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Composition properties of box brackets

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Abstract. In the homotopy theory of a 2-category with zeros and having a suspension functor we establish various composition properties of box brackets, including new formulae involving 2-sided matrix Toda brackets and classical Toda brackets. We are lead to define and study a new secondary homotopy operation called the box quartet operation. In the topological category this operation satisfies two triviality properties, one of which may be viewed as the foundation upon which an important classical mod zero result on Toda brackets rests. New insights and computations in the homotopy groups of spheres are obtained.

Introduction.

In his seminal 1962 book *Composition Methods in Homotopy Groups of Spheres* [**16**], Toda computed the homotopy groups of spheres through the 19-stem. His techniques involved use of the Toda bracket (or secondary composition) operation; this is a secondary homotopy operation $\{\alpha, \beta, \gamma\}$ defined for any triple composite $\bullet \xrightarrow{\gamma} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\alpha} \bullet$ that satisfies $\alpha \circ \beta \simeq o$ and $\beta \circ \gamma \simeq o$. Subsequently composition methods have been widely used and various extensions of the Toda bracket have been considered, such as matrix Toda brackets and long Toda brackets for example.

A recent trend has been the development of Toda bracket type operations in settings other than the topological category. For example the theory for an abstract 2-category with zeros has been explored in [**2**], [**4**], and [**5**], and that for a bicategory in [**3**]. Indeed new operations of Toda bracket type have emerged, of which the box bracket [**5**] and the 2-sided matrix Toda bracket [**4**] seem quite useful, and new formulae have been found. In fact it is to be noted that an early appearance[1] of the box bracket, but in a dual formulation under the name "ladder Toda bracket" and for use in study of Hopf algebra structure, occurs in the work

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[1]The second author is obliged to Professor Yutaka Hemmi for pointing out this reference.

of Zabrodsky [**17**].

In the present paper we begin in Section 1 by providing some additional fundamental results relating box brackets and 2-sided matrix Toda brackets in a 2-category with zeros. Key results obtained are stated in Theorems 1.2 and 1.5. In particular, new formulae among classical Toda brackets are proven. In Theorem 1.8 an equality is established which shows that a box bracket under double sided composition decomposes into a sum of 2-sided matrix Toda brackets. In Section 2 we establish some lemmas that are crucial to the remainder of the paper.

Now in [**16**] Toda devoted Chapter I to a succinct survey of the basic properties of the Toda bracket, including various composition properties. The statement of these latter may be facilitated by reference to an n-fold composite of maps

$$\bullet \xrightarrow{\alpha_n} \cdots \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_1} \bullet$$

for which all pair composites $\alpha_i \circ \alpha_{i+1}$ are null homotopic. For length $n = 3$, the definition of the Toda bracket $\{\alpha_1, \alpha_2, \alpha_3\}$ itself is obtained. For length $n = 4$, the very important formula (Proposition 1.4 of [**16**] or Theorem 4.3 i) of [**15**])

$$-\alpha_1 \circ \{\alpha_2, \alpha_3, \alpha_4\} = \{\alpha_1, \alpha_2, \alpha_3\} \circ E\alpha_4$$

arises. For length $n = 5$, Toda gives a more complicated result (Proposition 1.5 of [**16**] or Theorem 4.3 ii) of [**15**]) which states in rough form that

$$\{\{\alpha_1, \alpha_2, \alpha_3\}, E\alpha_4, E\alpha_5\} + \{\alpha_1, \{\alpha_2, \alpha_3, \alpha_4\}, E\alpha_5\} + \{\alpha_1, \alpha_2, \{\alpha_3, \alpha_4, \alpha_5\}\} \equiv 0$$

whenever defined. While many properties from [**16**] already have received attention in the abstract setting this last has not. One purpose of this paper is to examine this last result of Toda from the point of view of a new operation called the box quartet operation, and, in Theorem 4.15, this result of Toda is generalized to a formula involving two matrix Toda brackets and a box bracket in place of the three inner Toda brackets in the formula quoted above. Toda also proved fundamental formulae which give relations between coextensions and Toda brackets (Proposition 1.8 of [**16**]) and extensions and Toda brackets (Proposition 1.9 of [**16**]). These results of Toda are described precisely making use of the homotopies and extended to formulae involving matrix Toda brackets and box brackets in Propositions 4.5, 4.10 and 4.12, which are crucial to completing the proof of Theorem 4.15.

The box quartet operation is defined in Section 3 below; it may be regarded as an operation that in a certain sense increases stem dimension by 2. Significantly its definition may be given in any 2-category possessing zeros and a suspension functor.

We provide examples in the topological category $\mathscr{T}op_*$ to show that indeed the box quartet operation may be nontrivial (see Proposition 5.1). Furthermore in $\mathscr{T}op_*$ we show that the box quartet operation satisfies a triviality axiom in the form of the following theorem (see Theorem 4.13).

TRIVIALITY THEOREM. *In $\mathscr{T}op_*$ let*

$$\begin{tikzcd}
W \arrow[r, "w"] \arrow[d] & C \arrow[r, "g"] \arrow[d, "f"] & B \arrow[r, "r"] \arrow[d, "b"] & R \arrow[r] \arrow[d, "y"] & * \arrow[d] \\
* \arrow[r] & A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \arrow[r, "v"'] & V
\end{tikzcd}$$

be a homotopy commutative diagram in which all horizontal pair composites are null homotopic. Suppose $o \in \{r, g, w\}$ and $o \in \{v, s, a\}$. Then the box quartet operation $\mathscr{D} \subset \pi(\Sigma^2 W, V)$ of this diagram is defined and moreover $o \in \mathscr{D}$.

Obviously such a result may be formulated in the general 2-categorical context but we succeed in showing its validity only in $\mathscr{T}op_*$. Even there the proof is not immediate. Our approach uses a modified theory of extensions and coextensions and it is the lack of such in the general 2-category case that restricts our proof to $\mathscr{T}op_*$. Moreover difficulties with homotopy coherence arise and must be handled with care. We remark that what seems to be needed for the Triviality Theorem to be valid in the general case is some assumption of 3-dimensional structure as yet unidentified.

Box quartet operations in $\mathscr{T}op_*$ offer a very natural setting for consideration and clarification of Toda's result in Proposition 1.5 of [**16**]. As previously stated, we prove an extension of Toda's result in our Theorem 4.15, the proof of which is similar to our proof of the Triviality Theorem. In Section 5 and Section 6 we offer some sample computations in $\mathscr{T}op_*$. However it is not our intention to make extensive computations in this paper. Encouragingly we find our techniques capable of yielding new insights (see Proposition 6.1).

For two sets A and B, the symbol $A \sim B$ means that A and B have a common element.

1. Lemmas on 2-sided matrix Toda brackets.

In this section we work in a 2-category $\mathscr{C}$ with zeros. We refer to Section 4 of [**4**] for the definition and basic properties of the 2-sided matrix Toda bracket in such a setting. The two notations

$$\left\{ s\,,\ \begin{matrix} b \\ a \end{matrix}\,,\ \begin{matrix} g \\ f \end{matrix}\,,\ w \right\} = \overset{\cdot}{\underset{\cdot}{\square}} \left(\begin{tikzcd} W \arrow[r, "w"] & C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "b"] & \\ & A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \end{tikzcd} \right)$$

will be used interchangeably to denote the 2-sided matrix Toda bracket. We recall that by definition

$$\left\{ s\,,\ \begin{matrix} b \\ a \end{matrix}\,,\ \begin{matrix} g \\ f \end{matrix}\,,\ w \right\} \subset \mathscr{A}_{\mathscr{C}}(o\colon W \to Y)$$

consists of all composite 2-morphisms of the form

$$-(s \circ b)K + sFw + H(f \circ w) \colon o \Rightarrow o \colon W \to Y$$

for homotopies $H \colon o \Rightarrow s \circ a$, $F \colon a \circ f \Rightarrow b \circ g$ and $K \colon o \Rightarrow g \circ w$. Here we remark that $\mathscr{A}_{\mathscr{C}}(o\colon W \to Y)$ denotes the automorphism group in $\mathscr{C}$ of all self homotopies of the zero morphism $o\colon W \to Y$. The indeterminacy of the 2-sided matrix Toda bracket is established in Proposition 4.8 of [**4**].

In order to state Theorem 1.2 below we fix the following homotopy commutative diagram of 1-morphisms in $\mathscr{C}$.

$$\begin{tikzcd} W \arrow[r, "w"] & C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "b"] & \\ & A \arrow[r, "a"] \arrow[d, "k"'] & X \arrow[d, "p"] & \\ & R \arrow[r, "c"'] & P \arrow[r, "q"'] & Q \end{tikzcd} \tag{1.1}$$

Theorem 1.2. (1) *In diagram* (1.1) *suppose that* $g \circ w \simeq o$ *and* $q \circ c \simeq o$. *Then*

$$\{q \circ p, a, f \circ w\} \sim \overset{\cdot}{\underset{\cdot}{\square}} \left(\begin{tikzcd} W \arrow[r, "w"] & C \arrow[r, "g"] \arrow[d, "k \circ f"'] & B \arrow[d, "p \circ b"] & \\ & R \arrow[r, "c"'] & P \arrow[r, "q"'] & Q \end{tikzcd} \right)$$

as subsets of $\mathscr{A}_{\mathscr{C}}(o\colon W \to Q)$.

(2) *In diagram* (1.1) *suppose that* $g \circ w \simeq o$, $q \circ c \simeq o$ *and* $p \circ a \circ f \simeq o$. *Then*

$$\{q \circ p, a, f \circ w\} \sim q \circ \{p \circ b, g, w\} + \{q, c, k \circ f\} \circ w.$$

(3) *In diagram* (1.1) *suppose that* $g \circ w \simeq o$, $q \circ c \simeq o$, $p \circ a \circ f \simeq o$, $q \circ p \simeq o$ *and* $f \circ w \simeq o$. *Then*

$$q \circ \{p \circ b, g, w\} = -\{q, c, k \circ f\} \circ w.$$

PROOF.

(1) Select homotopies $K\colon o \Rightarrow g \circ w$, $F\colon a \circ f \Rightarrow b \circ g$, $L\colon c \circ k \Rightarrow p \circ a$ and $H\colon o \Rightarrow q \circ c$. Consider the composite homotopy

$$\theta := (q \circ p)[-bK + Fw] + [qL + Hk](f \circ w) = -(q \circ p \circ b)K + q[pF + Lf]w + H(k \circ f \circ w).$$

The first expression identifies θ as an element of $\{q \circ p, a, f \circ w\}$ while the second expression identifies θ as an element of $\left\{q\,, \begin{matrix} p \circ b \\ c \end{matrix}\,, \begin{matrix} g \\ k \circ f \end{matrix}\,, w\right\}$. Hence these two operations have θ as a common element.

(2) For, when $p \circ a \circ f \simeq o$ then

$$\left\{q\,, \begin{matrix} p \circ b \\ c \end{matrix}\,, \begin{matrix} g \\ k \circ f \end{matrix}\,, w\right\} = q \circ \{p \circ b, g, w\} + \{q, c, k \circ f\} \circ w$$

by Proposition 4.5(1) of [**4**] and so the result follows from Part (1).

(3) Note that $\{q \circ p, a, f \circ w\} = \{1_o\}$ since $q \circ p \simeq o$ and $f \circ w \simeq o$. Thus

$$1_o \in q \circ \{p \circ b, g, w\} + \{q, c, k \circ f\} \circ w$$

by Part (2). Consequently

$$q \circ \{p \circ b, g, w\} \sim -\{q, c, k \circ f\} \circ w.$$

Now there exist elements ρ and ζ so that

$$\{p \circ b, g, w\} = p \circ b \circ \mathscr{A}_{\mathscr{C}}(o\colon W \to B) + \rho + \mathscr{A}_{\mathscr{C}}(o\colon C \to P) \circ w$$

and

$$\{q, c, k \circ f\} = q \circ \mathscr{A}_{\mathscr{C}}(o\colon C \to P) + \zeta + \mathscr{A}_{\mathscr{C}}(o\colon R \to Q) \circ k \circ f$$

(see Proposition 8.2 of [**2**]). Because $q \circ p \circ b \simeq o$ and $k \circ f \circ w \simeq o$ it follows that $q \circ \{p \circ b, g, w\}$ and $-\{q, c, k \circ f\} \circ w$ both are cosets of the same subgroup in $\mathscr{A}_{\mathscr{C}}(o\colon W \to Q)$, namely $q \circ \mathscr{A}_{\mathscr{C}}(o\colon C \to P) \circ w$. But then, as they possess an element in common, these cosets must be equal, as claimed. $\square$

The following proposition is readily established; its proof is omitted.

PROPOSITION 1.3. *In the diagram*

$$\begin{tikzcd} W \arrow[r, "w"] & C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "b"] & \\ & A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \end{tikzcd}$$

let $g \circ w \simeq o$, $a \circ f \simeq b \circ g$ *and* $s \circ a \simeq o$.

(1) *If* $a \simeq o$ *then*

$$\boxdot\left(\begin{tikzcd} W \arrow[r, "w"] & C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "b"] & \\ & A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \end{tikzcd}\right) = s \circ \{b, g, w\} + \mathscr{A}_{\mathscr{C}}(o\colon A \to Y) \circ f \circ w.$$

(2) *If* $g \simeq o$ *then*

$$\boxdot\left(\begin{tikzcd} W \arrow[r, "w"] & C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "b"] & \\ & A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \end{tikzcd}\right) = s \circ b \circ \mathscr{A}_{\mathscr{C}}(o\colon W \to B) + \{s, a, f\} \circ w.$$

Next we fix a diagram of 1-morphisms in $\mathscr{C}$

$$\begin{tikzcd} U \arrow[r, "u"] & W \arrow[r, "w"] & C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "b"] & & \\ & & A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \arrow[r, "v"'] & V \end{tikzcd} \tag{1.4}$$

satisfying $a \circ f \simeq b \circ g$.

THEOREM 1.5.

(1) ("The 343 Lemma") *In diagram* (1.4) *above suppose that* $g \circ w \circ u \simeq o$, $s \circ b \circ g \circ w \simeq o$ *and* $v \circ s \circ a \simeq o$. *Then* $\{v \circ s, b \circ g, w \circ u\}$ *and*

$$v \circ \{s \circ b, g \circ w, u\} + \{v, s \circ a, f \circ w\} \circ u$$

each contain

$$\dot{\square}\left(\begin{tikzcd} U \arrow[r, "w\circ u"] & C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "b"] & \\ & A \arrow[r, "a"'] & X \arrow[r, "v\circ s"'] & V \end{tikzcd}\right)$$

as a subset. In particular the relation

$$\{v\circ s, b\circ g, w\circ u\} \sim v\circ\{s\circ b, g\circ w, u\} + \{v, s\circ a, f\circ w\}\circ u$$

is valid.

(2) *In diagram* (1.4) *suppose that* $g\circ w\circ u \simeq o$ *and* $s\circ a\simeq o$*. Then*

$$\{v\circ s, b\circ g, w\circ u\} \sim v\circ\{s\circ b, g\circ w, u\} + \mathscr{A}_{\mathscr{C}}(o\colon A\to V)\circ f\circ w\circ u.$$

(3) *In diagram* (1.4) *suppose that* $v\circ s\circ a \simeq o$ *and* $g\circ w\simeq o$*. Then*

$$\{v\circ s, b\circ g, w\circ u\} \sim v\circ s\circ b\circ \mathscr{A}_{\mathscr{C}}(o\colon U\to B) + \{v, s\circ a, f\circ w\}\circ u.$$

Proof. It is always the case that

$$\dot{\square}\left(\begin{tikzcd} U \arrow[r, "w\circ u"] & C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "b"] & \\ & A \arrow[r, "a"'] & X \arrow[r, "v\circ s"'] & V \end{tikzcd}\right) \subset \{v\circ s, b\circ g, w\circ u\}$$

and that

$$\dot{\square}\left(\begin{tikzcd} U \arrow[r, "w\circ u"] & C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "b"] & \\ & A \arrow[r, "a"'] & X \arrow[r, "v\circ s"'] & V \end{tikzcd}\right) \subset \dot{\square}\left(\begin{tikzcd} U \arrow[r, "u"] & W \arrow[r, "g\circ w"] \arrow[d, "f\circ w"'] & B \arrow[d, "s\circ b"] & \\ & A \arrow[r, "s\circ a"'] & Y \arrow[r, "v"'] & V \end{tikzcd}\right).$$

If $s\circ b\circ g\circ w\simeq o$ then

$$\dot{\square}\left(\begin{tikzcd} U \arrow[r, "u"] & W \arrow[r, "g\circ w"] \arrow[d, "f\circ w"'] & B \arrow[d, "s\circ b"] & \\ & A \arrow[r, "s\circ a"'] & Y \arrow[r, "v"'] & V \end{tikzcd}\right) = v\circ\{s\circ b, g\circ w, u\} + \{v, s\circ a, f\circ w\}\circ u$$

by Proposition 4.5(1) of [**4**]. This establishes Part (1) of the theorem.

