. A fundamental polygon of Λ.

If $G = \Lambda/\Gamma$ is an M^*-group of order n which acts on $X = \mathscr{D}/\Gamma$ then

$$\Lambda = g_1\Gamma \cup \ldots \cup g_n\Gamma,\ g_1 = 1_G,\ g_i \in \Lambda - \Gamma,\ i = 2, \ldots, n\,;$$

and a fundamental region $\mathscr{P}$ for Γ can be obtained by pasting, in an appropriate way, n copies of the fundamental polygon $\mathscr{Q}$ of Λ

$$\mathscr{P} = \bigcup_{i=1}^{n} g_i(\mathscr{Q})$$

The pasting is done having in mind the canonical epimorphism $\theta : \Lambda \to G$. Using this technique we will study geometrically the maximal planar q-trigonal Klein surfaces. From Theorem 5, we note they can be of three different topological types. The first case, that occurs when the surface X has three boundary components, was studied in [**6**]. Here we deal with the other two cases.

The techniques used in the sequel to construct canonical epimorphisms are detailed in [**5**, Ch. 2].

4.1.1 Maximal surfaces with four boundary components.

Let X be a maximal planar 1-trigonal Klein surface with $k = 4$. From Theorem 5 its automorphism group is S_4. Let us consider the following presentation

$$\begin{aligned} S_4 =&< R_1, R_2 : \ R_1^2 = R_2^3 = (R_1R_2)^4 = 1 > \\ &\text{where } \ R_1 = (1,4), \ R_2 = (1,2,3). \end{aligned}$$

The canonical epimorphism $\theta : \Lambda \to S_4$ such that $\ker\theta = \Gamma$ uniformizes X is given by

$$\begin{array}{rccl} \theta : & \Lambda & \to & S_4 \\ & c_1 & \longmapsto & R_2R_1^2R_2 = (3,4) \\ & c_2 & \longmapsto & 1 \\ & c_3 & \longmapsto & R_1R_2^2R_1R_2R_1R_2^2 = (2,3) \\ & c_4 & \longmapsto & R_1 = (1,4) \end{array}$$

The group homomorphism θ is onto because $R_2 = \theta(c_4)[\theta(c_1c_3)]^2\theta(c_4)$. Let us observe that $\ker\theta$ is a surface group because $\theta(c_3c_4)$ is an order 2 element of S_4 and $\theta(c_4c_1)$ is an element of order 3. Furthermore, the number of boundary components in $\ker\theta$ is $|S_4|/2\, ord\,(\theta(c_1c_3)) = 4$. Since $\Lambda/\Gamma \cong S_4$ is the automorphism group of the surface $X = \mathscr{D}/\Gamma$, then a fundamental region $\mathscr{P}$ for Γ is (see Figure 2).

$$\mathscr{P} = \bigcup_{i=1}^{24} g_i(\mathscr{Q}), \quad \text{for } \Lambda = g_1\Gamma \cup \ldots \cup g_{24}\Gamma$$

