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$\boldsymbol{Q}$-homology planes as cyclic covers of $\boldsymbol{A}^2$

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Abstract. This paper classifies all $\boldsymbol{Q}$-homology planes which appear as cyclic covers of $\boldsymbol{A}^2$.

1. Introduction.

A $\boldsymbol{Q}$-homology plane S is by definition a smooth affine algebraic surface over $\boldsymbol{C}$ such that $H_i(S, \boldsymbol{Q}) = 0$ for $i \geq 1$. A basic theorem proved by Gurjar, Pradeep and Shastri [**PrS**], [**GPrSII**], [**GPrIII**] is that such a plane is always rational.

Cyclic branch covers appear in the work of Zariski [**Zar1**], [**Zar2**] where he showed that cyclic branch cover of $\boldsymbol{A}^2$ ramified over an irreducible curve of degree p^e, for a prime p, has vanishing irregularity. Here we are interested in smooth cyclic ramified covers of affine space which have first and second Betti numbers trivial.

The boundary of a large nice compact subset of such a $\boldsymbol{Q}$-homology plane is a $\boldsymbol{Q}$-homology 3-sphere which is a cyclic cover of S^3 ramified over a link. Hence these $\boldsymbol{Q}$-homology planes are also interesting for the theory of 3-manifolds.

Not many examples of $\boldsymbol{Q}$-homology planes which are hypersurfaces are known. This paper classifies all $\boldsymbol{Q}$-homology planes which appear as cyclic covers of $\boldsymbol{A}^2$. Our proofs depend crucially on the theory of non-complete algebraic surfaces developed by Iitaka, Kawamata, Miyanishi, Fujita, Sugie and other Japanese mathematicians.

Our main result is the following:

Theorem. *Let $S = \{z^n - f(x, y) = 0\} \subset \boldsymbol{A}^3$ be a smooth affine algebraic surface branched over $\boldsymbol{A}^2$. S is a $\boldsymbol{Q}$-homology plane if and only if there exists a coordinate system (x, y) on $\boldsymbol{A}^2$ such that f belongs to one of the lists below*:

(1) $f(x, y) = \phi(\alpha y + \beta)$
where $\alpha, \beta, \phi \in \boldsymbol{C}[x]$, $\phi = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_r)$, all λ_i's are distinct

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complex numbers for $i = 1, \dots, r$, $r \geq 1$, $\sqrt{(\alpha)} = (\phi)$ *and* $(\alpha, \beta) = 1$ (*cf. Propositions* 4.1 *and* 5.2).

(2) $n = 2$; $f(x,y) = x(x^l y^2 + x^t g(x) y + (x^{2t} g(x)^2 - cx^k)/4x^l)$ *where* $g(x)$ *and* $(x^{2t} g(x)^2 - cx^k)/4x^l \in \boldsymbol{C}[x]$, $k \in \boldsymbol{Z}_{\geq 0}$, l, $t \in \boldsymbol{Z}_{>0}$, l is even, $c \in \boldsymbol{C}^*$, $g(0) \neq 0$, $((x^{2t} g(x)^2 - cx^k)/4x^l)(0) \neq 0$ *and the integers* k, l, t *satisfy the following relations*:
- $2t > l$ *if and only if* $k = l$ *and* $c \neq 0$,
- $2t = l$ *if and only if either* $\{k > l\}$ *or* $\{k = l$ *and* $c \neq g(0)^2\}$,
- $2t < l$ *if and only if* $\{k = 2t,\ c = g(0)^2,\ 2d \geq l$ *where* $\deg\{x^t g(x)\} = d$ *and* $2t + i = l$ *for largest* i *such that* $x^i | (g(x) - g(0))\}$

(*cf. Proposition* 4.4).

(3)

$$f(x,y) = \begin{cases} x(h^{\mu_1} + \lambda_1 x^{\mu_0}),\ \mu_1 \geq 2;\ \textit{or} \\ x\prod_{i=1}^{r}(h^{\mu_1} + \lambda_i x^{\mu_0}),\quad r \geq 2 \textit{ and } (n, 1 + \mu_0 r) = 1; \textit{or} \\ xh\prod_{i=2}^{r}(h^{\mu_1} + \lambda_i x^{\mu_0}),\ r \geq 2 \textit{ and } (n, \mu_0 + \mu_1 + \mu_0\mu_1(r-1)) = 1 \end{cases}$$

where
- $h = (x^l y + p(x))$, $p(x) \in \boldsymbol{C}[x]$, $p(0) \neq 0$, $l \in \boldsymbol{Z}_{>0}$,
- *for* $i = 0, 1$, $\mu_i \in \boldsymbol{Z}_{>0}$ *and* $(\mu_0, \mu_1) = 1$,
- *for* $i = 1, \dots, r$, $\lambda_i \in \boldsymbol{C}^*$ *are distinct constants*

(*cf. Proposition* 4.6).

(4)

$$f(x,y) = \begin{cases} x\prod_{i=1}^{r}(x^\alpha h^\beta + \lambda_i),\ (n, \beta) = 1 \textit{ and } \beta > 1 \text{ if } r = 1;\ \textit{or} \\ xh\prod_{i=2}^{r}(x^\alpha h^\beta + \lambda_i),\ (n, |\alpha - \beta|) = 1 \textit{ and } r \geq 2 \end{cases}$$

where $h = y$ *or* $h = x^l y + p(x)$ *in the first polynomial and* $h = x^l y + p(x)$ *in the second polynomial,* $p(x) \in \boldsymbol{C}[x], p(0) \neq 0; \alpha, \beta, l \in \boldsymbol{Z}_{>0}$ *and* $\lambda_i \in \boldsymbol{C}^*$ *are distinct* (*cf. Proposition* 4.8).

The structure of the paper is as follows. In Section 2 we collect some results for our reference. In Section 3 we study the branch loci and prove some useful results. In Section 4 and 5 we analyse the case of one or more lines in the branch locus respectively.

2. Preliminaries.

All algebraic varieties considered in this paper are defined over $\boldsymbol{C}$. The Euler-Poincare characteristic of a topological space X is denoted by $\chi(X)$. For a smooth quasi-projective variety Y the logarithmic Kodaira dimension is denoted by $\bar{\kappa}(Y)$. We denote the affine curve $\boldsymbol{A}^1 - \{l \text{ points}\}$ by $\boldsymbol{C}^{l*}$ for a positive integer l. A morphism $g : X \to B$ from a smooth algebraic surface X to a smooth algebraic curve B is called an F-fibration if a general fiber of g is isomorphic to F where F is an algebraic curve. We will mostly consider $F = \boldsymbol{A}^1$ or $\boldsymbol{C}^*$.

Following are some results which we use frequently:

LEMMA 2.1. *Let $Y \subset X$ be a closed algebraic subvariety of a variety X. Then*

$$\chi(X) = \chi(X - Y) + \chi(Y).$$

LEMMA 2.2. *If $U \subset X$ is a non-empty Zariski open subset in a normal irreducible algebraic variety X then the sequence $H_1(U, \boldsymbol{Z}) \to H_1(X, \boldsymbol{Z}) \to 0$ is exact.*

LEMMA 2.3. *If X is an algebraic curve (affine or projective) then $H_1(X, \boldsymbol{Z})$ is torsion free.*

LEMMA 2.4. *Euler characteristic of an affine algebraic curve does not exceed* 1.

LEMMA 2.5. *Suppose C is a smooth irreducible affine algebraic plane curve such that* $\chi(C) = 0$ (1 respectively), *then* $C \cong \boldsymbol{C}^*$ ($\boldsymbol{A}^1$ respectively).

LEMMA 2.6 (Iitaka's Easy Addition Theorem [**I**], Theorem 10.4). *Let $f : V \to W$ be a dominant morphism for two smooth algebraic varieties V and W. Then $\bar{\kappa}(V) \leq \bar{\kappa}(f^{-1}(x)) + \dim(W)$ where $x \in \bigcap_1^\infty W_m$ for certain Zariski-dense open sets W_m of W.*

LEMMA 2.7 (Kawamata's inequality [**K**], Theorem 1). *Let Y be a smooth quasi-projective algebraic surface and $Y \xrightarrow{f} B$ be a surjective morphism to a smooth algebraic curve B such that a general fiber of f is irreducible. Then*

$$\bar{\kappa}(Y) \geq \bar{\kappa}(B) + \bar{\kappa}(F).$$

LEMMA 2.8 (Suzuki-Zaidenberg [**Su77**], [**Z**]). *Let S be a smooth affine* *algebraic surface with a surjective morphism $g : S \to C$ with connected general fiber, where C is a smooth curve. Let F be a general fiber of g and let F_i be the singular fibers for $1 \leq i \leq l$. Then we have*

$$\chi(S) = \chi(C)\chi(F) + \sum_{1}^{l}(\chi(F_i) - \chi(F)).$$

Further, $\chi(F_i) \geq \chi(F)$ for all i. If the equality holds for some i then F is either isomorphic to $\boldsymbol{A}^1$ or $\boldsymbol{C}^*$ and $F_{i,\mathrm{red}}$ is isomorphic to F for all i if taken with reduced structures.

Lemma 2.9 (Abhyankar-Moh-Suzuki [**AM**], [**Su74**]). *Let $C \subset \boldsymbol{A}^2$ be a closed embedding of the affine line $\boldsymbol{A}^1$. Then there is an algebraic automorphism of $\boldsymbol{A}^2$ which maps C onto the line $\{x = 0\}$, where x, y are suitable affine coordinates on $\boldsymbol{A}^2$.*

The following theorem was proved by Gurjar and Parameswaran. We state without proof the part which is relevant to us.

