# Central and $L^p$-concentration of 1-Lipschitz maps into $R$-trees

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**Abstract.** In this paper, we study the Lévy-Milman concentration phenomenon of 1-Lipschitz maps from mm-spaces to $R$-trees. Our main theorems assert that the concentration to $R$-trees is equivalent to the concentration to the real line.

## 1. Introduction.

This paper is devoted to investigating the Lévy-Milman concentration phenomenon of 1-Lipschitz maps from mm-spaces (metric measure spaces) to $R$-trees. Here, an mm-space is a triple $(X, d_X, \mu_X)$ of a set $X$, a complete separable distance function $d_X$ on $X$, and a finite Borel measure $\mu_X$ on $(X, d_X)$. Let $\{(X_n, d_{X_n}, \mu_{X_n})\}_{n=1}^\infty$ be a sequence of mm-spaces and $\{(Y_n, d_{Y_n})\}_{n=1}^\infty$ a sequence of metric spaces. Given a sequence $\{f_n : X_n \to Y_n\}_{n=1}^\infty$ of 1-Lipschitz maps, we consider the following three properties:

(i) (Concentration property) There exist points $m_{f_n} \in Y_n$, $n \in \mathbf{N}$, such that

$$\mu_{X_n}(\{x_n \in X_n \mid d_{Y_n}(f_n(x_n), m_{f_n}) \geq \varepsilon\}) \to 0 \text{ as } n \to \infty$$

for any $\varepsilon > 0$.

(ii) (Central concentration property) The maps $f_n$, $n \in \mathbf{N}$, concentrate to the center of mass of the push-forward measure $(f_n)_*(\mu_{X_n})$. In other words, the concentration property (i) holds in the case where $m_{f_n}$ is the center of mass.

(iii) ($L^p$-concentration property) For a number $p > 0$, we have

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$$\int\int_{X_n \times X_n} d_{Y_n}(f_n(x_n), f_n(y_n))^p d\mu_{X_n}(x_n) d\mu_{X_n}(y_n) \to 0 \text{ as } n \to \infty.$$

Each target metric space $Y_n$, $n \in \mathbf{N}$, is called a *screen*. Chebyshev's inequality proves that the $L^p$-concentration (iii) implies the concentration property (i) for any $p > 0$. If each screen $Y_n$, $n \in \mathbf{N}$, is an Euclidean space $\mathbf{R}^k$, then the $L^p$-concentration (iii) for $p \geq 1$ yields the central concentration property (ii) (see Corollary 2.19). The central concentration (ii) is stronger than the concentration property (i). There is an example of maps $f_n$, $n \in \mathbf{N}$, with the concentration property (i), but not having the central concentration property (ii) (see Remark 2.17). In some special cases, the concentration (i) implies the central and $L^p$-concentration properties (ii) and (iii) (see [**3**, Subsection 2.4] and [**7**, Section 3(1/2).31]).

Vitali D. Milman first introduced the concentration and the central concentration properties (i) and (ii) for 1-Lipschitz functions (i.e., $Y_n = \mathbf{R}$ for all $n \in \mathbf{N}$) and emphasized their importance in his investigation of asymptotic geometric analysis (see [**11**]). Nowadays those properties are widely studied in large literature and blend with various areas of mathematics (see [**7**], [**9**], [**12**], [**13**], [**14**], [**16**], [**17**] and the references therein for further information). M. Gromov first considered the case of general screens in [**5**], [**6**], and [**7**, Chapter 3(1/2)]. See [**3**], [**4**], and [**10**] for other works of general screens. In [**7**], Gromov considered the concentration and central concentration properties (i) and (ii) for 1-Lipschitz maps by introducing *observable diameter* $\mathrm{ObsDiam}_Y(X; -\kappa)$ and *observable central radius* $\mathrm{ObsCRad}_Y(X; -\kappa)$ for an mm-space $X$, a metric space $Y$, and $\kappa > 0$ (see Section 2 for the precise definitions). The $L^2$-concentration property (iii) first appeared in Gromov's paper [**5**]. Motivated by [**5**], the author introduced in [**3**] the *observable $L^p$-variation* $\mathrm{Obs}\,L^p\text{-}\mathrm{Var}_Y(X)$ to study the property (iii) (see Section 2 for the definition). Note that given a sequence $\{X_n\}_{n=1}^\infty$ of mm-spaces and $\{Y_n\}_{n=1}^\infty$ of metric spaces, $\mathrm{ObsDiam}_{Y_n}(X_n; -\kappa)$ (resp., $\mathrm{ObsCRad}_{Y_n}(X_n; -\kappa)$, $\mathrm{Obs}\,L^p\text{-}\mathrm{Var}_{Y_n}(X_n)$) converges to zero as $n \to \infty$ for any $\kappa > 0$ if and only if any sequence $\{f_n : X_n \to Y_n\}_{n=1}^\infty$ of 1-Lipschitz maps (resp., central, $L^p$-)concentrates.

In this paper, we treat the case of $\mathbf{R}$-tree screens.

THEOREM 1.1. *Let $\{X_n\}_{n=1}^\infty$ be a sequence of mm-spaces. Then, the following* (1.1) *and* (1.2) *are equivalent to each other.*

$$\mathrm{ObsDiam}_{\mathbf{R}}(X_n; -\kappa) \to 0 \text{ as } n \to \infty \text{ for any } \kappa > 0. \qquad (1.1)$$

$$\sup\{\mathrm{ObsDiam}_T(X_n; -\kappa) \mid T \text{ is an } \mathbf{R}\text{-tree}\} \to 0 \text{ as } n \to \infty \text{ for any } \kappa > 0. (1.2)$$

Theorem 1.1 is a complete solution to Gromov's exercise in [**7**, Section 3(1/2).32]. In [**3**, Section 5], the author proved it only for simplicial tree screens. The implication $(1.2) \Rightarrow (1.1)$ is obvious. For the proof of the converse, we define the notion of a *median* for a finite Borel measure on an $\boldsymbol{R}$-tree in Section 3 and prove that any 1-Lipschitz maps $f_n$ from $X_n$ into $\boldsymbol{R}$-trees concentrate to medians of the push-forward measures $(f_n)_*(\mu_{X_n})$.

To study the central and $L^p$-concentration (ii) and (iii) into $\boldsymbol{R}$-trees, we estimate the distance between the center of mass and the median of a finite Borel measure on an $\boldsymbol{R}$-tree from above in Section 5. For this estimate, we partially extend K-T. Sturm's characterization of the center of mass on a simplicial tree to the case of an $\boldsymbol{R}$-tree (see Proposition 2.12 and Section 4). From the estimate, we bound $\mathrm{ObsCRad}_T(X; -\kappa)$ (resp., $\mathrm{Obs}\, L^p\text{-Var}_T(X)$) from above in terms of $\mathrm{ObsCRad}_{\boldsymbol{R}}(X; -\kappa)$ (resp., $\mathrm{Obs}\, L^p\text{-Var}_{\boldsymbol{R}}(X)$) (see Propositions 5.6 and 5.8). As a result, we obtain

THEOREM 1.2. *Let* $\{X_n\}_{n=1}^\infty$ *be a sequence of mm-spaces. Then, the following* (1.3) *and* (1.4) *are equivalent to each other.*

$$\mathrm{ObsCRad}_{\boldsymbol{R}}(X_n; -\kappa) \to 0 \text{ as } n \to \infty \text{ for any } \kappa > 0. \tag{1.3}$$

$$\sup\{\mathrm{ObsCRad}_T(X_n; -\kappa) \mid T \text{ is an } \boldsymbol{R}\text{-tree}\} \to 0 \text{ as } n \to \infty \text{ for any } \kappa > 0. \tag{1.4}$$

THEOREM 1.3. *Let* $\{X_n\}_{n=1}^\infty$ *be a sequence of mm-spaces and* $p \geq 1$. *Then, the following* (1.5) *and* (1.6) *are equivalent to each other.*

$$\mathrm{Obs}\, L^p\text{-Var}_{\boldsymbol{R}}(X_n) \to 0 \text{ as } n \to \infty. \tag{1.5}$$

$$\sup\{\mathrm{Obs}\, L^p\text{-Var}_T(X_n) \mid T \text{ is an } \boldsymbol{R}\text{-tree}\} \to 0 \text{ as } n \to \infty. \tag{1.6}$$

The condition (1.3) is stronger than (1.1) (see Lemma 2.16 and Remark 2.17), and (1.5) implies (1.3) (see Corollary 2.19). It seems that the conditions (1.3) and (1.5) are not equivalent, but we have no counterexample.

In our previous work, the author investigated the above properties (i), (ii), and (iii) for 1-Lipschitz maps into Hadamard manifolds (see [**3**, Theorems 1.3, 1.4, and Lemma 4.4]). The $L^2$-concentration property (iii) in that case is also studied by Gromov (see [**5**, Section 13]). Our theorems are thought of as a 1-dimensional analogue to these works.

