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Hyperbolic Schwarz map of the confluent hypergeometric differential equation

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Abstract. The hyperbolic Schwarz map is defined in [**SYY1**] as a map from the complex projective line to the three-dimensional real hyperbolic space by use of solutions of the hypergeometric differential equation. Its image is a flat front ([**GMM**], [**KUY**], [**KRSUY**]), and generic singularities are cuspidal edges and swallowtail singularities. In this paper, for the two-parameter family of the confluent hypergeometric differential equations, we study the singularities of the hyperbolic Schwarz map, count the number of swallowtails, and identify the further singularities, except those which are apparently of type A_5. This describes creations/eliminations of the swallowtails on the image surfaces, and gives a stratification of the parameter space according to types of singularities. Such a study was made for a 1-parameter family of hypergeometric differential equation in [**NSYY**], which counts only the number of swallowtails without identifying further singularities.

1. Introduction.

The *confluent hypergeometric differential equation* is defined as

$$xu'' + (\gamma - x)u' - \alpha u = 0. \tag{1.1}$$

It is regular singular at $x = 0$ and irregular singular at $x = \infty$. By a change of the unknown u by multiplying a non-zero function $\rho = \exp(-x/2)x^{\gamma/2}$, and a change of parameters

$$a = 2\alpha - \gamma, \quad b = \gamma^2 - 2\gamma, \tag{1.2}$$

this equation transforms to the SL-form:

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$$u'' - q(x)u = 0, \quad \text{where} \quad q = \frac{x^2 + 2ax + b}{4x^2}. \tag{E}$$

(Note that (E) is the SL-form of the Bessel equation if and only if $a = 0$.) Let us recall the definition of the hyperbolic Schwarz map associated with (E). For two linearly independent solutions u_0 and u_1 to this equation, we define the (multi-valued) map

$$\mathscr{S} : X = \boldsymbol{C} - \{0\} \ni x \longmapsto H(x) = U(x)\ {}^t\overline{U}(x), \tag{HS}$$

where

$$U = \begin{pmatrix} u_0 & u_0' \\ u_1 & u_1' \end{pmatrix}.$$

Its target can be regarded as the three-dimensional hyperbolic space $\boldsymbol{H}^3$ identified with the space of positive 2×2-hermitian matrices modulo diagonal ones. The map $\mathscr{S}$ is called the *hyperbolic Schwarz map*. Its image is a surface with singularity, which is known to be a flat front in $\boldsymbol{H}^3$ (*flat* means the vanishing of the Gaussian curvature of induced metric, for *front*, see Section 2.1). We remark that the ordinary (multi-valued) *Schwarz map* is defined as

$$S : X \ni x \longmapsto u_0(x) : u_1(x) \in \boldsymbol{P}^1,$$

and the (multi-valued) *derived Schwarz map* as

$$DS : X \ni x \longmapsto u_0'(x) : u_1'(x) \in \boldsymbol{P}^1,$$

where $\boldsymbol{P}^1$ denotes the complex projective line, the ideal boundary of $\boldsymbol{H}^3$. There is a one-parameter family of flat fronts in $\boldsymbol{H}^3$ such that the map $\mathscr{S}$ is an ordinary member, and the maps S and DS are two extreme ones. We refer to [**GMM**], [**KUY**], [**SYY2**] for these maps.

In this paper, we study the singularities on the image surface of the hyperbolic Schwarz map of the differential equation (E) with the coefficient q with real parameters a and b and show how the singularities depend on the parameters. It is well known that generic singularities of the image surfaces are cuspidal edges and swallowtails. We get criteria of the singularities appearing in the image surfaces in terms of the coefficient q, except for the singularity of type A_5. Thanks to those criteria, we identify further singularites, namely creations/ eliminations of swallowtail singularities. We thus get a stratification of the parameter space according to types of singularities. Refer to Figures 6 and 7. Since our proofs rely on purely algebraic treatment of ideals of the polynomial rings, to give a geometric idea, we show some pictures of image surfaces around the singularities.

The point $x = 0$ is a regular singular point of the differential equation and it is mapped into the boundary of $\boldsymbol{H}^3$. The behavior of the map around $x = \infty$ may be complicated because the differential equation is irregular singular at that point; the asymptotic behavior of the surface will be studied in the forthcoming paper [**SY2**].

We refer to [**E**], [**Y**] for the hypergeometric differential equation and its solutions.

2. Criteria on singularities of flat fronts.

2.1. Singularities of flat fronts.

A smooth map f from a domain $U \subset \boldsymbol{R}^2$ to a Riemannian 3-manifold N^3 is called a *front* if there exists a unit vector field ν: $U \to T_1N$ along the map f such that df and ν are perpendicular and the map ν: $U \to T_1N$ along f is an *immersion*, where T_1N is the unit tangent bundle of N. We call ν the *unit normal vector field* of f. Note that, if we identify T_1N with the unit cotangent bundle T_1N^*, the condition $df \perp \nu$ is equivalent to the corresponding map L: $U \to T_1^*N$ to be Legendrian with respect to the canonical contact structure T_1^*N. A point $p \in U$ is called a *singular point* of f if rank(df) is less than 2 at p; it is called a singular point of rank one if rank(df) is equal to one. Relative to the coordinates (u, v) on U, define a function λ by

$$\lambda(u, v) = \Omega(f_u, f_v, \nu),$$

where Ω is the volume form. A singular point in U is said to be nondegenerate if $d\lambda \neq 0$. Here, let us recall some terminologies in singularity theory: Let f_i be a map germ at p_i defined on the source space S_i into the target space T_i, for $i = 1, 2$. Then, they are said to be equivalent if there exist a local diffeomorphism ϕ from S_1 to S_2 with $\phi(p_1) = p_2$ and a local diffeomorphism ψ from T_1 to T_2, such that $\psi \circ f_1 = f_2 \circ \phi$. It is well-known that generic singularities of fronts are *cuspidal edges* and *swallowtails* [**A**]; refer also to [**SY1**] for an elementary description. Recall that the cuspidal edge is (the equivalence class of) the map germ

$$(u, v) \mapsto (u^2, u^3, v)$$

at the origin, and the swallowtail singularity is the map germ

$$(u,v) \mapsto (3u^4 + u^2v, 4u^3 + 2uv, v)$$

at the origin. The generic confluence of swallowtail singularities are classified into five types called A_4, a pair of cuspidal lips, a pair of cuspidal beaks, and two types where $\mathrm{rank}(df) = 0$; Refer to [**LLR**, p. 547] and also to [**IS**], [**IST**]. The first three are defined as the map germs at the origin as follows:

$$\begin{aligned} A_4\text{:} \qquad & (u,v) \mapsto (5u^5 + 2uv, 4u^5 + u^2v - v^2, v), \\ \text{Cuspidal lips:} \qquad & (u,v) \mapsto (u^3 + uv^2, 3u^4 + 2u^2v^2, v), \\ \text{Cuspidal beaks:} \qquad & (u,v) \mapsto (u^3 - uv^2, 3u^4 - 2u^2v^2, v). \end{aligned}$$