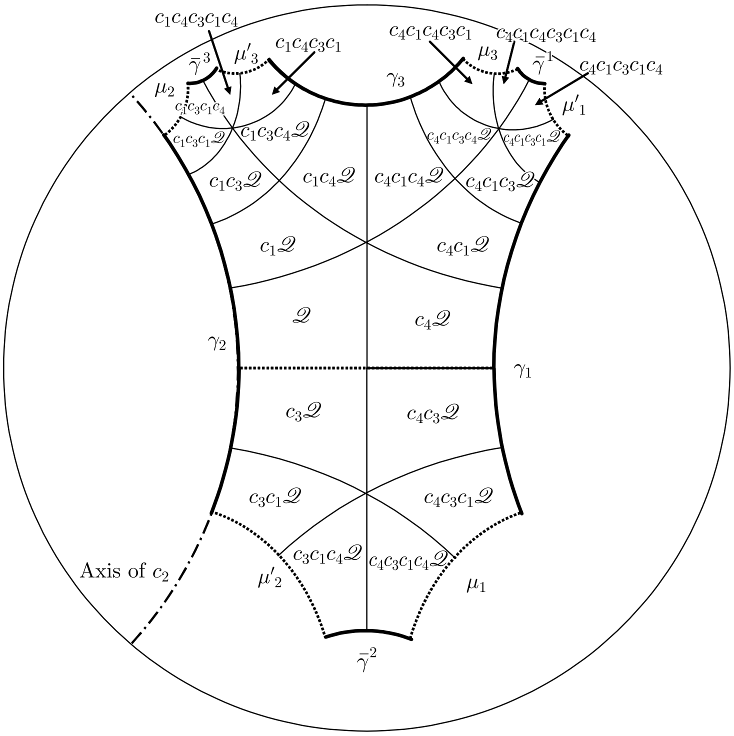

Figure 2.

The region $\mathscr{P}$ is a right-angled 12-gon. We label the sides in the perimeter anticlockwise order

$$\mathscr{P} : \mu_3,\ \gamma_3,\ \mu_3',\ \bar{\gamma}^3,\ \mu_2,\ \gamma_2,\ \mu_2',\ \bar{\gamma}^2,\ \mu_1,\ \gamma_1,\ \mu_1',\ \bar{\gamma}^1$$

where γ_2 is the side lying on λ_2, the axis of the reflection generator c_2 of Λ. Since we have the following relations between the length sides: $|\mu_i| = |\mu_i'| = 2l_3$, $i = 1, 2, 3$, then from [**8**] we can assert $\mathscr{P}$ is a fundamental region of an NEC group of signature $(0, +, [-], \{(-)^4\})$. Furthermore, we can obtain the hyperbolic transformations which are the side-pairings $f_i(\mu_i') = \mu_i$, having in mind the following

$$\begin{aligned}
\mu_1' &= c_4c_1c_3c_1(\lambda_3) \cup c_4c_1c_3c_1c_4(\lambda_3) & \mu_1 &= c_4c_3c_1(\lambda_3) \cup c_4c_3c_1c_4(\lambda_3)\\
\mu_2' &= c_3c_1(\lambda_3) \cup c_3c_1c_4(\lambda_3) & \mu_2 &= c_1c_3c_1(\lambda_3) \cup c_1c_3c_1c_4(\lambda_3)\\
\mu_3' &= c_1c_4c_3c_1(\lambda_3) \cup c_1c_4c_3c_1c_4(\lambda_3) & \mu_3 &= c_4c_1c_4c_3c_1(\lambda_3) \cup c_4c_1c_4c_3c_1c_4(\lambda_3)
\end{aligned}$$

then

$$f_1 = c_4(c_3c_1)^3c_4,\ f_2 = (c_1c_3)^3,\ f_3 = c_1c_4(c_1c_3)^3c_4c_1,$$

and it is easy to see $f_i \in \Gamma$ that is $\theta(f_i)$ is the identity in S_4.

On the other hand, the reflections on the sides γ_i are also elements of Γ. The reflection in γ_2 is $\tilde{c}_2 = c_2$. The side $\gamma_1 = c_4(\gamma_2)$, then the reflection in γ_1 is $\tilde{c}_1 = c_4c_2c_4$, a conjugate of c_2. The reflection with axis $\gamma_3 = c_4c_1c_4(\gamma_2)$ is also a conjugate of c_2, that is $\tilde{c}_3 = c_4c_1c_4c_2c_4c_1c_4$. In the same way we can obtain the reflections $\bar{c}_i$ in the sides $\bar{\gamma}^i$

$$\begin{aligned} \bar{c}_1 &= g_1c_2g_1^{-1}, & g_1 &= c_4c_1c_3c_1c_4 \\ \bar{c}_2 &= g_2c_2g_2^{-1}, & g_2 &= c_3c_1c_4 \\ \bar{c}_3 &= g_3c_2g_3^{-1}, & g_3 &= c_1c_4c_3c_1c_4 \end{aligned}$$