## 2.    Preliminaries.

### 2.1.    Basics of the concentration and the $L^p$-concentration.

#### 2.1.1   Observable diameter and separation distance.

Let $Y$ be a metric space and $\nu$ a Borel measure on $Y$ such that $m := \nu(Y) < +\infty$. We define for any $\kappa > 0$

$$\mathrm{diam}(\nu, m - \kappa) := \inf\{\mathrm{diam}\, Y_0 \mid Y_0 \subseteq Y \text{ is a Borel subset such that } \nu(Y_0) \geq m - \kappa\}$$

and call it the *partial diameter* of $\nu$.

DEFINITION 2.1 (Observable diameter).    Let $(X, d_X, \mu_X)$ be an mm-space with $m := \mu_X(X) < +\infty$ and $Y$ a metric space. For any $\kappa > 0$ we define the *observable diameter* of $X$ by

$$\mathrm{ObsDiam}_Y(X; -\kappa) := \sup\{\mathrm{diam}(f_*(\mu_X), m - \kappa) \mid f : X \to Y \text{ is a 1-Lipschitz map}\}.$$

The target metric space $Y$ is called the *screen*.

The idea of the observable diameter comes from quantum and statistical mechanics, that is, we think of $\mu_X$ as a state on a configuration space $X$ and $f$ is interpreted as an observable.

Let $(X, d_X, \mu_X)$ be an mm-space. For any $\kappa_1, \kappa_2 \geq 0$, we define the *separation distance* $\mathrm{Sep}(X; \kappa_1, \kappa_2) = \mathrm{Sep}(\mu_X; \kappa_1, \kappa_2)$ of $X$ as the supremum of the distance $d_X(A, B)$, where $A$ and $B$ are Borel subsets of $X$ satisfying that $\mu_X(A) \geq \kappa_1$ and $\mu_X(B) \geq \kappa_2$.

The proof of the following lemmas are easy and we omit it.

LEMMA 2.2 (cf. [**7**, Section 3(1/2).33]).    *Let $(X, d_X, \mu_X)$ and $(Y, d_Y, \mu_Y)$ be two mm-spaces. Assume that a 1-Lipschitz map $f : X \to \boldsymbol{R}$ satisfies $f_*(\mu_X) = \mu_Y$. Then we have*

$$\mathrm{Sep}(Y; \kappa_1, \kappa_2) \leq \mathrm{Sep}(X; \kappa_1, \kappa_2).$$

LEMMA 2.3.    *For any $\kappa > m/2$, we have $\mathrm{Sep}(X; \kappa, \kappa) = 0$.*

The relationships between the observable diameter and the separation distance are the following:

PROPOSITION 2.4 (cf. [**7**, Section 3(1/2).33]).    *Let $(X, d_X, \mu_X)$ be an mm-space and $0 < \kappa' < \kappa$. Then we have*

$$\mathrm{Sep}(X; \kappa, \kappa) \leq \mathrm{ObsDiam}_{\boldsymbol{R}}(X; -\kappa').$$

PROPOSITION 2.5 (cf. [**7**, Section 3(1/2).33]).  *For any $\kappa > 0$, we have*

$$\mathrm{ObsDiam}_{\mathbf{R}}(X; -2\kappa) \leq \mathrm{Sep}(X; \kappa, \kappa).$$

See [**4**, Subsection 2.2] for details of the proofs of the above propositions.

COROLLARY 2.6 (cf. [**7**, Section 3(1/2).33]).  *A sequence $\{X_n\}_{n=1}^{\infty}$ of mm-spaces satisfies that*

$$\mathrm{ObsDiam}_{\mathbf{R}}(X_n; -\kappa) \to 0 \text{ as } n \to \infty$$

*for any $\kappa > 0$ if and only if $\mathrm{Sep}(X_n; \kappa, \kappa) \to 0$ as $n \to \infty$ for any $\kappa > 0$.*

### 2.1.2 Observable $L^p$-variation.

Let $(X, d_X, \mu_X)$ be an mm-space and $(Y, d_Y)$ a metric space. Given a Borel measure $\nu$ on $Y$ and $p \in (0, +\infty)$, we put

$$V_p(\nu) := \left( \int \int_{Y \times Y} d_Y(x, y)^p d\nu(x) d\nu(y) \right)^{1/p}.$$

For a Borel measurable map $f : X \to Y$, we also put $V_p(f) := V_p(f_*(\mu_X))$.

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of mm-spaces and $\{Y_n\}_{n=1}^{\infty}$ a sequence of metric spaces. For any $p \in (0, +\infty)$, we say that a sequence $\{f_n : X_n \to Y_n\}_{n=1}^{\infty}$ of Borel measurable maps $L^p$-*concentrates* if $V_p(f_n) \to 0$ as $n \to \infty$.

Given an mm-space $X$ and a metric space $Y$ we define

$$\mathrm{Obs}\, L^p\text{-}\mathrm{Var}_Y(X) := \sup\{V_p(f) \mid f : X \to Y \text{ is a 1-Lipschitz map}\},$$

and call it the *observable $L^p$-variation* of $X$.

LEMMA 2.7.  *For any closed subset $A \subset X$, we have*

$$\mathrm{Obs}\, L^p\text{-}\mathrm{Var}_{\mathbf{R}}(A) \leq \mathrm{Obs}\, L^p\text{-}\mathrm{Var}_{\mathbf{R}}(X).$$

PROOF.  Let $f : A \to \mathbf{R}$ be an arbitrary 1-Lipschitz function. By [**1**, Theorem 3.1.2], there exists a 1-Lipschitz extension of $f$, say $\widetilde{f} : X \to \mathbf{R}$. Hence, we get

$$V_p(f) \leq V_p(\widetilde{f}) \leq \mathrm{Obs}\, L^p\text{-}\mathrm{Var}_{\mathbf{R}}(X).$$

This completes the proof.  □

See [**3**, Subsection 2.4] for the relationships between the observable diameter

and the observable $L^p$-variation.

### 2.2. Basics of $R$-trees.

Before reviewing the definition of $R$-trees, we recall some standard terminologies in metric geometry. Let $(X, d_X)$ be a metric space. A rectifiable curve $\eta : [0, 1] \to X$ is called a *geodesic* if its arclength coincides with the distance $d_X(\eta(0), \eta(1))$ and it has a constant speed, i.e., parametrized proportionally to the arc length. We say that $(X, d_X)$ is a *geodesic space* if any two points in $X$ are joined by a geodesic between them. Let $X$ be a geodesic space. A *geodesic triangle* in $X$ is the union of the image of three geodesics joining a triple of points in $X$ pairwise. A subset $A \subseteq X$ is called *convex* if every geodesic joining two points in $A$ is contained in $A$.

A complete metric space $(T, d_T)$ is called an $R$-*tree* if it satisfies the following properties:

(1) For all $z, w \in T$ there exists a unique unit speed geodesic $\phi_{z,w}$ from $z$ to $w$.
(2) The image of every simple path in $T$ is the image of a geodesic.

Denote by $[z, w]_T$ the image of the geodesic $\phi_{z,w}$. We also put $(z, w]_T := [z, w]_T \setminus \{z\}$ and $(z, w)_T := [z, w]_T \setminus \{z, w\}$. A complete geodesic space $T$ is an $R$-tree if and only if it is 0-hyperbolic, that is to say, every edge in any geodesic triangle in $T$ is included in the union of the other two edges. See [**2**] for another characterizations of $R$-trees. Given $z \in T$, we indicate by $\mathscr{C}_T(z)$ the set of all connected components of $T \setminus \{z\}$. We also denote by $\mathscr{C}'_T(z)$ the set of all $\{z\} \cup T'$ for $T' \in \mathscr{C}_T(z)$. Although the following lemma is somewhat standard, we prove it for completeness.

LEMMA 2.8.    *Each $T' \in \mathscr{C}_T(z)$ is convex.*

PROOF.    By the property (2) of $R$-trees, it is sufficient to prove that $T'$ is arcwise connected. Taking a point $v \in T'$, we put

$$A := \{w \in T' \mid v \text{ and } w \text{ are connected by a path in } T'\}.$$

It is easy to see that the set $A$ is closed in $T'$. Since every metric ball in $T$ is arcwise connected, the set $A$ is also open. Since $T'$ is connected, we get $T' = A$. This completes the proof.    □

A subset in an $R$-tree is called a *subtree* if it is a closed convex subset. Note that a subtree itself is an $R$-tree.

PROPOSITION 2.9.    *Every connected subset in an $R$-tree is convex.*

PROOF. Let $T$ be an $\boldsymbol{R}$-tree. Suppose that there exists a connected subset $T' \subseteq T$ which is not convex. Then, there are points $z, w \in T'$ and $\widetilde{z} \in (z, w)_T$ such that $\widetilde{z} \notin T'$. Since $T' = \bigcup \{T' \cap C \mid C \in \mathscr{C}_T(\widetilde{z})\}$ and each $C \in \mathscr{C}_T(\widetilde{z})$ is open, by the connectivity of $T'$, there is $C_0 \in \mathscr{C}_T(\widetilde{z})$ such that $T' \subseteq C_0$. Since $C_0$ is convex by Lemma 2.8, we get $\widetilde{z} \in [z, w]_T \subseteq C_0$. This is a contradiction since $\widetilde{z} \notin C_0$. This completes the proof. $\qquad \square$

### 2.3. Center of mass of a measure on a CAT(0)-space and observable central radius.

#### 2.3.1 Basics of the center of mass of a measure on a CAT(0)-space.

In this subsection, we review Sturm's works about measures on a CAT(0)-space. We refer to [**8**] and [**15**] for details. A geodesic metric space $X$ is called a CAT(0)-space if we have

$$d_X(x, \gamma(1/2))^2 \leq \frac{1}{2}\, d_X(x, y)^2 + \frac{1}{2}\, d_X(x, z)^2 - \frac{1}{4}\, d_X(y, z)^2$$

for any $x, y, z \in X$ and any minimizing geodesic $\gamma : [0, 1] \to X$ from $y$ to $z$. For example, Hadamard manifolds, Hilbert spaces, and $\boldsymbol{R}$-trees are all CAT(0)-spaces.