Each belongs to a family of the map germs

$$\begin{aligned} (u,v) \quad & \mapsto \quad (5u^5 + 2uv + 3cu^2, 4u^5 + u^2v + 2cu^3 - v^2, v), \\ (u,v) \quad & \mapsto \quad (u^3 + uv^2 + cu, 3u^4 + 2u^2v^2 + 2cu^2, v), \\ (u,v) \quad & \mapsto \quad (u^3 - uv^2 + cu, 3u^4 - 2u^2v^2 + 2cu^2, v), \end{aligned}$$

respectively, where c is the parameter. As c tends to zero, the two swallowtails merge and vanish; in the two former cases the two swallowtails are on the same cuspidal edge curve, while in the latter case they are on two different cuspidal edge curves. We furthermore introduce a family of the map germs of a higher-order singularity as

$$(u,v) \quad \mapsto \quad (6u^5 + 4cu^3 + 2uv, 5u^6 + 3cu^4 + u^2v, v),$$

which defines the singularity of type A_5 at the origin when $c = 0$:

$$A_5 : \quad (u,v) \mapsto (6u^5 + 2uv, 5u^6 + u^2v, v).$$

As c tends to zero, three swallowtails on the same cuspidal edge curves merge, and one swallowtail survives.

In this subsection, we review the criteria of these singularities, except that of type A_5, in terms of λ, and paraphrase them in terms of the coefficient q. We have no criterion for A_5-singularity yet.

Assume that the map f is of rank one at a point p. Then there exists a nonvanishing vector field η around p so that $(\eta f)(q) = 0$ for any singular point q around p.

Lemma 2.1 ([**KRSUY**], [**IS**]; see also [**SUY**]). *Let p be a nondegenerate* *singular point of the front f. Then, the map f at p is equivalent to*

(1) *a cuspidal edge if and only if $\eta(\lambda)(p) \neq 0$,*
(2) *a swallowtail singularity if and only if $\eta(\lambda)(p) = 0$ and $\eta\eta(\lambda)(p) \neq 0$, and*
(3) *a singularity of type A_4 if and only if $\eta(\lambda)(p) = 0$, $\eta\eta(\lambda)(p) = 0$, and $\eta\eta\eta(\lambda)(p) \neq 0$.*

When the map f is of rank one and degenerate, we have the following characterization of a pair of cuspidal lips or a pair of cuspidal beaks:

LEMMA 2.2 ([**IST**]). *Let p be a degenerate singular point of rank one of the front f. Then, it is equivalent to*

(1) *a pair of cuspidal lips if and only if $d\lambda(p) = 0$ and $\det(\mathrm{Hess}(\lambda)) > 0$, and*
(2) *a pair of cuspidal beaks if and only if $d\lambda(p) = 0$, $\det(\mathrm{Hess}(\lambda)) < 0$, and $\eta\eta\lambda(p) \neq 0$.*

We apply the criteria above to the hyperbolic Schwarz map $f = \mathscr{S}$ associated to the equation (E). The inner product $\langle\ ,\ \rangle$ on the tangent bundle $T\boldsymbol{H}^3$ is given as $\langle X, Y\rangle = \mathrm{tr}(X\tilde{Y})/2$, where $\tilde{Y}$ is the cofactor matrix of Y, and the cross-product at $p \in \boldsymbol{H}^3$ is given as $X \times Y = i(Xp^{-1}Y - Yp^{-1}X)/2$. Refer, *e.g.*, to [**KRSUY**, p. 319]. The normal vector field ν is given by the equation

$$\nu = U\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}{}^t\overline{U},$$

which we regard as a map to $T\boldsymbol{H}^3$. Then the function λ is equal to $2i\langle \nu, f' \times \overline{f'}\rangle$ up to a constant multiple. Here and in the following, f' denotes the derivative $\partial f/\partial x$ and $f'' = \partial^2 f/\partial x^2$, and so on. Since $f' = U'{}^t\overline{U}$ and $U' = U\begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}$, we see that

$$\lambda = q\overline{q} - 1. \tag{2.1}$$

By definition, the set of singular points is

$$CE := \{x \in X; q(x)\overline{q}(x) - 1 = 0\}.$$

A simple computation shows that the vector field η can be chosen as

$$\eta = i((1 + \overline{q})\partial_x - (1 + q)\partial_{\overline{x}})$$

around the point where $q \neq -1$ and

$$\eta = (1 - \overline{q})\partial_x + (1 - q)\partial_{\overline{x}}$$

around the point where $q \neq 1$. Then a computation using the first expression yields

$$
\begin{aligned}
\eta(\lambda) &= 2\,\mathrm{Re}\{i\overline{q}(1+\overline{q})q'\}, \\
\eta\eta(\lambda) &= 2\,\mathrm{Re}\{-(1+\overline{q})^2\overline{q}q'' + (1+\overline{q})(1+2q)q'\overline{q'}\}, \\
\eta\eta\eta(\lambda) &= 2\,\mathrm{Re}\{i(1+\overline{q})(-(1+\overline{q})^2\overline{q}q''' + (3+4\overline{q})\overline{q'}q'^2 \\
&\qquad + (2+3q+3\overline{q}+4q\overline{q})\overline{q'}q'' - (1+q)(1+3q)\overline{q''}q')\}.
\end{aligned} \tag{2.2}
$$

Now, we are prepared to paraphrase Lemmas 2.1 and 2.2 as follows:

Lemma 2.3.
(1) *A point $x \in X$ is a singular point of the hyperbolic Schwarz map $\mathscr{S}$ if and only if $|q(x)| = 1$,*
(2) *a singular point $x \in X$ of $\mathscr{S}$ is equivalent to the cuspidal edge if and only if $q'(x) \neq 0$ and $\overline{q'}(x) \neq (q'/q^3)(x)$,*
(3) *a singular point $x \in X$ of $\mathscr{S}$ is equivalent to the swallowtail if and only if $q'(x) \neq 0$, $\overline{q'}(x) = (q'/q^3)(x)$, and $\overline{q''}(x) \neq -(q'/q^3)'(x)/q(x)$,*
(4) *a singular point $x \in X$ of $\mathscr{S}$ is equivalent to A_4 if and only if $q'(x) \neq 0$, $\overline{q'}(x) = (q'/q^3)(x)$, and $\overline{q''}(x) = -(q'/q^3)'(x)/q(x)$, $\overline{q'''}(x) \neq ((q'/q^3)'/q)'(x)/q(x)$,*
(5) *any degenerate singular point of rank one cannot be cuspidal lips,*
(6) *and a degenerate singular point of rank one is equivalent to a pair of cuspidal beaks if and only if $q' = 0$, $q'' \neq 0$, and $\overline{q''}(x) \neq (q''/q^4)(x)$.*

Proof. The claim (1) is what we observed above. To see the claim (2), we rewrite the first identity of (2.2) by use of $\overline{q} = 1/q$:

$$\eta(\lambda) = iq(1+q)\left(\frac{q'}{q^3} - \overline{q'}\right),$$

from which we can see that $\eta(\lambda) \neq 0$ if and only if $\overline{q'} \neq q'/q^3$ when $1+q \neq 0$. We obtain the same condition also when $1-q \neq 0$; hence, we have (2). For (3), rewrite the second identity of (2.2) by use of $\overline{q} = 1/q$ and $\overline{q'} = q'/q^3$. Then we see

$$\eta\eta\lambda = -q(1+q)^2\left(\overline{q''} + \frac{q''}{q^4} - \frac{3q'^2}{q^5}\right),$$

which implies the third condition of (3) in case $1+q \neq 0$. In case $1-q \neq 0$, we have the same expression. The claim (4) is similarly shown.

Since q is a rational function of x, it is easy to see that

$$\det \mathrm{Hess}\lambda = -|q''|^2$$

on the singular set CE. This implies that $\det \mathrm{Hess}\lambda \leq 0$. Hence we have (5) and (6). (Note that $q'' = 0$ if and only if $2ax + 3b = 0$.) $\square$

We remark that the condition $\overline{q''}(x) \neq -(q'/q^3)'(x)/q(x)$ in (3) of the lemma above is rewritten as $\mathrm{Re}(2q''/q^2 - 3(q')^2/q^3)(x) \neq 0$, which is the expression given in [**KRSUY**].

2.2. Cuspidal edge.

In the following, we use the real coordinates (s,t): $x = s + it$ for the sake of simplicity. In these coordinates,

$$\eta = \mathrm{Im}(q)\partial_s + (1 + \mathrm{Re}(q))\partial_t \quad \text{or} \quad (1 - \mathrm{Re}(q))\partial_s + \mathrm{Im}(q)\partial_t.$$

The set CE is defined explicitly as

$$ce := 15t^4 - (4as - 30s^2 + 4a^2 - 2b)t^2 + (b + 2as + 5s^2)(-b - 2as + 3s^2) = 0. \quad (2.3)$$

It is a plane quartic curve symmetric relative to the change $t \to -t$ for each fixed (a,b). The point $x = 0$ is a singularity of the differential equation; the point $(s,t) = (0,0)$ is on CE only if $b = 0$.

The shape of the set CE depends on the parameter (a,b). Here, we notice that the polynomial ce has the homogeneity

$$ce(ks, kt, ka, k^2b) = k^4 ce(s,t,a,b),$$

that comes from $q(kx; ka, k^2b) = q(x; a, b)$. Hence, it is enough to consider the cases $a = 1$ and $a = 0$. When $a = 1$, the degenerate singular points, where $q' = 0$, are

$$(b; s,t) = (-1/3; 1/3, 0), (1/5; -1/5, 0), (0; 0, 0).$$

When $a = 0$, the equation still has the symmetry, it is enough to consider the case $b = \pm 1$. Remark that the case $b = 0$ was already excluded.

In each case, the polynomial ce in (s,t) can be transformed into a quadratic polynomial as follows, thus the shape of the set CE can be seen.
When $a = 0$, we have

$$\begin{aligned} ce &= 15t^4 + (30s^2 + 2b)t^2 + (5s^2 + b)(3s^2 - b) \\ &= 15T^2 + 2bS - b^2, \end{aligned}$$

where $T = t^2 + s^2, S = t^2 - s^2$. Since $T + S \geq 0$ and $T - S \geq 0$, the set CE in TS-plane is part (compact) of a parabola, if $b \neq 0$.
When $a = 1$, we have

$$\begin{aligned} ce &= 15t^4 + (30s^2 - 4s + 2b - 4)t^2 + (5s^2 + 2s + b)(3s^2 - 2s - b) \\ &= 15T^2 - 4sT - 4bs^2 + (2b - 4)T - 4bs - b^2, \end{aligned}$$

where $T = t^2 + s^2$. Since $T \geq s^2$, the set CE in Ts-plane is part (compact) of a hyperbola or an ellipse.

We consider the intersection $CE \cap \{t = 0\}$ to help understand a global view of the set CE. The intersection is defined by the equation

$$(b + 2as + 5s^2)(-b - 2as + 3s^2) = 0,$$

which is solved as

$$s = -\frac{a}{5} \pm \frac{\sqrt{a^2 - 5b}}{5}, \quad \frac{a}{3} \pm \frac{\sqrt{a^2 + 3b}}{3}.$$

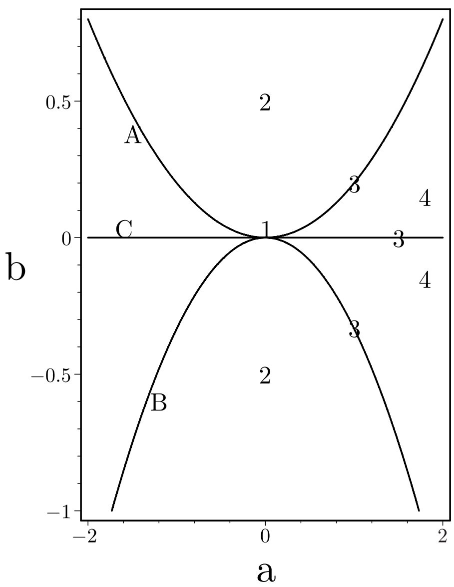


Figure 1. Stratification of ab-plane according to cardinalities of $CE \cap \{t = 0\}$. The curves A, B and C are given by $5b - a^2 = 0$, $3b + a^2 = 0$ and $b = 0$, respectively.