If $s \circ a \simeq o$ then

$$\square\left(\begin{tikzcd} U \arrow[r, "u"] & W \arrow[r, "g\circ w"] \arrow[d, "f\circ w"'] & B \arrow[d, "s\circ b"] & \\ & A \arrow[r, "s\circ a"'] & Y \arrow[r, "v"'] & V \end{tikzcd}\right) = v \circ \{s \circ b, g \circ w, u\} + \mathscr{A}_{\mathscr{C}}(o\colon A \to V) \circ f \circ w \circ u$$

by Proposition 1.3(1). This establishes part (2) of the theorem.

If $g \circ w \simeq o$ then

$$\square\left(\begin{tikzcd} U \arrow[r, "u"] & W \arrow[r, "g\circ w"] \arrow[d, "f\circ w"'] & B \arrow[d, "s\circ b"] & \\ & A \arrow[r, "s\circ a"'] & Y \arrow[r, "v"'] & V \end{tikzcd}\right) = v \circ s \circ b \circ \mathscr{A}_{\mathscr{C}}(o\colon U \to B) + \{v, s \circ a, f \circ w\} \circ u$$

by Proposition 1.3(2). This establishes Part (3) of the theorem. □

Remark 1.6. In Theorem 1.5(1) the conclusion

$$\{v \circ s, b \circ g, w \circ u\} \sim v \circ \{s \circ b, g \circ w, u\} + \{v, s \circ a, f \circ w\} \circ u$$

also may be obtained by applying Theorem 1.2(2) to the diagram:

$$\begin{tikzcd} U \arrow[r, "u"] & W \arrow[r, "g\circ w"] \arrow[d, "w"'] & B \arrow[d, "b"] & \\ & C \arrow[r, "b\circ g"] \arrow[d, "f"'] & X \arrow[d, "s"] & \\ & A \arrow[r, "s\circ a"'] & Y \arrow[r, "v"'] & V \end{tikzcd}$$

The next result, in the topological case, was given in Proposition 1.7 of [**1**]; in the 2-category case, it was given in Proposition 4.7 of [**4**].

Proposition 1.7 (“The 333 Lemma”). *Suppose given* 1*-morphisms*

$$W \xrightarrow{w} C \xrightarrow{f} A \xrightarrow{a} X \xrightarrow{p} P \xrightarrow{q} Q$$

with all 3*-fold composites null homotopic. Then the inclusion*

$$\{q \circ p, a, f \circ w\} \subset q \circ \{p, a \circ f, w\} + \{q, p \circ a, f\} \circ w$$

is valid.

PROOF. We note that the diagram

$$
\begin{tikzcd}
W \arrow[r, "w"] & C \arrow[r, "f"] & A \arrow[r, "a"] \arrow[d, "1_A"'] & X \arrow[d, "1_X"] & & \\
 & & A \arrow[r, "a"'] & X \arrow[r, "p"'] & P \arrow[r, "q"'] & Q
\end{tikzcd}
$$

satisfies the hypotheses of the 343 Lemma (Theorem 1.5(1) above). Moreover the equality

$$
\{q \circ p, a, f \circ w\} = \dot{\square} \left(\begin{tikzcd}
W \arrow[r, "f \circ w"] & A \arrow[r, "a"] \arrow[d, "1_A"'] & X \arrow[d, "1_X"] & \\
 & A \arrow[r, "a"'] & X \arrow[r, "q \circ p"'] & Q
\end{tikzcd} \right)
$$

holds. Thus the inclusion claimed in the proposition is valid. $\square$

PROPOSITION 1.8. *In the homotopy commutative diagram*

$$
\begin{tikzcd}
W \arrow[r, "w"] & C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[r, "r"] \arrow[d, "b"] & D \arrow[d, "d"] & \\
 & A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \arrow[r, "v"'] & Z
\end{tikzcd}
$$

let all horizontal pair composites be null homotopic. Then the equality

$$
v \circ \square\!\square \left(\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[r, "r"] \arrow[d, "b"] & D \arrow[d, "d"] \\
A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y
\end{tikzcd} \right) \circ w = \left\{ v, \begin{matrix} d \\ s \end{matrix}, \begin{matrix} r \\ b \end{matrix}, g \right\} \circ w + v \circ \left\{ s, \begin{matrix} b \\ a \end{matrix}, \begin{matrix} g \\ f \end{matrix}, w \right\}
$$

is valid in $\mathscr{A}_{\mathscr{C}}(o\colon W \to Z)$. Furthermore if $f \circ w \simeq o$ and $v \circ d \simeq o$ then

$$
v \circ \square\!\square \left(\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[r, "r"] \arrow[d, "b"] & D \arrow[d, "d"] \\
A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y
\end{tikzcd} \right) \circ w = \{1_o\}.
$$

PROOF. Let homotopies $K\colon o \Rightarrow r \circ g$, $G\colon s \circ b \Rightarrow d \circ r$, $F\colon a \circ f \Rightarrow b \circ g$, $H\colon o \Rightarrow s \circ a$, $T\colon o \Rightarrow g \circ w$ and $R\colon o \Rightarrow v \circ s$ be given. Then a straightforward argument, using the Interchange Law together with the equalities $oT = 1_o$ and $-Ro = 1_o$, yields the following equation.

$$
\begin{aligned}
&v(-dK + Gg + sF + Hf)w \\
&\qquad = (-(v \circ d)K + vGg + R(b \circ g))w + v(-(s \circ b)T + sFw + H(f \circ w))
\end{aligned}
$$

Here

$$
-dK + Gg + sF + Hf \in \square\!\square \left(\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[r, "r"] \arrow[d, "b"] & D \arrow[d, "d"] \\ A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \end{tikzcd} \right)
$$

$$
-(v \circ d)K + vGg + R(b \circ g) \in \left\{ v\,, \begin{matrix} d \\ s \end{matrix}, \begin{matrix} r \\ b \end{matrix}, g \right\}
$$

$$
-(s \circ b)T + sFw + H(f \circ w) \in \left\{ s\,, \begin{matrix} b \\ a \end{matrix}, \begin{matrix} g \\ f \end{matrix}, w \right\}
$$

and thus the first equality is obtained.

For the last statement of the proposition, observe that since $v \circ d \simeq o$ we have $\left\{ v\,, \begin{matrix} d \\ s \end{matrix}, \begin{matrix} r \\ b \end{matrix}, g \right\} = \left\{ v\,, \begin{matrix} d \\ s \end{matrix}, \begin{matrix} r \\ b \end{matrix} \right\} \circ g$ by Proposition 4.5(3) of [**4**], and since $f \circ w \simeq o$ we have $\left\{ s\,, \begin{matrix} b \\ a \end{matrix}, \begin{matrix} g \\ f \end{matrix}, w \right\} = s \circ \left\{ \begin{matrix} b \\ a \end{matrix}, \begin{matrix} g \\ f \end{matrix}, w \right\}$ by Proposition 4.5(2) of [**4**]. Hence the conclusion follows, for $g \circ w \simeq o$ and $v \circ s \simeq o$. □

Example 1.9. Consider a composite of 1-morphisms in $\mathscr{C}$

$$
W \xrightarrow{w} C \xrightarrow{g} B \xrightarrow{b} X \xrightarrow{s} Y \xrightarrow{v} Z
$$

with all pairwise composites null homotopic. Then always $v \circ \{s, b, g\} \circ w = \{1_o\}$. To see this, form the following homotopy commutative diagram

$$
\begin{tikzcd}
W \arrow[r, "w"] & C \arrow[r, "g"] \arrow[d] & B \arrow[r] \arrow[d, "b"] & * \arrow[d] & \\
& * \arrow[r] & X \arrow[r, "s"'] & Y \arrow[r, "v"'] & Z
\end{tikzcd}
$$

and apply Proposition 1.8. After suitable identifications the equality

$$
\begin{aligned}
&- v \circ \{s, b, g\} \circ w \\
&\qquad = [v \circ \{o, o, g\} + \{v, s, b\} \circ g] \circ w + v \circ [s \circ \{b, g, w\} + \{s, o, o\} \circ w] \\
&\qquad = \{1_o\}
\end{aligned}
$$

results.

2. Lemmas on box brackets.

LEMMA 2.1. *Let the diagram*

$$\begin{tikzcd} W \arrow[r, "w"] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{M}{\Longrightarrow}"] & C \arrow[r, "g"] \arrow[d, "f"] \arrow[dr, phantom, "\overset{F}{\Longrightarrow}"] & B \arrow[r, "r"] \arrow[d, "b"] \arrow[dr, phantom, "\overset{G}{\Longrightarrow}"] & R \arrow[d, "y"] \\ U \arrow[r, "u"'] & A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \end{tikzcd}$$

be homotopy commutative with homotopies as shown. Also assume that all horizontal pair composites are null homotopic with the following specific homotopies assigned.

$$K\colon o \Rightarrow g \circ w, \quad S\colon o \Rightarrow a \circ u, \quad T\colon o \Rightarrow r \circ g, \quad H\colon o \Rightarrow s \circ a$$

Define elements

$$\alpha := -bK + Fw + aM + Sh \in \square\square\begin{pmatrix} W \xrightarrow{w} C \xrightarrow{g} B \\ h\downarrow \qquad \downarrow f \quad \downarrow b \\ U \xrightarrow[u]{} A \xrightarrow[a]{} X \end{pmatrix}$$

and

$$\beta := -yT + Gg + sF + Hf \in \square\square\begin{pmatrix} C \xrightarrow{g} B \xrightarrow{r} R \\ f\downarrow \quad \downarrow b \quad \downarrow y \\ A \xrightarrow[a]{} X \xrightarrow[s]{} Y \end{pmatrix}.$$

Then the equality

$$s\alpha = y(-rK + Tw) + \beta w - (-sS + Hu)h$$

is valid. Note that each individual summand in this decomposition is an element of $\mathscr{A}_{\mathscr{C}}(o\colon W \to Y)$.

PROOF. The Interchange Law may be applied to the diagram

$$\begin{tikzcd} W \arrow[r, bend left, "o"] \arrow[r, bend right, "g\circ w"'] \arrow[r, phantom, "\Uparrow -K"] & B \arrow[r, bend left, "y\circ r"] \arrow[r, bend right, "s\circ b"'] \arrow[r, phantom, "\Uparrow G"] & Y \end{tikzcd}$$

to obtain the equality

$$-(y \circ r)K + G(g \circ w) = Go - (s \circ b)K = -(s \circ b)K$$

since $Go = 1_o$. Similarly $(s \circ a)M + H(u \circ h) = H(f \circ w)$. Hence we have

$$\begin{aligned}
y(-rK + Tw) &+ \beta w - (-sS + Hu)h \\
&= y(-rK + Tw) + (-yT + Gg + sF + Hf)w - (-sS + Hu)h \\
&= -(y \circ r)K + yTw - yTw + G(g \circ w) + sFw + H(f \circ w) - H(u \circ h) + sSh \\
&= -(y \circ r)K + G(g \circ w) + sFw + H(f \circ w) - H(u \circ h) + sSh \\
&= -(s \circ b)K + sFw + (s \circ a)M + H(u \circ h) - H(u \circ h) + sSh \\
&= -(s \circ b)K + sFw + (s \circ a)M + sSh \\
&= s(-bK + Fw + aM + Sh) \\
&= s\alpha
\end{aligned}$$

as claimed. $\square$

Corollary 2.2. (1) *In Lemma 2.1, if $\{s, a, u\} \circ h = \{1_o\} = y \circ \{r, g, w\}$ then*

$$\square\!\square\begin{pmatrix}\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[r, "r"] \arrow[d, "b"] & R \arrow[d, "y"] \\ A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \end{tikzcd}\end{pmatrix} \circ w = \left\{ s\,,\ \begin{matrix} b \\ a \end{matrix}\,,\ \begin{matrix} g \\ f \end{matrix}\,,\ w \right\} = s \circ \square\!\square\begin{pmatrix}\begin{tikzcd} W \arrow[r, "w"] \arrow[d, "h"'] & C \arrow[r, "g"] \arrow[d, "f"] & B \arrow[d, "b"] \\ U \arrow[r, "u"'] & A \arrow[r, "a"'] & X \end{tikzcd}\end{pmatrix}$$

as subsets of $\mathscr{A}_{\mathscr{C}}(o\colon W \to Y)$.

(2) *In Lemma 2.1, if $1_o \in \{s, a, u\} \circ h$ and $1_o \in y \circ \{r, g, w\}$ then*

$$\square\!\square\begin{pmatrix}\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[r, "r"] \arrow[d, "b"] & R \arrow[d, "y"] \\ A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \end{tikzcd}\end{pmatrix} \circ w \quad \sim \quad s \circ \square\!\square\begin{pmatrix}\begin{tikzcd} W \arrow[r, "w"] \arrow[d, "h"'] & C \arrow[r, "g"] \arrow[d, "f"] & B \arrow[d, "b"] \\ U \arrow[r, "u"'] & A \arrow[r, "a"'] & X \end{tikzcd}\end{pmatrix}$$

as subsets of $\mathscr{A}_{\mathscr{C}}(o\colon W \to Y)$.

Proof.

(1) For, in this case, always $\beta w = s\alpha$ in Lemma 2.1. Alternatively this part follows from Corollary 3.2 of [**4**]; it is also a consequence of Theorem 4.4 of [**5**].

(2) Since $1_o \in \{s, a, u\} \circ h$ the homotopies $S\colon o \Rightarrow a \circ u$ and $H\colon o \Rightarrow s \circ a$ may be selected so that $(-sS + Hu)h = 1_o$ in $\mathscr{A}_{\mathscr{C}}(o\colon W \to Y)$. Similarly, since $1_o \in y \circ \{r, g, w\}$ the homotopies $T\colon o \Rightarrow r \circ g$ and $K\colon o \Rightarrow g \circ w$ may be selected so that $y(-rK + Tw) = 1_o$ in $\mathscr{A}_{\mathscr{C}}(o\colon W \to Y)$. Hence we will have $s\alpha = \beta w$. This implies that Part (2) holds. $\square$

LEMMA 2.3. *Let*

$$\begin{tikzcd}
 & C \arrow[rr, "g"] \arrow[dr, "c"'] & & B \arrow[dr, "b"] & \\
W \arrow[ur, "w"] \arrow[dr, "h"'] & & Q \arrow[ur, "q"'] \arrow[dr, "p"] & & X \\
 & U \arrow[ur, "i"] \arrow[rr, "u"'] & & A \arrow[ur, "a"'] &
\end{tikzcd}$$

be a homotopy commutative diagram of 1*-morphisms in* $\mathscr{C}$*. If* $q \circ i \simeq o$ *then the relation*

$$1_o \in \Box\!\Box \left(\begin{array}{ccc} W \xrightarrow{w} & C \xrightarrow{g} & B \\ h\big\downarrow & \big\downarrow p\circ c & \big\downarrow b \\ U \xrightarrow[u]{} & A \xrightarrow[a]{} & X \end{array} \right)$$

holds.