Let $(X, d_X)$ be a metric space. We denote by $\mathscr{B}(X)$ the set of all finite Borel measures $\nu$ on $X$ with separable supports. We indicate by $\mathscr{B}^1(X)$ the set of all Borel measures $\nu \in \mathscr{B}(X)$ such that $\int_X d_X(x, y) d\nu(y) < +\infty$ for some (hence all) $x \in X$. We also indicate by $\mathscr{P}^1(X)$ the set of all probability measures in $\mathscr{B}^1(X)$. For any $\nu \in \mathscr{B}^1(X)$ and $z \in X$, we consider the function $h_{z,\nu} : X \to \boldsymbol{R}$ defined by

$$h_{z,\nu}(x) := \int_X \{d_X(x, y)^2 - d_X(z, y)^2\} d\nu(y).$$

Note that

$$\int_X \mid d_X(x, y)^2 - d_X(z, y)^2 \mid d\nu(y) \leq d_X(x, z) \int_X \{d_X(x, y) + d_X(z, y)\} d\nu(y) < +\infty.$$

A point $z_0 \in X$ is called the *center of mass* of the measure $\nu \in \mathscr{B}^1(X)$ if for any $z \in X$, $z_0$ is a unique minimizing point of the function $h_{z,\nu}$. We denote the point $z_0$ by $c(\nu)$. A metric space $X$ is said to be *centric* if every $\nu \in \mathscr{B}^1(X)$ has the center of mass.

PROPOSITION 2.10 (cf. [**15**, Proposition 4.3]). *Any complete* CAT(0)-*space*

*is centric.*

A simple variational argument yields the following lemma.

LEMMA 2.11 (cf. [**15**, Proposition 5.4]).   *Let $H$ be a Hilbert space. Then for each $\nu \in \mathscr{B}^1(H)$ with $m = \nu(X)$, we have*

$$c(\nu) = \frac{1}{m} \int_H y d\nu(y).$$

Let $(T, d_T)$ be an $\boldsymbol{R}$-tree and $\nu \in \mathscr{B}^1(T)$. For $z \in T$ and $T' \in \mathscr{C}'_T(z)$, we put

$$c_{z,T'}(\nu) := \int_{T'} d_T(z, w) d\nu(w) - \int_{T \setminus T'} d_T(z, w) d\nu(w).$$

Let us consider a (possibly infinite) simplicial tree $T_s$. Here, the length of each edge of $T_s$ is not necessarily equal to 1. We assume that every vertex of $T_s$ is an isolated point in the vertex set of $T_s$.

PROPOSITION 2.12 (cf. [**15**, Proposition 5.9]).   *Let $\nu \in \mathscr{B}^1(T_s)$ and $z \in T_s$. Then, $z = c(\nu)$ if and only if $c_{z,T'}(\nu) \leq 0$ for any $T' \in \mathscr{C}'_{T_s}(z)$.*

PROPOSITION 2.13 (cf. [**15**, Proposition 6.1]).   *Let $X$ be a complete CAT(0)-space and $\nu \in \mathscr{B}^1(X)$. Assume that the support of $\nu$ is contained in a closed convex subset $K$ of $X$. Then, we have $c(\nu) \in K$.*

Let $X$ be a metric space. For $\mu, \nu \in \mathscr{P}^1(X)$, we define the $L^1$-*Wasserstein distance* $d_1^W(\mu, \nu)$ between $\mu$ and $\nu$ as the infimum of $\int_{X \times X} d_X(x, y) d\pi(x, y)$, where $\pi \in \mathscr{P}^1(X \times X)$ runs over all *couplings* of $\mu$ and $\nu$, that is, measures $\pi$ with the property that $\pi(A \times X) = \mu(A)$ and $\pi(X \times A) = \nu(A)$ for any Borel subset $A \subseteq X$.

LEMMA 2.14 (cf. [**18**, Theorem 7.12]).   *A sequence $\{\mu_n\}_{n=1}^\infty \subseteq \mathscr{P}^1(X)$ converges to $\mu \in \mathscr{P}^1(X)$ with respect to the distance function $d_1^W$ if and only if the sequence $\{\mu_n\}_{n=1}^\infty$ converges weakly to the measure $\mu$ and*

$$\lim_{n \to \infty} \int_X d_X(x, y) d\mu_n(y) = \int_X d_X(x, y) d\mu(y)$$

*for some (and then any) $x \in X$.*

THEOREM 2.15 (cf. [**15**, Theorem 6.3]).   *Let $X$ be a CAT(0)-space. Given*

$\mu, \nu \in \mathscr{P}^1(X)$, *we have* $d_X(c(\mu), c(\nu)) \leq d_1^W(\mu, \nu)$.

### 2.3.2 Observable central radius.

Let $Y$ be a metric space and assume that $\nu \in \mathscr{B}^1(Y)$ has the center of mass. We denote by $B_Y(y, r)$ the closed ball in $Y$ centered at $y \in Y$ and with radius $r > 0$. For any $\kappa > 0$, putting $m := \nu(Y)$, we define the *central radius* $\mathrm{CRad}(\nu, m - \kappa)$ of $\nu$ as the infimum of $\rho > 0$ such that $\nu(B_Y(c(\nu), \rho)) \geq m - \kappa$.

Let $(X, d_X, \mu_X)$ be an mm-space with $\mu_X \in \mathscr{B}^1(X)$ and $Y$ a centric metric space. For any $\kappa > 0$, we define

$$\mathrm{ObsCRad}_Y(X; -\kappa) := \sup\{\mathrm{CRad}(f_*(\mu_X), m - \kappa) \mid f : X \to Y \text{ is a 1-Lipschitz map}\},$$

and call it the *observable central radius* of $X$.

LEMMA 2.16 (cf. [**7**, Section 3(1/2).31]). *For any $\kappa > 0$, we have*

$$\mathrm{diam}(\nu, m - \kappa) \leq 2\,\mathrm{CRad}(\nu, m - \kappa).$$

*In particular, we get*

$$\mathrm{ObsDiam}_Y(X; -\kappa) \leq 2\,\mathrm{ObsCRad}_Y(X; -\kappa).$$

REMARK 2.17. From the above lemma, we see that the central concentration implies the concentration. The converse is not true in general. For example, let us consider the mm-spaces $X_n := \{x_n, y_n\}$ with the distance function $d_{X_n}$ given by $d_{X_n}(x_n, y_n) := n$ and with the Borel probability measure $\mu_{X_n}$ given by $\mu_{X_n}(\{x_n\}) := 1 - 1/n$ and $\mu_{X_n}(\{y_n\}) := 1/n$. Then, 1-Lipschitz maps $f_n : X_n \to \mathbf{R}$ defined by $f_n(x) := d_{X_n}(x, x_n)$ satisfy that $c((f_n)_*(\mu_{X_n})) = 1$ for all $n \in \mathbf{N}$, whereas the maps $f_n$ concentrate to 0 and $\mathrm{ObsDiam}_{\mathbf{R}}(X_n; -\kappa) \to 0$ as $n \to \infty$ for any $\kappa > 0$.

LEMMA 2.18. *Let $\nu \in \mathscr{B}^1(\mathbf{R}^n)$ with $m := \nu(\mathbf{R}^n)$. Then, for any $p \geq 1$ and $\kappa > 0$, we have*

$$\mathrm{CRad}(\nu, m - \kappa) \leq \frac{V_p(\nu)}{(m\kappa)^{1/p}}. \tag{2.1}$$

*In the case of $p = 2$, we also have the better estimate*

$$\mathrm{CRad}(\nu, m - \kappa) \leq \frac{V_2(\nu)}{\sqrt{2m\kappa}}. \tag{2.2}$$

PROOF.    We shall prove that $\nu(\boldsymbol{R}^n \setminus B_{\boldsymbol{R}^n}(c(\nu), \rho_0)) \leq \kappa$  for  $\rho_0 := V_p(\nu)/$ $(m\kappa)^{1/p}$. Suppose that $\nu(\boldsymbol{R}^n \setminus B_{\boldsymbol{R}^n}(c(\nu), \rho_0)) > \kappa$. From Lemma 2.11, we get

$$\int_{\boldsymbol{R}^n} |c(\nu) - x|^p d\nu(x) \leq \frac{V_p(\nu)^p}{m}\,.$$

Hence, by Chebyshev's inequality, we see that

$$\frac{V_p(\nu)^p}{m} = \rho_0^p \kappa < \int_{\boldsymbol{R}^n} |c(\nu) - x|^p d\nu(x) \leq \frac{V_p(\nu)^p}{m}\,,$$

which is a contradiction. Therefore, we obtain $\nu(B_{\boldsymbol{R}^n}(c(\nu), \rho_0)) \geq m - \kappa$ and so (2.1).