When $(a, b) = (0, 0)$, four roots coincide; in this case, the coefficient reduces to $q = 1/4$, which we exclude from our consideration since $\mathscr{S}$ does not define a surface. On the line $b = 0$, the roots are $2a/3$, 0(double), $-2a/5$. On the curve $a^2 = 5b$, the first equation has double roots and, on the curve $a^2 = -3b$, the second equation has double roots. Thus, we have the stratification of the parameter plane as in Figure 1. Each numeral in the figure shows the cardinality of the intersection $CE \cap \{t = 0\}$; these are invariant under the change $a \to -a$.

In Figure 2, we exhibit the set CE for the case $(a, b) = (1, -0.31)$ and $(a, b) = (0, -1)$. Refer to Figures 3 and 4 for other cases.

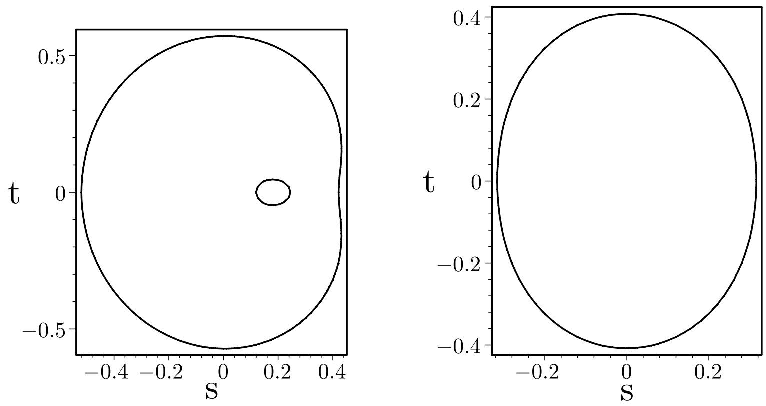


Figure 2. The curve CE when $(a, b) = (1, -0.31)$ (Left) and $(a, b) = (0, -1)$ (Right).

2.3. Swallowtail singularities.

To find swallowtail singularities using Lemma 2.3, we need to solve the equation $q^3\overline{q'} - q' = 0$. This equation turns out to be

$$swr = 0 \quad \text{and} \quad swi = 0,$$

where

$$\begin{aligned} swr = {} & 118at^6s - 12a^2 + 130bt^6 + 394at^4s^3 + (60a^2 + 354b)t^4s^2 \\ & + 6a(-7b + 12a^2)t^4s + (16a^4 - 6b^2)t^4 + 402at^2s^5 + (60a^2 + 414b)t^2s^4 \\ & + 12a(4a^2 + 11b)t^2s^3 + 36b(4a^2 + b)t^2s^2 + 6ab(4a^2 + 11b)t^2s \\ & + 6b^2(2a^2 + b)t^2 + 126as^7 + (-12a^2 + 126b)s^6 - 6a(4a^2 + 3b)s^5 \\ & - (16a^4 + 48a^2b + 6b^2)s^4 - 10ab(4a^2 + 3b)s^3 - 6b^2(6a^2 + b)s^2 - 14b^3as - 2b^4 \end{aligned}$$

$$\begin{aligned} swi = t[&126at^6 + 402at^4s^2 + (48a^2 - 12b)t^4s + 6a(4a^2 - b)t^4 \\ &+ 394at^2s^4 + 40bt^2s^3 - 12a(4a^2 - 9b)t^2s^2 - 32a^4 + 24b^2 + 48a^2bt^2s \\ &- 2ab(4a^2 - 9b)t^2 + 118as^6 - (48a^2 + 12b)s^5 - 6a(12a^2 + 13b)s^4 \\ &- (32a^4 + 24b^2 + 144a^2b)s^3 - 6ab(12a^2 + 13b)s^2 - 12b^2(4a^2 + b)s - 10b^3a] \end{aligned}$$

For a point in the set $SW = \{(s,t); ce = swr = swi = 0\}$ to be a swallowtail singularity, it is necessary to check the third condition in (3) of Lemma 2.3. We denote by *swexc* the numerator of the real part of $2q''/q^2 - 3(q')^2/q^3$, which is a polynomial of (s,t) of total degree nine; its explicit expression is given in the appendix.

We first compute the exceptional set $SWE := \{ce = swr = swi = swexc = 0\}$. By relying on the primary decomposition of the corresponding ideal $\langle ce, swr, swi, swexc\rangle$ in the polynomial ring $\boldsymbol{R}[s,t,b]$ ($a = 1$ is assumed), we have the following. (Refer to, e.g., [**Jac**] for primary decomposition.)

Lemma 2.4. *Assume $a = 1$. Then the set defined by the ideal $\langle ce, swr, swi, swexc\rangle$ is the union of the sets defined by the ideals*

(1) $[b, s^2 + t^2]$,
(2) $[b - 1, 4t^2 + 1, 2s + 1]$,
(3) $[25b^2 - 10b - 27, 25t^2 + 10b + 7, 10s + 5b + 1]$,
(4) $[27b^2 - 70b - 21, t, 8s + 3b - 3]$,
(5) $[P_1, P_2, P_3]$,

where

$$\begin{aligned} P_1 &= 490056b^5 - 91665b^4 + 51270b^3 - 6414b^2 - 147b - 1, \\ P_2 &= 3011952t^2 + 2889334480b^4 - 5676546615b^3 + 3561741692b^2 - 68910105b \\ &\quad + 3483983, \\ P_3 &= 16063744s + 3965974658b^4 - 830462220b^3 + 605893890b^2 - 1627824686 \\ &\quad + 17102389. \end{aligned}$$