PROOF. Note that $g \circ w \simeq q \circ i \circ h \simeq o$ and $a \circ u \simeq b \circ q \circ i \simeq o$; thus the box bracket

$$\mathscr{B} := \Box\!\Box \left(\begin{array}{ccc} W \xrightarrow{w} & C \xrightarrow{g} & B \\ h\big\downarrow & \big\downarrow p\circ c & \big\downarrow b \\ U \xrightarrow[u]{} & A \xrightarrow[a]{} & X \end{array} \right)$$

is well-defined. By our hypotheses we may select homotopies $L\colon i \circ h \Rightarrow c \circ w$, $R\colon a \circ p \Rightarrow b \circ q$, $M\colon u \Rightarrow p \circ i$, $N\colon q \circ c \Rightarrow g$, and $K\colon q \circ i \Rightarrow o$. Define new homotopies by

$$H := Nw + qL - Kh\colon o \Rightarrow g \circ w$$

$$G := -aM - Ri - bK\colon o \Rightarrow a \circ u$$

$$F := pL + Mh\colon u \circ h \Rightarrow p \circ c \circ w$$

$$T := bN + Rc\colon a \circ p \circ c \Rightarrow b \circ g.$$

Observe that $Tw + aF = b(Nw + qL) + (Ri + aM)h$ by the Interchange Law. Next consider the element $\theta := -bH + Tw + aF + Gh$. Plainly θ is an element of $\mathscr{B}$. On the other hand we may write

$$\theta = b[-H + Nw + qL] + [Ri + aM + G]h = bKh - bKh = 1_o.$$

Therefore $1_o \in \mathscr{B}$. □

3. The box quartet operation.

In this section $\mathscr{C}$ is to be a 2-category with zeros that has a suspension functor $\Sigma\colon \mathscr{C} \to \mathscr{C}$ in the sense of Definition 1.4 of [**4**]. Recall that such a Σ is a 2-functor and that for any pair (W, X) of objects of $\mathscr{C}$ there is a bijection

$$d\colon \mathscr{A}_{\mathscr{C}}(o\colon W \to X) \to \mathrm{H}\mathscr{C}(\Sigma W, X)$$

given by

$$\begin{tikzcd} W \arrow[r, bend left, "o"] \arrow[r, bend right, "o"'] \arrow[r, phantom, "\Downarrow \xi"] & X \end{tikzcd} \quad \mapsto \quad [\mu_\xi]\colon \Sigma W \to X$$

where $\mathrm{H}\mathscr{C}$ designates the associated homotopy category of $\mathscr{C}$ and [ ] is used to denote homotopy class. Consequently $\mathrm{H}\mathscr{C}(\Sigma W, X)$ receives a group structure by means of the bijection d. In particular

$$\mu_{\xi+\eta} \simeq \mu_\xi + \mu_\eta$$

$$\mu_{1_o} \simeq o$$

$$\mu_{-\xi} \simeq -\mu_\xi$$

and

$$\mu_{s\xi} \simeq s \circ \mu_\xi$$

$$\mu_{\xi p} \simeq \mu_\xi \circ \Sigma p$$

where $s\colon X \to Y$ and $p\colon P \to W$.

We fix a homotopy commutative diagram

$$\begin{array}{ccccccccc} W & \xrightarrow{w} & C & \xrightarrow{g} & B & \xrightarrow{r} & R & \xrightarrow{z} & Z \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle k} \\ U & \xrightarrow[u]{} & A & \xrightarrow[a]{} & X & \xrightarrow[s]{} & Y & \xrightarrow[v]{} & V \end{array} \tag{3.1}$$

in which all horizontal pair composites are null homotopic. This allows us to consider three box brackets, namely

$$\square\!\square\left(\begin{array}{ccccc} W & \xrightarrow{w} & C & \xrightarrow{g} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle b} \\ U & \xrightarrow[u]{} & A & \xrightarrow[a]{} & X \end{array}\right), \quad \square\!\square\left(\begin{array}{ccccc} C & \xrightarrow{g} & B & \xrightarrow{r} & R \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle y} \\ A & \xrightarrow[a]{} & X & \xrightarrow[s]{} & Y \end{array}\right), \quad \square\!\square\left(\begin{array}{ccccc} B & \xrightarrow{r} & R & \xrightarrow{z} & Z \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle k} \\ X & \xrightarrow[s]{} & Y & \xrightarrow[v]{} & V \end{array}\right)$$

which we denote $\mathscr{A}$, Γ, and $\mathscr{B}$ respectively.

Definition 3.2. A triple of elements $(\alpha, \gamma, \beta) \in \mathscr{A} \times \Gamma \times \mathscr{B}$ will be said to be *coherent* if $\gamma w = s\alpha$ and $v\gamma = \beta g$. These conditions imply the homotopy relations $\mu_\gamma \circ \Sigma w \simeq s \circ \mu_\alpha$ and $v \circ \mu_\gamma \simeq \mu_\beta \circ \Sigma g$ respectively. Consequently the diagram

$$\begin{array}{ccccc} \Sigma W & \xrightarrow{\Sigma w} & \Sigma C & \xrightarrow{\Sigma g} & \Sigma B \\ {\scriptstyle \mu_\alpha}\downarrow & & \downarrow{\scriptstyle \mu_\gamma} & & \downarrow{\scriptstyle \mu_\beta} \\ X & \xrightarrow[s]{} & Y & \xrightarrow[v]{} & V \end{array}$$

will be homotopy commutative in $\mathscr{C}$. We let $\mathscr{D}_{(\alpha,\gamma,\beta)}$ denote the box bracket of this diagram and define

$$\mathscr{D} := \bigcup \{\mathscr{D}_{(\alpha,\gamma,\beta)} \mid (\alpha, \gamma, \beta) \in \mathscr{A} \times \Gamma \times \mathscr{B} \text{ is a coherent triple}\}.$$

Note that $\mathscr{D}$ is a subset of $\mathscr{A}_{\mathscr{C}}(o\colon \Sigma W \to V) \cong \mathrm{H}\mathscr{C}(\Sigma^2 W, V)$. We call this operation the *box quartet operation*. We will say that $\mathscr{D}$ is *defined* if it is nonempty, and is *trivial* if it contains 1_o.

Note. Originally we considered using *quaternary box bracket* rather than *box quartet operation* for the name of the operation $\mathscr{D}$. However forthcoming work on long box brackets convinces us that the name *quaternary box bracket* should be reserved for an operation that raises stem dimension by 3 (that is, with values in $\mathrm{H}\mathscr{C}(\Sigma^3 W, V)$). The operation $\mathscr{D}$ raises stem dimension by 2 (that is, has values in $\mathrm{H}\mathscr{C}(\Sigma^2 W, V)$). We note that in $\mathscr{T}op_*$ another operation that raises stem dimension by 2 is the quaternary Toda bracket (for which see [**13**], Section 5 of [**4**] or [**3**]) but the two operations should not be confused. The box quartet operation is defined in the 2-category setting starting with a "quartet" of boxes; on the other hand, the quaternary Toda bracket has been defined only in $\mathscr{T}op_*$ and arises from a 4-fold composite of maps with all pair composites null homotopic.

REMARK 3.3. If the *strong edge conditions*

$$y \circ \{r, g, w\} = \{1_o\} = \{s, a, u\} \circ h$$
$$k \circ \{z, r, g\} = \{1_o\} = \{v, s, a\} \circ f$$

are satisfied in diagram (3.1) then by Corollary 2.2(1) the equations $s \circ \mathscr{A} = \Gamma \circ w$ and $v \circ \Gamma = \mathscr{B} \circ g$ respectively are valid. Thus in this case at least one coherent triple of elements (α, γ, β) must exist, for $\Gamma \neq \emptyset$. Hence $\mathscr{D}$ will be defined.

PROPOSITION 3.4. *Assume that the strong edge conditions of Remark* 3.3 *are satisfied for diagram* (3.1). *Also assume that*

$$1_o \in s \circ \square\square \begin{pmatrix} W \xrightarrow{w} C \xrightarrow{g} B \\ h\downarrow \quad\ \downarrow f \quad \downarrow b \\ U \xrightarrow[u]{} A \xrightarrow[a]{} X \end{pmatrix} \quad \text{and} \quad 1_o \in \square\square \begin{pmatrix} B \xrightarrow{r} R \xrightarrow{z} Z \\ b\downarrow \quad\ \downarrow y \quad \downarrow k \\ X \xrightarrow[s]{} Y \xrightarrow[v]{} V \end{pmatrix} \circ g.$$

Then a coherent triple of elements $(\alpha, \gamma, \beta) \in \mathscr{A} \times \Gamma \times \mathscr{B}$ *exists satisfying the conditions* $s \circ \mu_\alpha = o = \mu_\gamma \circ \Sigma w$ *and* $v \circ \mu_\gamma = o = \mu_\beta \circ \Sigma g$. *Consequently the inclusion*

$$\{\mu_\beta, \Sigma g, \Sigma w\} - \{v, \mu_\gamma, \Sigma w\} + \{v, s, \mu_\alpha\} \subset \mathscr{D}$$

is valid. (*For a related result in* $\mathscr{T}op_*$, *see Theorem* 4.15 *below.*)

PROOF. By hypothesis we may select elements

$$\alpha = -bK + Fw + aM + Sh \in \square\square \begin{pmatrix} W \xrightarrow{w} C \xrightarrow{g} B \\ h\downarrow \quad\ \downarrow f \quad \downarrow b \\ U \xrightarrow[u]{} A \xrightarrow[a]{} X \end{pmatrix} = \mathscr{A}$$

and

$$\beta = -kP + Nr + vG + Lb \in \square\square \begin{pmatrix} B \xrightarrow{r} R \xrightarrow{z} Z \\ b\downarrow \quad\ \downarrow y \quad \downarrow k \\ X \xrightarrow[s]{} Y \xrightarrow[v]{} V \end{pmatrix} = \mathscr{B}$$

such that

$$
\begin{aligned}
s(-bK + Fw + aM + Sh) &= 1_o \in \mathscr{A}_{\mathscr{C}}(o\colon W \to Y) \\
(-kP + Nr + vG + Lb)g &= 1_o \in \mathscr{A}_{\mathscr{C}}(o\colon C \to V).
\end{aligned} \tag{3.5}
$$

Further we select arbitrary homotopies $T\colon o \Rightarrow r \circ g$ and $H\colon o \Rightarrow s \circ a$ and define

$$
\gamma := -yT + Gg + sF + Hf \in \square\square \begin{pmatrix} \begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[r, "r"] \arrow[d, "b"] & R \arrow[d, "y"] \\ A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \end{tikzcd} \end{pmatrix} = \Gamma.
$$

The relations (3.5) imply that $s \circ \mu_\alpha = o$ and $\mu_\beta \circ \Sigma g = o$. It remains to verify that $s\alpha = \gamma w$ and $\beta g = v\gamma$. Now by the strong edge conditions we have $y(-rK + Tw) = 1_o$ since $y \circ \{r, g, w\} = \{1_o\}$, and $(-sS + Hu)h = 1_o$ since $\{s, a, u\} \circ h = \{1_o\}$. Applying Lemma 2.1 we conclude that $s\alpha = \gamma w$. Similarly, by the strong edge conditions, we have $k(-zT + Pg) = 1_o$ and $(-vH + La)f = 1_o$, so, by another application of Lemma 2.1, we may conclude that $\beta g = v\gamma$.

Finally the last statement of the proposition holds because by definition $\mathscr{D}_{(\alpha,\gamma,\beta)} \subset \mathscr{D}$ while the equality

$$
\mathscr{D}_{(\alpha,\gamma,\beta)} = \{\mu_\beta, \Sigma g, \Sigma w\} - \{v, \mu_\gamma, \Sigma w\} + \{v, s, \mu_\alpha\}
$$

is valid by Proposition 3.3(3) of [**5**]. □

Definition 3.6. We will say that diagram (3.1) satisfies the *inner weak edge conditions* if the following conditions hold:

$$
\begin{aligned}
\{s, a, u\} \circ h &= \{1_o\}, \quad 1_o \in y \circ \{r, g, w\} \\
k \circ \{z, r, g\} &= \{1_o\}, \quad 1_o \in \{v, s, a\} \circ f
\end{aligned}
$$

It satisfies the *outer weak edge conditions* if the following conditions hold:

$$
\begin{aligned}
1_o &\in \{s, a, u\} \circ h, \quad y \circ \{r, g, w\} = \{1_o\} \\
1_o &\in k \circ \{z, r, g\}, \quad \{v, s, a\} \circ f = \{1_o\}
\end{aligned}
$$

Plainly the strong edge conditions imply the weak ones.

Proposition 3.7. *If diagram* (3.1) *satisfies either the inner or outer weak edge conditions of Definition* 3.6 *then* $\mathscr{D}$ *is defined.*

Proof. Let us assume that the inner weak edge conditions hold. Then since $1_o \in y \circ \{r, g, w\}$ there exist homotopies $T\colon o \Rightarrow r \circ g$ and $K\colon o \Rightarrow g \circ w$ such that $y(-rK + Tw) = 1_o$ in $\mathscr{A}_{\mathscr{C}}(o\colon W \to Y)$. And since $1_o \in \{v, s, a\} \circ f$ there exist homotopies $L\colon o \Rightarrow v \circ s$ and $H\colon o \Rightarrow s \circ a$ such that $(-vH + La)f = 1_o$ in

$\mathscr{A}_{\mathscr{C}}(o\colon C \to V)$. We further select arbitrary homotopies as follows.

$$M : u \circ h \Rightarrow f \circ w \quad G : s \circ b \Rightarrow y \circ r \quad P : o \Rightarrow z \circ r$$
$$F : a \circ f \Rightarrow b \circ g \quad N : v \circ y \Rightarrow k \circ z \quad S : o \Rightarrow a \circ u$$

These homotopies allow us to define the required homotopies; namely, we set

$$\alpha := -bK + Fw + aM + Sh \in \mathscr{A}$$
$$\gamma := -yT + Gg + sF + Hf \in \Gamma$$
$$\beta := -kP + Nr + vG + Lb \in \mathscr{B}.$$

Now we have $y(-rK + Tw) = 1_o$ and $(-sS + Hu)h = 1_o$ (this latter relation holds since $(-sS + Hu)h \in \{s, a, u\} \circ h = \{1_o\}$). Thus by Lemma 2.1 we may conclude that $s\alpha = \gamma w$ in $\mathscr{A}_{\mathscr{C}}(o\colon W \to Y)$. A similar argument shows that $v\gamma = \beta g$ in $\mathscr{A}_{\mathscr{C}}(o\colon C \to V)$. Hence (α, γ, β) is a coherent triple and thus $\mathscr{D}$ is defined.

The proof that $\mathscr{D}$ is defined when the outer weak edge conditions hold is similar; we omit the details. $\square$

4. The box quartet operation in $\mathscr{T}\!op_*$.

In this section we consider special features of the topological box quartet operation.

We recall that $\mathscr{T}\!op_*$ denotes the 2-category whose objects, 1-morphisms and 2-morphisms are based topological spaces, based maps and track classes of based homotopies respectively. We shall continue to use the notation $o\colon X \to Y$ to denote a zero map in $\mathscr{T}\!op_*$ (as in a general 2-category with zeros) but for the homotopy class of the zero map we permit ourselves to use either o or 0 (the latter in accordance with usual practice in $\mathscr{T}\!op_*$). The 2-morphisms in $\mathscr{T}\!op_*$ take the form

$$\{F\}\colon f \Rightarrow g\colon X \to Y$$

where $\{F\}$ denotes track class with representative homotopy $F\colon f \Rightarrow g$. However as is customary we often work directly with the topological homotopies themselves. This causes some slight conflict with notational usage in previous sections but is easily understood in context. Of course the usual suspension ΣX of a space X also constitutes a suspension functor in the 2-categorical sense.

It will be convenient to utilize the double mapping cylinder functor $\mathscr{M}$ on $\mathscr{T}\!op_*$. Recall that if $A \xleftarrow{f} C \xrightarrow{g} B$ is a cotriad of maps then by construction of the double mapping cylinder $\mathscr{M}(f, g)$ there is a homotopy pushout square

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{D}{\Longrightarrow}"] & B \arrow[d, "i_1"] \\ A \arrow[r, "i_0"] & \mathscr{M}(f,g) \end{tikzcd} \tag{4.1}$$

with homotopy D, referred to as the *defining homotopy* of $\mathscr{M}(f,g)$, given by

$$D(c,t) = [c,t] \in \mathscr{M}(f,g) \quad \text{for } c \in C \text{ and } 0 \leq t \leq 1.$$

There is an obvious quotient map $\kappa \colon \mathscr{M}(f,g) \to \Sigma C$.