Since

$$\int_{\boldsymbol{R}^n} |c(\nu) - x|^2 d\nu(x) = \frac{V_2(\nu)^2}{2m}\,,$$

the same argument yields (2.2). This completes the proof.                    $\square$

COROLLARY 2.19.    *Let $X$ be an mm-space with $\mu_X \in \mathscr{B}^1(X)$. Then, for any $p \geq 1$, we have*

$$\mathrm{ObsCRad}_{\boldsymbol{R}^n}(X; -\kappa) \leq \frac{1}{(m\kappa)^{1/p}} \mathrm{Obs}\, L^p\text{-}\mathrm{Var}_{\boldsymbol{R}^n}(X). \qquad (2.3)$$

*In the case of $p = 2$, we also have the better estimate*

$$\mathrm{ObsCRad}_{\boldsymbol{R}^n}(X; -\kappa) \leq \frac{1}{\sqrt{2m\kappa}} \mathrm{Obs}\, L^2\text{-}\mathrm{Var}_{\boldsymbol{R}^n}(X). \qquad (2.4)$$

COROLLARY 2.20.    *Let $X$ be an mm-space. Then, for any $p \geq 1$ and $\kappa > 0$, we have*

$$\mathrm{Sep}(X; \kappa, \kappa) \leq \frac{2}{(m\kappa)^{1/p}} \mathrm{Obs}\, L^p\text{-}\mathrm{Var}_{\boldsymbol{R}}(X). \qquad (2.5)$$

*In the case of $p = 2$, we also have*

$$\mathrm{Sep}(X; \kappa, \kappa) \leq \sqrt{\frac{2}{m\kappa}} \mathrm{Obs}\, L^2\text{-}\mathrm{Var}_{\boldsymbol{R}}(X). \qquad (2.6)$$

PROOF.    Assume first that there is a 1-Lipschitz function $f : X \to \boldsymbol{R}$ such that $f_*(\mu_X) \notin \mathscr{B}^1(\boldsymbol{R})$. From Hölder's inequality, we have $\int_{\boldsymbol{R}} |x - y|^p df_*(\mu_X)(y) = +\infty$ for any $x \in X$. This implies $V_p(f) = +\infty$ and so $\operatorname{Obs} L^p\text{-Var}_{\boldsymbol{R}}(X) = +\infty$.

We consider the other case that $f_*(\mu_X) \in \mathscr{B}^1(\boldsymbol{R})$ for any 1-Lipschitz function $f : X \to \boldsymbol{R}$. Combining Proposition 2.4 with Lemma 2.16 and (2.3), we have

$$\operatorname{Sep}(X; \kappa, \kappa) \le \frac{2}{(m\kappa')^{1/p}} \operatorname{Obs} L^p\text{-Var}_{\boldsymbol{R}}(X)$$

for any $\kappa > \kappa' > 0$. Letting $\kappa' \to \kappa$, we have (2.5). Replacing (2.3) with (2.4) in the above argument, we also obtain (2.6). □

## 3.   Existence of a median in an $\boldsymbol{R}$-tree.

Let $T$ be an $\boldsymbol{R}$-tree and $\nu$ a finite Borel measure on $T$ with $m := \nu(T) < +\infty$. A median of $\nu$ is a point $z \in T$ such that there exist two subtrees $T', T'' \subseteq T$ such that $T = T' \cup T''$, $T' \cap T'' = \{z\}$, $\nu(T') \ge m/3$, and $\nu(T'') \ge m/3$.

EXAMPLE 3.1.    For $i = 1, 2, 3$, let $T_i := \{(i, r) \mid r \in [0, +\infty)\}$ be three copies of $[0, +\infty)$ equipped with the usual Euclidean distance function. The *tripod* $T$ is the metric space obtained by gluing together all these spaces $T_i$, $i = 1, 2, 3$, at their origins with the intrinsic distance function. We consider the Borel probability measure $\nu$ defined by $\nu(\{(i, 1)\}) := 1/3$ for all $i = 1, 2, 3$. We easily see that the origin of $T$ is a median of the measure $\nu$. We note that there is no point in $T$ separating $T$ into two subtrees with measures greater than $1/3$.

The existence of a median of a finite Borel measure on a simplicial tree is proved in [**3**, Proposition 5.2]. The purpose of this section is to prove the existence of a median of a finite Borel measure on an $\boldsymbol{R}$-tree, which is needed for the proofs of our main theorems. Although the proof of the existence is similar to the proof for the case of a simplicial tree, we give it for completeness:

PROPOSITION 3.2.    *Every finite Borel measure on an $\boldsymbol{R}$-tree has a median.*

PROOF.    Let $\nu$ be a finite Borel measure on an $\boldsymbol{R}$-tree with $m := \nu(T)$. Assume that a point $z \in T$ satisfies that $\nu(T') < m/3$ for any $T' \in C_T'(z)$, then it is easy to check that $z$ is a median of $\nu$. So, we assume that for any $z \in T$ there exists $T(z) \in \mathscr{C}_T'(z)$ such that $\nu(T(z)) \ge m/3$. If for some $z \in T$, there exists $T' \in \mathscr{C}_T'(z) \setminus \{T(z)\}$ such that $\nu(T') \ge m/3$, then this $z$ is a median of $\nu$. Thereby, we also assume that $\nu(T') < m/3$ for any $z \in T$ and $T' \in \mathscr{C}_T'(z) \setminus \{T(z)\}$.

Fixing a point $z_0 \in T$, we assume that there exists $z \in T(z_0) \setminus \{z_0\}$ such that

$z_0 \in T(z)$. Put

$$t_0 := \inf\{t \in (0, d_T(z_0, z)] \mid z_0 \in T(\phi_{z_0,z}(t))\}.$$

CLAIM 3.3.   $\phi_{z_0,z}(t_0)$ *is a median of* $\nu$.

PROOF.   Assume first that $t_0 = 0$. Then, taking a monotone decreasing sequence $\{t_n\}_{n=1}^\infty \subseteq (0, d_T(z_0, z)]$ such that $t_n \to 0$ as $n \to \infty$ and $z_0 \in T(\phi_{z_0,z}(t_n))$ for any $n \in \mathbf{N}$, we shall show that $\bigcap_{n=1}^\infty T(\phi_{z_0,z}(t_n)) \subseteq (T \setminus T(z_0)) \cup \{z_0\}$. If it is, we conclude that the point $z_0 = \phi_{z_0,z}(0)$ is a median of $\nu$ as follows: From the uniqueness of $T(\phi_{z_0,z}(t_n))$, we have $T(\phi_{z_0,z}(t_{n+1})) \subseteq T(\phi_{z_0,z}(t_n))$ for each $n \in \mathbf{N}$. Thus, we get $\nu(\bigcap_{n=1}^\infty T(\phi_{z_0,z}(t_n))) = \lim_{n\to\infty} \nu(T(\phi_{z_0,z}(t_n))) \geq m/3$.

Suppose that there exists $w \in (T(z_0) \setminus \{z_0\}) \cap \bigcap_{n=1}^\infty T(\phi_{z_0,z}(t_n))$. Note that $(z_0, z]_T \cap (z_0, w]_T \neq \emptyset$. Actually, if $(z_0, z]_T \cap (z_0, w]_T = \emptyset$, then it follows from the property (2) of $\mathbf{R}$-trees that $[z, w]_T = [z_0, z]_T \cup [z_0, w]_T$. Especially, we have $z_0 \in [z, w]_T$. Since $T(z_0) \setminus \{z_0\}$ is convex by virtue of Lemma 2.8, $[z, w]_T$ does not contain the point $z_0$. This is a contradiction. Thus, there exists $t \in (0, d_T(z_0, z)]$ such that $\phi_{z_0,z}(t) \in (z_0, z]_T \cap (z_0, w]_T$. We pick $n_0 \in \mathbf{N}$ with $t_{n_0} < t$. Since $w \in (T(z_0) \setminus \{z_0\}) \cap \bigcap_{n=1}^\infty T(\phi_{z_0,z}(t_n)) \subseteq T(\phi_{z_0,z}(t_{n_0})) \setminus \{z_0\}$, we get $\phi_{z_0,z}(t) \in (z_0, w]_T \subseteq T(\phi_{z_0,z}(t_{n_0})) \setminus \{z_0\}$. Thereby, we get $\phi_{z_0,z}(t) \in T(\phi_{z_0,z}(t_{n_0})) \setminus \{\phi_{z_0,z}(t_{n_0})\}$. Therefore, since $z_0 \in T(\phi_{z_0,z}(t_{n_0})) \setminus \{\phi_{z_0,z}(t_{n_0})\}$ and $T(\phi_{z_0,z}(t_{n_0})) \setminus \{\phi_{z_0,z}(t_{n_0})\}$ is convex, we obtain

$$\phi_{z_0,z}(t_{n_0}) \in [z_0, \phi_{z_0,z}(t)]_T \subseteq T(\phi_{z_0,z}(t_{n_0})) \setminus \{\phi_{z_0,z}(t_{n_0})\}.$$

This is a contradiction. Therefore, we have $\bigcap_{n=1}^\infty T(\phi_{z_0,z}(t_n)) \subseteq (T \setminus T(z_0)) \cup \{z_0\}$.