By this lemma, the exceptional real points are

$$\begin{aligned} &p_1: \quad b = b_1; \quad (s,t) \sim (0.3291502622, 0.2516350726), \\ &p_2: \quad b = b_2; \quad (s,t) \sim (0.4768336246, 0), \\ &p_3: \quad b = b_3; \quad (s,t) \sim (-0.1696551154, 0.03711109674), \\ &p_4: \quad b = b_4; \quad (s,t) \sim (-0.6990558469, 0), \end{aligned}$$

where b_1 is one of solutions of the equation $25b^2 - 10b - 27 = 0$ of (3):

$$b_1 = \frac{1-\sqrt{28}}{5} \sim -0.8583005244;$$

Note that the second solution $(1+\sqrt{28})/5$ is excluded because the value t cannot be real. The values b_2 and b_4 are solutions of the equation $27b^2 - 70b - 21 = 0$ of (4):

$$b_2 = \frac{35-\sqrt{1792}}{27} \sim -0.2715563324, \quad b_4 = \frac{35+\sqrt{1792}}{27} \sim 2.8641489250,$$

and b_3 is the unique real solution of the equation $P_1 = 0$:

$$b_3 \sim 0.2081942455.$$

In the case $a = 0$, we see that when $b = 1$ the swallowtail points are $(s,t) = (0, \pm 1/\sqrt{5}), (\pm 1/\sqrt{3}, 0)$ and that when $b = -1$ there exist no swallowtail points.

2.4. Types of confluence of swallowtail singularities.

The types of the above exceptional points can be identified by using the criteria in Lemma 2.3. Let C be one of the ideals in the previous lemma for the cases (3)–(5) when $a = 1$. By computing the primary decomposition of the ideal generated by the polynomials in C and the numerator *num* of the expression of $\eta\eta\eta(\lambda)$ in (2.2), we can see that p_1 and p_3 are of type A_4 and that $\eta\eta\eta(\lambda) = 0$ for p_2 and p_4. The polynomial *num* is given in the appendix.

When b passes through b_2 (resp. b_4), three swallowtail points get together to the point p_2 (resp. p_4) and then reappears a single swallowtail point. At these extreme values of b, the derivatives $\eta^k(\lambda)$ vanishes for $0 \le k \le 3$ as we have seen, while we can examine $\eta^4(\lambda) \neq 0$. These strongly suggest that these are of singularity is A_5. We remark that this type of confluence is not generic in Arnold's sense.

When $a = 0$, no confluence occurs.

We next treat degenerate points of rank one, which can be a pair of cuspidal beaks in view of Lemma 2.3. In fact, by solving the equation $q' = 0$ and checking $q'' \neq 0$ and $q^3(x)\overline{q'}(x) - q'(x) \neq 0$, we can show that $(s,t) = (1/3, 0)$ when $b = -1/3$ and $(s,t) = (-1/5, 0)$ when $b = 1/5$ are actually cuspidal beaks.

Remark 2.5. When b passes through the value 0, three swallowtail points get together to the origin and then reappear three swallowtail points again. Such a phenomenon was observed also in the study of Gauss hypergeometric equation

([**NSYY**]). A similar phenomenon is known in the study of confluence of swallowtail points by Arnold; refer to the third case in the list of classification given in [**LLR**, p. 547]. However, the present type of confluence seems to be different from Arnold's since the point $x = 0$ is a singularity of the equation and the map $\mathscr{S}$ itself is multi-valued at this point.

2.5. Figures of the cuspidal edge.

Summarizing the above, in the case $a = 1$, we have critical values of b where the shape of the cuspidal edge and the location of the swallowtail points on it make changes: $b = b_1, -1/3, b_2, 1/5, b_3, b_4$.

In Figures 3–4, we exhibit the shape of the cuspidal edge for several values of b including these. The balck ball indicates a swallowtail point, the white quadrangle a pair of cuspidal beaks, the white ball a singularity of type A_4, and the crossed ball a singularity seemingly of type A_5.

When $a = 0$, we show two figures in Figure 5.

In Figure 6, we give a finer stratification of the ab-plane according to the cardinality of swallowtail points; since they are invariant under the change $a \to -a$, they are marked only in the right half of the figure. The curves are named as $A : 5b - a^2 = 0$, $B : 3b + a^2 = 0$, $C : b = 0$, as in Figure 1, and

$$E1 : \; b = b_1 a^2, \quad E2 : \; b = b_2 a^2, \quad E3 : \; b = 0.4a^2, \quad E4 : \; b = b_4 a^2;$$

Here we remark that the two curves $b = b_3 a^2$ and $b = (1/5)a^2$ are very nearly situated; so, we draw $b = 0.4a^2$ instead of $b = b_3 a^2$ so that the stratification is better observed. In Figure 7, the stratification is transported to the $\alpha\gamma$-plane. The lines C_1 and C_2 are the pullback of the line C.