Now if

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{F}{\Longrightarrow}"] & B \arrow[d, "b"] \\ A \arrow[r, "a"] & X \end{tikzcd} \tag{4.2}$$

is a homotopy commutative square with homotopy F then we may define a map $\mu_F \colon \mathscr{M}(f,g) \to X$ by setting $\mu_F[x,t] = F(x,t)$ for $x \in C$, $0 \leq t \leq 1$, with $\mu_F \circ i_0 = a$ and $\mu_F \circ i_1 = b$. The homotopy class of μ_F depends only on the track class of F. Thus a well-defined function

$$hom_{\mathscr{T}\!op_*}(a \circ f, b \circ g) \to \pi(\mathscr{M}(f,g), X), \quad \{F\} \mapsto \mu_{\{F\}} := [\mu_F],$$

is obtained, where $hom_{\mathscr{T}\!op_*}(a \circ f, b \circ g)$ denotes the set of all track classes $a \circ f \Rightarrow b \circ g$. If $A = \{*\}$ and $B = \{*\}$ in (4.2) then we have

$$\mathscr{M}(f,g) = \Sigma C, \quad hom_{\mathscr{T}\!op_*}(a \circ f, b \circ g) = \mathscr{A}_{\mathscr{T}\!op_*}(o \colon C \to X),$$

and this function becomes the bijection

$$d \colon \mathscr{A}_{\mathscr{T}\!op_*}(o \colon C \to X) \to \pi(\Sigma C, X)$$

involved in the definition of the suspension functor Σ in $\mathscr{T}\!op_*$. These observations are made to point out that the usage of the notation μ_F for double mapping cylinders is consistent with its usuage for the suspension functor.

As a special case of a double mapping cylinder let us recall that the diagram

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\overset{D_g}{\Longrightarrow}"] & B \arrow[d, "i_1^g"] \\ * \arrow[r] & \mathscr{M}(o,g) := C_g \end{tikzcd}$$

defines the *mapping cone* C_g of g. Thus (in this paper) a mapping cone always has vertex at parameter $t=0$.

Next suppose given $W \xrightarrow{w} C \xrightarrow{g} B$ with null homotopy $K\colon o \Rightarrow g\circ w$. Then extension and coextension in the classical sense with respect to this data are given as follows. The *coextension* $\zeta_K : \Sigma W \to C_g$ *of* w is defined by

$$\zeta_K[x,t] = \begin{cases} [w(x), 2t], & 0 \le t \le \frac{1}{2} \\ i_1^g K(x, 2-2t), & \frac{1}{2} \le t \le 1 \end{cases}$$

and the *extension* $\mu_K : C_w \to B$ *of* g is defined by $\mu_K[x,t] = K(x,t)$ for $x \in W$, $0 \le t \le 1$. The homotopy classes of ζ_K and μ_K depend only on the track class of K. Note that the relations

$$\begin{aligned} \kappa \circ \zeta_K &\simeq \Sigma w\colon \Sigma W \to \Sigma C \\ \mu_K \circ i_1^w &= g\colon C \to B \end{aligned} \tag{4.3}$$

are valid. Also there is a homotopy commutative square

$$\begin{tikzcd} C_w \arrow[r, "\kappa"] \arrow[d, "\mu_K"'] & \Sigma W \arrow[d, "-\zeta_K"] \\ B \arrow[r, "i_1^g"] & C_g \end{tikzcd} \tag{4.4}$$

which is a homotopy pushout, as follows from Lemma 3.3 of [**6**].

In the following proposition, the result of Proposition 1.9 of [**16**] is described precisely making use of the homotopies involved.

Proposition 4.5. *Consider maps $W \xrightarrow{w} C \xrightarrow{g} B \xrightarrow{r} R$ and suppose null homotopies $K : o \Rightarrow g\circ w$ and $T : o \Rightarrow r \circ g$ are given. Then the square*

$$\begin{tikzcd} C_w \arrow[r, "\kappa"] \arrow[d, "\mu_K"'] & \Sigma W \arrow[d, "-\mu_{-r\{K\}+\{T\}w}"] \\ B \arrow[r, "r"] & R \end{tikzcd}$$

is homotopy commutative. Of course the track $-r\{K\}+\{T\}w$ is just an element of the Toda bracket $\{r,g,w\} \subset \mathscr{A}_{\mathscr{T}op_*}(o\colon W \to R)$. If moreover the homotopies K and T satisfy $-r\{K\}+\{T\}w = 1_o$ then $r\circ\mu_K = o$ up to homotopy.

Proof. We consider the diagram

$$\begin{tikzcd}
C_w \arrow[r, "\kappa"] \arrow[d, "\mu_K"'] & \Sigma W \arrow[d, "-\zeta_K"] \\
B \arrow[r, "i_1^g"] \arrow[dr, "r"'] & C_g \arrow[d, "\mu_T"] \\
& R
\end{tikzcd}$$

which is homotopy commutative by (4.3) and (4.4). Furthermore we see that $\mu_{-r\{K\}+\{T\}w} = \mu_T \circ \zeta_K$ up to homotopy, as indicated in the following diagram.

$$\begin{tikzcd}
* \arrow[d] & W \arrow[l] \arrow[r] \arrow[d, "w"'] \arrow[r, phantom, shift right=3, "\overset{-K}{\Longrightarrow}"] & * \arrow[d] \\
* \arrow[dr] & C \arrow[l] \arrow[r, "g"] \arrow[d, phantom, "\overset{T}{\Longrightarrow}"] & B \arrow[dl, "r"] \\
& R &
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\Sigma W \arrow[d, "\zeta_K"] \arrow[dd, bend left=60, "\mu_{-r\{K\}+\{T\}w}"] \\
C_g \arrow[d, "\mu_T"] \\
R
\end{tikzcd}$$

This establishes the homotopy commutativity of the square in the proposition. For the last statement in the proposition we note that $\mu_{-r\{K\}+\{T\}w} = \mu_{1_o} = o$ and hence the relation $r \circ \mu_K = o$ must hold up to homotopy by homotopy commutativity of the square. $\square$

We modify and extend the classical definitions above to obtain the *coextension construction*:

$$\begin{tikzcd}
W \arrow[r, "w"] \arrow[d, "h"'] \arrow[rr, bend left=50, "o"] \arrow[dr, phantom, "\overset{M}{\Longrightarrow}"] & C \arrow[r, "g"] \arrow[d, "f"] \arrow[r, phantom, shift left=4, "\Uparrow -K"] & B \\
U \arrow[r, "u"] & A &
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\Sigma W \arrow[r, dotted, "coext(M{,}K)"] \arrow[dr, "\Sigma w"'] & \mathscr{M}(i_1^u \circ f, g) \arrow[d, "\kappa"] \\
& \Sigma C
\end{tikzcd} \tag{4.6}$$

and the *extension construction*:

$$\begin{tikzcd}
& C \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{F}{\Longrightarrow}"] & B \arrow[d, "b"] \\
U \arrow[r, "u"] \arrow[rr, bend right=50, "o"'] & A \arrow[r, "a"] \arrow[d, phantom, "\Uparrow S"] & X \\
& &
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\mathscr{M}(i_1^u \circ f, g) \arrow[r, dotted, "ext(S{,}F)"] & X \\
A \arrow[u, "\mathrm{inc}"] \arrow[ur, "a"'] &
\end{tikzcd} \tag{4.7}$$

Here inc: $A \to \mathscr{M}(i_1^u \circ f, g)$ denotes the composite $A \xrightarrow{i_1^u} C_u \xrightarrow{i_0} \mathscr{M}(i_1^u \circ f, g)$. Observe that each construction requires a pair of homotopies as defining data, as indicated above on the left in each instance. Now for the explicit definitions, we let $ext(S,F)$ arise from the square

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "i_1^u\circ f"'] \arrow[dr, phantom, "\overset{F}{\Longrightarrow}"] & B \arrow[d, "b"] \\ C_u \arrow[r, "\mu_S"] & X \end{tikzcd}$$

with homotopy F (recall that the equality $\mu_S \circ i_1^u = a\colon A \to X$ holds by (4.3) above) while $coext(M,K)$ is induced functorially under the double mapping cylinder functor $\mathscr{M}$ as follows.

$$\begin{tikzcd} * \arrow[d] \arrow[dr, phantom, "\overset{-D_u}{\Longleftarrow}"] & U \arrow[l] \arrow[d, "u"'] \arrow[dr, phantom, "\overset{-M}{\Longleftarrow}"] & W \arrow[l, "h"'] \arrow[r] \arrow[d, "w"] \arrow[dr, phantom, "\overset{-K}{\Longrightarrow}"] & * \arrow[d] \\ C_u & A \arrow[l, "i_1^u"] & C \arrow[l, "f"] \arrow[r, "g"'] & B \end{tikzcd} \quad \overset{\mathscr{M}}{\rightsquigarrow} \quad \begin{tikzcd} \Sigma W \arrow[d, "coext(M,K)"] \\ \mathscr{M}(i_1^u\circ f, g) \end{tikzcd}$$

We remark that while we often regard $coext(M,K)$ and $ext(S,F)$ as maps, they really are well-defined only up to homotopy; of course their homotopy classes depend only on the track classes of the homotopies involved.

Remark 4.8. By definition elements of the box bracket

$$\Box\!\Box \left(\begin{tikzcd} W \arrow[r, "w"] \arrow[d, "h"'] & C \arrow[r, "g"] \arrow[d, "f"] & B \arrow[d, "b"] \\ U \arrow[r, "u"'] & A \arrow[r, "a"'] & X \end{tikzcd} \right) \subset \mathscr{A}_{\mathscr{T}op_*}(o\colon W \to X)$$

have form

$$\theta = -b\{K\} + \{F\}w + a\{M\} + \{S\}h$$

for homotopies $S\colon o \Rightarrow a\circ u$, $K\colon o \Rightarrow g\circ w$, $M\colon u\circ h \Rightarrow f\circ w$ and $F\colon a\circ f \Rightarrow b\circ g$. We recall from Proposition 6.3 of [**5**] that under the group isomorphism

$$d\colon \mathscr{A}_{\mathscr{T}op_*}(o\colon W \to X) \to \mathrm{H}\mathscr{T}op_*(\Sigma W, X) := \pi(\Sigma W, X)$$

the homotopy class $d(\theta)$ is represented by the extension-coextension composite

$$\Sigma W \xrightarrow{coext(M,K)} \mathscr{M}(i_1^u\circ f, g) \xrightarrow{ext(S,F)} X.$$

Also note that our practice is to use the same symbol to denote both the box bracket and its image in $\pi(\Sigma W, X)$ under d. In context this abuse of notation is easily understood.

PROPOSITION 4.9. *For the data in diagram* (4.6) *above the relation*

$$[coext(M,K)] \in \square\square\left(\begin{tikzcd} W \arrow[r, "w"] \arrow[d, "h"'] & C \arrow[r, "g"] \arrow[d, "f"] & B \arrow[d, "i_1"] \\ U \arrow[r, "u"'] & A \arrow[r, "\mathrm{inc}"'] & \mathscr{M}(i_1^u \circ f, g) \end{tikzcd}\right) \subset \pi(\Sigma W, \mathscr{M}(i_1^u \circ f, g))$$

is valid.

PROOF. Consider the homotopy commutative diagram:

$$\begin{tikzcd} W \arrow[r, "w"] \arrow[d, "h"'] & C \arrow[r, "g"] \arrow[d, "f"] & B \arrow[d, "i_1"] \\ U \arrow[r, "u"'] & A \arrow[r, "\mathrm{inc}"'] & \mathscr{M}(i_1^u \circ f, g) \end{tikzcd}$$

Because $(\mathrm{inc}) \circ u = i_0 \circ i_1^u \circ u$ with $i_1^u \circ u$ null homotopic, the box bracket of this diagram is defined. Let $D\colon i_0 \circ (i_1^u \circ f) \Rightarrow i_1 \circ g$ be the defining homotopy for the double mapping cylinder $\mathscr{M}(i_1^u \circ f, g)$. Then the composite track

$$\mathscr{L} := -i_1\{K\} + \{D\}w + \mathrm{inc}\{M\} + i_0\{D_u\}h \in \mathscr{A}_{\mathscr{T}op_*}(o\colon W \to \mathscr{M}(i_1^u \circ f, g))$$

defines an element of this box bracket. Moreover $[coext(M,K)] \in d(\mathscr{L})$ is valid. This latter follows directly from the definition above of $coext(M,K)$ as a class functorially induced by the functor $\mathscr{M}$. Thus the proposition holds. $\square$

In the following proposition, we generalize Proposition 1.8 of [**16**] to a formula involving the box bracket.

PROPOSITION 4.10. *In* $\mathscr{T}op_*$ *let the diagram*

$$\begin{tikzcd} W \arrow[r, "w"] \arrow[d, "h"'] & C \arrow[r, "g"] \arrow[d, "f"] & B \arrow[r, "r"] \arrow[d, "b"] & R \\ U \arrow[r, "u"'] & A \arrow[r, "a"'] & X & \end{tikzcd}$$

be homotopy commutative with all horizontal pair composites null homotopic. Let homotopies $K\colon o \Rightarrow g \circ w$ *and* $T\colon o \Rightarrow r \circ g$ *be given satisfying* $-r\{K\} + \{T\}w = 1_o$

in $\mathscr{A}_{\mathscr{T}\!op_*}(o\colon W \to R)$. *Let homotopies* $M\colon u\circ h \Rightarrow f\circ w$, $F\colon a\circ f \Rightarrow b\circ g$ *and* $H\colon o \Rightarrow a\circ u$ *be selected arbitrarily and set*

$$\alpha := -b\{K\} + \{F\}w + a\{M\} + \{H\}h \in \square\!\square\left(\begin{tikzcd} W \arrow[r, "w"] \arrow[d, "h"'] & C \arrow[r, "g"] \arrow[d, "f"] & B \arrow[d, "b"] \\ U \arrow[r, "u"'] & A \arrow[r, "a"'] & X \end{tikzcd}\right).$$

Additionally assume that $\{\mathrm{inc}, a, u\}\circ \Sigma h = 0$ *in* $\pi(\Sigma W,\ \mathscr{M}(i_1^a\circ b, r))$ (*or equivalently require* $\{\mathrm{inc}, a, u\}\circ h = \{1_o\}$ *in* $\mathscr{A}_{\mathscr{T}\!op_*}(o\colon W \to \mathscr{M}(i_1^a\circ b, r))$) *where* $\mathrm{inc}\colon X \to \mathscr{M}(i_1^a\circ b, r)$ *is the inclusion map. Then the square*

$$\begin{tikzcd} \Sigma W \arrow[r, "\Sigma w"] \arrow[d, "\mu_\alpha"'] & \Sigma C \arrow[d, "coext(F{,}T)"] \\ X \arrow[r, "\mathrm{inc}"'] & \mathscr{M}(i_1^a\circ b, r) \end{tikzcd}$$

is homotopy commutative.