We consider the other case that $t_0 > 0$. Take a monotone increasing sequence $\{t_n\}_{n=1}^\infty \subseteq (0, +\infty)$ such that $t_n \to t_0$ as $n \to \infty$ and $z_0 \notin T(\phi_{z_0,z}(t_n))$ for each $n \in \mathbf{N}$. Then, the same proof in the case of $t_0 = 0$ implies that $\nu(\bigcap_{n=1}^\infty T(\phi_{z_0,z}(t_n))) \geq m/3$ and $\bigcap_{n=1}^\infty T(\phi_{z_0,z}(t_n)) \subseteq (T \setminus T(\phi_{z_0,z}(t_0))) \cup \{\phi_{z_0,z}(t_0)\}$. Therefore, $\phi_{z_0,z}(t_0)$ is a median of $\nu$. This completes the proof of the claim.   $\square$

We next assume that $z_0 \notin T(z)$ for any $z \in T(z_0)$. We denote by $\Gamma$ the set of all unit speed geodesics $\gamma : [0, L(\gamma)] \to T(z_0)$ such that $\gamma(0) = z_0$ and $\gamma([t, L(\gamma)]) \subseteq T(\gamma(t))$ for any $t \in [0, L(\gamma)]$. Because of the assumption, we easily see

CLAIM 3.4.   *For any* $z \in T(z_0)$, *we have* $\phi_{z_0,z} \in \Gamma$.

CLAIM 3.5.   *For any* $\gamma, \gamma' \in \Gamma$ *with* $L(\gamma) \leq L(\gamma')$, *we have*

$$[\gamma(0), \gamma(L(\gamma))]_T \subseteq [\gamma'(0), \gamma'(L(\gamma'))]_T.$$

PROOF.   Suppose that

$$t_0 := \sup\{t \in [0, L(\gamma)] \mid [\gamma(0), \gamma(t)]_T \subseteq [\gamma'(0), \gamma'(L(\gamma'))]_T\} < L(\gamma).$$

Then, we have $\gamma(t) \notin [\gamma'(0), \gamma'(L(\gamma'))]_T$ for any $t > t_0$. Actually, if $\gamma(t) \in [\gamma'(0), \gamma'(L(\gamma'))]_T$, then we have $\gamma(t) = \gamma'(t)$. Thus, $[\gamma(t_0), \gamma(t)]_T = [\gamma'(t_0), \gamma'(t)]_T$ by the property (2) of $\boldsymbol{R}$-trees. Thereby, we get $[\gamma(0), \gamma(t)]_T \subseteq [\gamma'(0), \gamma'(L(\gamma'))]_T$. Since $t > t_0$, this contradicts the definition of $t_0$. Therefore, from the property (2) of $\boldsymbol{R}$-trees, we have

$$[\gamma(L(\gamma)), \gamma'(L(\gamma'))]_T = [\gamma(t_0), \gamma(L(\gamma))]_T \cup [\gamma'(t_0), \gamma'(L(\gamma'))]_T. \qquad (3.1)$$

Since $\gamma, \gamma' \in \Gamma$, we have $\gamma(L(\gamma)), \gamma'(L(\gamma')) \in T(\gamma(t_0)) \setminus \{\gamma(t_0)\}$. So, from the convexity of $T(\gamma(t_0)) \setminus \{\gamma(t_0)\}$, we get $[\gamma(L(\gamma)), \gamma'(L(\gamma'))]_T \subseteq T(\gamma(t_0)) \setminus \{\gamma(t_0)\}$. This is a contradiction, because $\gamma(t_0) \in [\gamma(L(\gamma)), \gamma'(L(\gamma'))]_T$ by (3.1). This completes the proof of the claim. $\qquad \square$

Putting $\alpha := \sup\{L(\gamma) \mid \gamma \in \Gamma\}$, we shall show that $\alpha < +\infty$. If $\alpha < +\infty$, we finish the proof of the proposition as follows: By the completeness of $\boldsymbol{R}$-trees and Claim 3.5, there exists unique $\gamma \in \Gamma$ with $L(\gamma) = \alpha$. We also note that $\alpha > 0$ by Claim 3.4. Thus, there exists a monotone increasing sequence $\{t_n\}_{n=1}^\infty$ of positive numbers such that $t_n \to \alpha$ as $n \to \infty$. We easily see that $T(\gamma(t_{n+1})) \subseteq T(\gamma(t_n))$ for any $n \in \boldsymbol{N}$ and $\bigcap_{n=1}^\infty T(\gamma(t_n)) = \{\gamma(L(\gamma))\}$. Since $\nu(T(\gamma(t_n))) \geq m/3$, the point $\gamma(L(\gamma))$ is a median of $\nu$.

Suppose that $\alpha = +\infty$. Then, taking a sequence $\{\gamma_n\}_{n=1}^\infty \subseteq \Gamma$ such that $L(\gamma_n) < L(\gamma_{n+1})$ for any $n \in \boldsymbol{N}$ and $L(\gamma_n) \to +\infty$ as $n \to \infty$, we obtain $\bigcap_{n=1}^\infty T(\gamma_n(L(\gamma_n))) = \emptyset$. Since $T(\gamma_{n+1}(L(\gamma_{n+1}))) \subseteq T(\gamma_n(L(\gamma_n)))$ for any $n \in \boldsymbol{N}$, we have

$$0 = \nu\left(\bigcap_{n=1}^\infty T(\gamma_n(L(\gamma_n)))\right) = \lim_{n \to \infty} \nu(T(\gamma_n(L(\gamma_n)))) \geq \frac{m}{3},$$

which is a contradiction. This completes the proof of the proposition. $\qquad \square$

## 4.   The necessity in Proposition 2.12 for $\boldsymbol{R}$-trees.

In order to prove the main theorems, we extend the necessity in Proposition 2.12 to $\boldsymbol{R}$-trees:

PROPOSITION 4.1.   *Let $T$ be an $\boldsymbol{R}$-tree and $\nu \in \mathscr{B}^1(T)$. Then, we have*

$c_{c(\nu),T'}(\nu) \leq 0$ *for any* $T' \in \mathscr{C}'_T(c(\nu))$.

PROOF.    For simplicity, we assume that $\nu(T) = 1$. We shall approximate the measure $\nu$ by a measure whose support lies on a simplicial tree in $T$. Given $n \in \mathbf{N}$, there exists a compact subset $K_n \subseteq T$ such that $\nu(T \setminus K_n) < 1/n$ and $\int_{T \setminus K_n} d_T(c(\nu), w) d\nu(w) < 1/n$. Take a $(1/n)$-net $\{z_i^n\}_{i=1}^{l_n}$ of $K_n$ with mutually different elements such that $d_T(c(\nu), z_1^n) < 1/n$. We then take a sequence $\{A_i^n\}_{i=1}^{l_n}$ of mutually disjoint Borel subset of $K_n$ such that $z_i^n \in A_i^n$, $A_i^n \subseteq B_T(z_i^n, 1/n)$, and $K_n = \bigcup_{i=1}^{l_n} A_i^n$. Define the Borel probability measure $\nu_n$ on $\{z_i^n\}_{i=1}^{l_n}$ by $\nu_n(\{z_1^n\}) := \nu(A_1^n) + \nu(T \setminus K_n)$ and $\nu_n(\{z_i^n\}) := \nu(A_i^n)$ for $i \geq 2$.

CLAIM 4.2.    $d_1^W(\nu_n, \nu) \to 0$ *as* $n \to \infty$.

PROOF.    We shall show that

$$\lim_{n \to \infty} \int_T d_T(c(\nu), w) d\nu_n(w) = \int_T d_T(c(\nu), w) d\nu(w). \qquad (4.1)$$

Since

$$\int_T d_T(c(\nu), w) d\nu_n(w) = \sum_{i=1}^{l_n} d_T(c(\nu), z_i^n) \nu(A_i^n) + d_T(c(\nu), z_1^n) \nu(T \setminus K_n),$$

we have

$$\left| \int_T d_T(c(\nu), w) d\nu_n(w) - \sum_{i=1}^{l_n} d_T(c(\nu), z_i^n) \nu(A_i^n) \right| < \frac{1}{n^2}. \qquad (4.2)$$

By $A_i^n \subseteq B_T(z_i^n, 1/n)$, we get

$$\left| \sum_{i=1}^{l_n} d_T(c(\nu), z_i^n) \nu(A_i^n) - \int_{K_n} d_T(c(\nu), w) d\nu(w) \right| \qquad (4.3)$$

$$= \left| \sum_{i=1}^{l_n} \int_{A_i^n} \{ d_T(c(\nu), w) - d_T(c(\nu), z_i^n) \} d\nu(w) \right| \leq \sum_{i=1}^{l_n} \int_{A_i^n} d_T(w, z_i^n) d\nu(w) \leq \frac{1}{n}.$$

Hence, combining (4.2) with (4.3) and

$$\left| \int_{K_n} d_T(c(\nu), w) d\nu(w) - \int_T d_T(c(\nu), w) d\nu(w) \right| \leq \int_{T \setminus K_n} d_T(c(\nu), w) d\nu(w) < \frac{1}{n},$$

we obtain (4.1). The same way as the above proof shows that the sequence $\{\nu_n\}_{n=1}^{\infty}$ converges weakly to the measure $\nu$. Therefore, by using Lemma 2.14, this completes the proof of the claim. $\qquad\square$

Applying Claim 4.2 to Theorem 2.15, we get $c(\nu_n) \to c(\nu)$ as $n \to \infty$. Since the convex hull in $T$ of the set $\{z_i^n\}_{i=1}^{l_n}$ is a simplicial tree with finite vertex set and $c(\nu_n)$ is contained in the convex hull by Proposition 2.13, it follows from Proposition 2.12 that $c_{\widetilde{T},c(\nu_n)}(\nu_n) \leq 0$ for any $\widetilde{T} \in \mathscr{C}'_T(c(\nu_n))$. Let $T' \in \mathscr{C}'_T(c(\nu))$.