Lemma 2.10 (Gurjar-Parameswaran [**GP1**], Section 5, Case 1). *Suppose X is a smooth rational affine algebraic surface with $\chi(X) = 0$. Then one of* (1) *or* (2) *is true*:

(1) *There is a morphism from X onto $\boldsymbol{C}^*$ with connected general fiber with the following two properties*:

- (1a) *All the fibers are irreducible and mutually diffeomorphic if taken with reduced structure.*
- (1b) *Either $X \to \boldsymbol{C}^*$ is a C^∞ fiber bundle or the general fiber of this map is isomorphic to $\boldsymbol{C}$ or $\boldsymbol{C}^*$.*

(2) *There is a morphism from X to a curve of general type with the following two properties*:

- (2a) *A general fiber of this map is isomorphic to $\boldsymbol{C}$ or $\boldsymbol{C}^*$.*
- (2b) *If the general fiber is $\boldsymbol{C}^*$ then all the fibers are irreducible and isomorphic to $\boldsymbol{C}^*$ if taken with reduced structure.*

The following result is about the number of affine lines on surfaces with $\bar{\kappa} = 0$.

Lemma 2.11 (Gurjar-Parameswaran [**GP2**], Section 1, Theorem). *Let X be a $\boldsymbol{Q}$-homology plane with $\bar{\kappa}(X) = 0$. Then the following assertions are true.*

(i) *If X is not NC-minimal, then X contains a unique contractible curve C. Moreover C is smooth with $\bar{\kappa}(X - C) = 0$.*

(ii) *If X is NC-minimal and not the surface $H[k, -k]$ in Fujita's classification*

[**F**, (8.64)], then X has no contractible curves.

(iii) *If X is NC-minimal and is isomorphic to $H[k,-k]$ with $k \geq 2$, then there is a unique contractible curve C on X and it is smooth. Further, $\bar{\kappa}(X-C)=0$.*

(iv) *The surface $X = H[1,-1]$ has exactly two contractible curves, say C and L. Further, both the curves are smooth, $\bar{\kappa}(X-C)=0$ and $\bar{\kappa}(X-L)=1$. The curves C and L intersect each other transversally in exactly two points.*

We include the following result about the uniqueness of a $\boldsymbol{C}^*$-fibration on any smooth affine surface V with $\bar{\kappa}(V)=1$.

LEMMA 2.12 (Gurjar-Miyanishi [**GM**], Lemma 2.4). *Let V and W be smooth affine surfaces with $\bar{\kappa}(V)=\bar{\kappa}(W)=1$ with a dominant morphism $f: W \to V$. Let ϕ and ψ be $\boldsymbol{C}^*$-fibrations on V and W. Then f maps the fibers of ψ into fibers of ϕ.*

The following lemma is the relevant part of Miyanishi-Sugie [**MS**, Lemmas 2.10, 2.11, 2.14, 2.15] (see also, [**KK**, Lemma 2.8] and [**M2**, Chapter 3, Section 4.6]).

LEMMA 2.13. *Let X be a $\boldsymbol{Q}$-homology plane with a $\boldsymbol{C}^*$-fibration $\phi: X \to C$. Then we have*:

(1) *C is either $\boldsymbol{P}^1$ or $\boldsymbol{A}^1$.*

(2) *If $C \cong \boldsymbol{P}^1$ then every fiber of ϕ is irreducible, and there is exactly one fiber isomorophic to $\boldsymbol{A}^1$. Let $F_0,\dots,F_r$ be all the singular fibers with respective multiplicities $m_0,\dots,m_r$, where $F_{0,\mathrm{red}} \cong \boldsymbol{A}^1$ and $F_{i,\mathrm{red}} \cong \boldsymbol{C}^*$ for $i>0$. Then $\bar{\kappa} = 1, 0$ or $-\infty$ if and only if*

$$(r-1)-\sum_{i=1}^{r}\frac{1}{m_i} > 0, = 0 \text{ or } < 0, \ \textit{respectively}$$

where it is understood that the L.H.S is -1 if $r=0$.

(3) *If $C \cong \boldsymbol{A}^1$, ϕ is untwisted and if $F_1,\dots,F_r$ are all its singular fibers then all the fibers are irreducible except one, say F_1, which consists of two irreducible components $F_1 = \nu_1 G_1 + \nu_2 G_2$ such that either G_1 and G_2 are both $\boldsymbol{A}^1$ and intersect each other transversally in one point or $G_1 \cong \boldsymbol{C}^*, G_2 \cong \boldsymbol{A}^1$ and they are disjoint. Let $m_1 = \min(\nu_1,\nu_2)$ in the case $G_1 \cong G_2 \cong \boldsymbol{A}^1$ and $m_1 = \nu_1$ in the case $G_1 \cong \boldsymbol{C}^*, G_2 \cong \boldsymbol{A}^1$. Also suppose that $m_2F_2,\dots,$ m_rF_r are the other singular fibers. Then $\bar{\kappa} = 1, 0$ or $-\infty$ if and only if*

$$(r-1)-\sum_{i=1}^{r}\frac{1}{m_i} > 0, = 0 \text{ or } < 0, \ \textit{respectively.}$$

Note that $r \geq 1$, so the above sum is always well defined.

(4) *If $C \cong \boldsymbol{A}^1$, ϕ is twisted and if $F_i = m_i A_i$ $(0 \leq i \leq r)$ are its singular fibers, where $A_0 \cong \boldsymbol{A}^1$ and $A_i \cong \boldsymbol{C}^*$ for $1 \leq i \leq r$ then the following assertions hold*:

(4a) *Let $N = r - (1/2) - \sum_{i=1}^{r}(1/m_i)$ in the case where X is NC-minimal where it is understood that $N = -(1/2)$ if $r = 0$. Then $\bar{\kappa}(X) = 1, 0$ or $-\infty$ if and only if $N > 0, = 0$ or < 0, respectively.*

(4b) *$H_1(X, \boldsymbol{Z})$ is an extension of $\prod_{i=0}^{r} \boldsymbol{Z}/m_i\boldsymbol{Z}$ by $\boldsymbol{Z}/2\boldsymbol{Z}$.*

LEMMA 2.14 (Saito [**Sa**], p. 332). *Let f be an irreducible polynomial in $\boldsymbol{C}[x, y]$ such that a general fiber of the map $\boldsymbol{A}^2 \xrightarrow{f} \boldsymbol{A}^1$ is a $\boldsymbol{C}^*$. Then, after a suitable change of coordinates, f is reduced to either one of the following two forms*:

(1) *$f = x^\alpha y^\beta + 1$, where α, $\beta \in \boldsymbol{Z}_{>0}$ and $(\alpha, \beta) = 1$ or*
(2) *$f = x^\alpha(x^l y + P(x))^\beta + 1$, where α, β, $l \in \boldsymbol{Z}_{>0}$ and $(\alpha, \beta) = 1$ and $P(x) \in \boldsymbol{C}[x]$ with $\deg P(x) < l$ and $P(0) \neq 0$.*

LEMMA 2.15 (Miyanishi [**M1**], Theorem 2.1). *Let $\rho : \boldsymbol{C}^2 \to \boldsymbol{P}^1$ be a $\boldsymbol{C}^*$-fibration parametrized by $\boldsymbol{P}^1$ and $\mu_0 A_0$, $\mu_1 A_1$ be the singular fibers of ρ with $A_0 \cong \boldsymbol{A}^1$ and $A_1 \cong \boldsymbol{C}^*$. Then, the pencil associated to ρ is given as follows*:

$$\Lambda = (yx^{r+1} - p(x))^{\mu_1} + \lambda x^{\mu_0} \ ; \ \lambda \in \boldsymbol{P}^1$$

where $p(x) \in \boldsymbol{C}[x]$, $\deg p(x) \leq r$, $p(0) \neq 0$, μ_0, $\mu_1 \in \boldsymbol{Z}_{>0}$ and $(\mu_0, \mu_1) = 1$. Furthermore, we understand that $\mu_1 = 1$ when there is no multiple fiber whose reduced form is isomorphic to $\boldsymbol{C}^*$.

REMARK 2.16. A $\boldsymbol{C}^*$-fibration on $\boldsymbol{C}^2$ has atmost two singular fibers.

To state the next results we need the following definitions.

DEFINITION 2.17. An affine algebraic surface defined over $\boldsymbol{C}$ is called ML_0 if it has two G_a-actions such that the general orbits for the two actions are transverse to each other. Such a surface is called ML_1 if it has a unique G_a-action.

DEFINITION 2.18. For an algebraic variety X, we define the number $\rho(X) =$ rank of $\mathrm{Pic}(X)_{\boldsymbol{Q}}$ to be the Picard number of X.

LEMMA 2.19 (Gurjar, Masuda, Miyanishi, Russell [**GMMR**], Theorem 2.1). *Let X be an ML_0 surface with $\rho(X) = 0$. Let C be a curve isomorphic to the affine line on X. Then there exists an $\boldsymbol{A}^1$-fibration $f : X \to \boldsymbol{A}^1$ and C is a fiber component of f.*

We need the following theorem about uniqueness of $\boldsymbol{A}^1$-fibrations on $\boldsymbol{Q}$-homology planes which are ML_1.