THEOREM 6. *The family of maximal* 1*-trigonal planar Klein surfaces with* 4 *boundary components is a uniparametric family.*

PROOF. Let us fix $\ell \in (0, \infty)$. Construct the quadrilateral $\mathscr{Q}_\ell$ and then the polygon $\mathscr{P}_\ell$ as above. The polygon $\mathscr{P}_\ell$ is a fundamental region of an NEC group Γ_ℓ with signature $(0, +, [-], \{(-)^4\})$. The side lengths relations in $\mathscr{P}_\ell$ can be easily obtained from (6) and Figure 2

$$|\gamma_i| = 6\ell,\ |\bar{\gamma}^i| = 2\ell,\ |\mu_i| = 2l_3,\ \text{where}\ \sinh l_3 = \frac{1}{2\sinh\ell}.$$

The only thing to prove is the surface X uniformized by Γ_ℓ is 1-trigonal, that is, there exists an order three automorphism ϕ of X, such that the quotient $X/\phi = D/\Gamma^*$ and $\sigma(\Gamma^*) = (0, +, [3], \{(-)^2\})$. To prove that it is sufficient to consider the subpolygon $\mathscr{N}_\ell$

$$\mathscr{N}_\ell : \mathscr{Q}_\ell \cup c_1\mathscr{Q}_\ell \cup c_1c_3\mathscr{Q}_\ell \cup c_3c_1\mathscr{Q}_\ell \cup c_1c_3c_1\mathscr{Q}_\ell \cup c_3c_1c_4\mathscr{Q}_\ell \cup c_1c_3c_1c_4\mathscr{Q}_\ell$$

which is a fundamental region of Γ^*. Furthermore, $\mathscr{P}_\ell = \mathscr{N}_\ell \cup c_1c_4\mathscr{N}_\ell \cup (c_1c_4)^2\mathscr{N}_\ell$ and $c_1c_4\Gamma_\ell$ (an element in Λ/Γ_ℓ) is a 1-trigonal automorphism (see Figure 3). $\square$

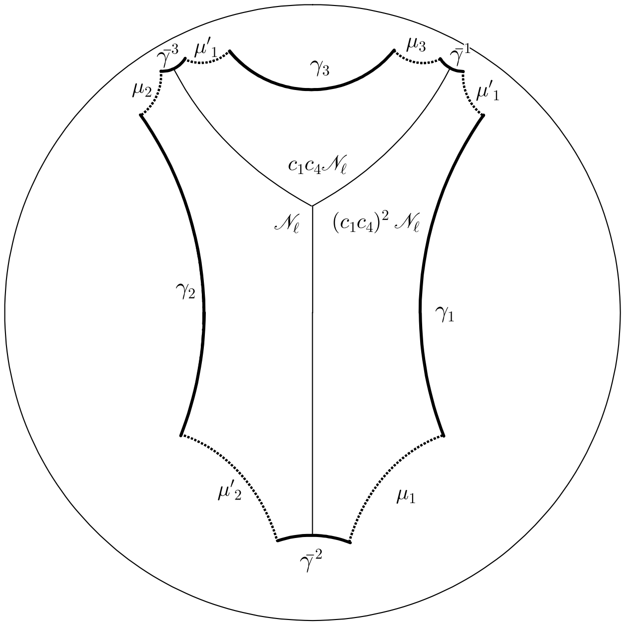

Figure 3.

4.1.2 Maximal surfaces with twelve boundary components.

Let X be a maximal planar 3-trigonal Klein surface with $k = 12$ boundary components, then from Theorem 5 its automorphism group is $A_5 \times \boldsymbol{Z}_2$. Let us consider the following generators of $A_5 \times \boldsymbol{Z}_2$

$$R_1 = (1,4)(2,5)U,\ R_2 = (1,5)(2,4)U \ \text{ and } \ R_3 = (1,4)(2,3)U$$

where U is the generator of $\boldsymbol{Z}_2$ and the set $\{R_iU,\ i = 1,\dots,3\}$ generates A_5. The canonical epimorphism $\theta : \Lambda \to A_5 \times \boldsymbol{Z}_2$, such that $\ker\theta$ is the group which uniformizes X, is given by

$$\begin{array}{rccl} \theta: & \Lambda & \to & A_5 \times \boldsymbol{Z}_2 \\ & c_1 & \longmapsto & R_3 \\ & c_2 & \longmapsto & 1 \\ & c_3 & \longmapsto & R_2 \\ & c_4 & \longmapsto & R_1 \end{array}$$