Assume first that $c(\nu_n) \in T \setminus T'$ for infinitely many $n \in \boldsymbol{N}$. Then, taking $T_n \in \mathscr{C}'_T(c(\nu_n))$ with $T' \subseteq T_n$, we have

$$c_{T',c(\nu)}(\nu_n) \leq c_{T_n,c(\nu_n)}(\nu_n) + d_T(c(\nu_n),c(\nu)) \leq d_T(c(\nu_n),c(\nu)).$$

Therefore, we obtain $c_{T',c(\nu)}(\nu) = \lim_{n\to\infty} c_{T',c(\nu)}(\nu_n) \leq 0$.

We consider the other case that $c(\nu_n) \in T'$ for any $n \in \boldsymbol{N}$. Let $z_n \in [c(\nu),c(\nu_1)]_T$ be the unique point such that

$$d_T(z_n,c(\nu_n)) = \inf\{d_T(z,c(\nu_n)) \mid z \in [c(\nu),c(\nu_1)]_T\}.$$

Since $c(\nu_n) \in T'$ for all $n \in \boldsymbol{N}$ and $c(\nu_n) \to c(\nu)$ as $n \to \infty$, by taking a subsequence, we may assume that $d_T(c(\nu),z_{n+1}) \leq d_T(c(\nu),z_n)$ for any $n \in \boldsymbol{N}$. For each $n \geq 2$, we take $T_n \in \mathscr{C}'_T(z_n)$ and $\widetilde{T}_n \in \mathscr{C}'_T(c(\nu_n))$ such that $c(\nu_1) \in T_n$ and $c(\nu_1) \in \widetilde{T}_n$. Observe that $T_n \subseteq T_{n+1}$. Since $T_n \subseteq \widetilde{T}_n$, we have

$$c_{T_n,z_n}(\nu_n) \leq c_{\widetilde{T},c(\nu_n)}(\nu_n) + d_T(c(\nu_n),z_n) \leq d_T(c(\nu_n),z_n). \qquad (4.4)$$

We also easily see

CLAIM 4.3.    $T' \setminus \{c(\nu)\} = \bigcup_{n=2}^{\infty} T_n$.

The same proof as Claim 4.2 implies that

$$\sup\left\{\left|\int_A d_T(z_n,w)d\nu_n(w) - \int_A d_T(z_n,w)d\nu(w)\right| \,\Big|\, A \subseteq T \text{ is a Borel subset}\right\} \to 0 \text{ as } n \to \infty.$$

Combining this with (4.4) and Claim 4.3, we obtain

$$c_{T',c(\nu)}(\nu) = \lim_{n\to\infty} c_{T_n,z_n}(\nu) = \lim_{n\to\infty} c_{T_n,z_n}(\nu_n) \leq \lim_{n\to\infty} d_T(c(\nu_n),z_n) = 0.$$

This completes the proof of the proposition. $\qquad\square$

The author does not know whether the converse of Proposition 4.1 holds or not.

## 5.   Proof of the main theorems.

Combining Proposition 3.2 with the same proof as [**3**, Lemma 5.3] implies the following proposition:

PROPOSITION 5.1.    *Let $T$ be an $\boldsymbol{R}$-tree and $\nu$ a finite Borel measure. Then, for any $\kappa > 0$, we have*

$$\nu\left( B_T\left( m_\nu, \mathrm{Sep}\left( \nu; \frac{m}{3}, \frac{\kappa}{2} \right) \right) \right) \geq m - \kappa, \tag{5.1}$$

*where $m_\nu$ is a median of the measure $\nu$. In particular, letting $X$ be an mm-space, we have*

$$\mathrm{ObsDiam}_T(X; -\kappa) \leq 2\,\mathrm{Sep}\left( X; \frac{m}{3}, \frac{\kappa}{2} \right). \tag{5.2}$$

Proposition 5.1 together with Corollary 2.6 yields Theorem 1.1. The following way to prove Theorem 1.1 is much easier and more straightforward than the above way, that is, to prove the existence of a median of a measure on an $\boldsymbol{R}$-tree.

PROOF OF THEOREM 1.1.    *Our goal is to prove the following inequality*:

$$\mathrm{ObsDiam}_T(X; -\kappa) \leq 3\,\mathrm{Sep}\left( X; \frac{\kappa}{3}, \frac{\kappa}{3} \right) + 4\,\mathrm{ObsDiam}_{\boldsymbol{R}}(X; -\kappa) \tag{5.3}$$

*for any $\kappa > 0$ with $\kappa \leq m/2$. Let $f : X \to T$ be an arbitrary 1-Lipschitz map. Fixing a point $z_0 \in T$, we shall consider the function $g : T \to \boldsymbol{R}$ defined by $g(z) := d_T(z, z_0)$. Since $g \circ f : X \to \boldsymbol{R}$ is a 1-Lipschitz function, by the definition of the observable diameter, there is an interval $A = [s, t] \subseteq [0, +\infty)$ such that $\mathrm{diam}\, A \leq \mathrm{ObsDiam}_{\boldsymbol{R}}(X; -\kappa)$ and $(g \circ f)_*(\mu_X)(A) \geq m - \kappa$. Observe that the set $g^{-1}(A)$ is the annulus $\{ z \in T \mid s \leq d_T(z, z_0) \leq t \}$. We denote by $\mathscr{C}$ the set of all connected components of the set $g^{-1}(A) \setminus \{z_0\}$.*

CLAIM 5.2.    *For any $T' \in \mathscr{C}$, we have $\mathrm{diam}\, T' \leq 2\,\mathrm{diam}\, A$.*

PROOF.    Given any $z_1, z_2 \in T'$, we shall show that $\phi_{z_0, z_1}(s) = \phi_{z_0, z_2}(s)$.

Suppose that $\phi_{z_0,z_1}(s) \neq \phi_{z_0,z_2}(s)$. Then, putting $s_0 := \sup\{t \in [0,+\infty) \mid \phi_{z_0,z_1}(t) = \phi_{z_0,z_2}(t)\}$, we have $0 < s_0 < s$. By the definition of $s_0$ and the property (2) of $\boldsymbol{R}$-trees, we have $(\phi_{z_0,z_1}(s_0), z_1]_T \cap (\phi_{z_0,z_2}(s_0), z_2]_T = \emptyset$. Therefore, by the property (2) of $\boldsymbol{R}$-trees, we get

$$[z_1, z_2]_T = [\phi_{z_0,z_1}(s_0), z_1]_T \cup [\phi_{z_0,z_2}(s_0), z_2]_T.$$

Hence, since $T'$ is convex by virtue of Proposition 2.9, the points $z_1$ and $z_2$ must be included in different components in $\mathscr{C}_T(\phi_{z_0,z_1}(s_0))$. This is a contradiction, since $T' = \bigcup \{C \cap T' \mid C \in \mathscr{C}_T(\phi_{z_0,z_1}(s_0))\}$ and $T'$ is connected. Thus, we have $\phi_{z_0,z_1}(s) = \phi_{z_0,z_2}(s)$. Consequently, we obtain

$$d_T(z_1, z_2) \leq d_T(z_1, \phi_{z_0,z_1}(s)) + d_T(\phi_{z_0,z_2}(s), z_2) \leq 2(t-s) \leq 2\,\mathrm{ObsDiam}_{\boldsymbol{R}}(X; -\kappa).$$

This completes the proof of the claim. $\qquad\square$

We shall consider the relation $\sim$ on $\mathscr{C}$ defined by $T' \sim T''$ if and only if $d_T(T', T'') \leq \mathrm{Sep}(X; \kappa/3, \kappa/3)$.