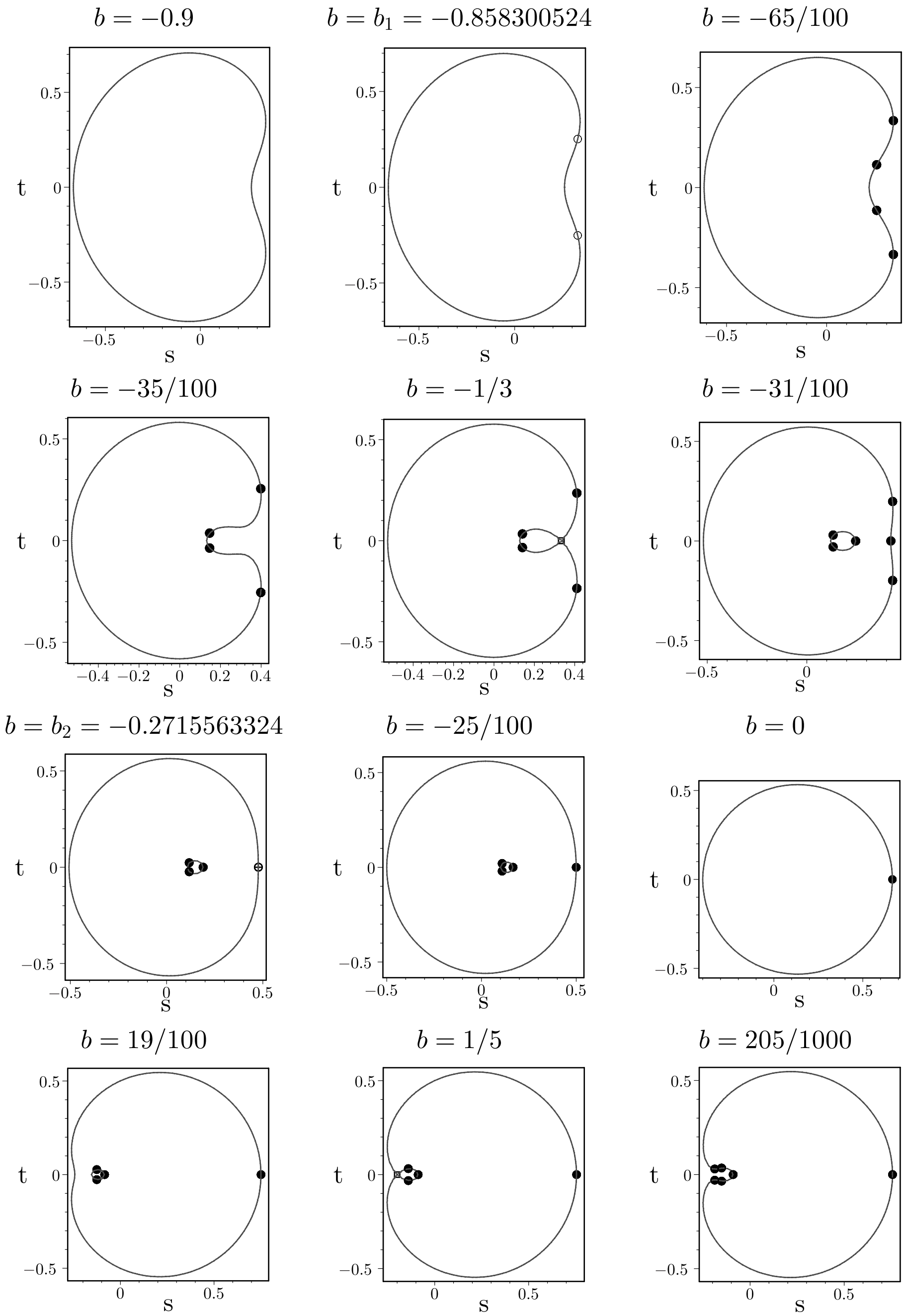


Figure 3. Shapes of CE when $a = 1$.

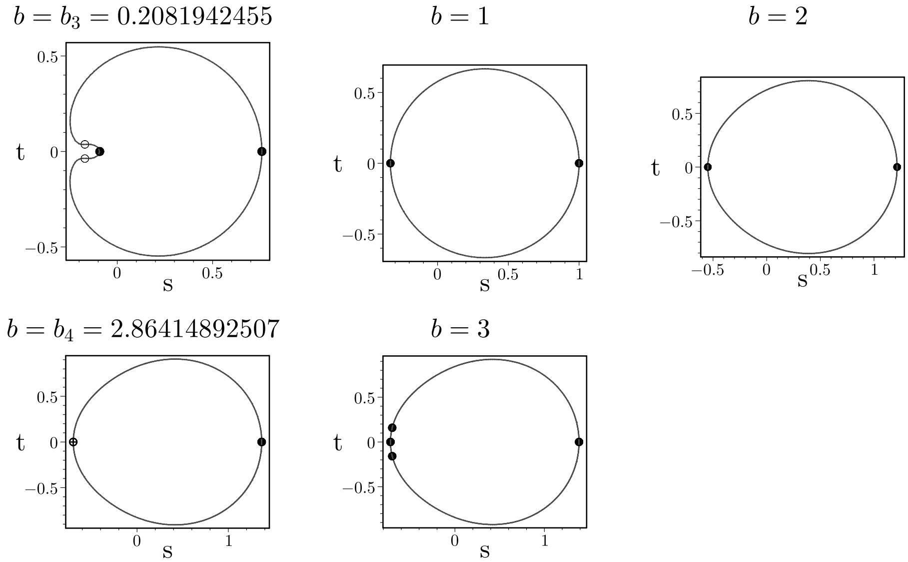


Figure 4. Shapes of CE when $a = 1$ continued.

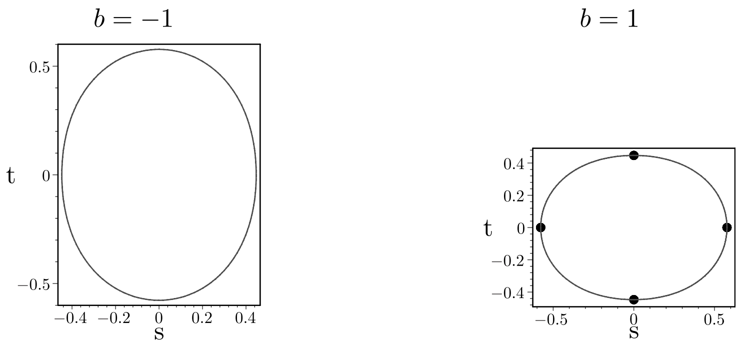


Figure 5. Shapes of CE when $a = 0$.

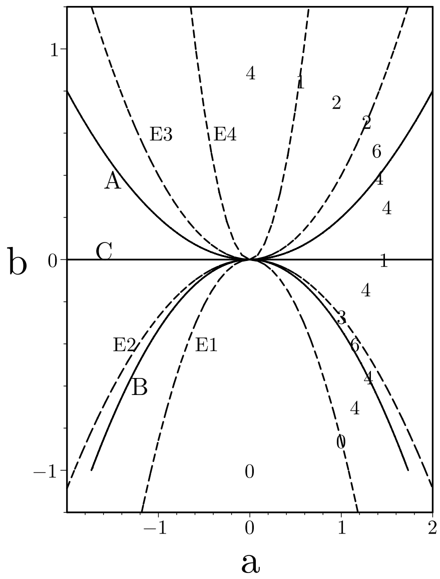


Figure 6. Stratification of ab-plane and cardinalities of swallow-tail points.