Proof. We consider the 3-box diagram

$$\begin{tikzcd} W \arrow[r, "w"] \arrow[d, "h"'] & C \arrow[r, "g"] \arrow[d, "f"] & B \arrow[r, "r"] \arrow[d, "b"] & R \arrow[d, "i_1"] \\ U \arrow[r, "u"'] & A \arrow[r, "a"'] & X \arrow[r, "\mathrm{inc}"'] & \mathscr{M}(i_1^a\circ b, r) \end{tikzcd}$$

which is homotopy commutative with all horizontal pair composites null homotopic. Let $D\colon i_0\circ i_1^a\circ b = (\mathrm{inc})\circ b \Rightarrow i_1\circ r$ be the defining homotopy for the double mapping cylinder $\mathscr{M}(i_1^a\circ b, r)$. We also have the homotopy $i_0 D_a\colon o \Rightarrow (\mathrm{inc})\circ a$ with $D_a\colon o \Rightarrow i_1^a\circ a$ the defining homotopy for the mapping cone C_a. By hypothesis, homotopies $K\colon o\Rightarrow g\circ w$ and $T\colon o \Rightarrow r\circ g$ satisfying $-r\{K\} + \{T\}w = 1_o$ are given. Then by Lemma 2.1

$$(\mathrm{inc})\alpha = (-i_1\{T\} + \{D\}g + (\mathrm{inc})\{F\} + i_0\{D_a\}f)w$$

since $i_1(-r\{K\} + \{T\}w) = i_1(1_o) = 1_o$ and

$$(-(\mathrm{inc})H + i_0\{D_a\}u)h \in \{\mathrm{inc}, a, u\}\circ h = \{1_o\}$$

in $\mathscr{A}_{\mathscr{T}\!op_*}(o\colon W \to \mathscr{M}(i_1^a\circ b, r))$. But by Proposition 4.9 we know that

$$-i_1\{T\} + \{D\}g + (\mathrm{inc})\{F\} + i_0\{D_a\}f \in \Box\!\Box \left(\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[r, "r"] \arrow[d, "b"] & R \arrow[d, "i_1"] \\ A \arrow[r, "a"'] & X \arrow[r, "\mathrm{inc}"'] & \mathscr{M}(i_1^a \circ b, r) \end{tikzcd} \right)$$

defines $coext(F,T)$. Thus $(\mathrm{inc}) \circ \mu_\alpha = coext(F,T) \circ \Sigma w$ up to homotopy and the proposition is established. $\square$

PROPOSITION 4.11. *In the diagram in* $\mathscr{T}op_*$

$$\begin{tikzcd} W \arrow[r, "w"] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{F}{\Longrightarrow}"] & C \arrow[r, "b"] \arrow[d, "a"] & B \\ U \arrow[r, "u"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\overset{L}{\Longrightarrow}"] & A \arrow[d, "\alpha"] & \\ U' \arrow[r, "u'"'] & A' & \end{tikzcd}$$

let homotopies be given as indicated. If the square containing L *is a homotopy pushout then the square*

$$\begin{tikzcd} \mathscr{M}(h,w) \arrow[r, "\xi"] \arrow[d, "\mu_F"'] & \mathscr{M}(\rho \circ h, b \circ w) \arrow[d, "\theta"] \\ A \arrow[r, "i_0 \circ \alpha"'] & \mathscr{M}(\alpha \circ a, b) \end{tikzcd}$$

is also a homotopy pushout where ξ *is the evident map and* θ *by definition represents the class functorially induced under the double mapping cylinder functor* $\mathscr{M}$ *as follows.*

$$\begin{tikzcd} U' \arrow[d, "u'"'] & U \arrow[l, "\rho"'] \arrow[d, "u"'] \arrow[l, phantom, shift left=0, ""] & W \arrow[l, "h"'] \arrow[r, "b \circ w"] \arrow[d, "w"] & B \arrow[d, "1"] \\ A' & A \arrow[l, "\alpha"] \arrow[ul, phantom, "\overset{-L}{\Longleftarrow}"] & C \arrow[l, "a"] \arrow[r, "b"'] \arrow[ul, phantom, "\overset{-F}{\Longleftarrow}"] & B \end{tikzcd} \quad \overset{\mathscr{M}}{\rightsquigarrow} \quad \begin{tikzcd} \mathscr{M}(\rho \circ h, b \circ w) \arrow[d, "\theta"] \\ \mathscr{M}(\alpha \circ a, b) \end{tikzcd}$$

PROOF. We apply Lemma 2.1 of [**7**] and observe that it is only necessary to identify $\mathscr{M}(\alpha \circ a, b)$ as the homotopy colimit of the diagram:

$$\begin{tikzcd} & U \arrow[dl, "\rho"'] \arrow[dr, "u"] & & C \arrow[dl, "a"'] \arrow[dr, "b"] & \\ U' & & A & & B \end{tikzcd}$$

Now this homotopy colimit may be constructed by taking successive homotopy pushouts. Thus, in view of our hypothesis that the square containing L is a homotopy pushout, the result is immediate. □

In the following proposition, we generalize Proposition 1.9 of [**16**] to a formula involving the matrix Toda bracket.

PROPOSITION 4.12. *In $\mathscr{T}op_*$ let the diagram*

$$\begin{tikzcd} & B \arrow[r, "r"] \arrow[d, "b"'] & R \arrow[r] \arrow[d, "y"] & * \arrow[d] \\ A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \arrow[r, "v"'] & V \end{tikzcd}$$

be homotopy commutative with all horizontal pair composites null homotopic. Suppose in particular that the homotopies $H\colon o \Rightarrow s \circ a$ and $L\colon o \Rightarrow v \circ s$ satisfy $-v\{H\} + \{L\}a = 1_o$ in $\mathscr{A}_{\mathscr{T}op_*}(o\colon A \to V)$. If $G\colon s \circ b \Rightarrow y \circ r$ is any homotopy then there exists an element

$$\beta \in \left\{ v\,,\ \begin{matrix} y \\ s \end{matrix}\,,\ \begin{matrix} r \\ b \end{matrix} \right\} \subset \mathscr{A}_{\mathscr{T}op_*}(o\colon B \to V) \cong \pi(\Sigma B, V)$$

so that the square

$$\begin{tikzcd} \mathscr{M}(i_1^a \circ b, r) \arrow[r, "\kappa"] \arrow[d, "ext(H,G)"'] & \Sigma B \arrow[d, "\mu_\beta"] \\ Y \arrow[r, "v"'] & V \end{tikzcd}$$

is homotopy commutative.

PROOF. Also let $K\colon o \Rightarrow v \circ y$ be an arbitrary homotopy. Form the diagram

$$\begin{tikzcd} B \arrow[r, "r"] \arrow[d, "i_1^a \circ b"'] \arrow[dr, phantom, "\overset{G}{\Longrightarrow}"] & R \arrow[r] \arrow[d, "y"] & * \\ C_a \arrow[r, "\mu_H"] \arrow[d] \arrow[dr, phantom, "\overset{D_{\mu_H}}{\Longrightarrow}"] & Y \arrow[d, "i_1^{\mu_H}"] & \\ * \arrow[r] & C_{\mu_H} & \end{tikzcd}$$

from which, by Proposition 4.11, a homotopy pushout square

$$\begin{tikzcd}
\mathscr{M}(i_1^a \circ b, r) \arrow[r, "\kappa"] \arrow[d, "ext(H,G)"'] & \Sigma B \arrow[d, "\theta"] \\
Y \arrow[r, "i_0 \circ i_1^{\mu_H}"'] & \mathscr{M}(i_1^{\mu_H} \circ y, o)
\end{tikzcd}$$

is obtained. Next, recalling the definition of θ from Proposition 4.11, we have the following induced maps

$$\begin{tikzcd}
* \arrow[d] & C_a \arrow[l] \arrow[d, "\mu_H"'] \arrow[l, phantom, shift right=3, "\overset{-D_{\mu_H}}{\Leftarrow}"] & B \arrow[l, "i_1^a \circ b"'] \arrow[r] \arrow[d, "r"] \arrow[l, phantom, shift right=3, "\overset{-G}{\Leftarrow}"] & * \arrow[d] \\
C_{\mu_H} \arrow[drr, bend right, "\varepsilon"'] & Y \arrow[l, "i_1^{\mu_H}"] \arrow[dr, "v"'] & R \arrow[l, "y"] \arrow[r] \arrow[d, phantom, "\overset{-K}{\Rightarrow}"] & * \arrow[dl] \\
 & & V &
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\Sigma B \arrow[d, "\theta"] \\
\mathscr{M}(i_1^{\mu_H} \circ y, o) \arrow[d, "\eta"] \\
V
\end{tikzcd}$$

where ε exists since $v \circ \mu_H = o$ by Proposition 4.5. We note that $\eta \circ i_0 \circ i_1^{\mu_H} = v$. Also

$$-\{K\}r + v\{G\} + \varepsilon\{D_{\mu_H}\}(i_1^a \circ b) = -\{K\}r + v\{G\} + (\varepsilon\{D_{\mu_H}\}i_1^a)b$$

with $\varepsilon\{D_{\mu_H}\}i_1^a$ being a homotopy $o \Rightarrow v \circ s$. Clearly in its latter form this composite track class implies that $\eta \circ \theta$ represents an element β of $\left\{ v\,, \begin{matrix} y \\ s \end{matrix}\,, \begin{matrix} r \\ b \end{matrix} \right\}$ and consequently the proposition is established. $\square$

THEOREM 4.13. *In $\mathscr{T}op_*$ let*

$$\begin{tikzcd}
W \arrow[r, "w"] \arrow[d] & C \arrow[r, "g"] \arrow[d, "f"] & B \arrow[r, "r"] \arrow[d, "b"] & R \arrow[r] \arrow[d, "y"] & * \arrow[d] \\
* \arrow[r] & A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \arrow[r, "v"'] & V
\end{tikzcd}$$

be a homotopy commutative diagram in which all horizontal pair composites are null homotopic. Suppose $o \in \{r, g, w\}$ and $o \in \{v, s, a\}$. Then the box quartet operation $\mathscr{D} \subset \pi(\Sigma^2 W, V)$ of this diagram is defined and moreover $o \in \mathscr{D}$.

PROOF. The inner weak edge conditions clearly hold in this diagram so by Proposition 3.7 its associated box quartet operation $\mathscr{D}$ is defined.

Since $o \in \{r, g, w\}$ we may select homotopies $K : o \Rightarrow g \circ w$ and $T : o \Rightarrow r \circ g$ satisfying $-r\{K\} + \{T\}w = 1_o$. Since $o \in \{v, s, a\}$ we may select homotopies $H : o \Rightarrow s \circ a$ and $L : o \Rightarrow v \circ s$ so that $-v\{H\} + \{L\}a = 1_o$. Also let

$$F\colon a\circ f \Rightarrow b\circ g, G\colon s\circ b \Rightarrow y\circ r, M\colon o \Rightarrow f\circ w \text{ and } N\colon o \Rightarrow v\circ y$$

be arbitrary homotopies. Set

$$\alpha := -b\{K\} + \{F\}w + a\{M\} \in \left\{ \begin{matrix} b \\ a \end{matrix} , \begin{matrix} g \\ f \end{matrix} , w \right\}.$$

Next we observe that the following homotopy commutative diagram may be constructed.

$$\begin{tikzcd} \Sigma W \arrow[rr, "\Sigma w"] \arrow[ddd, "\mu_\alpha"'] & & \Sigma C \arrow[rr, "\Sigma g"] \arrow[d, "coext(F,T)"'] & & \Sigma B \arrow[ddd, "\mu_\beta"] \\ & & \mathscr{M}(i_1^a\circ b, r) \arrow[urr, "\kappa"'] \arrow[dd, "ext(H,G)"] & & \\ & & & & \\ X \arrow[rr, "s"'] \arrow[uurr, "\mathrm{inc}"] & & Y \arrow[rr, "v"'] & & V \end{tikzcd} \tag{4.14}$$

For, by Proposition 4.10, $(\mathrm{inc})\circ\mu_\alpha \simeq coext(F,T)\circ\Sigma w$, and by Proposition 4.12, there exists an element $\beta \in \left\{ v , \begin{matrix} y \\ s \end{matrix} , \begin{matrix} r \\ b \end{matrix} \right\}$ such that $v\circ ext(H,G) \simeq \mu_\beta\circ\kappa$. Also the relations $\kappa\circ coext(F,T) \simeq \Sigma g$ and $ext(H,G)\circ \mathrm{inc} = s$ hold. Hence diagram (4.14) is homotopy commutative. Furthermore, by Remark 4.8, setting

$$\gamma := ext(H,G)\ \circ coext(F,T)$$

defines an element of the box bracket $\square\square\left(\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[r, "r"] \arrow[d, "b"] & R \arrow[d, "y"] \\ A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \end{tikzcd}\right)$. Thus (α,γ,β) is a coherent triple and accordingly the box bracket

$$\mathscr{D}_{(\alpha,\gamma,\beta)} = \square\square\left(\begin{tikzcd} \Sigma W \arrow[r, "\Sigma w"] \arrow[d, "\mu_\alpha"'] & \Sigma C \arrow[r, "\Sigma g"] \arrow[d, "\mu_\gamma"] & \Sigma B \arrow[d, "\mu_\beta"] \\ X \arrow[r, "s"'] & Y \arrow[r, "v"'] & V \end{tikzcd}\right)$$

is a subset of $\mathscr{D}$. But plainly the composite $\kappa\circ(\mathrm{inc})$ is null homotopic. By Lemma 2.3 this latter fact implies that $o \in \mathscr{D}_{(\alpha,\gamma,\beta)} \subset \mathscr{D}$ as asserted. □

We are now in a position to prove a theorem which generalizes Proposition 1.5 of [**16**].

Theorem 4.15. *In $\mathscr{T}\!op_*$ let*

$$\begin{tikzcd} W \arrow[r, "w"] \arrow[d] & C \arrow[r, "g"] \arrow[d, "f"] & B \arrow[r, "r"] \arrow[d, "b"] & R \arrow[r] \arrow[d, "y"] & * \arrow[d] \\ * \arrow[r] & A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \arrow[r, "v"'] & V \end{tikzcd}$$

be a homotopy commutative diagram where all horizontal pair composites are null homotopic. Assume $\{r, g, w\} = o$*,* $\{v, s, a\} = o$*,* $o \in s \circ \left\{ \begin{matrix} b \\ a \end{matrix} , \begin{matrix} g \\ f \end{matrix} , w \right\}$ *and* $o \in \left\{ v , \begin{matrix} y \\ s \end{matrix} , \begin{matrix} r \\ b \end{matrix} \right\} \circ \Sigma g$*. Then there exists a coherent triple of elements*

$$(\alpha, \gamma, \beta) \in \left\{ \begin{matrix} b \\ a \end{matrix} , \begin{matrix} g \\ f \end{matrix} , w \right\} \times \square\square \left(\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[r, "r"] \arrow[d, "b"] & R \arrow[d, "y"] \\ A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \end{tikzcd} \right) \times \left\{ v , \begin{matrix} y \\ s \end{matrix} , \begin{matrix} r \\ b \end{matrix} \right\}$$

with the property that $\mu_\beta \circ \Sigma g = o$*,* $v \circ \mu_\gamma = o$*,* $\mu_\gamma \circ \Sigma w = o$*,* $s \circ \mu_\alpha = o$ *and*

$$o \in \{\mu_\beta, \Sigma g, \Sigma w\} - \{v, \mu_\gamma, \Sigma w\} + \{v, s, \mu_\alpha\}$$

in $\pi(\Sigma^2 W, V)$*.*

PROOF. We have $o \in s \circ \left\{ \begin{matrix} b \\ a \end{matrix} , \begin{matrix} g \\ f \end{matrix} , w \right\}$ and $o \in \left\{ v , \begin{matrix} y \\ s \end{matrix} , \begin{matrix} r \\ b \end{matrix} \right\} \circ \Sigma g$ by assumption. Hence we may select homotopies

$$\begin{array}{lll} M : o \Rightarrow f \circ w & F : a \circ f \Rightarrow b \circ g & N : o \Rightarrow v \circ y \\ K : o \Rightarrow g \circ w & G : s \circ b \Rightarrow y \circ r & L : o \Rightarrow v \circ s \end{array}$$

satisfying the conditions:

$$\begin{aligned} s(-b\{K\} + \{F\}w + a\{M\}) &= 1_o \\ (-\{N\}r + v\{G\} + \{L\}b)g &= 1_o \end{aligned} \tag{4.16}$$

Also we choose homotopies $T : o \Rightarrow r \circ g$ and $H : o \Rightarrow s \circ a$ arbitrarily. The hypothesis $\{v, s, a\} = o$ implies that $-v\{H\} + \{L\}a = 1_o$. Hence by Lemma 2.1 and by (4.16) we deduce that

$$v(-y\{T\} + \{G\}g + s\{F\} + \{H\}f) = (-\{N\}r + v\{G\} + \{L\}b)g = 1_o.$$

We define tracks

$$\alpha := -b\{K\} + \{F\}w + a\{M\} \in \left\{ \begin{matrix} b \\ a \end{matrix}, \begin{matrix} g \\ f \end{matrix}, w \right\}$$

and

$$\gamma := -y\{T\} + \{G\}g + s\{F\} + \{H\}f \in \square\square \left(\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[r, "r"] \arrow[d, "b"] & R \arrow[d, "y"] \\ A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \end{tikzcd} \right).$$

It follows that the relations $s \circ \mu_\alpha = o$ and $v \circ \mu_\gamma = o$ are valid up to homotopy.

