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A generalization of Miyachi's theorem

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Abstract. The classical Hardy theorem on $\boldsymbol{R}$, which asserts f and the Fourier transform of f cannot both be very small, was generalized by Miyachi in terms of $L^1 + L^\infty$ and $\log^+$-functions. In this paper we generalize Miyachi's theorem for $\boldsymbol{R}^d$ and then for other generalized Fourier transforms such as the Chébli-Trimèche and the Dunkl transforms.

1. Introduction.

Classical Hardy's theorem [**7**] asserts the following: let $a, b > 0$ and f a measurable function on $\boldsymbol{R}$ satisfying $|f(x)| \leq Ce^{-ax^2}$ and $|\hat{f}(y)| \leq Ce^{-b\lambda^2}$. Then $f \equiv 0$ if $ab > 1/4$, f is a constant multiple of e^{-ax^2} if $ab = 1/4$, and there are infinitely many f if $ab < 1/4$. Considerable attention has been devoted to finding generalizations to new contexts for Hardy's theorem. Especially, M. Cowling and J. Price [**5**] obtained an L^p version of the theorem. As further generalizations, A. Bonami, B. Demange, P. Jaming [**2**] extend it in a Beurling form, and Miyachi [**9**] obtains an $L^1 + L^\infty$ version: Let $ab = 1/4$ and $f \in L^1(\boldsymbol{R})$ satisfy

$$e^{ax^2} f(x) \in L^1(\boldsymbol{R}) + L^\infty(\boldsymbol{R})$$

and

$$\int_{-\infty}^{+\infty} \log^+ \frac{|\hat{f}(\lambda)e^{b\lambda^2}|}{C}\, d\lambda < \infty$$

for some $C > 0$. Then f is a constant multiple of e^{-ax^2}, where $L^1(\boldsymbol{R}) + L^\infty(\boldsymbol{R})$ is the set of functions of the form $f = f_1 + f_2$, $f_1 \in L^1(\boldsymbol{R})$, $f_2 \in L^\infty(\boldsymbol{R})$, and $\log^+ x = \log x$ if $x > 1$ and $\log^+ x = 0$ if $x \leq 1$. In this paper we shall generalize Miyachi's

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theorem for $\boldsymbol{R}^d$ and furthermore, we shall obtain an analogue in a general measure space $(X, d\omega)$ equipped with Fourier, Radon, and dual Radon transforms. As a special case of this general setting, we can deduce Miyachi's theorem for the Chébli-Trimèche, the Dunkl transforms, and the Jacobi-Dunkl transform. Especially, we can obtain alternative proofs in [**6**] for the Jacobi transform and in [**3**] for the Dunkl transform.

2. Miyachi's theorem on $\boldsymbol{R}^d$.

For $l = 1, 2, \cdots$, we denote by $\mathscr{F}_l$ the Fourier transform on $\boldsymbol{R}^l$. Miyachi's theorem is generalized on $\boldsymbol{R}^d$ as follows.

THEOREM 1. *Let* $a_i, b_i > 0$ *and* $a_i b_i = 1/4$ *for* $1 \leq i \leq d$. *We put* $A = \mathrm{diag}(a_1, a_2, \cdots, a_d)$ *and* $B = \mathrm{diag}(b_1, b_2, \cdots, b_d)$. *If a measurable function on* $\boldsymbol{R}^d$ *satisfies*

$$e^{\langle Ax,x\rangle} f(x) \in L^1(\boldsymbol{R}^d) + L^\infty(\boldsymbol{R}^d) \tag{1}$$

and

$$\int_{\boldsymbol{R}^d} \log^+ \frac{|\mathscr{F}_d f(\lambda) e^{\langle B\lambda,\lambda\rangle}|}{C} d\lambda < \infty \tag{2}$$

for some $C > 0$, *then* f *is a constant multiple of* $e^{-\langle Ax,x\rangle}$.