We can see $\ker\theta$ is a surface group because

$$\begin{aligned} ord\,\theta(c_3c_4) &= ord(R_2R_1) = ord(\,(1,2)(4,5)) = 2, \\ ord\,\theta(c_4c_1) &= ord(R_1R_3) = ord(\,(2,3,5)) = 3. \end{aligned}$$

On the other hand, since $R_3R_2 = (1,5,4,3,2)$ has order 5, then the number of boundary components in $\ker\theta$ is

$$\frac{|A_5 \times \boldsymbol{Z}_2|}{2\,ord\,\phi(c_1c_3)} = \frac{120}{2*5} = 12.$$

As in the previous case we can obtain a fundamental polygon $\mathscr{R}$ for the surface group $\Gamma = \ker\theta$ by pasting, in a suitable way, 120 images of the quadrilateral $\mathscr{Q}$ by the elements of the group $\Lambda/\Gamma \cong A_5 \times \boldsymbol{Z}_2$. In order to describe $\mathscr{R}$, let us consider the subpolygon $\mathscr{L}$, where g denotes the transformation $(c_1c_3c_1)c_4(c_1c_3c_1)$ (see Figure 4, where the $\mathscr{Q}$'s have been omitted in the most of images of the original quadrilateral).

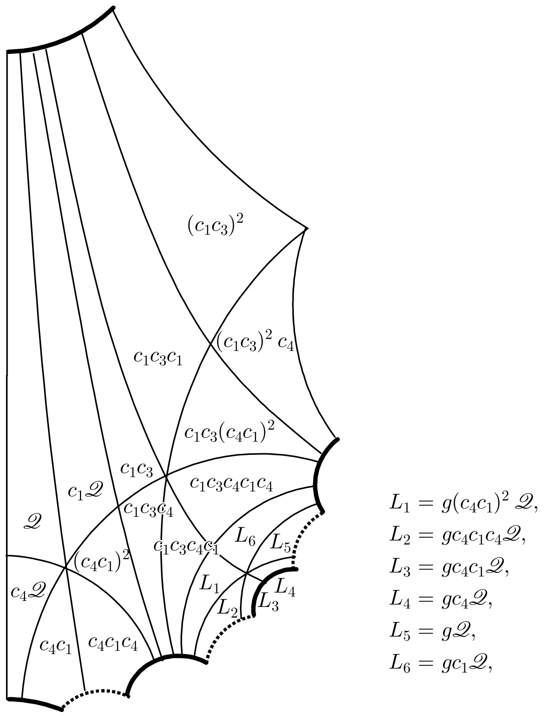

Figure 4. The polygon $\mathscr{L}$.

$$\begin{aligned}\mathscr{L} \ :\ & \mathscr{Q} \cup c_1\mathscr{Q} \cup c_1c_3\mathscr{Q} \cup c_1c_3c_1\mathscr{Q} \cup (c_1c_3)^2\mathscr{Q} \\ & \cup c_4\mathscr{Q} \cup c_4c_1\mathscr{Q} \cup c_4c_1c_4\mathscr{Q} \cup (c_4c_1)^2\mathscr{Q} \\ & \cup c_1c_3c_4\mathscr{Q} \cup c_1c_3c_4c_1\mathscr{Q} \cup c_1c_3c_4c_1c_4\mathscr{Q} \cup c_1c_3(c_4c_1)^2\mathscr{Q} \cup (c_1c_3)^2c_4\mathscr{Q} \\ & \cup g(c_4c_1)^2\mathscr{Q} \cup gc_4c_1c_4\mathscr{Q} \cup gc_4c_1\mathscr{Q} \cup gc_4\mathscr{Q} \cup g\mathscr{Q} \cup gc_1\mathscr{Q}\end{aligned}$$