Lemma 2.20 (Gurjar, Masuda, Miyanishi, Russell [**GMMR**], Theorem 3.10). *Let X be a $\boldsymbol{Q}$-homology plane. Suppose X is an ML_1 surface and not isomorphic to one of the surfaces constructed in Example* 2.21 *below* (*Example* 3.8 *and* 3.9 *op.cit.*). *Then any affine line on X is a fiber of the unique $\boldsymbol{A}^1$-fibration $f : X \to \boldsymbol{A}^1$. In other words, there are no affine lines which lie transversally to the unique $\boldsymbol{A}^1$-fibration $f : X \to \boldsymbol{A}^1$.*

The example referred to in Lemma 2.20 is the following:

Example 2.21 ([**GMMR**], Example 3.8, 3.9). Consider the surface X constructed as follows. Let V_0 be a Hirzerbruch surface of degree $n \geq 0$ with the $\boldsymbol{P}^1$-fibration $p_0 : V_0 \to \boldsymbol{P}^1$ with general fiber ℓ. Let M_0 and M_1 be disjoint sections (so $M_0^2 = -M_1^2$ and $\mid (M_i^2) \mid = n$). Choose three fibers $\ell_0, \ell_1, \ell_\infty$. Let $\sigma : V \to V_0$ be a sequence of blowing-ups which produce the following degenerate fibers Γ_i from ℓ_i for $i = 0, 1$ (Γ_0 and Γ_1 meet M_0' and M_1' as in the figure below):

$$\Gamma_0 : \quad M_0' \;-\; \underset{\overline{H}}{(-m_1)} \;-\; \underset{E_0}{(-1)} \;-\; \underset{E_1}{(-2)} \;-\; \dots \;-\; \underset{E_{m_1-1}}{(-2)} \;-\; M_1'$$

$$\Gamma_1 : \quad M_0' - (-a_1) - \;\dots\; - (-a_s) - \underset{F_0}{(-1)} - (-b_t) \;-\; \dots \;- (-b_1) - M_1'$$

where $a_i \geq 2 (1 \leq i \leq s)$, $b_j \geq 2 (1 \leq j \leq t)$, $\overline{H} = \sigma'(\ell_0)$ and $M_k' = \sigma'(M_k)$ for $k = 0, 1$. Let m_2 be the multiplicity of the component F_0 in the fiber $\sigma^*(\ell_1)$ and let $D = M_0' + M_1' + \ell_\infty + (\sigma^*(\ell_0)_{\text{red}} - (\overline{H} + E_0)) + (\sigma^*(\ell_1)_{\text{red}} - F_0)$ and let $X = V - D$. Let $H = \overline{H} \cap X$. Suppose that $m_1 \geq 2$ and $m_2 \geq 2$. Then the following assertions hold:

(1) X is an ML_1 surface.

(2) H is an affine line and it lies transversally to a unique $\boldsymbol{A}^1$-fibration $f : X \to \boldsymbol{A}^1$.

(3) $\bar{\kappa}(X - H) = 0$ if and only if $m_1 = m_2 = 2$ and $\bar{\kappa}(X - H) = 1$ otherwise.

In [**GMMR**] Example 3.8 is a special case of Example 3.9 corresponding to $m_1 = m_2 = 2$.

Definition 2.22. Suppose $\overline{X}$ is a smooth complete algebraic surface with a $\boldsymbol{P}^1$-fibration $\phi : \overline{X} \to B$ where B is a smooth complete curve, such that there is an open set $X \subset \overline{X}$ on which $\phi|_X$ is a $\boldsymbol{C}^*$-fibration. If $D := \overline{X} - X$ is the boundary divisor of X then

(1) Define D_h as the union of those irreducible components of D on which ϕ is non-constant. We call D_h as the horizontal component of D.
(2) An X-component of a fiber F of ϕ is an irreducible component of F which is not in D.
(3) For a fiber F of ϕ define $\sigma(F)$ as the number of X-components of F.
(4) Define a 'rivet' to be a connected component of $F \cap D$ if it meets D_h in more than one points, or if it is a node of D_h.
(5) If $\phi|_X$ is a twisted fibration it is also called 'gyoza' by Fujita.
(6) A subgraph Γ' of a graph Γ with vertices $\{v_1, \ldots, v_r\}$ is called a *linear chain* if $\beta_\Gamma(v_1) = 1$, $\beta_\Gamma(v_i) = 2$ and $(v_{i-1}, v_i)_\Gamma = (v_i, v_{i+1})_\Gamma = 1$ for $2 \leq i \leq r-1$ where $\beta_\Gamma(v)$ is the number of edges in Γ connecting v to other vertices and $(v, v')_\Gamma$ is the number of edges between v and v' in Γ. If $\beta_\Gamma(v_r) \geq 2$, Γ' is called a *twig*.

Lemma 2.23 ([**F**], Lemma 7.6). *Assume that $\overline{X}$ is a smooth complete algebraic surface, B is a smooth complete algebraic curve and $\phi : \overline{X} \to B$ is a $\boldsymbol{P}^1$-fibration. Let there be an open set $X \subset \overline{X}$ such that the restriction $\phi|_X$ is a $\boldsymbol{C}^*$-fibration. Let $D := \overline{X} - X$ be the boundary divisor of X. Assume that F is a fiber of ϕ such that $\sigma(F) = 1$ and F does not contain a rivet. Then*
(1) *$F \cong \boldsymbol{P}^1$ and F meets D_h at two different points, or*
(2) *F looks like a twig $[A, 1, B]$ as in* [**F**](4.7), the X-component of F is the unique (-1)-curve in F, and D_h meets the highest and the lowest components of F, or
(3) *ϕ is twisted (Fujita calls it 'gyoza') and $\phi(F)$ is a branch point of $D_h \to B$.*

3. Branch locus and other results.

Using Euler characteristic calculations we prove in this section that the ramification locus must consist of disjoint curves, atleast one of which is an $\boldsymbol{A}^1$.

Throughout the rest of the paper we will assume the following notation. For $n > 1$ and $f(x,y) \in \boldsymbol{C}[x,y]$, $S := \{z^n - f(x,y) = 0\}$ is a $\boldsymbol{Q}$-homology plane branched over $\boldsymbol{A}^2$. Since S is smooth $f(x,y)$ is a reduced polynomial in $\boldsymbol{C}[x,y]$ whose zero locus is a smooth and possibly reducible curve in $\boldsymbol{A}^2$. We define $C := \{f(x,y) = 0\}$ to be the branch locus. Suppose that $\psi : S \to \boldsymbol{A}^2$ is the map given by $(x,y,z) \mapsto (x,y)$. It is a finite map ramified over C. For an irreducible component C' of C we will denote $\psi^{-1}(C') \subset S$ by C' itself when there is no scope of confusion.

We begin by proving a few results about C :

Lemma 3.1. $\chi(C) = 1$.

Proof.

$$\begin{aligned}
&\chi(S) = \chi(S - \pi^{-1}(C)) + \chi(\pi^{-1}(C)) \\
&\chi(\boldsymbol{C}^2) = \chi(\boldsymbol{C}^2 - C) + \chi(C) = 1 \\
&\chi(S - \pi^{-1}(C)) = n \cdot \chi(\boldsymbol{C}^2 - C) = n(1 - \chi(C)) \\
&\Rightarrow \chi(S) = n(1 - \chi(C)) + \chi(\pi^{-1}(C))
\end{aligned}$$

but $\chi(\pi^{-1}(C)) = \chi(C)$ and $\chi(S) = 1$ since S is a $\boldsymbol{Q}$-homology plane

$$\begin{aligned}
&\Rightarrow 1 = n - (n-1)\chi(C) \\
&\Rightarrow \chi(C) = 1
\end{aligned}$$

since $n > 1$. □

Lemma 3.2. *If the curve C is irreducible then $S \cong \boldsymbol{A}^2$.*

Proof. If C is irreducible then $\chi(C) = 1$ along with Lemma 2.5 implies that $C \cong \boldsymbol{A}^1$ and by Lemma 2.9 we can assume it to be $\{x = 0\}$. Clearly a branch covering of $\boldsymbol{A}^2$ over the line $\{x = 0\}$ is $\boldsymbol{A}^2$ itself. □

We now assume that the curve C is reducible. Since C is smooth we can write it as a disjoint union of smooth irreducible curves:

$$C = C_0 \amalg C_1 \amalg \ldots \amalg C_r$$

for some $r \geq 1$.

Lemma 3.3. *At least one of the curves C_i is isomorphic to $\boldsymbol{A}^1$ which we assume to be C_0 after reindexing and that it is the coordinate axis $\{x = 0\}$. The other curves are given by $C_i := \{xg_i + 1 = 0\}$ where $g_i(x, y) \in \boldsymbol{C}[x, y]$ for $i = 1, \ldots, r$.*

Proof. By repeated use of Lemma 2.1 we get

$$\chi(C) = \sum_{i=0}^{r} \chi(C_i)$$

and

$$\chi(C) = 1$$

by Lemma 3.1. Therefore not all $\chi(C_i) \leq 0$. Hence at least one of the $\chi(C_i)$ is 1 and by Lemma 2.5 it must be an $\boldsymbol{A}^1$. By appealing to Lemma 2.9 we get the rest of the statement. □

LEMMA 3.4. *Let $\mathscr{C} = \mathscr{C}_0 \amalg \mathscr{C}_1 \amalg \ldots \amalg \mathscr{C}_l$ be the irreducible decomposition of a smooth affine plane curve $\mathscr{C}$ with $\chi(\mathscr{C}) = 1$ such that $\mathscr{C}_0 \cong \boldsymbol{A}^1$ and $\mathscr{C}_i \ncong \boldsymbol{A}^1$ for $1 \leq i \leq l$. Then $\mathscr{C}_i \cong \boldsymbol{C}^*$ for $i \geq 1$. In particular $\mathscr{C}_i$ are rational curves.*

PROOF.

$$\chi(\mathscr{C}) = \sum_{i=0}^{i=l} \chi(\mathscr{C}_i) \qquad \Rightarrow 1 = 1 + \sum_{i=1}^{i=l} \chi(\mathscr{C}_i)$$

$$\Rightarrow \sum_{i=1}^{i=l} \chi(\mathscr{C}_i) = 0. \tag{1}$$

However, $\mathscr{C}_i$ are smooth irreducible plane curves $\ncong \boldsymbol{A}^1$, hence

$$\chi(\mathscr{C}_i) \leq 0. \tag{2}$$

By (1) and (2), $\chi(\mathscr{C}_i) = 0$ for $1 \leq i \leq l$. Therefore by Lemma 2.5, $\mathscr{C}_i \cong \boldsymbol{C}^*$ as was required to prove. □

LEMMA 3.5. *Suppose $\mathscr{C} = \mathscr{C}_1 \amalg \ldots \amalg \mathscr{C}_l \amalg \mathscr{D}$ is the irreducible decomposition of a smooth affine plane curve $\mathscr{C}$ with $\chi(\mathscr{C}) = 1$ such that $\mathscr{C}_i \cong \boldsymbol{A}^1$ $\forall i$ and $\mathscr{D} := \{G(x, y) = 0\}$ is a rational curve. Then there exists a coordinate system (x, y) in $\boldsymbol{A}^2$ in which the following is true*:

(a) $\mathscr{C}_i := \{x - \lambda_i = 0\}$ *for distinct* λ_i.