PROOF. For $(x_1, x') = (x_1, x_2, \cdots, x_d)$ and $\lambda' = (\lambda_2, \cdots, \lambda_d)$, we shall consider

$$G(x_1, \lambda') = \mathscr{F}_{d-1}(f(x_1, \cdot))(\lambda') = \int_{\boldsymbol{R}^{d-1}} f(x_1, x') e^{-i\langle \lambda', x'\rangle} dx',$$

where $dx' = dx_2 \cdots dx_d$. Since $f \in L^1(\boldsymbol{R}^d)$ by (1), $G(x_1, \lambda')$ is well-defined and is in $L^1(\boldsymbol{R}) \otimes L^\infty(\boldsymbol{R}^{d-1})$. Moreover, since $f = e^{-(a_1 x_1^2 + a_2 x_2^2 + \cdots + a_d x_d^2)}(u_1 + u_2)$, where $u_1 \in L^1(\boldsymbol{R}^d)$ and $u_2 \in L^\infty(\boldsymbol{R}^d)$ by (1), it follows that

$$|G(x_1, \lambda')| \leq e^{-a_1 x_1^2} \sum_{k=1}^{2} \int_{\boldsymbol{R}^{d-1}} e^{-(a_2 x_2^2 + \cdots + a_d x_d^2)} |u_k(x_1, x')| dx'.$$

and thus, as a function of x_1, $e^{a_1 x_1^2} G(x_1, \lambda') \in L^1(\boldsymbol{R}) + L^\infty(\boldsymbol{R})$. We note that $\mathscr{F}_1(G(\cdot, \lambda'))(\lambda_1) = \mathscr{F}_d f(\lambda)$ and substitute it in (2). Then Fubini's theorem implies that there exists a subset E of $\boldsymbol{R}^{d-1}$ with positive measure such that for

$\lambda' = (\lambda_2, \cdots, \lambda_d) \in E$,

$$\int_{-\infty}^{+\infty} \log^+ \frac{|\mathscr{F}_1(G(\cdot, \lambda'))(\lambda_1)e^{b_1\lambda_1^2}|}{Ce^{-(b_2\lambda_2^2+\cdots+b_d\lambda_d^2)}}\, d\lambda_1 < \infty.$$

Therefore, Miyachi's theorem on $\boldsymbol{R}$ yields that $G(x_1, \lambda') = C(\lambda')e^{-a_1x_1^2}$ for $\lambda' \in E$. Hence $\mathscr{F}_d f(\lambda) = \mathscr{F}_1(G(\cdot, \lambda'))(\lambda_1) = C(\lambda')e^{-b_1\lambda_1^2}$ for $\lambda' \in E$. Since $\mathscr{F}_d f(\lambda)$ has a holomorphic extension on $\boldsymbol{C}^n$ (see (1)), we can prolong the precedent relation as $\mathscr{F}_d f(\lambda) = C(\lambda')\, e^{-b_1\lambda_1^2}$ for all $\lambda' \in \boldsymbol{R}^{d-1}$. Hence $G(x_1, \lambda') = C(\lambda')e^{-a_1x_1^2}$ for all $\lambda' \in \boldsymbol{R}^{d-1}$. Here we put $h(x) = e^{a_1x_1^2}f(x)$. Then, as a function of $x' = (x_2, \cdots, x_d)$, it belongs to $L^1(\boldsymbol{R}^{d-1})$ (see (1)) and $\mathscr{F}_{d-1}(h(x_1, \cdot))(\lambda') = e^{a_1x_1^2}\mathscr{F}_{d-1}(f(x_1, \cdot))(\lambda') = e^{a_1x_1^2}G(x_1, \lambda') = C(\lambda')$ for $\lambda' \in \boldsymbol{R}^{d-1}$. Since $C(\lambda')$ is independent of x_1, it follows that h is also independent of x_1. We obtain that f is of the form $f(x_1, x_2, \cdots, x_d) = e^{-a_1x_1^2}h(x_2, \cdots, x_d)$. We can obtain the desired result by repeating this argument. □