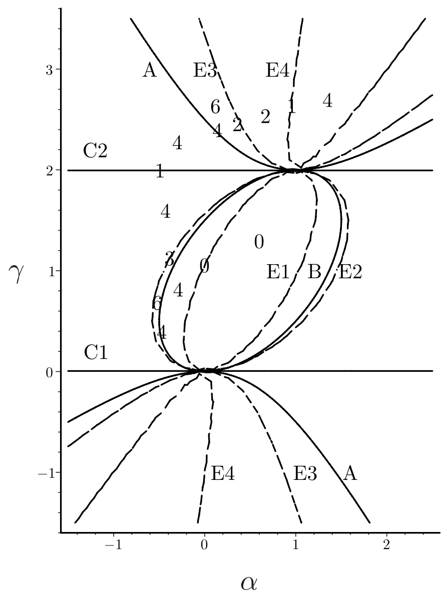


Figure 7. Stratification of $\alpha\gamma$-plane and cardinalities of swallow-tail points.

3. Surfaces for particular values of the parameter *b*.

We show the image surface of $\mathscr{S}$ for some parameters $(a, b) = (1, b)$. Since $\mathscr{S}$ is multi-valued in general, we cut the (s, t)-plane along the negative s-axis. The hyperbolic Schwarz map is defined by use of solutions

$$u_0 = \rho(x)\ {}_1F_1(\alpha, \gamma; x) \quad \text{and} \quad u_1 = \rho(x) x^{1-\gamma}\ {}_1F_1(\alpha - \gamma + 1, 2 - \gamma; x),$$

where

$${}_1F_1(\alpha, \gamma; x) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)n!} x^n$$

is the confluent hypergeomtric function and $\rho(x) = \exp(-x/2)x^{\gamma/2}$ is the multiplier that changes the equation (1.1) into the SL-form (E). For a given b, we have generally two sets of (α, γ) determined by (1.2); however, the both define the same map up to interchange of u_0 and u_1.

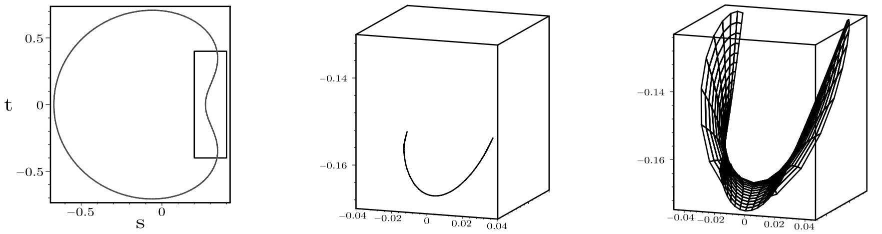
In drawing the surfaces in Figures 8–9, we chose five values of b from the five intervals separated by the exceptional values b_i $(1 \leq i \leq 4)$. Areas of drawing are chosen to be the quadrangles shown in the left figures, where the cuspidal edge curves and the swallowtail points are drawn. In the middle columns, we draw the images of the cuspidal edge curves lying in the quadrangles. In the second row when $b = -0.31$ and $b = -0.25$, the curve in the middle column is the image of the cuspidal edge curve in the thin quadrangle with dotted frame. Each surface in the right column is the image of the quadrangles. A remark is in order: Though the V shaped curve in the middle figure (in the first row) when $b = -0.31$ and that when $b = -0.25$ look similar, it carries three swallowtails in the former case and one swallowtail in the latter case.

The following is the list of quadrangles relative to the coordinate $x = s + it$.

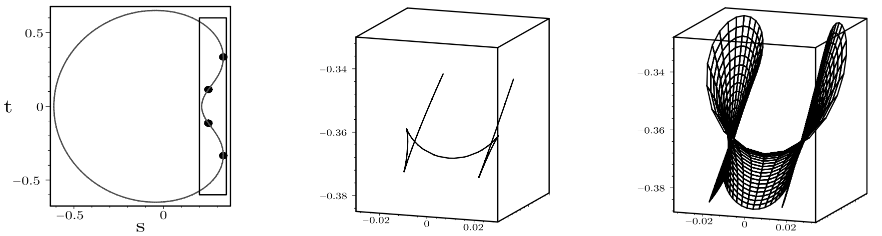
b	$s-$ interval	$t-$ interval
-0.9	$[0.2, 0.4]$	$[-0.4, 0.4]$
-0.65	$[0.2, 0.35]$	$[-0.6, 0.6]$
-0.31	$[0.1, 0.5]$	$[-0.5, 0.5]$
	$[0.40, 0.44]$	$[-0.3, 0.3]$
-0.25	$[0.05, 0.56]$	$[-0.42, 0.42]$
	$[0.07, 0.2]$	$[-0.04, 0.04]$
	$[0.40, 0.54]$	$[-0.4, 0.4]$
1	$[0.7, 1.1]$	$[-0.5, 0.5]$

Drawing is done by Maple 9.5.

$b = -0.9 \quad |ST| = 0$



$b = -0.65 \quad |ST| = 4$



$b = -0.31 \quad |ST| = 6$

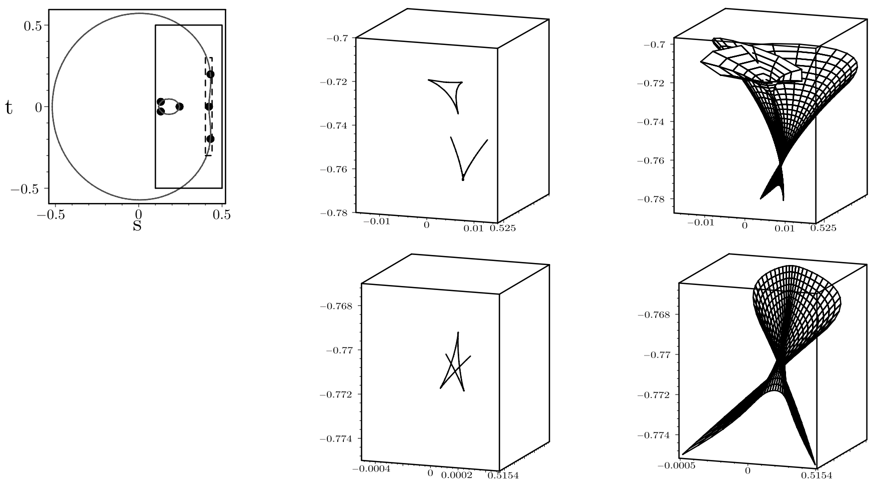


Figure 8. Pictures of image surfaces when $a = 1$. (1)