(b) $\mathscr{D} \cong \boldsymbol{C}^{l*}$.

PROOF. (*a*) By Lemma 2.9, $\mathscr{C}_1 = \{x = 0\}$. Consider the map $\theta : \boldsymbol{A}^2 \xrightarrow{x} \boldsymbol{A}^1$. It is clear that $\mathscr{C}_i$ are contained in fibers of θ since otherwise we get non-trivial maps from $\boldsymbol{A}^1 \to \boldsymbol{C}^*$.

(b) $\chi(\mathscr{C}) = 1$
$\Rightarrow \chi(\mathscr{D}) = 1 - l$.

But $\mathscr{D}$ is rational and irreducible, hence the conclusion follows easily. □

LEMMA 3.6. *Let $\eta(y) \in \boldsymbol{C}[y]$ be a polynomial such that $\{z^n - \eta(y) = 0\} \cong \boldsymbol{A}^1$ where $n \geq 2$. Then $\eta(y)$ is a linear polynomial.*

PROOF. Let X be the curve $\{z^n - \eta(y) = 0\}$. The map $X \xrightarrow{y} \boldsymbol{A}^1$ is a finite morphism and for any $y = y_0$ such that $\eta(y_0) \neq 0$, there are n distinct inverse images. The morphism y extends to $\phi : \boldsymbol{P}^1 \mapsto \boldsymbol{P}^1$ such that $\infty \mapsto \infty$ with ramification index n. The map ϕ is also ramified over each of the roots of $\eta(y)$ with ramification index $= n$. Suppose that $d = \deg(\eta(y)) \geq 2$. Then the Hurwitz ramification formula gives us the following where the point ∞ and two roots of η each contribute $(n-1)$:

$$\begin{aligned}&2g(\boldsymbol{P}^1) - 2 = -2(n) + (n-1) + (n-1) + (n-1) + (\text{non-neg terms})\\ &\Rightarrow n - 1 + (\text{non-neg terms}) = 0\end{aligned}$$

but this is impossible since $n - 1 \geq 1$. Therefore $d = 1$ and $\eta(y)$ is linear. □

REMARK 3.7. Conversely if $\eta \in \boldsymbol{C}[y]$ is linear then $\{z^n - \eta(y) = 0\} \cong \boldsymbol{A}^1$.

LEMMA 3.8. *Suppose that for all but finitely many $x = \lambda \in \boldsymbol{C}$, the curve*

$$\{z^n - x(xg(x,y) + 1) = 0\} \cong \boldsymbol{C}^*$$

where $n \geq 2$ and $g(x,y) \in \boldsymbol{C}[x,y]$. Then $n = 2$ and $g(x,y)$ has y-degree $= 2$.

PROOF. Let the y-degree of $g(x,y)$ be m. It is not 1 since then for a fixed value of $x \in \boldsymbol{C}$ outside a finite set, the curve $\{z^n - x(xg(x,y) + 1) = 0\}$ will be isomorphic to $\boldsymbol{A}^1$ and not $\boldsymbol{C}^*$. Therefore $m \geq 2$. At a general $x = \lambda$, we can rewrite the above equation as $\{z^n - (y - a_1) \cdots (y - a_m) = 0\}$ where $a_i \in \boldsymbol{C}$, $\forall i = 1, \ldots, m$. Let $\mathscr{C}_\lambda = \{z^n - (y - a_1) \cdots (y - a_m) = 0\}$. Consider the map $\phi : \mathscr{C}_\lambda \to \boldsymbol{A}^1$ given by $(z, y) \mapsto y$. There are m points of ramification of ϕ, namely a_i for $i = 1, \ldots, m$. The map ϕ extends to a map $\tilde{\phi}$ on the smooth minimal compactification of $\mathscr{C}_\lambda$. We apply Riemann-Hurwitz formula to $\tilde{\phi}$ to get the following calculation:

$$\begin{aligned}&-2 = n(-2) + m(n-1) + (\geq 0)\\ &\Rightarrow -2 \geq -2n + mn - m\\ &\Rightarrow \quad 0 \geq n(m-2) - (m-2)\\ &\Rightarrow \quad 0 \geq (n-1)(m-2)\\ &\Rightarrow \quad m = 2\end{aligned}$$

since $n \geq 2$ and $m \geq 2$.

The equation for $\mathscr{C}_\lambda$ is now $\{z^n - (y - a_1)(y - a_2) = 0\}$. After completing square in y and a linear change of variables in y the equation becomes $\{y^2 + z^n + c = 0\}$. Consider the map $(y, z) \mapsto z$ on this latter curve. By similar arguments as above we see that $n = 2$. The lemma follows. □

LEMMA 3.9. *Suppose that* $S := \{z^n - xh(x, y) = 0\}$ *is a* $\boldsymbol{Q}$*-homology plane such that the map* $S \to \boldsymbol{A}^1$ *given by* x *is a* $\boldsymbol{C}^*$*-fibration. Then the following are true*:

(a) $n = 2$.

(b) $h(x, y) = x^l y^2 + x^t g(x)y + (x^{2t}g^2 - cx^k)/4x^l$ *where* $g(x) \in \boldsymbol{C}[x]$, $k \in \boldsymbol{Z}_{\geq 0}$, l, $t \in \boldsymbol{Z}_{>0}$, $c \in \boldsymbol{C}^*$, $h(0, 0) \neq 0$, $g(0) \neq 0$ *and the integers* k, l, t *satisfy the following relations*:

(i) $2t > l$ *if and only if* $k = l$,

(ii) $2t = l$ *if and only if either* $\{k > l\}$ *or* $\{k = l$ *and* $c \neq g(0)^2\}$,

(iii) $2t < l$ *if and only if* $\{k = 2t$, $c = g(0)^2$, $2d \geq l$ *where* $\deg\{x^t g(x)\} = d$ *and* $2t + i = l$ *for largest* i *such that* $x^i | (g(x) - g(0))\}$.

(c) $\bar{\kappa}(S) = 0$.

(d) *S is not NC-minimal.*

(e) *l is even.*

PROOF. Let $\phi : S \xrightarrow{x} \boldsymbol{A}^1$ be the $\boldsymbol{C}^*$-fibration in the hypothesis. By Lemma 3.8 it follows that $n = 2$ and h has y-degree $= 2$. Therefore it can be assumed that $h(x, y) = g_2(x)y^2 + g_1(x)y + g_0(x)$ where $g_i \in \boldsymbol{C}[x]$ $\forall i = 0, 1, 2$ and the polynomial $g_2 \not\equiv 0$. By Lemma 3.3 the polynomial $h(x, y)$ is of the form $xh_1(x, y) + 1$ therefore $x \mid g_1$ and $x \mid g_2$ and $g_0(0) \neq 0$. We claim that g_2 is a monomial. For if g_2 had a root $x = \alpha \neq 0$ then the reduced form of the fiber $\phi^{-1}(\alpha)$ would be an $\boldsymbol{A}^1$. This is a contradiction by Suzuki's formula applied to ϕ since $\chi(S) = 1$ and $\phi^{-1}(0) \cong \boldsymbol{A}^1$. It can be assumed without loss of generality and after a linear change of variables that $g_2 = x^l$ where $l > 0$.

$\phi^{-1}(0)$ is a singular fiber isomorphic to $\boldsymbol{A}^1$ with multiplicity 2. A direct application of Lemma 2.8 tells that ϕ does not have any reducible singular fiber outside $x = 0$. This implies that the quadratic in y and z, $z^2 - x(g_2(x)y^2 + g_1(x)y + g_0(x))$, with coefficients in $\boldsymbol{C}[x]$, is not factorizable at any $x = \lambda \neq 0$, which means $G = g_2(x)y^2 + g_1(x)y + g_0(x)$ has no repeated roots at any $x \neq 0$. Therefore the discriminant $D = g_1^2 - 4g_2g_0 = g_1^2 - 4x^l g_0$ has no roots except possibly $x = 0$. Since D does not have any root other than $x = 0$, it follows that $D = cx^k$ for some $c \in \boldsymbol{C}^*$ and $k \in \boldsymbol{Z}_{\geq 0}$. This implies that $g_0 = (g_1^2 - cx^k)/4x^l$.

We have the constraints $g_0 \in \boldsymbol{C}[x]$, $g_0(0) \neq 0$ and $x \mid g_1$. Assume that $g_1 = x^t g(x)$ where $t \in \boldsymbol{Z}_{>0}$, $g \in \boldsymbol{C}[x]$ and x is not a factor of $g(x)$. Then $g_0 = (x^{2t}g^2 - cx^k)/4x^l$.

We determine the relation between the integers we have introduced so far so that the above constraints are satisfied. Let $x^t g(x) = \sum_{i=t}^{d} a_i x^i$ with $a_i \in \boldsymbol{C}$, $a_t \neq 0$ and $a_d \neq 0$. Either we have $2t > l$, $2t < l$ or $2t = l$.

Case 1: $2t > l$. In this case $x^{2t}g^2/4x^l$ is a polynomial vanishing at $x = 0$ so we must have $k = l$ for g_0 to be a polynomial and $c \neq 0$ for $g_0(0)$ to be a non-zero constant. This is also sufficient which proves part b(i).

Case 2: $2t = l$. In this case, $x^{2t}g^2/4x^l = g^2/4$ is a polynomial which doesn't vanish at $x = 0$. So if moreover $k > l$ then the conditions $g_0 \in \boldsymbol{C}[x]$ and $g_0(0) \neq 0$ are automatically satisfied. If $k < l$ then $c = 0$ is necessary to make g_0 a polynomial but we know that $c \neq 0$. Hence $k < l$ is not possible. Finally in the case when $k = l$ we must have $c \neq a_t^2$ to ensure $g_0(0)$ is non-zero constant. These conditions are also sufficient as can be seen from the equations. This proves part b(ii).