then

$$\mathscr{R} = \mathscr{L} \cup h(\mathscr{L}) \cup h^2(\mathscr{L}) \cup c_3(\mathscr{L}) \cup c_3h(\mathscr{L}) \cup c_3h^2(\mathscr{L})$$

where $h = (c_1c_3)^2c_1c_4(c_1c_3)^{-2}$ is an order three elliptic transformation. The region $\mathscr{R}$ is a 44-sided right-angled polygon (see Figure 5).

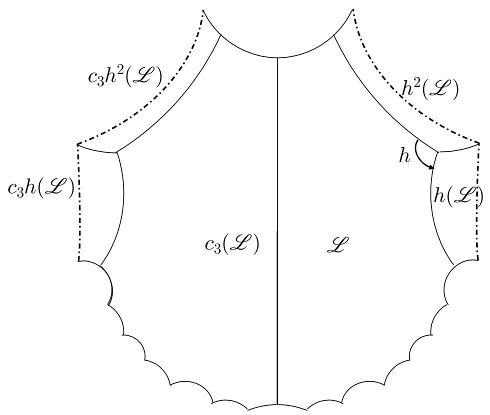

Figure 5. The polygon $\mathscr{R}$.

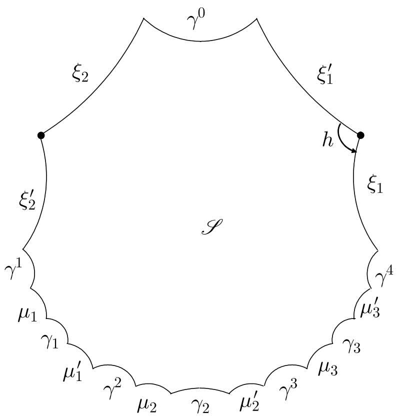

Figure 6. The polygon $\mathscr{S}$.

THEOREM 7. *The family of maximal* 3*-trigonal planar Klein surfaces with* 12 *boundary components is a uniparametric family.*

PROOF. Let us fix $\ell \in (0, \infty)$. Construct the quadrilateral $\mathscr{Q}_\ell$ and the polygon $\mathscr{R}_\ell$ as above. Then, $\mathscr{R}_\ell$ is a fundamental polygon of an NEC group Γ_ℓ with signature $(0, +, [-], \{(-)^{12}\})$.

Now, we have to prove the quotient surface X uniformized by Γ_ℓ is 3-trigonal, that is, there exists an order three automorphism ϕ of X, such that the quotient $X/\phi = D/\Gamma^*$ and $\sigma(\Gamma^*) = (0, +, [3, 3], \{(-)^4\})$ (see Proposition 4). To prove that, it is sufficient to consider the subpolygon

$$\mathscr{S} = \mathscr{L} \cup c_3(\mathscr{L})$$

whose perimeter can be labelled

$$\xi_1, \xi_1', \gamma^0, \xi_2, \xi_2', \gamma^1, \mu_1, \gamma_1, \mu_1', \gamma^2, \mu_2, \gamma_2, \mu_2', \gamma^3, \mu_3, \gamma_3, \mu_3', \gamma^4,$$

with the following side-pairings

$$\begin{array}{ll} h(\xi_1') = \xi_1, & c_3 h^2 c_3^{-1}(\xi_2') = \xi_2 \\ g_1(\mu_1') = \mu_1 & g_1 = c_3 g_3 c_3 \\ g_2(\mu_2') = \mu_2 & g_2 = c_3 c_4 c_1 c_3 c_1 c_4 \\ g_3(\mu_3') = \mu_3 & g_3 = g c_4 c_1 c_4 c_3 g^{-1} \end{array}$$

and the reflections in the respective γ-sides. The polygon $\mathscr{S}$ is a fundamental region for Γ^* and $\phi = h$ is a 3-trigonal automorphism. □

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