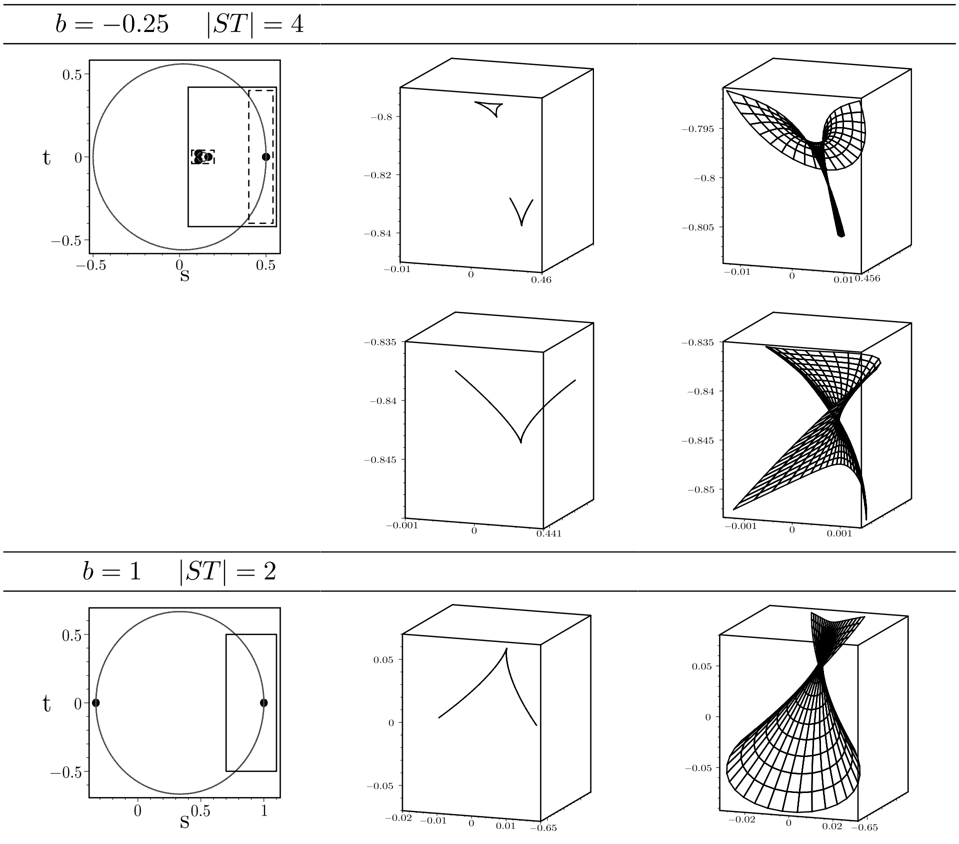


Figure 9. Pictures of image surfaces when $a = 1$. (2)

Appendix.

Two polynomials *swexc* and *num* are given as follows.

$$
\begin{aligned}
swexc \quad = \quad & 32t^6a^4 - 88t^8a^2 + 64a^5t^4s - 144s^5abt^2 - 656s^3at^4b + 576s^3at^2b^2 \\
& + 16s^9a + 104s^8a^2 + 240s^7a^3 + 224s^6a^4 + 64a^5s^5 + 24s^8b + 24s^2b^4 \\
& + 72s^4b^3 + 72s^6b^2 - 24b^4t^2 + 24bt^8 - 72b^2t^6 + 72b^3t^4 + 208s^3ab^3 \\
& + 384s^5ab^2 + 208s^7ab + 672s^5a^3b + 600s^4a^2b^2 + 600s^6a^2b - 96s^5at^4 \\
& - 128s^3at^6 + 160s^6a^2t^2 + 528s^5a^3t^2 + 336s^3a^3t^4 + 480s^4a^4t^2 \\
& - 80s^4a^2t^4 - 48ast^8 + 48a^3st^6 + 288a^4s^2t^4 + 408s^4a^2t^2b \\
& + 576s^3a^3t^2b - 224a^2s^2t^6 + 104a^2s^2b^3 + 224a^4s^4b + 240a^3s^3b^2
\end{aligned}
$$

$$+128a^5s^3t^2+16asb^4-304at^6bs-32a^4t^4b-24a^2t^6b+88a^2t^2b^3$$
$$+24a^2t^4b^2-96s^2bt^6-96s^6bt^2-240s^4bt^4+72s^4b^2t^2-72s^2b^2t^4$$
$$+144s^2b^3t^2-96a^3t^4bs+192a^4t^2bs^2-216a^2t^4bs^2+624a^2t^2b^2s^2$$
$$+240a^3t^2b^2s+192at^4b^2s+144at^2b^3s$$

$$\begin{aligned}
num \quad = \quad & -81t^{12}+162s^2t^{10}-(810b+90)st^{10}+(45b-108)t^{10}+1377s^4t^8\\
& -(810b+450)s^3t^8+(-45b+204)s^2t^8-(135b^2+1068b+648)st^8\\
& +(-99b^2+324b+192)t^8+2268s^6t^6+(2268b-900)s^5t^6\\
& +(-630b+1896)s^4t^6-(540b^2+2592+1296b)s^3t^6\\
& +(324b^2-1296b+768)s^2t^6-b(81b^2+958b-768)st^6\\
& +b^2(27b+400)t^6+1377s^8t^4+(3564b-900)s^7t^4\\
& +(-1170b+3384)s^6t^4+(-810b^2+2520b-3888)s^5t^4\\
& +(2598b^2-5832b+1152)s^4t^4+b(273b^2-4594b+2304)s^3t^4\\
& -b^2(985b-1936)s^2t^4-b^3(25b-556)st^4+28b^4t^4+162s^{10}t^2\\
& +(1134b-450)s^9t^2+(-855b+2436)s^8t^2\\
& +(-540b^2+4656b-2592)s^7t^2+(3828b^2-6480b+768)s^6t^2\\
& +b(789b^2-6314b+2304)s^5t^2-b^2(2323b-2672)s^4t^2\\
& -2b^3(127b-676)s^3t^2+272b^4s^2t^2+11b^5st^2-81s^{12}+(-162b-90)s^{11}\\
& +(636-225b)s^{10}+(-135b^2+1908b-648)s^9\\
& +(192-2268b+1653b^2)s^8+b(435b^2-2678b+768)s^7\\
& -b^2(-1136+1311b)s^6-b^3(229b-796)s^5+268b^4s^4+35b^5s^3
\end{aligned}$$

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