Case 3: $2t < l$. In this case we must have $k = 2t$ and $c = a_t^2$ for g_0 to be a polynomial. Let $i \geq 1$ be the smallest integer such that $a_{t+i} \neq 0$. Then $x^{2t}g(x)^2 = x^{2t}(a_t^2 + 2a_t a_{t+i}x^i + \cdots)$ and we must have $2t + i = l$ for $g_0(0) \neq 0$. We must also have $2d \geq l$ since $g_0(0)$ is a polynomial. This proves part b(iii) and finishes the proof of part (b) of the lemma. So we have finally : $\{2t < l\} \Leftrightarrow \{k = 2t,\ c = a_t^2,\ 2t + i = l$ for largest i such that $x^i|(g(x) - g(0))\}$.

The only singular fiber of ϕ is $\phi^{-1}(0) \cong \boldsymbol{A}^1$. By Lemma 2.13(3, 4) we see that ϕ is a twisted fibration. Let $U = S - \phi^{-1}(0)$ be an open set in S. Restricted to U, ϕ is a twisted $\boldsymbol{C}^*$-fibration over $\boldsymbol{C}^*$ with no singular fiber. Hence it has $\boldsymbol{C}^* \times \boldsymbol{C}^*$ as an etale double cover. Therefore $\bar{\kappa}(U) = 0$ since log-Kodaira dimension doesn't change under etale maps. Therefore $\bar{\kappa}(S) \leq 0$ since $\bar{\kappa}$ is a non-decreasing function under restriction to an open set. If $\bar{\kappa}(S) = -\infty$ then by Proposition 4.1 we see that the y-degree of f has to be 1. This is a contradiction since the y-degree of f is clearly 2 by part (b). Hence $\bar{\kappa}(S) = 0$.

Apply Lemma 2.13(4a), with $r = 0$ to see that if S is NC-minimal then $N = -1/2$ hence $\bar{\kappa}(S) = -\infty$. This is a contradiction to part (c). Hence S is not NC-minimal proving part (d).

The fibration ϕ is twisted as seen above. Suppose now that l is odd. Consider S as a curve defined over the function field $\boldsymbol{C}(x)$. To find out the number of divisors at infinity we homogenise the defining polynomial of S by introducing a variable u and get:

$$z^2 - x(x^l y^2 + x^t g(x) y u + g_0 u^2)$$

which, at $u = 0$ becomes $z^2 - x^{l+1}y^2$. This latter polynomial defines a reducible divisor since $l + 1$ is even. Hence ϕ is untwisted, a contradiction. Hence l is even.

This completes the proof. □

Lemma 3.10. *The surface $H[1,-1]$ is not an n-cover of $\boldsymbol{A}^2$.*

Proof. Assume that $H[1,-1]$ is an n-cover of $\boldsymbol{A}^2$. We look at the natural action of the group $G = \boldsymbol{Z}/n\boldsymbol{Z}$ on $H[1,-1]$. By Lemma 2.11(iv) there are two lines on this surface such that they intersect each other transversally at two points. One of these lines is C_0. Let us call the other one L. Under G-action L must map to itself as $H[1,-1]$ does not have any other line K with the property that its complement $S-K$ has the same $\bar{\kappa}$ as $S-L$. Next we observe that L has two fixed points, the points of its intersection with C_0. So we get an automorphism of L with two fixed points. Any such automorphism on an $\boldsymbol{A}^1$ is identity. Therefore L is pointwise fixed under G-action, a contradiction since L is not in the branch locus (it intersects C_0) and only the branch locus can be pointwise fixed by G. Hence the lemma follows. □

Lemma 3.11. *Suppose $\phi : X - D \to B$ is a $\boldsymbol{C}^*$-fibration on a smooth affine surface X to a curve B where $D \subset X$ is an embedding of $\boldsymbol{A}^1$ in X. Then ϕ extends to a map $\phi' : X \to B'$, for a curve B' if $\bar{\kappa}(X) \neq -\infty$.*

Proof. The map ϕ is a rational map on X. Either (a) the closure of all but finitely many fibers intersect D in one point or, (b) the general fibers of ϕ intersect D in distinct points or, (c) closure of only finitely many but atleast two fibers intersect D or, (d) exactly one of the closure of the fibers of ϕ intersects D or, (e) all the fibers of ϕ are closed in X. In case (a) we blow up X along the base points until we get a morphism on a variety $Y \supset X$. Now note that Y has infinitely many affine lines, namely the proper transforms of D and the closure of the fibers of ϕ. Hence $\bar{\kappa}(X) = -\infty$, a contradiction, so this case does not occur. In case (b), X contains infinitely many contractible curves since the closure of the general fibers of ϕ are contractible, hence $\bar{\kappa}(X) = -\infty$, a contradiction, so this case also does not occur. In case (c) we still have three or more contractible curves on X hence $\bar{\kappa}(X) = -\infty$, so this case also does not occur. In case (d) we extend ϕ by mapping D to the image of the fiber of ϕ whose closure it intersects. In case (e) we resolve the indeterminacy on X and restrict the obtained morphism to X to get an extension of ϕ. This proves the lemma. □

Lemma 3.12. *Suppose $S = \{z^n - f(x,y) = 0\} \subset \boldsymbol{A}^3$ is a $\boldsymbol{Q}$-homology plane with branch locus $C = C_0 \amalg \ldots \amalg C_r$ and $C_0 \cong \boldsymbol{A}^1$. If $\phi : S \to \boldsymbol{A}^1$ is a $\boldsymbol{C}^*$-fibration on S such that C_0 is a full fiber then, in a suitable choice of coordinates on $\boldsymbol{C}^2$, ϕ is given by the function x.*

Proof. Assume that $C_0 = \{x = 0\} \subset \boldsymbol{C}^2$. We call $\pi^{-1}(C_i)$ as C_i again. The divisor $C_0 \subset S$ is n-torsion hence there exists a function h on S such that

$(h) = nC_0$. Then (h/x) has no poles or zeroes on the surface. But S is a $\boldsymbol{Q}$-homology plane therefore it has no global non-constant invertible functions. Therefore upto a constant multiple h is x. The lemma follows. □

LEMMA 3.13. *Suppose X is a $\boldsymbol{Q}$-homology plane with $\bar{\kappa}(X) \neq -\infty$ and has a $\boldsymbol{C}^*$-fibration $\phi : X \to \boldsymbol{P}^1$. Then X has atleast three singular fibers including a fiber isomorphic to $\boldsymbol{A}^1$ possibly with some multiplicity.*

PROOF. Follows by the formula (2) of Lemma 2.13. □

LEMMA 3.14. *Suppose X is a $\boldsymbol{Q}$-homology plane with $\bar{\kappa}(X) \neq -\infty$ and has an untwisted $\boldsymbol{C}^*$-fibration $\phi : X \to \boldsymbol{A}^1$. Then X has atleast two singular fibers.*

PROOF. Follows by the formula (3) of Lemma 2.13. □

4. One line and one (or more) $\boldsymbol{C}^*$'s in the branch locus.

We recall the notation. $S = \{z^n - f(x,y) = 0\} \subset \boldsymbol{A}^3$ is a $\boldsymbol{Q}$-homology plane with branch locus $C = C_0 \amalg \ldots \amalg C_r$. In this section we assume that the ramification locus consists of exactly one line, i.e, $C_0 \cong \boldsymbol{A}^1$, $C_i \cong \boldsymbol{C}^*$ for $i = 1, \ldots, r$. We investigate S depending on whether $\bar{\kappa}(S) = -\infty, 0$ or 1. Since S contains an $\boldsymbol{A}^1$ it is not of general type [**MT**].

4.1. The case $\bar{\kappa}(S) = -\infty$.

PROPOSITION 4.1. *Suppose $S = \{z^n - f(x,y) = 0\} \subset \boldsymbol{A}^3$ is a smooth affine algebraic surface with $\bar{\kappa}(S) = -\infty$ and branch locus $C = C_0 \amalg \ldots \amalg C_r$, $C_0 \cong \boldsymbol{A}^1$, $C_i \cong \boldsymbol{C}^*$ where C_0 is defined by x and C_i are defined by $xg_i(x,y) + 1$. Then S is $\boldsymbol{Q}$-homology plane if and only if $f(x,y) = x(x^k y + h(x))$ where $h \in \boldsymbol{C}[x]$ such that $h(0) \neq 0$ and $k \in \boldsymbol{Z}_{\geq 1}$. In particular, $r = 1$.*

PROOF. Assume that S is a $\boldsymbol{Q}$-homology plane. If $\bar{\kappa}(S) = -\infty$ then there is an $\boldsymbol{A}^1$-fibration on S. Note that S is either ML_0 or ML_1. We consider both these cases.

Case 1: Suppose S is ML_0. Then by Lemma 2.19 we get an $\boldsymbol{A}^1$-fibration $\phi : S \to \boldsymbol{A}^1$ such that C_0 is a fiber component. But S is a $\boldsymbol{Q}$-homology plane, so all fibers of ϕ are irreducible and the reduced form of each fiber is isomorphic to $\boldsymbol{A}^1$. Therefore C_0 is the full fiber possibly with some multiplicity. Hence we can assume by Lemma 3.12 that ϕ is defined by x on S. At a general point $x = \lambda$ the fiber is isomorphic to $\boldsymbol{A}^1$ and is given by the algebra $\boldsymbol{C}[x,y,z]/(x - \lambda, z^n - f(x,y)) \cong \boldsymbol{C}[y,z]/(z^n - f(\lambda,y))$. By Lemma 3.6, $f(\lambda,y)$ is linear in y. Therefore the number of irreducible factors of f is two. Suppose $f(x,y) =$

$x(xh_1(x)y + xh_0(x) + 1)$ is an irreducible decomposition where $h_0, h_1 \in \boldsymbol{C}[x]$. If h_1 has a root at $x = \lambda \neq 0$ then the fiber $\phi^{-1}(\lambda)$ is disjoint union of n copies of $\boldsymbol{A}^1$. This is impossible. Hence $x = 0$ is the only root of h_1. Rename $xh_0(x) + 1$ as $h(x)$ and assume that $xh_1(x) = x^k$ without loss of generality. Therefore $f = x(x^k y + h(x))$, $h \in \boldsymbol{C}[x]$ and $h(0) \neq 0$. This settles the case of S being ML_0.