3. Generalization of Miyachi's theorem.

Let $(X, d\omega, \sigma)$ be a topological space with a positive measure $d\omega$ and a distance function $\sigma : X \to \boldsymbol{R}_+$. In what follows we assume the existence of an integral operator $\mathscr{R}^*$ that satisfies the following properties: $\mathscr{R}^*$ is of the form

$$\mathscr{R}^* f(y) = \int_X f(x)d\nu_y(x),\ \ f \in C_c(X), \tag{3}$$

(A1) ν_y is a positive measure with the support in $\{x \in X |\ \|y\| \le \sigma(x)\}$,
(A2) $\mathscr{R}^*$ is bounded from $L^1(X, d\omega)$ to $L^1(\boldsymbol{R}^d)$,
(A3) $\mathscr{R}^*$ gives an isomorphism between $\mathscr{S}(X)$ and $\mathscr{S}(\boldsymbol{R}^d)$, where $\mathscr{S}(X)$ is a suitable Schwartz class on X and $\mathscr{S}(\boldsymbol{R}^d)$ is the Schwartz class on $\boldsymbol{R}^d$.

We here define the dual operator $\mathscr{R} : \mathscr{S}'(\boldsymbol{R}^d) \to \mathscr{S}'(X)$ of $\mathscr{R}^*$ by

$$\int_{\boldsymbol{R}^d} \mathscr{R}^* f(y)g(y)dy = \int_X f(x)\mathscr{R}g(x)d\omega(x). \tag{4}$$

We furthermore assume that

(A4) $\mathscr{R}$ is of the form

$$\mathscr{R}f(x) = \int_{\boldsymbol{R}^d} f(y)d\mu_x(y),\ \ f \in C(\boldsymbol{R}^d), \tag{5}$$

where μ_x is a positive measure on $\boldsymbol{R}^d$ with support in $\{y \in \boldsymbol{R}^d |\ \|y\| \le \sigma(x)\}$.

Then we see that $\mathscr{R}(1)(x) \le C_0$ for all $x \in X$ because of (4) and (A2). We denote the Euclidean Fourier transform on $\boldsymbol{R}^d$ by $\mathscr{F}_d$ and put

$$\mathscr{F}_X = \mathscr{F}_d \circ \mathscr{R}^*.$$

We call $\mathscr{F}_X$ a generalized Fourier transform on X, which is an isomorphism between $\mathscr{S}(X)$ and $\mathscr{S}(\boldsymbol{R}^d)$ (see (A3)). We define Gaussian type functions $h_a(x)$ and $h^*_{-a}(x)$, $x \in X, a \in \boldsymbol{R}_+$, as follows:

$$h_a(x) = (\mathscr{R}^*)^{-1}(e^{-a\|\cdot\|^2})(x) \quad \text{and} \quad h^*_{-a}(x) = \mathscr{R}(e^{a\|\cdot\|^2})(x).$$

Since $e^{-a\|\cdot\|^2} \in \mathscr{S}(\boldsymbol{R}^d)$, $h_a(x)$ is well-defined (see (A3)), positive on X and $\mathscr{F}_X h_a(y) = Ce^{-\|y\|^2/4a}$. Since $h^*_{-a}(x) \le e^{a\sigma(x)^2}\mathscr{R}(1)(x) \le C_0 e^{a\sigma(x)^2}$, $h^*_{-a}(x)$ is well-defined and positive on X. Especially, $(h^*_{-a})^{-1}(x)$ is positive on X.