CLAIM 5.3. *The relation $\sim$ is an equivalence relation on $\mathscr{C}$.*

PROOF. We only prove "$T_1 \sim T_2$, $T_2 \sim T_3 \Rightarrow T_1 \sim T_3$" for different $T_i \in \mathscr{C}$, $i = 1, 2, 3$. Taking points $z_i \in T_i$ for $i = 1, 2, 3$, we put $s_{12} := \max\{t \in [0,+\infty) \mid \phi_{z_0,z_1}(t) = \phi_{z_0,z_2}(t)\}$ and $s_{23} := \max\{t \in [0,+\infty) \mid \phi_{z_0,z_2}(t) = \phi_{z_0,z_3}(t)\}$. Since $T_i$ are different to each other, we easily see that $s_{12} \leq s$, $s_{23} \leq s$,

$$\begin{aligned}
d_T(T_1, T_2) &= d_T(\phi_{z_0,z_1}(s), \phi_{z_0,z_1}(s_{12})) + d_T(\phi_{z_0,z_1}(s_{12}), \phi_{z_0,z_2}(s)) \\
&= 2(s - s_{12}) \\
&\leq \mathrm{Sep}(X; \kappa/3, \kappa/3),
\end{aligned}$$

and similarly $d_T(T_2, T_3) = 2(s - s_{23}) \leq \mathrm{Sep}(X; \kappa/3, \kappa/3)$. If $s_{12} \leq s_{23}$, we then obtain

$$\begin{aligned}
d_T(T_1, T_3) &\leq d_T(\phi_{z_0,z_1}(s), \phi_{z_0,z_3}(s)) \\
&\leq d_T(\phi_{z_0,z_1}(s), \phi_{z_0,z_1}(s_{12})) + d_T(\phi_{z_0,z_2}(s_{12}), \phi_{z_0,z_2}(s_{23})) \\
&\quad + d_T(\phi_{z_0,z_3}(s_{23}), \phi_{z_0,z_3}(s)) \\
&= (s - s_{12}) + (s_{23} - s_{12}) + (s - s_{23}) \\
&= 2(s - s_{12}) \\
&\leq \mathrm{Sep}(X; \kappa/3, \kappa/3).
\end{aligned}$$

In the case of $s_{23} \leq s_{12}$, the same conclusion also follows from the same argument as the above. This completes the proof of the claim. $\square$

Let $\mathscr{C}_0$ denote the set of equivalence classes of $\mathscr{C}/\sim$. Suppose that $f_*(\mu_X)(\bigcup_{T' \in S} T') < \kappa/3$ for any $S \in \mathscr{C}_0$. Since $f_*(\mu_X)(g^{-1}(A)) \geq m - \kappa \geq \kappa$, we have $\mathscr{C}' \subseteq \mathscr{C}_0$ such that

$$\frac{\kappa}{3} \leq f_*(\mu_X)\left( \bigcup_{S' \in \mathscr{C}'} \bigcup_{T' \in S'} T' \right) < \frac{2\kappa}{3}.$$

Hence, by putting $\mathscr{C}'' := \mathscr{C}_0 \setminus \mathscr{C}'$, we get

$$\begin{aligned}
\mathrm{Sep}\left( X; \frac{\kappa}{3}, \frac{\kappa}{3} \right) &< d_T\left( \bigcup_{S' \in \mathscr{C}'} \bigcup_{T' \in S'} T', \bigcup_{S'' \in \mathscr{C}''} \bigcup_{T'' \in S''} T'' \right) \\
&\leq \mathrm{Sep}\left( f_*(\mu_X); \frac{\kappa}{3}, \frac{\kappa}{3} \right) \\
&\leq \mathrm{Sep}\left( X; \frac{\kappa}{3}, \frac{\kappa}{3} \right),
\end{aligned}$$

which is a contradiction. Thereby, there exists $S \in \mathscr{C}_0$ such that $f_*(\mu_X)(\bigcup_{T' \in S} T') \geq \kappa/3$. For a subset $A \subseteq T$ and $r > 0$, we put $A_r := \{ z \in T \mid d_T(z, A) \leq r \}$.

CLAIM 5.4. $f_*(\mu_X)((\bigcup_{T' \in S} T')_{\mathrm{Sep}(X;\kappa/3,\kappa/3)}) \geq m - 2\kappa/3$.

PROOF. Suppose that $f_*(\mu_X)((\bigcup_{T' \in S} T')_{\mathrm{Sep}(X;\kappa/3,\kappa/3)}) < m - 2\kappa/3$. Then, we have a contradiction since

$$\begin{aligned}
\mathrm{Sep}\left( X; \frac{\kappa}{3}, \frac{\kappa}{3} \right) &< d_T\left( \bigcup_{T' \in S} T', T \setminus \left( \bigcup_{T' \in S} T' \right)_{\mathrm{Sep}(X;\kappa/3,\kappa/3)+\varepsilon} \right) \\
&\leq \mathrm{Sep}\left( f_*(\mu_X); \frac{\kappa}{3}, \frac{\kappa}{3} \right) \\
&\leq \mathrm{Sep}\left( X; \frac{\kappa}{3}, \frac{\kappa}{3} \right)
\end{aligned}$$

for any sufficiently small $\varepsilon > 0$. $\square$

Combining Claims 5.2 and 5.4, we obtain

$$\mathrm{diam}(f_*(\mu_X), m - \kappa) \leq \mathrm{diam}\left(\left(\bigcup_{T' \in S} T'\right)_{\mathrm{Sep}(X;\kappa/3,\kappa/3)}\right)$$

$$\leq 3\,\mathrm{Sep}\left(X; \frac{\kappa}{3}, \frac{\kappa}{3}\right) + 4\,\mathrm{ObsDiam}_{\boldsymbol{R}}(X; -\kappa)$$

and so (5.3). This completes the proof of the theorem. □

Note that the inequality (5.3) yields a slightly worse estimate for the observable diameter $\mathrm{ObsDiam}_T(X; -\kappa)$ than (5.2).

Let $T$ be an $\boldsymbol{R}$-tree and $\nu \in \mathscr{B}^1(T)$ with $m := \nu(X)$. Taking a median $m_\nu \in T$ of the measure $\nu$, we let $T_\nu$ an element in $\mathscr{C}'_T(c(\nu))$ with $m_\nu \in T_\nu$. We then define the function $\varphi_\nu : T \to \boldsymbol{R}$ by $\varphi_\nu(w) := d_T(c(\nu), w)$ if $w \in T_\nu$ and $\varphi_\nu(w) := -\,d_T(c(\nu), w)$ otherwise. The function $\varphi_\nu$ is clearly the 1-Lipschitz function.

LEMMA 5.5. *Let $T$ be an $\boldsymbol{R}$-tree and $\nu \in \mathscr{B}^1(T)$. Then, the function $\varphi_\nu : T \to \boldsymbol{R}$ satisfies that $c((\varphi_\nu)_*(\nu)) \leq 0$,*

$$|c((\varphi_\nu)_*(\nu))| \leq \mathrm{CRad}((\varphi_\nu)_*(\nu), m - \kappa) + \mathrm{Sep}\left((\varphi_\nu)_*(\nu); \frac{m}{3}, \frac{\kappa}{2}\right) \qquad (5.4)$$

$$+ \mathrm{Sep}((\varphi_\nu)_*(\nu); m - \kappa, m - \kappa),$$

*and*

$$\mathrm{CRad}(\nu, m - \kappa) \leq \mathrm{CRad}((\varphi_\nu)_*(\nu), m - \kappa) + \mathrm{Sep}\left(\nu; \frac{m}{3}, \frac{\kappa}{2}\right) \qquad (5.5)$$

$$+ \mathrm{Sep}\left((\varphi_\nu)_*(\nu); \frac{m}{3}, \frac{\kappa}{2}\right) + \mathrm{Sep}((\varphi_\nu)_*(\nu); m - \kappa, m - \kappa)$$

*for any $\kappa > 0$.*

PROOF. Combining Lemma 2.11 with Proposition 4.1, we have

$$\nu(T)c((\varphi_\nu)_*(\nu)) = \int_T \varphi_\nu(z)\ d\nu(z) = \int_{T_\nu} d_T(c(\nu), z)d\nu(z) - \int_{T \setminus T_\nu} d_T(c(\nu), z)d\nu(z)$$

$$= c_{T_\nu, c(\nu)}(\nu) \leq 0.$$

Put $r_1 := \mathrm{CRad}((\varphi_\nu)_*(\nu), m - \kappa)$ and $r_2 := \mathrm{Sep}((\varphi_\nu)_*(\nu); m/3, \kappa/2)$. From (5.1), we observe that $(\varphi_\nu)_*(\nu)(B_{\boldsymbol{R}}(\varphi_\nu(m_\nu), r_2)) \geq \nu(B_T(m_\nu, r_2)) \geq m - \kappa$. Thus, we get

$$d_{\boldsymbol{R}}(B_{\boldsymbol{R}}(c((\varphi_\nu)_*(\nu)), r_1), B_{\boldsymbol{R}}(\varphi_\nu(m_\nu), r_2)) \leq \mathrm{Sep}((\varphi_\nu)_*(\nu); m - \kappa, m - \kappa) \quad (5.6)$$

and so (5.4). The above inequality (5.6) together with $c((\varphi_\nu)_*(\nu)) \leq 0$ yields that

$$d_T(c(\nu), m_\nu) = \varphi_\nu(m_\nu) \leq |c((\varphi_\nu)_*(\nu)) - \varphi_\nu(m_\nu)|$$
$$\leq r_1 + r_2 + \mathrm{Sep}((\varphi_\nu)_*(\nu); m - \kappa, m - \kappa) =: r_3.$$