Now observe that the hypotheses of Proposition 4.10 hold for the diagram

$$\begin{tikzcd} W \arrow[r, "w"] \arrow[d] & C \arrow[r, "g"] \arrow[d, "f"] & B \arrow[r, "r"] \arrow[d, "b"] & R \\ * \arrow[r] & A \arrow[r, "a"'] & X & \end{tikzcd}$$

and the homotopies M, F, K, and T; this is due to the presence of the one-point space in the lower left corner of the diagram and because $\{r, g, w\} = o$. Thus the square

$$\begin{tikzcd} \Sigma W \arrow[r, "\Sigma w"] \arrow[d, "\mu_\alpha"'] & \Sigma C \arrow[d, "coext(F,T)"] \\ X \arrow[r, "\text{inc}"'] & \mathscr{M}(i_1^a \circ b, r) \end{tikzcd}$$

is homotopy commutative. Next we consider the diagram

$$\begin{tikzcd} & B \arrow[r, "r"] \arrow[d, "b"'] & R \arrow[r] \arrow[d, "y"] & * \arrow[d] \\ A \arrow[r, "a"'] & X \arrow[r, "s"'] & Y \arrow[r, "v"'] & V \end{tikzcd}$$

with the homotopies H, L, G and N. Since $\{v, s, a\} = o$ we may apply Proposition 4.12 to obtain a homotopy commutative square

$$\begin{tikzcd} \mathscr{M}(i_1^a \circ b, r) \arrow[r, "\kappa"] \arrow[d, "ext(H,G)"'] & \Sigma B \arrow[d, "\mu_\beta"] \\ Y \arrow[r, "v"'] & V \end{tikzcd}$$

for some $\beta \in \left\{ \begin{matrix} v\,, & y\,, & r \\ & s & b \end{matrix} \right\} \subset \mathscr{A}_{\mathscr{T}\!op^*}(o\colon B \to V)$. (Note that we have not claimed that β equals $-\{N\}r + v\{G\} + \{L\}b$.) Consequently we may form the following homotopy commutative diagram

$$\begin{tikzcd}
\Sigma W \arrow[r, "\Sigma w"] \arrow[dd, "\mu_\alpha"'] & \Sigma C \arrow[r, "\Sigma g"] \arrow[d, "coext(F,T)"'] & \Sigma B \arrow[dd, "\mu_\beta"] \\
 & \mathscr{M}(i_1^a \circ b, r) \arrow[ur, "\kappa"'] \arrow[d, "ext(H,G)"] & \\
X \arrow[r, "s"'] \arrow[ur, "\mathrm{inc}"] & Y \arrow[r, "v"'] & V
\end{tikzcd}$$

in which the triangles are homotopy commutative by (4.6) and (4.7). Furthermore the relation $\mu_\gamma = ext(H,G) \circ coext(F,T)$ holds since

$$\gamma = -y\{T\} + \{G\}g + s\{F\} + \{H\}f$$

(see Remark 4.8). Thus (α,γ,β) is a coherent triple. Moreover, since $\kappa \circ (\mathrm{inc}) = o$, it follows by Proposition 2.3 that $o \in \mathscr{D}_{(\alpha,\gamma,\beta)}$. Finally, because $s \circ \mu_\alpha = o$ and $\mu_\gamma \circ \Sigma w = o$, we have

$$\mathscr{D}_{(\alpha,\gamma,\beta)} = \{\mu_\beta, \Sigma g, \Sigma w\} - \{v, \mu_\gamma, \Sigma w\} + \{v, s, \mu_\alpha\}$$

by Proposition 3.3(3) of [**5**]. This completes the proof of the theorem. $\square$

Corollary 4.17 (cf. Proposition 1.5 of [**16**]). *In $\mathscr{T}\!op_*$ let*

$$\begin{tikzcd}
W \arrow[r, "w"] \arrow[d] & C \arrow[r, "g"] \arrow[d] & B \arrow[r] \arrow[d, "b"] & * \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & * \arrow[r] & X \arrow[r, "s"'] & Y \arrow[r, "v"'] & V
\end{tikzcd}$$

be a homotopy commutative diagram in which all horizontal pair composites are null homotopic. If $o \in s \circ \{b, g, w\}$ and $o \in \{v, s, b\} \circ \Sigma g$ then there exists a triple of elements

$$(\alpha, \gamma, \beta) \in \{b, g, w\} \times \{s, b, g\} \times \{v, s, b\}$$

such that $\mu_\beta \circ \Sigma g = o$, $v \circ \mu_\gamma = o$, $\mu_\gamma \circ \Sigma w = o$, $s \circ \mu_\alpha = o$ and

$$o \in \{\mu_\beta, \Sigma g, \Sigma w\} + \{v, \mu_\gamma, \Sigma w\} + \{v, s, \mu_\alpha\}.$$

PROOF. The relations $\left\{\begin{matrix} b \\ o \end{matrix}, \begin{matrix} g \\ o \end{matrix}, w\right\} = \{b, g, w\}$ and $\left\{v, \begin{matrix} o \\ s \end{matrix}, \begin{matrix} o \\ b \end{matrix}\right\} = \{v, s, b\}$ are valid by Proposition 8.3 of [**2**]. Thus we see that Theorem 4.15 is applicable to the 4-box diagram in the statement of this corollary. Also

$$\square\square\left(\begin{tikzcd} C \arrow[r, "g"] \arrow[d] & B \arrow[r] \arrow[d, "b"] & * \arrow[d] \\ * \arrow[r] & X \arrow[r, "s"'] & Y \end{tikzcd}\right) = -\{s, b, g\}$$

by Proposition 3.1 of [**5**]. Therefore by Theorem 4.15 there is a triple of elements

$$(\alpha, -\gamma, \beta) \in \{b, g, w\} \times -\{s, b, g\} \times \{v, s, b\}$$

such that

$$o \in \{\mu_\beta, \Sigma g, \Sigma w\} - \{v, \mu_{-\gamma}, \Sigma w\} + \{v, s, \mu_\alpha\}.$$

Note that $\mu_{-\gamma} = -\mu_\gamma$. Hence by Proposition 4.18 below we have

$$-\{v, \mu_{-\gamma}, \Sigma w\} = -\{v, -\mu_\gamma, \Sigma w\} = \{v, \mu_\gamma, \Sigma w\}$$

and the corollary is established. $\square$

PROPOSITION 4.18. *In $\mathscr{T}op_*$ let $\Sigma W \xrightarrow{\Sigma w} \Sigma C \xrightarrow{f} Y \xrightarrow{v} V$ be a composite of maps with $f \circ \Sigma w = o$ and $v \circ f = o$. Then $-\{v, f, \Sigma w\} = \{v, -f, \Sigma w\}$.*

PROOF. Note that $(-f) \circ \Sigma w = f \circ (-1_{\Sigma C}) \circ \Sigma w = f \circ \Sigma w \circ (-1_{\Sigma W}) = o$ and $v \circ (-f) = v \circ f \circ (-1_{\Sigma C}) = o$ so that $\{v, -f, \Sigma w\}$ is also well-defined. We recall that the Toda bracket satisfies the basic properties $\{\alpha, \beta, \gamma\} \circ \delta \subset \{\alpha, \beta, \gamma \circ \delta\}$ and $\{\alpha, \beta, \gamma \circ \delta\} \subset \{\alpha, \beta \circ \gamma, \delta\}$. Using these properties we have

$$\begin{aligned} -\{v, f, \Sigma w\} &= \{v, f, \Sigma w\} \circ (-1_{\Sigma^2 W}) \subset \{v, f, \Sigma w \circ (-1_{\Sigma W})\} \\ &= \{v, f, (-1_{\Sigma C}) \circ \Sigma w\} \subset \{v, f \circ (-1_{\Sigma C}), \Sigma w\} = \{v, -f, \Sigma w\}. \end{aligned}$$

But then also $\{v, -f, \Sigma w\} = -(-\{v, -f, \Sigma w\}) \subset -\{v, -(-f), \Sigma w\} = -\{v, f, \Sigma w\}$ and consequently $-\{v, f, \Sigma w\} = \{v, -f, \Sigma w\}$, as claimed. $\square$

5. Some computations in $\mathscr{T}op_*$.

In this section we provide a few computations in $\mathscr{T}op_*$ for the purpose of illustrating basic aspects and points of the theory. We follow [**16**] for names of elements in the homotopy groups of spheres. We begin by showing that the box quartet operation may be nontrivial.

PROPOSITION 5.1. *Let $n \geq 9$. In $\mathscr{T}\!op_*$ the box quartet operation $\mathscr{D}$ for the diagram*

$$\begin{tikzcd} S^{n+8} \arrow[r, "2\iota_{n+8}"] \arrow[d] & S^{n+8} \arrow[r] \arrow[d, "8\sigma_{n+1}"] & * \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & S^{n+1} \arrow[r, "2\iota_{n+1}"'] & S^{n+1} \arrow[r, "\eta_n"'] & S^n \arrow[r, "2\iota_n"'] & S^n \end{tikzcd}$$

is defined and satisfies

$$\mathscr{D} = \begin{cases} \eta_n \circ \mu_{n+1} + \{2\sigma_n \circ \nu_{n+7}\}, & n = 9,\, 10 \\ \eta_n \circ \mu_{n+1}, & n \geq 11 \end{cases}$$

in

$$\pi^n_{n+10} = \begin{cases} \boldsymbol{Z}/8 \oplus \boldsymbol{Z}/2 = \{\sigma_9 \circ \nu_{16}\} \oplus \{\eta_9 \circ \mu_{10}\}, & n = 9 \\ \boldsymbol{Z}/4 \oplus \boldsymbol{Z}/2 = \{\sigma_{10} \circ \nu_{17}\} \oplus \{\eta_{10} \circ \mu_{11}\}, & n = 10 \\ \boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2 = \{\sigma_{11} \circ \nu_{18}\} \oplus \{\eta_{11} \circ \mu_{12}\}, & n = 11 \\ \boldsymbol{Z}/2 = \{\eta_n \circ \mu_{n+1}\}, & n \geq 12. \end{cases}$$

In particular $\mathscr{D}$ is nontrivial.

PROOF. The value of π^n_{n+10} is given in Theorem 7.3 of [**16**]. We observe that the strong edge conditions hold, for

$$\{2\iota_n, \eta_n, 2\iota_{n+1}\} \circ 8\sigma_{n+1} = \eta_n^2 \circ 8\sigma_{n+1} = 0.$$

Hence by Remark 3.3 the box quartet operation $\mathscr{D}$ is defined.

In view of Proposition 5.2 below the following identifications are valid.

$$\mathscr{A} = \square\square \left(\begin{tikzcd} S^{n+8} \arrow[r, "2\iota_{n+8}"] \arrow[d] & S^{n+8} \arrow[r] \arrow[d, "8\sigma_{n+1}"] & * \arrow[d] \\ * \arrow[r] & S^{n+1} \arrow[r, "2\iota_{n+1}"'] & S^{n+1} \end{tikzcd} \right) = -\{2\iota_{n+1}, 8\sigma_{n+1}, 2\iota_{n+8}\} = 0 \quad (n \geq 8)$$

$$
\begin{aligned}
\Gamma = \square\!\square & \left(\begin{array}{ccccc} S^{n+8} & \longrightarrow & * & \longrightarrow & * \\ {\scriptstyle 8\sigma_{n+1}}\downarrow & & \downarrow & & \downarrow \\ S^{n+1} & \xrightarrow[2\iota_{n+1}]{} & S^{n+1} & \xrightarrow[\eta_n]{} & S^n \end{array} \right) \\
&= \{\eta_n, 2\iota_{n+1}, 8\sigma_{n+1}\} \\
&= \mu_n + \{\nu_n^3, \eta_n \circ \varepsilon_{n+1}\} \quad (n \geq 9)
\end{aligned}
$$

$$
\mathscr{B} = \square\!\square \left(\begin{array}{ccccc} * & \longrightarrow & * & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ S^{n+1} & \xrightarrow[\eta_n]{} & S^n & \xrightarrow[2\iota_n]{} & S^n \end{array} \right) = 0 \quad (n \geq 3)
$$

Hence all coherent triples for $\mathscr{D}$ are of the form $(o, \gamma, o) \in \mathscr{A} \times \Gamma \times \mathscr{B}$ with

$$
\gamma = \mu_n + x\eta_n \circ \varepsilon_{n+1} + y\nu_n^3
$$

for some integers x, y. For any such γ we have

$$
\mathscr{D}_{(o,\gamma,o)} = \square\!\square \left(\begin{array}{ccccc} S^{n+9} & \xrightarrow{2\iota_{n+9}} & S^{n+9} & \longrightarrow & * \\ {\scriptstyle o}\downarrow & & \downarrow{\scriptstyle \gamma} & & \downarrow \\ S^{n+1} & \xrightarrow[\eta_n]{} & S^n & \xrightarrow[2\iota_n]{} & S^n \end{array} \right) = -\{2\iota_n, \gamma, 2\iota_{n+9}\}.
$$

Now the indeterminacy of the Toda bracket $\{2\iota_n, \gamma, 2\iota_{n+9}\}$ is seen to be

$$
2\iota_n \circ \pi_{n+10}^n + \pi_{n+10}^n \circ 2\iota_{n+10} = \begin{cases} \{2\sigma_n \circ \nu_{n+7}\}, & n = 9,\ 10 \\ 0, & n \geq 11 \end{cases}
$$

by examination of the structure of π_{n+10}^n. Furthermore by Corollary 3.7 of [**16**] the Toda bracket $\{2\iota_n, \gamma, 2\iota_{n+9}\}$ contains the element $\gamma \circ \eta_{n+9}$. But

$$
\begin{aligned}
\gamma \circ \eta_{n+9} &= \mu_n \circ \eta_{n+9} + x\eta_n \circ \varepsilon_{n+1} \circ \eta_{n+9} + y\nu_n^3 \circ \eta_{n+9} \\
&= \eta_n \circ \mu_{n+1} + x\eta_n^2 \circ \varepsilon_{n+2} + y\nu_n^3 \circ \eta_{n+9}
\end{aligned}
$$

and this last reduces to $\eta_n \circ \mu_{n+1}$ since $\eta_n^2 \circ \varepsilon_{n+2} = 0$ $(n \geq 9)$ by Proposition 5.2(1) below and $\nu_n \circ \eta_{n+3} = 0$ $(n \geq 6)$ by (5.9) of [**16**]. Thus

$$
\mathscr{D} = \{2\iota_n, \gamma, 2\iota_{n+9}\} = \begin{cases} \eta_n \circ \mu_{n+1} + \{2\sigma_n \circ \nu_{n+7}\}, & n = 9,\ 10 \\ \eta_n \circ \mu_{n+1}, & n \geq 11 \end{cases}.
$$

Moreover by the structure of π^n_{n+10} we see $\eta_n \circ \mu_{n+1} \notin \{2\sigma_n \circ \nu_{n+7}\}$ and thus $o \notin \mathscr{D}$. $\square$

PROPOSITION 5.2. *(1) $\eta_n^2 \circ \varepsilon_{n+2} = 0$ for $n \geq 9$.*
(2) $\{2\iota_n, 8\sigma_n, 2\iota_{n+7}\} = 0$ for $n \geq 9$.
(3) $\{\eta_n, 2\iota_{n+1}, 8\sigma_{n+1}\} = \mu_n + \{\nu_n^3,\ \eta_n \circ \varepsilon_{n+1}\}$ for $n \geq 9$.

PROOF.

(1) We have $\eta_n^2 \circ \varepsilon_{n+2} = 4(\nu_n \circ \sigma_{n+3})$ for $n \geq 5$ by (7.10) of [**16**]. Also the relation $\nu_9 \circ \sigma_{12} = \pm 2\sigma_9 \circ \nu_{16}$ follows from (7.19) of [**16**]. Hence

$$\eta_9^2 \circ \varepsilon_{11} = 4(\nu_9 \circ \sigma_{12}) = \pm 8(\sigma_9 \circ \nu_{16}) = 0$$

since $\sigma_9 \circ \nu_{16}$ is an element of order 8 by Theorem 7.3 of [**16**].

(2) The indeterminacy of $\{2\iota_n, 8\sigma_n, 2\iota_{n+7}\}$ is $2\iota_n \circ \pi^n_{n+8} + \pi^n_{n+8} \circ 2\iota_{n+8} = 0$ for $n \geq 9$ by Theorem 7.1 of [**16**]. We remark that $2\sigma_9 = E^2\sigma'$ by (5.16) of [**16**]. Then by Corollary 3.7 of [**16**] we have

$$\{2\iota_9, 8\sigma_9, 2\iota_{16}\} = \{2\iota_9, 4E^2\sigma', 2\iota_{16}\} = \{2\iota_9, 4E^2\sigma', 2\iota_{16}\}_1 \ni 4E^2\sigma' \circ \eta_{16} = 0.$$

It follows that $\{2\iota_n, 8\sigma_n, 2\iota_{n+7}\} \ni 0$ and hence $\{2\iota_n, 8\sigma_n, 2\iota_{n+7}\} = 0$ for $n \geq 9$.