Theorem 2. *Let $a > 0$ and $ab = 1/4$. If a measurable function f on X satisfies $f = f_1 + f_2$ such that*

$$h^*_{-a}(x) f_1(x) \in L^1(X, d\omega) \text{ and } h_a^{-1}(x) f_2(x) \in L^\infty(X, d\omega) \tag{6}$$

and

$$\int_{\boldsymbol{R}^d} \log^+ \frac{|\mathscr{F}_X f(\lambda) e^{b\|\lambda\|^2}|}{C}\, d\lambda < \infty$$

for some $C > 0$, then f is a constant multiple of h_a.

Proof. We show that $\mathscr{R}^*(f)$ satisfies

$$e^{a\|y\|^2}\mathscr{R}^* f(y) \in L^1(\boldsymbol{R}^d) + L^\infty(\boldsymbol{R}^d).$$

Actually, by (6) $f(x) = f_1(x) + f_2(x) = (h^*_{-a})^{-1}(x) v_1(x) + h_a(x) v_2(x)$, where $v_1 \in L^1(X, d\omega)$ and $v_2 \in L^\infty(X, d\omega)$. Then

$$\begin{aligned}\|e^{a\|y\|^2}\mathscr{R}^*f_1\|_{L^1(\boldsymbol{R}^d)} &= \int_{\boldsymbol{R}^d} e^{a\|y\|^2}\big|\mathscr{R}^*((h^*_{-a})^{-1}v_1)(y)\big|dy \\ &\le \int_{\boldsymbol{R}^d} e^{a\|y\|^2}\mathscr{R}^*((h^*_{-a})^{-1}|v_1|)(y)dy \\ &= \int_X \mathscr{R}(e^{a\|\cdot\|^2})(x)(h^*_{-a})^{-1}(x)|v_1(x)|d\omega(x) \text{ (see(4))} \\ &= \int_X |v_1(x)|d\omega(x)\end{aligned}$$

and

$$e^{a\|y\|^2}|\mathscr{R}^*f_2(y)| \le \|v_2\|_\infty e^{a\|y\|^2}\mathscr{R}^*h_a(y) = \|v_2\|_\infty.$$

Hence $e^{a\|y\|^2}\mathscr{R}^*f(y)$ belongs to $L^1(\boldsymbol{R}^d)+L^\infty(\boldsymbol{R}^d)$. Since $\mathscr{F}_X f = \mathscr{F}_d(\mathscr{R}^*f)$, it follows from Miyachi's theorem on $\boldsymbol{R}^d$ (see Theorem 1) that $\mathscr{R}^*f(y) = Ce^{-a\|y\|^2}$. Hence we obtain that $f = Ch_a$. □

Remark 3. If there exists a function $\gamma(x)$ on X such that

$$h_a(x)h^*_{-a}(x) \le \gamma(x),$$

then the condition (6) in Theorem 2 can be replaced by

$$h_a^{-1}(x)f(x) \in L^1(X,\gamma d\omega) + L^\infty(X,d\omega). \tag{7}$$

4. Examples.

4.1. Chébli-Trimèche hypergroups.

We refer to [**1**], [**10**] and [**12**] for general notations and basic facts on Chébli-Trimèche hypergroups. Let $\alpha > -1/2$ and $A(x) = x^{2\alpha+1}B(x)$ be a Chébli-Trimèche function on $\boldsymbol{R}_+$. Then $(X, d\omega, \sigma) = (\boldsymbol{R}_+, A(x)dx, |\cdot|)$ and the Chébli-Trimèche transform $\mathscr{F}_X$ gives an isometry between $L^2(X, d\omega)$ and $L^2(\boldsymbol{R}, d\mu)$ where $d\mu(\lambda) = |C(\lambda)|^{-2}d\lambda$ and supported on even functions on $\boldsymbol{R}$. We recall that $\mathscr{F}_X = \mathscr{F}\circ\mathscr{R}^*$ where $\mathscr{R}^*$ is the Weyl type integral transform