Therefore, putting $r_4 := \mathrm{Sep}(\nu; m/3, \kappa/2)$, we obtain

$$\nu(B_T(c(\nu), r_3 + r_4)) \geq \nu(B_T(m_\nu, r_4)) \geq m - \kappa$$

and so (5.5). This completes the proof. $\qquad\square$

PROPOSITION 5.6.    *Let $T$ be an $\mathbf{R}$-tree and $X$ an mm-space with $\mu_X \in \mathscr{B}^1(X)$. Then, for any $\kappa > 0$ we have*

$$\mathrm{ObsCRad}_T(X; -\kappa) \leq \mathrm{ObsCRad}_{\mathbf{R}}(X; -\kappa) + 2\,\mathrm{Sep}\left(X; \frac{m}{3}, \frac{\kappa}{2}\right)$$
$$+ \mathrm{Sep}(X; m - \kappa, m - \kappa).$$

PROOF.    This follows from Lemmas 2.2 and 5.5. $\qquad\square$

PROOF OF THEOREM 1.2.    Proposition 5.6 together with Corollary 2.6 and Lemma 2.16 directly implies the theorem. $\qquad\square$

LEMMA 5.7.    *Let $T$ be an $\mathbf{R}$-tree and $\nu \in \mathscr{B}^1(T)$. Then, for any $p \geq 1$ and $\kappa > 0$, we have*

$$V_p(\nu) \leq 2m^{2/p}\left\{\mathrm{CRad}((\varphi_\nu)_*(\nu), m - \kappa) + \mathrm{Sep}\left((\varphi_\nu)_*(\nu); \frac{m}{3}, \frac{\kappa}{2}\right)\right. \qquad (5.7)$$
$$\left. + \mathrm{Sep}((\varphi_\nu)_*(\nu); m - \kappa, m - \kappa)\right\} + 2V_p(\varphi_\nu).$$

*In the case of $p = 2$, we also have the better estimate*

$$V_2(\nu)^2 \leq 4m^2\left\{\mathrm{CRad}((\varphi_\nu)_*(\nu), m - \kappa) + \mathrm{Sep}\left((\varphi_\nu)_*(\nu); \frac{m}{3}, \frac{\kappa}{2}\right)\right. \qquad (5.8)$$
$$\left. + \mathrm{Sep}((\varphi_\nu)_*(\nu); m - \kappa, m - \kappa)\right\}^2 + 2V_2(\varphi_\nu)^2.$$

PROOF.    From the triangle inequality, we have

$$V_p(\nu) \leq 2\left(\int\int_{T \times T} d_T(c(\nu), z)^p d\nu(z)d\nu(w)\right)^{1/p} = 2\left(m\int_T d_T(c(\nu), z)^p d\nu(z)\right)^{1/p}. \qquad (5.9)$$

Putting $c_\nu := c((\varphi_\nu)_*(\nu))$, we also get

$$\left(\int_T d_T(c(\nu), z)^p d\nu(z)\right)^{1/p} = \left(\int_T |\varphi_\nu(z)|^p d\nu(z)\right)^{1/p} \tag{5.10}$$

$$\leq m^{1/p}|c_\nu| + \left(\int_{\mathbf{R}} |c_\nu - r|^p d(\varphi_\nu)_*(\nu)(r)\right)^{1/p}$$

$$\leq m^{1/p}|c_\nu| + \frac{V_p(\varphi_\nu)}{m^{1/p}},$$

where in the last inequality we used Lemma 2.11. Combining (5.9) with (5.10), we obtain (5.7).

In the case of $p = 2$, we have

$$\int_T d_T(c(\nu), z)^2 d\nu(z) = \int_{\mathbf{R}} |r|^2 d(\varphi_\nu)_*(\nu)(r) \tag{5.11}$$

$$= m|c_\nu|^2 + \int_{\mathbf{R}} |r - c_\nu|^2 d(\varphi_\nu)_*(\nu)(r)$$

$$= m|c_\nu|^2 + \frac{V_2(\varphi_\nu)^2}{2m},$$

where in the second and the last equalities we used Lemma 2.11. Substituting (5.11) to (5.9), we obtain (5.8). This completes the proof. $\square$

PROPOSITION 5.8.    *Let $T$ be an $\mathbf{R}$-tree and $X$ an mm-space. Then, for any $p \geq 1$, we have*

$$\mathrm{Obs}\, L^p\text{-}\mathrm{Var}_T(X) \leq 2\{2^{1/p}(1 + 2 \cdot 2^{1/p}) + 1\}\mathrm{Obs}\, L^p\text{-}\mathrm{Var}_{\mathbf{R}}(X). \tag{5.12}$$

*In the case of $p = 2$, we also have the better estimate*

$$\mathrm{Obs}\, L^2\text{-}\mathrm{Var}_T(X)^2 \leq (38 + 16\sqrt{2})\mathrm{Obs}\, L^2\text{-}\mathrm{Var}_{\mathbf{R}}(X)^2. \tag{5.13}$$

PROOF.    Assume first that $f_*(\mu_X) \in \mathscr{B}^1(T)$ for any 1-Lipschitz map $f : X \to T$. Then, Lemma 2.2 together with Lemma 2.3 and (5.7) implies that

$$\operatorname{Obs} L^p\text{-}\mathrm{Var}_T(X) \le 2m^{2/p}\bigg\{\operatorname{ObsCRad}_{\boldsymbol{R}}(X;-\kappa) + \operatorname{Sep}\Big(X; \frac{m}{3}, \frac{\kappa}{2}\Big)\bigg\}$$
$$+ 2\operatorname{Obs} L^p\text{-}\mathrm{Var}_{\boldsymbol{R}}(X)$$
$$\le 2m^{2/p}\bigg\{\operatorname{ObsCRad}_{\boldsymbol{R}}(X;-\kappa) + \operatorname{Sep}\Big(X; \frac{\kappa}{2}, \frac{\kappa}{2}\Big)\bigg\}$$
$$+ 2\operatorname{Obs} L^p\text{-}\mathrm{Var}_{\boldsymbol{R}}(X)$$

for any $0 < \kappa < m/2$. Hence, applying the inequalities (2.3) and (2.5) to this inequality, we get

$$\operatorname{Obs} L^p\text{-}\mathrm{Var}_T(X) \le 2\{m^{1/p}\kappa^{-1/p}(1 + 2 \cdot 2^{1/p}) + 1\}\operatorname{Obs} L^p\text{-}\mathrm{Var}_{\boldsymbol{R}}(X)$$

for any $0 < \kappa < m/2$. Letting $\kappa \to m/2$, we get (5.12). In the case of $p = 2$, from (5.8), we have

$$\operatorname{Obs} L^2\text{-}\mathrm{Var}_T(X)^2 \le 4m^2\bigg\{\operatorname{ObsCRad}_{\boldsymbol{R}}(X;-\kappa) + \operatorname{Sep}\Big(X; \frac{\kappa}{2}, \frac{\kappa}{2}\Big)\bigg\}^2$$
$$+ 2\operatorname{Obs} L^2\text{-}\mathrm{Var}_{\boldsymbol{R}}(X)^2$$

for any $0 < \kappa < m/2$. Therefore, substituting the inequalities (2.4) and (2.6) to this inequality, we get

$$\operatorname{Obs} L^2\text{-}\mathrm{Var}_T(X)^2 \le 2\{m\kappa^{-1}(2\sqrt{2} + 1)^2 + 1\}\operatorname{Obs} L^2\text{-}\mathrm{Var}_{\boldsymbol{R}}(X)^2$$

for any $0 < \kappa < m/2$. Letting $\kappa \to m/2$, we obtain (5.13).

We consider the other case that there exists a 1-Lipschitz map $f : X \to T$ with $f_*(\mu_X) \notin \mathscr{B}^1(T)$. By using Hölder's inequality and Fubini's theorem, we have $V_p(f) = +\infty$. Taking $x_0 \in X$, we put $f_n := f|_{B_X(x_0,n)}$ for each $n \in \boldsymbol{N}$. From Lemma 2.7 and the above proof, we have

$$V_p(f_n) \le \operatorname{Obs} L^p\text{-}\mathrm{Var}_T(B_X(x_0,n)) \le 2\{2^{1/p}(1 + 2 \cdot 2^{1/p}) + 1\}\operatorname{Obs} L^p\text{-}\mathrm{Var}_{\boldsymbol{R}}(B_X(x_0,n))$$
$$\le 2\{2^{1/p}(1 + 2 \cdot 2^{1/p}) + 1\}\operatorname{Obs} L^p\text{-}\mathrm{Var}_{\boldsymbol{R}}(X).$$

Since $V_2(f_n) \to V_2(f) = +\infty$ as $n \to \infty$, this implies $\operatorname{Obs} L^p\text{-}\mathrm{Var}_{\boldsymbol{R}}(X) = +\infty$. This completes the proof.                                                                                □

PROOF OF THEOREM 1.3.    Proposition 5.8 directly implies the theorem.   □

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