Case 2: Suppose S is ML_1. We observe that S is not one of the surfaces in the Example 2.21 (Example 3.9 of [**GMMR**]) by Lemma 4.2. By Lemma 2.20 we get a unique $\boldsymbol{A}^1$-fibration on S with C_0 as a fiber. By similar analysis as above we get the same list of surfaces.

This completes the proof of the "only if" part.

Conversely, the equation $\{z^n - x(x^k y + h(x)) = 0\}$ defines a $\boldsymbol{Q}$-homology plane since it has an $\boldsymbol{A}^1$-fibration defined by x. The last fact is seen by using an exact sequence from Suzuki's paper [**Su77**, Lemme 7]

$$H_1(F, \boldsymbol{R}) \to H_1(X, \boldsymbol{R}) \to H_1(B, \boldsymbol{R}) \to 0$$

where a smooth surface X has an F-fibration over a smooth curve B and F is an irreducible general fiber. In our case $X = S$, $F \cong \boldsymbol{A}^1$ and $B \cong \boldsymbol{A}^1$ so $H_1(F, \boldsymbol{R}) = H_1(B, \boldsymbol{R}) = (0)$. Therefore $H_1(S, \boldsymbol{R}) = (0)$ proving that S is a $\boldsymbol{Q}$-homology plane. □

Lemma 4.2. *The surfaces of Example* 2.21 *are not cyclic covers of* $\boldsymbol{A}^2$. *In particular, the pair* (S, C_0) *of Theorem* 4.1 *is not isomorphic to any of the surfaces of Example* 2.21 (*Example* 3.9 *of* [**GMMR**]).

Proof. Suppose (S, C_0) is one of the surfaces in Example 2.21. If C_0 is a fiber of an $\boldsymbol{A}^1$-fibration $\phi : S \to \boldsymbol{A}^1$ then by Lemma 3.12, ϕ is defined by x. It follows by the methods of the last proposition that S will be defined by the polynomials of Proposition 4.1. Since the surfaces of Example 2.21 are exceptions to Lemma 2.20, they do not have the property of C_0 occuring as a fiber of any $\boldsymbol{A}^1$-fibration on S. So we assume C_0 is not a fiber ϕ.

By Example 2.21, property (2) it follows that $C_0 = H$ (in the notation of the said example) and it is transversal to an $\boldsymbol{A}^1$-fibration $\psi : S \to \boldsymbol{A}^1$. Since S is ML_1, $\boldsymbol{Z}/n\boldsymbol{Z}$ maps fibers of ψ to fibers of ψ itself. We note that C_1 is also transversal to ϕ and intersects all but perhaps one fiber, say F'. So each fiber except F' has two fixed points under the action of the group $\boldsymbol{Z}/n\boldsymbol{Z}$ on S (action is $z \mapsto \omega z$ where ω is an n^{th}-root of unity). It is clear that identity is the only automorphism of $\boldsymbol{A}^1$ fixing two points. Hence these fibers are pointwise fixed. This implies that $S - F'$ is pointwise fixed by the action of $\boldsymbol{Z}/n\boldsymbol{Z}$ which is a contradiction since only the branch curves C_i should be pointwise fixed. It follows that the surfaces of the Example 2.21 are not cyclic branch covers of $\boldsymbol{A}^2$. □

4.2. The case $\bar{\kappa}(S) = 0$.

PROPOSITION 4.3. *Suppose $S = \{z^n - f(x, y) = 0\} \subset \boldsymbol{A}^3$ is a $\boldsymbol{Q}$-homology plane which is a branched cover of the plane with branch locus $C = C_0 \amalg \ldots \amalg C_r$, $C_0 \cong \boldsymbol{A}^1$, $C_i \cong \boldsymbol{C}^*$. If $\bar{\kappa}(S) = 0$ then S is isomorphic to one of the surfaces given by Lemma* 3.9.

PROOF. We consider the cases of Lemma 2.11.

Step 1: Case (i) of Lemma 2.11. If S is not NC-minimal then S has a unique contractible curve such that its complement is of $\bar{\kappa} = 0$. So $C_0 \subset S$ is that curve and $\bar{\kappa}(S - C_0) = 0$. As $\chi(S - C_0) = 0$ we apply Lemma 2.10 to get the following cases:

Step 1.1: There is map $\phi : S - C_0 \to \boldsymbol{C}^*$ with connected general fibers. Either ϕ is a C^∞-fiber bundle or a general fiber is isomorphic to $\boldsymbol{C}$ or $\boldsymbol{C}^*$. The general fiber cannot be $\boldsymbol{C}$ as then $\bar{\kappa}(S) = -\infty$ and by assumption $\bar{\kappa}(S) = 0$.

Step 1.1.1: Suppose ϕ is a C^∞-fiber bundle with general fiber of general type. Then by Kawamata's inequality $\bar{\kappa}(S - C_0) \geq 1$, a contradiction. So this case does not occur.

Step 1.1.2: Suppose $\phi : S - C_0 \to \boldsymbol{C}^*$ has $\boldsymbol{C}^*$ as the general fiber. Then ϕ extends to $\bar{\phi} : S \to \boldsymbol{A}^1$ by Lemma 3.11. C_0 is not horizontal to ϕ as otherwise $\bar{\phi}$ will have many lines implying that $\bar{\kappa}(S) = -\infty$. By Lemma 3.12, ϕ is given by x so we get a possible list of surfaces for S by Lemma 3.9.

Step 2: Case (ii) of Lemma 2.11 does not occur as the lemma says S has no contractible curves but C_0 is a contractible curve on S.

Step 3: Case (iii) of Lemma 2.11. S is NC-minimal. $S \cong H[k, -k]$ with $k \geq 2$ and C_0 is the unique contractible curve on S with $\bar{\kappa}(S - C_0) = 0$. Since $\chi(S - C_0) = 0$, we apply Lemma 2.10 exactly as in Step 1 to get the same list of surfaces as in Lemma 3.9. But the same lemma asserts that these surfaces are not NC-minimal so they do not occur here.

Step 4: Case (iv) of Lemma 2.11 gives $S \cong H[1, -1]$. But by Proposition 3.10 we see that $H[1, -1]$ cannot be a cyclic branch cover of $\boldsymbol{A}^2$.

The proposition is now proved. □

Proposition 4.4. *The surfaces given by the Lemma* 3.9 *are* $\boldsymbol{Q}$*-homology planes.*

Proof. In a smooth compactification of S such that the boundary divisor has simple normal crossings, the irreducible components of the divisor are linearly independent. For, the fibration on S, given by x, is twisted, hence the union of the 2-section at infinity and each fiber minus one irreducible component is a divisor whose irreducible components are linearly independent. Therefore S is a $\boldsymbol{Q}$-homology plane. □

4.3. The case $\bar{\kappa}(S) = 1$.

A $\boldsymbol{Q}$-homology plane of $\bar{\kappa} = 1$ always has a $\boldsymbol{C}^*$-fibration with base either $\boldsymbol{P}^1$ or $\boldsymbol{A}^1$. We consider both these cases.

Proposition 4.5. *Suppose* $S = \{z^n - f(x,y) = 0\} \subset \boldsymbol{A}^3$ *is a* $\boldsymbol{Q}$*-homology plane with branch locus* $C = C_0 \amalg \ldots \amalg C_r$, $C_0 \cong \boldsymbol{A}^1$, $C_i \cong \boldsymbol{C}^*$ *and* $\bar{\kappa}(S) = 1$ *such that* S *has a* $\boldsymbol{C}^*$*-fibration onto* $\boldsymbol{P}^1$*. Then*

(1) $f(x,y) = x\prod_{i=1}^{r}(h(x,y)^{\mu_1} + \lambda_i x^{\mu_0})$ *with* $\mu_1 > 1$ *if* $r = 1$; *or*

(2) $f(x,y) = xh(x,y)\prod_{i=2}^{r}(h(x,y)^{\mu_1} + \lambda_i x^{\mu_0})$ *with* $r \geq 2$

where

- $\mu_i \in \boldsymbol{Z}_{>0}$ *for* $i = 0, 1$ *and* $(\mu_0, \mu_1) = 1$,
- $\lambda_i \in \boldsymbol{C}^*$ *are distinct constants for* $i = 1, \ldots, r$,
- $h(x,y) = (x^l y + p(x))$, $p(x) \in \boldsymbol{C}[x]$, $p(0) \neq 0$ *and* $l \in \boldsymbol{Z}_{>0}$.

Proof. Let $\phi : S \to \boldsymbol{P}^1$ be the $\boldsymbol{C}^*$-fibration. There is an action of the cyclic group $G := \boldsymbol{Z}/n\boldsymbol{Z}$ on S by $(x,y,z) \mapsto (x,y,\zeta z)$ where ζ is an n^{th}-root of unity. The generator of G acts on S producing another $\boldsymbol{C}^*$-fibration say $\tilde{\phi}$, which by Lemma 2.12 is the same as ϕ upto an automorphism of the base $\boldsymbol{P}^1$. So the G-action permutes the fibers of ϕ and gives an automorphism of the base which we call ξ.