$$\mathscr{R}^*f(y) = \int_y^\infty f(x)K(x,y)A(x)dx, \quad y \ge 0$$

and $\mathscr{R}$ is the Riemann-Liouville type integral transform

$$\mathscr{R}f(x) = \int_0^x f(y)K(x,y)dy, \quad x \geq 0.$$

The multiplicative functions on the hypergroup coincide with spherical functions $\phi_\lambda(x)$, $\lambda \in \boldsymbol{C}$. Let $\mathscr{S}_e(\boldsymbol{R})$ denote the space of even rapidly decreasing functions on $\boldsymbol{R}$ and $\mathscr{S}(X) = \phi_0(x)\mathscr{S}_e(\boldsymbol{R})$, where ϕ_0 is the spherical function with $\lambda = 0$. Then $\mathscr{R}^*$ satisfies the assumptions (A1) - (A4) in Section 3 (see [**12**]), where $d\nu_y(x) = K(x,y)\chi_{[y,\infty)}(x)A(x)dx,\ y \geq 0,$ and $d\mu_x(y) = K(x,y)\chi_{[0,x]}(y)dy,\ x \geq 0.$ Hence, by introducing the Gaussian type functions h_a and h^*_{-a}, Theorem 2 holds for the Chébli-Trimèche transform $\mathscr{F}_X$. We shall obtain $\gamma(x)$ in Remark 3. We recall from [**12**] that

$$h_a(x) \leq C\frac{1}{\sqrt{B(x)}}\, e^{-ax^2}.$$

On the other hand, since $e^{ax^2} = e^{a(x-\rho/2a)^2 + \rho x - \rho^2/4a}$,

$$h^*_{-a}(x) \leq Ce^{a(x-\rho/2a)^2}\mathscr{R}(e^{\rho y})(x) \leq Ce^{a(x-\rho/2a)^2}\phi_{i\rho}(x) \leq Ce^{ax^2-\rho x}.$$

Hence

$$h_a(x)h^*_{-a}(x) \leq C\frac{1}{\sqrt{B(x)}}\, e^{-\rho x}$$

and

$$C\frac{1}{\sqrt{B(x)}}\, e^{-\rho x}d\omega(x) = Cx^{2\alpha+1}\sqrt{B(x)}e^{-\rho x}dx$$

In particular, if X is the Bessel-Kingman hypergroup, then $A(x) = x^{2\alpha+1}$ and $\gamma d\omega = Cx^{2\alpha+1}dx$. If X is the Jacobi hypergroup, then $A(x) = c(\sinh x)^{2\alpha+1} \cdot(\cosh x)^{2\beta+1}$, $\alpha \geq \beta \geq -1/2$, and $\gamma d\omega = C(\tanh x)^{\alpha+1/2}x^{\alpha+1/2}dx$. Hence the condition (7) in Remark 3 is nothing but the one used in [**6**, Theorem 3.1].

4.2. Dunkl analysis.

We refer to [**8**] and [**11**] for general notations and basic facts in Dunkl transform. Let $(X, d\omega, \sigma) = (\boldsymbol{R}^d, \omega_k(x)dx, \|\cdot\|)$ where k is a multiplicity function on the root system. Then the Dunkl transform $\mathscr{F}_X$ gives an isometry between $L^2(X, d\omega)$ and $L^2(\boldsymbol{R}^d, C_k d\omega)$. We recall that $\mathscr{F}_X = \mathscr{F}_d \circ \mathscr{R}^*$. Here $\mathscr{R}^*$ and its dual operator $\mathscr{R}$ are given as

$$\mathscr{R}^* f(y) = \int_{\boldsymbol{R}^d} f(x) d\nu_y(x),$$

where ν_y is a positive measure on $\boldsymbol{R}^d$ with the support in $\{x \in \boldsymbol{R}^d | \, \|x\| \geq \|y\|\}$ and