(3) By Theorem 7.1 of [**16**] $\pi^{n+1}_{n+9} = (\boldsymbol{Z}/2)^2 = \{\overline{\nu}_{n+1}\} \oplus \{\varepsilon_{n+1}\}$ for $n \geq 9$. The indeterminacy of $\{\eta_n, 2\iota_{n+1}, 8\sigma_{n+1}\}$ is

$$\eta_n \circ \pi^{n+1}_{n+9} + \pi^n_{n+2} \circ 8\sigma_{n+2} = \{\eta_n \circ \overline{\nu}_{n+1}, \eta_n \circ \varepsilon_{n+1}\} = \{\nu_n^3, \eta_n \circ \varepsilon_{n+1}\}$$

for $2 \cdot \pi^n_{n+2} = 0$ $(n \geq 3)$ and $\eta_n \circ \overline{\nu}_{n+1} = \nu_n^3$ $(n \geq 5)$ by (7.3) of [**16**]. Now it may be deduced from Lemma 6.5 of [**16**] that $\mu_n \in \{\eta_n, 2\iota_{n+1}, 8\sigma_{n+1}\}$ for $n \geq 8$ (one must recall that $8\sigma_{n+1} = E^{n-4}\sigma'''$). Hence, calculating the indeterminancies of the Toda brackets for $n \geq 9$, we have

$$\{\eta_n, 2\iota_{n+1}, 8\sigma_{n+1}\} = \mu_n + \{\nu_n^3,\ \eta_n \circ \varepsilon_{n+1}\}$$

as claimed. $\square$

REMARK 5.3. With respect to Proposition 5.1 it is interesting to note that for the very similar diagram

$$\begin{tikzcd}
S^{n+8} \arrow[r, "2\iota_{n+8}"] \arrow[d] & S^{n+8} \arrow[r, "8\sigma_{n+1}"] \arrow[d] & S^{n+1} \arrow[r] \arrow[d, "2\iota_{n+1}"] & * \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & * \arrow[r] & S^{n+1} \arrow[r, "\eta_n"'] & S^n \arrow[r, "2\iota_n"'] & S^n
\end{tikzcd}$$

one has $o \in \mathscr{D}$. This is a consequence of Theorem 4.13 above. This example shows the sensitivity in the value of $\mathscr{D}$ to the placement of the maps in the 4-box diagram which defines $\mathscr{D}$.

Proposition 5.4. *In $\mathscr{T}op_*$ the box quartet operation $\mathscr{D}$ for the diagram*

$$\begin{tikzcd} S^{28} \arrow[r, "\varepsilon_{20}"] \arrow[d] & S^{20} \arrow[r, "\nu_{17}"] \arrow[d] & S^{17} \arrow[r, "\eta_{16}"] \arrow[d, "\varepsilon_9"] & S^{16} \arrow[r] \arrow[d, "\nu_6\circ\sigma_9"] & * \arrow[d] \\ * \arrow[r] & * \arrow[r] & S^9 \arrow[r, "\nu_6"'] & S^6 \arrow[r, "\nu'"'] & S^3 \end{tikzcd}$$

is defined and trivial in

$$\pi^3_{30} = \boldsymbol{Z}/4 \oplus (\boldsymbol{Z}/2)^3 = \{\mu'_3\} \oplus \{\nu' \circ \delta_6\} \oplus \{\nu' \circ \overline{\mu}_6 \circ \sigma_{23}\} \oplus \{\varepsilon_3 \circ \overline{\sigma}_{11}\}.$$

Proof. The value of π^3_{30} is given in Theorem 2(a) of [**12**]. The 4-box diagram in this proposition has been considered in part in Theorem 22 of [**8**]. Details given there show that the diagram is homotopy commutative and that all horizontal pair composites are null homotopic. Since $\{\eta_{16}, \nu_{17}, \varepsilon_{20}\} \subset \pi^{16}_{29} = 0$ by Theorem 7.7 of [**16**] we readily see that this diagram satisfies the strong edge conditions and hence by Theorem 4.13 its box quartet operation $\mathscr{D}$ is defined and trivial. □

6. More about relations.

The value of a box quartet operation is determined by the relations and values of those elements, Toda brackets and box brackets that occur in its definition. Conversely when the value of a box quartet operation is known (or even when the box quartet operation is known to be defined only) then information may be extracted from the box quartet operation about the relations and values of those elements, Toda brackets and box brackets. In this regard Proposition 3.7, Theorem 4.13 and Theorem 4.15 are quite helpful. We illustrate this observation by examining in detail the box quartet operation in Proposition 5.4.

Let $\mathscr{D}$ denote the box quartet operation for the 4-box diagram in Proposition 5.4. By Theorem 22 of [**8**]

$$\Gamma = \square\square\left(\begin{tikzcd} S^{20} \arrow[r, "\nu_{17}"] \arrow[d] & S^{17} \arrow[r, "\eta_{16}"] \arrow[d, "\varepsilon_9"] & S^{16} \arrow[d, "\nu_6\circ\sigma_9"] \\ * \arrow[r] & S^9 \arrow[r, "\nu_6"'] & S^6 \end{tikzcd}\right) = \left\{\begin{matrix}\nu_6\circ\sigma_9 \\ \nu_6\end{matrix}, \begin{matrix}\eta_{16} \\ \varepsilon_9\end{matrix}, \nu_{17}\right\} = \overline{\varepsilon}_6$$

and by Theorem 23 of [**8**]

$$\mathscr{B} = \square\!\square \left(\begin{array}{ccccc} S^{17} & \xrightarrow{\eta_{16}} & S^{16} & \longrightarrow & * \\ {\scriptstyle \varepsilon_9}\big\downarrow & & \big\downarrow{\scriptstyle \nu_6\circ\sigma_9} & & \big\downarrow \\ S^9 & \xrightarrow[\nu_6]{} & S^6 & \xrightarrow[\nu']{} & S^3 \end{array} \right) = \left\{ \nu' , \begin{matrix} \nu_6\circ\sigma_9 \\ \nu_6 \end{matrix} , \begin{matrix} \eta_{16} \\ \varepsilon_9 \end{matrix} \right\} = \overline{\varepsilon}_3 .$$

Also

$$\mathscr{A} = \square\!\square \left(\begin{array}{ccccc} S^{28} & \xrightarrow{\varepsilon_{20}} & S^{20} & \xrightarrow{\nu_{17}} & S^{17} \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_9} \\ * & \longrightarrow & * & \longrightarrow & S^9 \end{array} \right) = \{\varepsilon_9, \nu_{17}, \varepsilon_{20}\}.$$

The indeterminacy of $\{\varepsilon_9, \nu_{17}, \varepsilon_{20}\}$ is $\varepsilon_9 \circ \pi^{17}_{29} + \pi^9_{21} \circ \varepsilon_{21} = 0$, for $\pi^{17}_{29} = 0$ and $\pi^9_{21} = 0$ by Theorem 7.6 of [**16**]. Hence $\{\varepsilon_9, \nu_{17}, \varepsilon_{20}\} \subset \pi^9_{29}$ consists of a single element. Now $\pi^9_{29} = \boldsymbol{Z}/8 = \{\overline{\kappa}_9\}$ by page 48 of [**11**]. We note that

$$2\{\varepsilon_9, \nu_{17}, \varepsilon_{20}\} = \{\varepsilon_9, \nu_{17}, \varepsilon_{20}\} \circ 2\iota_{29} = \varepsilon_9 \circ \{\nu_{17}, \varepsilon_{20}, 2\iota_{28}\} \subset \varepsilon_9 \circ \pi^{17}_{29} = 0$$

by Theorem 7.6 of [**16**]. Hence we may put $\{\varepsilon_9, \nu_{17}, \varepsilon_{20}\} = 4x\overline{\kappa}_9$ for $x = 0$ or 1. By Proposition 5.4 we know $\mathscr{D}$ is defined and we have shown that there is exactly one coherent triple $(4x\overline{\kappa}_9, \overline{\varepsilon}_6, \overline{\varepsilon}_3) \in \mathscr{A} \times \Gamma \times \mathscr{B}$ to be considered in the definition of $\mathscr{D}$. Consequently $\mathscr{D}$ reduces to a single box bracket:

$$\mathscr{D} = \square\!\square \left(\begin{array}{ccccc} S^{29} & \xrightarrow{\varepsilon_{21}} & S^{21} & \xrightarrow{\nu_{18}} & S^{18} \\ {\scriptstyle 4x\overline{\kappa}_9}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\varepsilon}_6} & & \big\downarrow{\scriptstyle \overline{\varepsilon}_3} \\ S^9 & \xrightarrow[\nu_6]{} & S^6 & \xrightarrow[\nu']{} & S^3 \end{array} \right) \subset \pi^3_{30}.$$

In particular the relation $\overline{\varepsilon}_6 \circ \varepsilon_{21} = \nu_6 \circ (4x\overline{\kappa}_9)$ must be valid. Furthermore we can conclude that $x = 1$ since it follows from Proposition 6.2(2) below that $\overline{\varepsilon}_6 \circ \varepsilon_{21} \neq 0$. This yields the previously unrecorded relations $\overline{\varepsilon}_6 \circ \varepsilon_{21} = 4\nu_6 \circ \overline{\kappa}_9$ and $\{\varepsilon_9, \nu_{17}, \varepsilon_{20}\} = 4\overline{\kappa}_9$. But note that in Proposition 6.2(2) below, we show (independently of the present discussion) that the relation $\overline{\varepsilon}_5 \circ \varepsilon_{20} = 4\nu_5 \circ \overline{\kappa}_8$ holds.

It might also be pointed out that the relation $\nu' \circ \overline{\varepsilon}_6 \neq 0$ similarly implies $\left\{ \nu' , \begin{matrix} \nu_6\circ\sigma_9 \\ \nu_6 \end{matrix} , \begin{matrix} \eta_{16} \\ \varepsilon_9 \end{matrix} \right\} \neq 0$ in $\pi^3_{18} = \boldsymbol{Z}/2 = \{\overline{\varepsilon}_3\}$ and hence $\left\{ \nu' , \begin{matrix} \nu_6\circ\sigma_9 \\ \nu_6 \end{matrix} , \begin{matrix} \eta_{16} \\ \varepsilon_9 \end{matrix} \right\} = \overline{\varepsilon}_3$ (a fact previously obtained in Theorem 23 of [**8**]).

PROPOSITION 6.1. *The relations* $\overline{\varepsilon}_6 \circ \varepsilon_{21} = 4\nu_6 \circ \overline{\kappa}_9$ *in* π^6_{29}, $\{\varepsilon_9, \nu_{17}, \varepsilon_{20}\} = 4\overline{\kappa}_9$ *in* π^9_{29} *and*

$$o \in \square\square\left(\begin{tikzcd} S^{29} \arrow[r, "\varepsilon_{21}"] \arrow[d, "4\overline{\kappa}_9"'] & S^{21} \arrow[r, "\nu_{18}"] \arrow[d, "\overline{\varepsilon}_6"] & S^{18} \arrow[d, "\overline{\varepsilon}_3"] \\ S^{9} \arrow[r, "\nu_6"'] & S^{6} \arrow[r, "\nu'"'] & S^{3} \end{tikzcd}\right) \subset \pi^3_{30}$$

hold.

PROPOSITION 6.2. *The following relations hold in the* 23*-stem.*
(1) $\overline{\varepsilon}_3 \circ \varepsilon_{18} = \overline{\varepsilon}_3 \circ \overline{\nu}_{18} = \varepsilon_3 \circ \overline{\varepsilon}_{11} = \varepsilon_3 \circ \eta_{11} \circ \kappa_{12} = \eta_3 \circ \varepsilon_4 \circ \kappa_{12} = 2\,\overline{\alpha}$ *in* π^3_{26}.
(2) $\overline{\varepsilon}_5 \circ \varepsilon_{20} = \overline{\varepsilon}_5 \circ \overline{\nu}_{20} = \varepsilon_5 \circ \overline{\varepsilon}_{13} = 4\nu_5 \circ \overline{\kappa}_8 = \nu^3_5 \circ \kappa_{14}$ *in* π^5_{28}
(3) $\overline{\nu}_6 \circ \overline{\varepsilon}_{14} = 4\nu_6 \circ \overline{\kappa}_9$ *in* π^6_{29}.

PROOF.
(1) By Lemma 6.4 of [**16**] and Proposition 17(4) of [**8**] we have

$$\overline{\varepsilon}_3 \circ \varepsilon_{18} + \overline{\varepsilon}_3 \circ \overline{\nu}_{18} = \overline{\varepsilon}_3 \circ (\varepsilon_{18} + \overline{\nu}_{18}) = \overline{\varepsilon}_3 \circ \sigma_{18} \circ \eta_{25} = 0.$$

It follows that $\overline{\varepsilon}_3 \circ \varepsilon_{18} = \overline{\varepsilon}_3 \circ \overline{\nu}_{18}$. We remark that $\pi^3_{26} = \boldsymbol{Z}/4 = \{\overline{\alpha}\}$ by Theorem 1.1(a) of [**10**] and that $2\,\overline{\alpha} = \eta_3 \circ \varepsilon_4 \circ \kappa_{12}$ by (3.1) of [**10**]. Moreover the suspension homomorphism $E^\infty : \pi^3_{26} \to {}^S\pi_{23}$ (the 23-stem of the stable homotopy groups of spheres) is a monomorphism. Then the relation $\eta_n \circ \kappa_{n+1} = \overline{\varepsilon}_n$ for $n \geq 6$ by (10.23) of [**16**] implies that

$$E^\infty(\overline{\varepsilon}_3 \circ \varepsilon_{18}) = \overline{\varepsilon} \circ \varepsilon = \eta \circ \kappa \circ \varepsilon = \eta \circ \varepsilon \circ \kappa = E^\infty(\eta_3 \circ \varepsilon_4 \circ \kappa_{12}).$$

Hence we have $\overline{\varepsilon}_3 \circ \varepsilon_{18} = \eta_3 \circ \varepsilon_4 \circ \kappa_{12}$. We remark that $\eta_3 \circ \varepsilon_4 = \varepsilon_3 \circ \eta_{11}$ by (7.5) of [**16**].

(2) By Proposition 3.1(2) of [**12**] the relation $4\nu_5 \circ \overline{\kappa}_8 = \eta_5 \circ \varepsilon_6 \circ \kappa_{14}$ holds. Moreover by Lemma 15.4 of [**11**], we have $4\,\overline{\kappa}_8 = \nu^2_8 \circ \kappa_{14}$ and hence

$$4\nu_5 \circ \overline{\kappa}_8 = \nu_5 \circ 4\,\overline{\kappa}_8 = \nu^3_5 \circ \kappa_{14}.$$

(3) We see that $\overline{\nu}_6 \circ \overline{\varepsilon}_{14} = \overline{\nu}_6 \circ \eta_{14} \circ \kappa_{15} = \nu^3_6 \circ \kappa_{15} = 4\nu_6 \circ \overline{\kappa}_9$ by Lemma 6.3 of [**16**]. □

In the remainder of this section we give some additional basic relations which involve the elements $\overline{\varepsilon}_n$ and which we feel are not easily proven. In particular, in Proposition 6.6, we study Toda brackets of the type $\{\varepsilon_n, \nu_{n+8}, \varepsilon_{n+11}\}$ for $n \geq 5$ (cf. Proposition 6.1). The proof of Proposition 6.6 gives a good example of the use of Theorem 4.15 (actually, Corollary 4.17).

PROPOSITION 6.3. *The following relations hold in the* 22*-stem.*
(1) $\overline{\mu}' \circ \nu_{22} = 0$ *and* $\{\overline{\varepsilon}_3, 2\iota_{18}, \nu_{18}^2\} = \{\overline{\varepsilon}_3, 2\nu_{18}, \nu_{21}\} = \varepsilon_3 \circ \kappa_{11}$ *in* π_{25}^3.
(2) $\{\overline{\varepsilon}_5, 2\iota_{20}, \nu_{20}^2\} = \varepsilon_5 \circ \kappa_{13}$ *and* $\{\overline{\varepsilon}_5, 2\nu_{20}, \nu_{23}\} = \varepsilon_5 \circ \kappa_{13} + \{\nu_5 \circ \overline{\zeta}_8\}$ *in* π_{27}^5.
(3) $\kappa_7 \circ \varepsilon_{21} = \sigma' \circ \overline{\varepsilon}_{14} + \varepsilon_7 \circ \kappa_{15}$ *in* π_{29}^7.

PROOF.
(1) We see $E^2(\overline{\mu}' \circ \nu_{22}) = E^2(\overline{\mu}') \circ \nu_{24} = 2\overline{\zeta}_5 \circ \nu_{24} \subset 2\pi_{27}^5 = 0$ by Lemma 12.4 of [**16**] and Theorem B of [**9**]. Since $E^2 : \pi_{25}^3 \to \pi_{27}^5$ is a monomorphism, the relation $\overline{\mu}' \circ \nu_{22} = 0$ follows.