Claim is that the branch curves $C_0, \ldots, C_r$ are fibers of ϕ. Suppose C_i, for some i, is not in a fiber of ϕ (henceforth we say that it is horizontal to ϕ). Then the induced automorphism ξ, on $\boldsymbol{P}^1$ is identity. This is because except for a finite number of fibers, others intersect C_i and hence have a fixed point under the G-action, namely the point of intersection with C_i. It follows that these fibers are stable under the G-action. Therefore all but finitely many points of $\boldsymbol{P}^1$ are fixed, hence ξ is identity. So the fibers of ϕ are acted upon by G as automorphisms with a fixed point. Since a general fiber of ϕ is a $\boldsymbol{C}^*$ therefore $n = 2$ and the quotient by G of such a fiber is $\boldsymbol{A}^1$. This latter fact is easy to see by looking at the ring of invariants. So on $\boldsymbol{A}^2$, the quotient of S by G, we get an $\boldsymbol{A}^1$ fibration with base $\boldsymbol{P}^1$. But this is a contradiction by Suzuki's formula :

$$\begin{aligned}&\chi(\boldsymbol{A}^2) = \chi(\boldsymbol{P}^1)\chi(\boldsymbol{A}^1) + (\text{non-neg terms})\\ &\Rightarrow\ 1 = 2 + (\geq 0).\end{aligned}$$

Therefore for $i = 0, \ldots, r$, C_i are fibers of ϕ.

So the fibers of ϕ are permuted by the G-action while the branch curves C_i, which are also fibers, are left pointwise fixed. It is possible that this permutation is the identity permutation. In any case, ϕ induces a $\boldsymbol{C}^*$-fibration on $\boldsymbol{A}^2 = S/G$ with base $\boldsymbol{P}^1$. We call this fibration ϕ'.

$$\begin{array}{ccc} S & \xrightarrow{\ \psi\ } & \boldsymbol{A}^2 \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ \boldsymbol{P}^1 & \xrightarrow{\ \xi\ } & \boldsymbol{P}^1 \end{array}$$

If D is a fiber of ϕ with multiplicity μ such that its image in $\boldsymbol{A}^2$ is D' with multiplicity μ', ramification index of ξ at $\phi(D)$ is d and ramification index of ψ on D is d' then

$$\mu d = \mu' d'. \tag{3}$$

Since $C_0 \cong \boldsymbol{A}^1$, it is a singular fiber of ϕ' therefore ϕ' has atmost one other singular fiber since a $\boldsymbol{C}^*$-fibration on $\boldsymbol{A}^2$ can have at most two singular fibers by Remark 2.16.

By Lemma 2.15 we can choose coordinates on $\boldsymbol{A}^2$ such that the pencil associated to ϕ' is given by:

$$\Lambda = (x^l y + p(x))^{\mu_1} + \lambda x^{\mu_0};\ \lambda \in \boldsymbol{P}^1,$$

where $p(x) \in \boldsymbol{C}[x]$ and $p(0) \neq 0$. This pencil has singular fibers at $\lambda = \infty$ and at $\lambda = 0$ if $\mu_1 > 1$. So the polynomial which defines C_0 is given by $\lambda = \infty$, i.e. x. The defining polynomials for the other fibers are given by various other values of λ. So C_i can be assumed to be given by substituting $\lambda_i \in \boldsymbol{C}$ in Λ. If none of the branch curves other than C_0 is singular for ϕ' then we get the first polynomial in the proposition. If a branch curves C_i is singular for ϕ' then it is defined by $x^l y + p(x)$, i.e., reduced of the polynomial Λ at $\lambda = 0$. This gives us the second equation in the proposition.

Suppose that $r = 1$ in the first equation of the proposition. If further $\mu_1 = 1$ then x will give an $\boldsymbol{A}^1$-fibration on S forcing $\bar{\kappa}(S) = -\infty$, a contradiction. Therefore $\{r = 1\} \Rightarrow \{\mu_1 > 1\}$ in the first equation in the proposition.

Suppose that $r = 1$ in the second equation. Since h is linear in y we get an $\boldsymbol{A}^1$-fibration on S given by x, which is also not possible since $\bar{\kappa}(S) = 1$. Therefore in the second equation $r \geq 2$. $\square$

PROPOSITION 4.6. *Suppose* $S = \{z^n - f(x,y) = 0\} \subset \boldsymbol{A}^3$ *is a smooth affine algebraic surface with branch locus* $C = C_0 \amalg \ldots \amalg C_r$, $C_0 \cong \boldsymbol{A}^1$ *is defined by* x, $C_i \cong \boldsymbol{C}^*$, $\bar{\kappa}(S) = 1$ *and with a* $\boldsymbol{C}^*$*-fibration to* $\boldsymbol{P}^1$. *Then* S *is a* $\boldsymbol{Q}$*-homology plane if and only if:*

(1) $f(x,y) = x(h(x,y)^{\mu_1} + \lambda_1 x^{\mu_0})$ *with* $\mu_1 \geq 2$; *or*
(2) $f(x,y) = x\prod_{i=1}^{r}(h(x,y)^{\mu_1} + \lambda_i x^{\mu_0})$ *with* $r \geq 2$ *and* $(n, 1 + \mu_0 r) = 1$; *or*
(3) $f(x,y) = xh(x,y)\prod_{i=2}^{r}(h(x,y)^{\mu_1} + \lambda_i x^{\mu_0})$ *with* $r \geq 2$ *and* $(n, \mu_0 + \mu_1 + \mu_0\mu_1(r-1)) = 1$

where

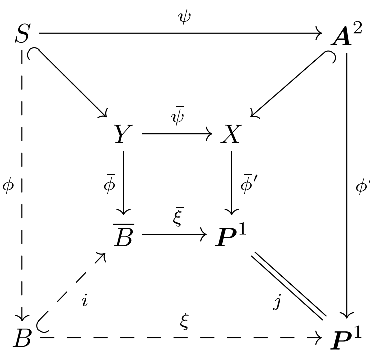
- $h(x,y) = (x^l y + p(x))$, $p(x) \in \boldsymbol{C}[x]$, $p(0) \neq 0$, $l \in \boldsymbol{Z}_{>0}$,
- for $i = 0, 1$, $\mu_i \in \boldsymbol{Z}_{>0}$ *and* $(\mu_0, \mu_1) = 1$,
- for $i = 1, \ldots, r$, $\lambda_i \in \boldsymbol{C}^*$ *are distinct constants.*

PROOF. For the 'if' case we already have a potential list of such surfaces by Proposition 4.5. We work with this list to prune it further.

Step 1: The polynomials defining the $\boldsymbol{C}^*$'s in the branch locus belong to the linear system on $\boldsymbol{A}^2$ with base $\boldsymbol{P}^1$ given by :

$$\Lambda = (x^l y + p(x))^{\mu_1} + \lambda x^{\mu_0};\ \lambda \in \boldsymbol{P}^1.$$

Let $\phi' : \boldsymbol{A}^2 \to \boldsymbol{P}^1$ be the fibration given by the above linear system. Let $\psi : S \to \boldsymbol{A}^2$ be the map given by projection along z. Let $X \supset \boldsymbol{A}^2$ and $Y \supset S$ be smooth compactifications such that ϕ' extends to $\bar{\phi}' : X \to \boldsymbol{P}^1$ as a $\boldsymbol{P}^1$-fibration and ψ extends to $\bar{\psi} : Y \to X$. We can choose Y such that $Y \backslash S$ is a normal crossings divisor and G action extends to Y. The above notations are shown in the diagram below:



For the map $\bar{\phi}' \circ \bar{\psi} : Y \to \boldsymbol{P}^1$ let $\overline{B}$ be the normalization of $\boldsymbol{P}^1$ in the function field of Y, $B = \bar{\phi}(S)$, $\phi = \bar{\phi}|_S$, i the inclusion map, ξ the induced map from $\bar{\xi}$ and j is identity map.

Step 2:
Claim: $\bar{\phi}$ is a $\boldsymbol{P}^1$-fibration and ϕ is a $\boldsymbol{C}^*$-fibration.
We find out the fibers of the map $\phi' \circ \psi$.

Case A: $f(x,y) = x\prod_{i=1}^{r}(h(x,y)^{\mu_1} + \lambda_i x^{\mu_0})$ where $r \geq 1$.
The inverse image by ψ of a branch curve C_i is irreducible. Let F_λ be a fiber of ϕ', different from the branch curves, and given by $h^{\mu_1} + \lambda x^{\mu_0}$. Its inverse image in S corresponds to the ring $A := \boldsymbol{C}[x,y,z]/(z^n - f(x,y), h^{\mu_1} + \lambda x^{\mu_0})$. Since C_0 is given by $\{x = 0\}$ in $\boldsymbol{A}^2$ and F_λ is disjoint from C_0 therefore x is invertible in A. So $A = \boldsymbol{C}[x, 1/x, y, z]/(z^n - f(x,y), h^{\mu_1} + \lambda x^{\mu_0})$. Since x is a unit, we replace $h = x^l y + p(x)$ by y and simplify $f(x,y)$ in the ideal to get $A = \boldsymbol{C}[x, 1/x, y, z]/(z^n - \prod(\lambda_i - \lambda)x^{1+r\mu_0}, y^{\mu_1} + \lambda x^{\mu_0})$. Since $(\mu_0, \mu_1) = 1$ the above fiber has $(n, 1 + r\mu_0)$ irreducible components. Observe that this is true even if $\lambda = 0$. Each of the components is of the type $R = \boldsymbol{C}[x, 1/x, y, z]/(z^a - c_1 x^b, y^{\mu_1} + \lambda x^{\mu_0})$ which is isomorphic to $\boldsymbol{C}^*$ by the parametrization $x = t^{a\mu_1}$, $y = c_2 t^{a\mu_0}$ and $z = c_3 t^{b\mu_1}$ where a, b are some positive integers such that $(a,b) = 1$ and for $i = 1,2,3$, c_i are appropriately chosen non-zero complex numbers. Hence a general fiber of $\phi' \circ \psi$ is a disjoint union of $\boldsymbol{C}^*$'s.