$$\mathscr{R} f(x) = \int_{\boldsymbol{R}^d} f(y) d\mu_x(y),$$

where μ_x is a positive measure of probability on $\boldsymbol{R}^d$ with the support in $\{y \in \boldsymbol{R}^d | \, \|y\| \leq \|x\|\}$ (see [**11**]). Let $\mathscr{S}(\boldsymbol{R}^d)$ denote the Schwartz space on $\boldsymbol{R}^d$. Then $\mathscr{R}^*$ satisfies the assumptions (A1) - (A4) in Section 3 (see [**11**]). Moreover, we note that $h_a(x) = e^{-a\|x\|^2}$ and

$$h^*_{-a}(x) \leq e^{a\|x\|^2} \mathscr{R}(1)(x) = e^{a\|x\|^2}.$$

Hence $h_a(x) h^*_{-a}(x) \leq 1$ and $\gamma(x) = 1$ in (7). Hence we can obtain the corresponding Theorem 2 for the Dunkl transform.

4.3. Jacobi-Dunkl analysis.

We refer to [**4**] for general notations and basic facts in Jacobi-Dunkl transform. Let $\alpha \geq \beta \geq -1/2$ and $\alpha \neq -1/2$. We put $\rho = \alpha + \beta + 1 > 0$ and $A_{\alpha,\beta}(x) = 2^{2\rho} (\sinh |x|)^{2\alpha+1} (\cosh x)^{2\beta+1}$. Then $(X, d\omega, \sigma) = (\boldsymbol{R}, A_{\alpha,\beta}(x)dx, |\cdot|)$ and the Jacobi-Dunkl transform $\mathscr{F}_X$ gives an isometry between $L^2(X, d\omega)$ and $L^2(\boldsymbol{R}, d\mu)$. Here $d\mu$ is of the form

$$d\mu(\lambda) = \frac{|\lambda|}{8\pi\sqrt{\lambda^2 - \rho^2} \left| c(\sqrt{\lambda^2 - \rho^2}) \right|^2} 1_{\boldsymbol{R} \backslash]-\rho,\rho[}(\lambda) d\lambda,$$

where $1_{\boldsymbol{R} \backslash]-\rho,\rho[}$ is the characteristic function of $\boldsymbol{R} \backslash]-\rho, \rho[$ and $c(\lambda)$ is a meromorphic function on $\boldsymbol{C}$. We recall that $\mathscr{F}_X = \mathscr{F} \circ \mathscr{R}^*$ where $\mathscr{R}^*$ the Jacobi-Dunkl dual intertwining operator defined by

$$\mathscr{R}^* f(y) = \int_{|x| \geq |y|} K(x,y) f(x) A_{\alpha,\beta}(x) dx$$

and $\mathscr{R}$ is the Jacobi-Dunkl intertwining operator defined by

$$\mathscr{R} f(x) = \begin{cases} \displaystyle\int_{-|x|}^{|x|} K(x,y) f(y) dy & \text{if } x \in \boldsymbol{R} \backslash \{0\}, \\ f(0) & \text{if } x = 0. \end{cases}$$

Let $\mathscr{S}(\boldsymbol{R})$ denote the Schwartz space on $\boldsymbol{R}$ and $\mathscr{S}(X) = (\cosh x)^{-2\rho}\mathscr{S}(\boldsymbol{R})$. Then $\mathscr{R}^*$ satisfies the assumptions (A1) - (A4) in Section 3, where $d\nu_y(x) = K(x,y)\chi_{[|y|,\infty)}(x)A_{\alpha,\beta}(x)dx$ and $d\mu_x(y) = K(x,y)\chi_{[0,|x|]}(x)A_{\alpha,\beta}(x)dx$. Hence, by introducing the Gaussian type functions h_a and h^*_{-a}, Theorem 2 holds for the Jacobi-Dunkl transform $\mathscr{F}_X$. As in Jacobi hypergroup case, the condition (7) in Remark 3 is given by $\gamma d\omega = C(\tanh|x|)^{\alpha+\frac{1}{2}}|x|^{\alpha+\frac{1}{2}}dx$.

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