By Theorem B of [**9**] $\pi_{25}^3 = \boldsymbol{Z}/2 = \{\varepsilon_3 \circ \kappa_{11}\}$. We observe that

$$\overline{\varepsilon}_3 \circ \pi_{25}^{18} = \{\overline{\varepsilon}_3 \circ \sigma_{18}\} = 0$$

by Proposition 5.15 of [**16**] and Proposition 17(4) of [**8**];

$$\pi_{19}^3 \circ \nu_{19}^2 = \{\mu_3 \circ \sigma_{12} \circ \nu_{19}^2,\ \eta_3 \circ \overline{\varepsilon}_4 \circ \nu_{19}^2\} = 0$$

by Theorem 12.6, (7.20), Lemma 12.10, and (5.9) of [**16**];

$$\pi_{22}^3 \circ \nu_{22} = \{\overline{\mu}' \circ \nu_{22},\ \nu' \circ \mu_6 \circ \sigma_{15} \circ \nu_{22}\} = 0$$

by Theorem 12.9 and (7.20) of [**16**]. Hence the Toda brackets $\{\overline{\varepsilon}_3, 2\iota_{18}, \nu_{18}^2\}$ and $\{\overline{\varepsilon}_3, 2\nu_{18}, \nu_{21}\}$ have trivial indeterminacies. We see that

$$\{\overline{\varepsilon}_3, 2\iota_{18}, \nu_{18}^2\} \circ \eta_{25} = \overline{\varepsilon}_3 \circ \{2\iota_{18}, \nu_{18}^2, \eta_{24}\} = \overline{\varepsilon}_3 \circ \varepsilon_{18}$$

and

$$\{\overline{\varepsilon}_3, 2\nu_{18}, \nu_{21}\} \circ \eta_{25} = \overline{\varepsilon}_3 \circ \{2\nu_{18}, \nu_{21}, \eta_{24}\} = \overline{\varepsilon}_3 \circ \varepsilon_{18}$$

by (7.6) of [**16**]. Thus the result follows.

(2) We have to calculate the indeterminacies of the Toda brackets involved. We have

$$\overline{\varepsilon}_5 \circ \pi_{27}^{20} + \pi_{21}^5 \circ \nu_{21}^2 = \{\overline{\varepsilon}_5 \circ \sigma_{20},\ \mu_5 \circ \sigma_{14} \circ \nu_{21}^2,\ \eta_5 \circ \overline{\varepsilon}_6 \circ \nu_{21}^2\} = 0$$

by Theorem 12.6 of [**16**]. Also

$$\overline{\varepsilon}_5 \circ \pi_{27}^{20} + \pi_{24}^5 \circ \nu_{24} = \{\overline{\varepsilon}_5 \circ \sigma_{20},\ \overline{\zeta}_5 \circ \nu_{24},\ \nu_5 \circ \mu_8 \circ \sigma_{17} \circ \nu_{24}\} = \{\nu_5 \circ \overline{\zeta}_8\}$$

by Theorem 12.9 of [**16**] and Proposition 2.1(3) of Part II of [**12**].

(3) By (6.1) of [**16**], Proposition 2.6(4) of [**14**] and Corollary 5(1) of [**8**], we have

$$\begin{aligned}\kappa_7 \circ \varepsilon_{21} \in \kappa_7 \circ \{\eta_{21}, 2\iota_{22}, \nu_{22}^2\} &\subset \{\kappa_7 \circ \eta_{21}, 2\iota_{22}, \nu_{22}^2\}\\ &= \{\sigma' \circ \overline{\nu}_{14} + \overline{\varepsilon}_7, 2\iota_{22}, \nu_{22}^2\}\\ &\subset \{\sigma' \circ \overline{\nu}_{14}, 2\iota_{22}, \nu_{22}^2\} + \{\overline{\varepsilon}_7, 2\iota_{22}, \nu_{22}^2\}\end{aligned}$$

and with the result of (1) we see also

$$\begin{aligned}\sigma' \circ \overline{\varepsilon}_{14} + \varepsilon_7 \circ \kappa_{15} &\in \sigma' \circ \{\overline{\nu}_{14}, 2\iota_{22}, \nu_{22}^2\} + \{\overline{\varepsilon}_7, 2\iota_{22}, \nu_{22}^2\}\\ &\subset \{\sigma' \circ \overline{\nu}_{14}, 2\iota_{22}, \nu_{22}^2\} + \{\overline{\varepsilon}_7, 2\iota_{22}, \nu_{22}^2\}.\end{aligned}$$

Examining the indeterminacies of the Toda brackets $\{\sigma' \circ \overline{\nu}_{14}, 2\iota_{22}, \nu_{22}^2\}$ and $\{\overline{\varepsilon}_7, 2\iota_{22}, \nu_{22}^2\}$ we have

$$\begin{aligned}&\sigma' \circ \overline{\nu}_{14} \circ \pi_{29}^{22} + \overline{\varepsilon}_7 \circ \pi_{29}^{22} + \pi_{23}^{7} \circ \nu_{23}^2\\ &\quad = \{\sigma' \circ \overline{\nu}_{14} \circ \sigma_{22},\ \overline{\varepsilon}_7 \circ \sigma_{22},\ \sigma' \circ \mu_{14} \circ \nu_{23}^2,\ \mu_7 \circ \sigma_{16} \circ \nu_{23}^2,\ E\zeta' \circ \nu_{23}^2,\ \eta_7 \circ \overline{\varepsilon}_8 \circ \nu_{23}^2\}\\ &\quad = 0\end{aligned}$$

by Theorem 12.6 and Lemma 10.7 of [**16**], Proposition 17(4) of [**8**], Proposition 2.2(4) of [**14**], and (7.20) and (12.4) of [**16**]. It follows that

$$\kappa_7 \circ \varepsilon_{21} = \sigma' \circ \overline{\varepsilon}_{14} + \varepsilon_7 \circ \kappa_{15}. \qquad \square$$

Proposition 6.4. *$\eta_5 \circ \varepsilon_6^2 = 0$ and $\varepsilon_5 \circ E\theta = 0$ in π_{25}^5.*

Proof. By Lemma 12.10, (7.5), (7.3) and (7.13) of [**16**], we have

$$\eta_5 \circ \varepsilon_6^2 = \eta_5 \circ \varepsilon_6 \circ \overline{\nu}_{14} = \varepsilon_5 \circ \eta_{13} \circ \overline{\nu}_{14} = \varepsilon_5 \circ \nu_{13}^3 = 0$$

which establishes the first statement. By (7.30), Proposition 2.5, Lemma 12.10 and (7.20) of [**16**] and the properties of the Whitehead product, we have

$$\varepsilon_5 \circ E\theta = \varepsilon_5 \circ \Delta(\iota_{27}) = \varepsilon_5 \circ [\iota_{13}, \iota_{13}] = [\varepsilon_5, \varepsilon_5] = [\iota_5, \iota_5] \circ \varepsilon_9^2 = 0$$

and this establishes the second statement. $\square$

Proposition 6.5. *In π_{25}^5, the following relations hold.*

(1) $\{\{\nu_5^2, 2\iota_{11}, \eta_{11}\}, \nu_{13}, \varepsilon_{16}\} = \{\varepsilon_5, \nu_{13}, \varepsilon_{16}\}$ *and the indeterminacy of the Toda*

bracket is $\varepsilon_5 \circ \pi_{25}^{13} + \pi_{17}^5 \circ \varepsilon_{17} = \{\nu_5 \circ \eta_8 \circ \mu_9 \circ \sigma_{18}\}$.

(2) $\{\nu_5^2, \{2\iota_{11}, \eta_{11}, \nu_{12}\}, \varepsilon_{16}\} = \{\nu_5^2 \circ \kappa_{11},\ \nu_5 \circ \eta_8 \circ \mu_9 \circ \sigma_{18}\}$.

(3) $\{\nu_5^2, 2\iota_{11}, \{\eta_{11}, \nu_{12}, \varepsilon_{15}\}\} = \{\nu_5^2 \circ \kappa_{11}\}$.

PROOF.

(1) We see that $\varepsilon_5 \in \{\nu_5^2, 2\iota_{11}, \eta_{11}\}$ by (7.6) of [**16**]. The indeterminacy of this Toda bracket is

$$\nu_5^2 \circ \pi_{13}^{11} + \pi_{12}^5 \circ \eta_{12} = \{\nu_5^2 \circ \eta_{11}^2,\ \sigma''' \circ \eta_{12}\} = 0$$

as follows from Propositions 5.3 and 5.15, (5.9) and (7.4) of [**16**]. Hence we have $\{\nu_5^2, 2\iota_{11}, \eta_{11}\} = \varepsilon_5$. The indeterminacy of $\{\varepsilon_5, \nu_{13}, \varepsilon_{16}\}$ is

$$\begin{aligned}\varepsilon_5 \circ \pi_{25}^{13} + \pi_{17}^5 \circ \varepsilon_{17} &= \{\varepsilon_5 \circ E\theta,\ \nu_5^4 \circ \varepsilon_{17},\ \nu_5 \circ \mu_8 \circ \varepsilon_{17},\ \nu_5 \circ \eta_8 \circ \varepsilon_9^2\}\\ &= \{\nu_5 \circ \eta_8 \circ \mu_9 \circ \sigma_{18}\}\end{aligned}$$

by Theorem 7.6 and (7.18) of [**16**], Proposition 2.13(7) of [**14**] and Proposition 5.8 above.

(2) Since $\{2\iota_{11}, \eta_{11}, \nu_{12}\} \subset \pi_{16}^{11} = 0$ by Proposition 5.9 of [**16**], we have

$$\begin{aligned}\{\nu_5^2, \{2\iota_{11}, \eta_{11}, \nu_{12}\}, \varepsilon_{16}\} &= \nu_5^2 \circ \pi_{25}^{11} + \pi_{17}^5 \circ \varepsilon_{17}\\ &= \{\nu_5^2 \circ \kappa_{11},\ \nu_5^2 \circ \sigma_{11}^2,\ \nu_5 \circ \eta_8 \circ \varepsilon_9^2,\ \nu_5 \circ \mu_8 \circ \varepsilon_{17},\ \nu_5^4 \circ \varepsilon_{17}\}\\ &= \{\nu_5^2 \circ \kappa_{11},\ \nu_5 \circ \eta_8 \circ \mu_9 \circ \sigma_{18}\}\end{aligned}$$

by Theorems 10.3 and 7.6, Lemma 12.3 and (7.18) of [**16**], Proposition 7(2) of [**8**], Proposition 2.13(7) of [**14**] and Proposition 6.4 above.

(3) We have $\{\eta_{11}, \nu_{12}, \varepsilon_{15}\} \ni \nu_{11} \circ \sigma_{14} \circ \nu_{21} = 0$ by Theorem 12 of [**8**] and (7.20) of [**16**]. The indeterminacy of the Toda bracket is

$$\eta_{11} \circ \pi_{24}^{12} + \pi_{16}^{11} \circ \varepsilon_{16} = \{\eta_{11} \circ \theta,\ \eta_{11} \circ E\theta'\} = \{\eta_{11} \circ \theta\}$$

by Theorem 7.6 and Proposition 5.9 of [**16**], and Proposition 2.2(8) of [**14**]. It follows that $\{\eta_{11}, \nu_{12}, \varepsilon_{15}\} = \{\eta_{11} \circ \theta\}$. We see

$$\{\nu_5^2, 2\iota_{11}, \eta_{11} \circ \theta\} \supset \{\nu_5^2, 2\iota_{11}, \eta_{11}\} \circ E\theta = \varepsilon_5 \circ E\theta = 0$$

by Proposition 6.4 above. The indeterminacy of the first Toda bracket is

$$\nu_5^2 \circ \pi_{25}^{11} + \pi_{12}^5 \circ \eta_{12} \circ E\theta = \{\nu_5^2 \circ \sigma_{11}^2,\ \nu_5^2 \circ \kappa_{11},\ \sigma''' \circ \eta_{12} \circ E\theta\} = \{\nu_5^2 \circ \kappa_{11}\}.$$

It follows that $\{\nu_5^2, 2\iota_{11}, \{\eta_{11}, \nu_{12}, \varepsilon_{15}\}\} = \{\nu_5^2 \circ \kappa_{11}\}$. □

PROPOSITION 6.6. *The following relations hold in the* 20*-stem.*
(1) $\{\varepsilon_5, \nu_{13}, \varepsilon_{16}\} = \nu_5^2 \circ \kappa_{11} + \{\nu_5 \circ \eta_8 \circ \mu_9 \circ \sigma_{18}\}$ *in* π_{25}^5.
(2) $\{\varepsilon_6, \nu_{14}, \varepsilon_{17}\} = \nu_6^2 \circ \kappa_{12} = 2\,\overline{\kappa}'$ *in* π_{26}^6.
(3) $\{\varepsilon_n, \nu_{n+8}, \varepsilon_{n+11}\} = \nu_n^2 \circ \kappa_{n+6} = 4\,\overline{\kappa}_n$ *in* π_{n+20}^n *for* $n = 7, 8, 9, 10$ *and* $n \geq 14$.

PROOF.
(1) By page 46 of [**11**] we have

$$\pi_{25}^5 = (\boldsymbol{Z}/2)^3 = \{\nu_5 \circ \overline{\mu}_8\} \oplus \{\nu_5^2 \circ \kappa_{11}\} \oplus \{\nu_5 \circ \eta_8 \circ \mu_9 \circ \sigma_{18}\}.$$

By an application of Corollary 4.17 above (or Proposition 1.5 of [**16**]), we have the following formula

$$0 \in \{\{\nu_5^2, 2\iota_{11}, \eta_{11}\}, \nu_{13}, \varepsilon_{16}\} + \{\nu_5^2, \{2\iota_{11}, \eta_{11}, \nu_{12}\}, \varepsilon_{16}\} + \{\nu_5^2, 2\iota_{11}, \{\eta_{11}, \nu_{12}, \varepsilon_{15}\}\}.$$

This implies by Proposition 6.5 that

$$0 \in \{\varepsilon_5, \nu_{13}, \varepsilon_{16}\} + \{\nu_5^2 \circ \kappa_{11},\ \nu_5 \circ \eta_8 \circ \mu_9 \circ \sigma_{18}\}.$$

Thus there exist integers x and y such that

$$\{\varepsilon_5, \nu_{13}, \varepsilon_{16}\} \ni x\,\nu_5^2 \circ \kappa_{11} + y\,\nu_5 \circ \eta_8 \circ \mu_9 \circ \sigma_{18}.$$

We see by Corollary 5(1) of [**8**] and Proposition 6.2(2) that

$$\{\varepsilon_5, \nu_{13}, \varepsilon_{16}\} \circ \nu_{25} = \varepsilon_5 \circ \{\nu_{13}, \varepsilon_{16}, \nu_{24}\} = \varepsilon_5 \circ \overline{\varepsilon}_{13} = 4\nu_5 \circ \overline{\kappa}_8.$$

On the other hand, we have

$$(x\,\nu_5^2 \circ \kappa_{11} + y\,\nu_5 \circ \eta_8 \circ \mu_9 \circ \sigma_{18}) \circ \nu_{25} = x\,\nu_5^3 \circ \kappa_{14} = 4x\,\nu_5 \circ \overline{\kappa}_8.$$

It follows that $4x\,\nu_5 \circ \overline{\kappa}_8 = 4\,\nu_5 \circ \overline{\kappa}_8 \neq 0$ and hence $x \equiv 1 \pmod 2$. Therefore

$$\{\varepsilon_5, \nu_{13}, \varepsilon_{16}\} = \nu_5^2 \circ \kappa_{11} + \{\nu_5 \circ \eta_8 \circ \mu_9 \circ \sigma_{18}\}.$$

(2) and (3): We see $\nu_6 \circ \eta_9 = 0$ by (5.9) of [**16**]. Hence by Lemmas 15.3 and 15.4 of [**11**] we have

$$\{\varepsilon_6, \nu_{14}, \varepsilon_{17}\} \ni \nu_6^2 \circ \kappa_{12} = 2\,\overline{\kappa}',$$
$$\{\varepsilon_n, \nu_{n+8}, \varepsilon_{n+11}\} \ni \nu_n^2 \circ \kappa_{n+6} = 4\,\overline{\kappa}_n \text{ for } n \geq 7.$$

The indeterminacy of the Toda bracket is $\varepsilon_n \circ \pi_{n+20}^{n+8} + \pi_{n+12}^n \circ \varepsilon_{n+12}$ which is zero for $n = 6, 7, 8, 9, 10$ and $n \geq 14$ by Theorem 7.6 of [**16**]. $\square$

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