Case B: $f(x,y) = xh(x,y)\prod_{i=2}^{r}(h(x,y)^{\mu_1} + \lambda_i x^{\mu_0})$ where $r \geq 2$.
Similar to above a general fiber of $\psi \circ \phi'$ is given by the ring

$$A := \frac{\boldsymbol{C}[x, \frac{1}{x}, y, z]}{(z^n - y(\prod(\lambda_i - \lambda))x^{1+(r-1)\mu_0}, y^{\mu_1} + \lambda x^{\mu_0})}.$$

We eliminate y to get

$$A = \frac{\boldsymbol{C}[x, \frac{1}{x}, z]}{\left(\left(\frac{z^n}{\prod(\lambda_i - \lambda)x^{1+(r-1)\mu_0}}\right)^{\mu_1} + \lambda x^{\mu_0}\right)} \cong \frac{\boldsymbol{C}[x, \frac{1}{x}, z]}{(z^{n\mu_1} - \lambda' x^{\mu_0 + \mu_1(1+(r-1)\mu_0)})}.$$

The curve defined by A has $(n\mu_1, \mu_0 + \mu_1(1 + (r-1)\mu_0)) = (n, \mu_0 + \mu_1 + \mu_0\mu_1(r-1))$ irreducible and disjoint factors. Each of the irreducible components is given by $R = \boldsymbol{C}[x, 1/x, z]/(z^a - x^b)$ which is isomorphic to a $\boldsymbol{C}^*$ by the parametrization $x = t^a$ and $z = t^b$ where $(a,b) = 1$.

So we have proved in all cases that a general fiber of the map $\phi' \circ \psi$ is a disjoint union of finitely many $\boldsymbol{C}^*$'s.

Now suppose that a general fiber of $\phi' \circ \psi$ over a point $p \in \boldsymbol{P}^1$ is $\coprod_{i=1}^u F_i$ where each $F_i \cong \boldsymbol{C}^*$. Then the fiber of the map $\bar{\phi}' \circ \bar{\psi}$ over the same point p is $\coprod_{i=1}^u \bar{F}_i$ where $\bar{F}_i$ is the closure of F_i in Y. Since Y is complete, smooth and since $\bar{\psi}$ extends ψ with G-action it follows that $\bar{F}_i \cong \boldsymbol{P}^1$. So the Stein factorization $\bar{\phi}$ of $\bar{\phi}' \circ \bar{\psi}$ is a $\boldsymbol{P}^1$-fibration and a general fiber of $\bar{\phi}$ is obtained by taking closure in Y of a $\boldsymbol{C}^* \subset S$. Since a fiber of ϕ is nothing but intersection of a fiber of $\bar{\phi}$ with S therefore a general fiber of ϕ is a $\boldsymbol{C}^*$.

Step 3:

Claim: $B = \bar{B}$.

Suppose $p \in \bar{B} \backslash B,\ q = \bar{\xi}(p),\ T = \bar{\phi}^{-1}(p),\ W = \bar{\phi}'^{-1}(q)$ and $Z = \phi'^{-1}(q)$. It is clear that $T \subset Y \backslash S$ and $\bar{Z} \subset W$. By the map $\bar{\psi}$, T surjects on W hence contains Z, i.e., image of T intersects $\boldsymbol{A}^2$. This is a contradiction since S is dense in Y and from the properness of ψ it follows that the full inverse image of $\boldsymbol{A}^2$ in Y is S.

Step 4: A necessary condition for the surface S to be a $\boldsymbol{Q}$-homology plane is that $\bar{B} \cong \boldsymbol{P}^1$ which by Step 3 is the same as $B \cong \boldsymbol{P}^1$. In Steps 5 and 6 we find out those polynomials which satisfy this condition.

Step 5: Suppose $r = 1$, that is C_0 and C_1 are the only branch curves. Then equation (1) is the only one allowed for S. It is clear that C_1 is not a singular fiber of ϕ'. Since $\mu_1 > 1$ we know that there has to be two singular fibers of ϕ' including C_0. Call the other one D. By Step 2 above, the number of irreducible curves in the inverse image of D is the same as that of a general fiber say F, which is $d := (n, 1 + \mu_0)$. Hence the map $B \to \boldsymbol{P}^1$ is a degree d map with exactly two points of ramifications, namely the images of C_0 and C_1, and these points are totally ramified. It follows by Riemann-Hurwitz that $B \cong \boldsymbol{P}^1$. So all the polynomials for $r = 1$ in the Proposition 4.5 are such that $\overline{B} \cong \boldsymbol{P}^1$.

Step 6: Now suppose that $r \geq 2$. We claim that $\overline{B} \cong \boldsymbol{P}^1$ if and only if the fibers of $\bar{\psi} \circ \bar{\phi}'$ are irreducible. To see this suppose that the above fibers are irreducible. Then we get an injective map from $\overline{B}$ to $\boldsymbol{P}^1$ which forces $\overline{B} \cong \boldsymbol{P}^1$. Conversely, suppose that $\overline{B} \cong \boldsymbol{P}^1$. We know that ξ has three or more totally ramified points, namely the images of the branch curves. So again by an application of Riemann-Hurwitz on ξ we get that ξ must be an isomorphism. This clearly implies that fibers of $\bar{\psi} \circ \bar{\phi}'$ are irreducible. Our claim is proved. We also conclude that fibers of $\psi \circ \phi'$ are also irreducible. This is a checkable criterion for the following type of polynomials given by Proposition 4.5:

Type 1: $f(x,y) = x\prod_{i=1}^{r}(h(x,y)^{\mu_1} + \lambda_i x^{\mu_0})$.

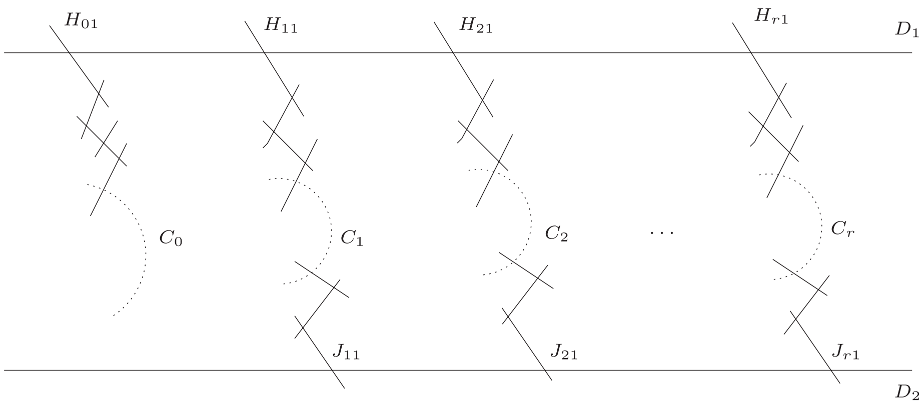
Type 2: $f(x,y) = xh(x,y)\prod_{i=2}^{r}(h(x,y)^{\mu_1} + \lambda_i x^{\mu_0})$.

For the polynomial of Type 1 we know by Step 2 that the above fiber is irreducible if and only if $(n, 1 + r\mu_0) = 1$.

For the polynomial of Type 2 it follows from Step 2 that a general fiber of $\phi' \circ \psi$ is irreducible if and only if $(n, \mu_0 + \mu_1 + \mu_0\mu_1(r-1)) = 1$. So this is the necessary condition for a polynomial of Type 2 to give rise to a $\boldsymbol{Q}$-homology plane.

Step 7: Now we prove the converse that the above polynomials indeed define a $\boldsymbol{Q}$-homology planes.

The following is the boundary divisor of S in $\overline{S}$ where the dotted curves are in S and are not part of the divisor. They are shown here only for fixing ideas.



The fibration $\bar{\phi}$ is a $\boldsymbol{P}^1$-fibration, the fibers containing C_i are linear chains for $i = 1, \ldots, r$ by Lemma 2.23 and the curves D_1, H_{ij}, J_{ij} are linearly independent in $\mathrm{Pic}(\overline{S})$. So along with D_2, there is atmost one relation between the irreducible components of the divisor $\overline{S} - S$. If there is no relation among these divisors, then S is a $\boldsymbol{Q}$-homology plane, but if there is a relation then $\Gamma(S, \mathscr{O}_S)^*/\boldsymbol{C}^* \cong \boldsymbol{Z}$. We work with the unit, say u, which generates this free group supposing that S is not a $\boldsymbol{Q}$-homology plane. Note that it is non-constant on S. If σ is the generator of $G = \boldsymbol{Z}/n\boldsymbol{Z}$ then we prove that $\sigma(u) \neq \omega u$ for some root of unity ω. For, if $\sigma(u) = \omega u$ then $\sigma(u^n) = u^n$. This implies that u^n is G-invariant and therefore descends to the quotient $\boldsymbol{A}^2$ of S as a unit. Since all the units on $\boldsymbol{A}^2$ are constants, it follows that u^n is a constant therefore u is a constant, a contradiction. Hence $\sigma(u) = c/u$ for $c \in \boldsymbol{C}^*$ which can be assumed to be 1 after substituting $u/\sqrt{c}$ for u.

If we restrict u to the fibers of ϕ it is a non-constant unit on them. Since σ

takes u to $1/u$, the points at ∞ of a general fiber of ϕ, which is a $\boldsymbol{C}^*$, are interchanged. Hence $\sigma(D_1) = D_2$ and vice-versa. But the points at ∞ of the branch curves remain fixed. Hence applying this to C_1 we get a contradiction. Therefore S is a $\boldsymbol{Q}$-homology plane. □

Proposition 4.7. *Suppose $S = \{z^n - f(x,y) = 0\} \subset \boldsymbol{A}^3$ is a $\boldsymbol{Q}$-homology plane with branch locus $C = C_0 \amalg \ldots \amalg C_r$, $C_0 \cong \boldsymbol{A}^1$, $C_i \cong \boldsymbol{C}^*$. Suppose that $\bar{\kappa}(S) = 1$ and S has a $\boldsymbol{C}^*$-fibration to $\boldsymbol{A}^1$, then*

(1) $f(x,y) = x\prod_{i=1}^{r}(x^{\alpha}h(x,y)^{\beta} + \lambda_i)$ *where* $\beta > 1$ *if* $r = 1$; *or*

(2) $f(x,y) = xh(x,y)\prod_{i=2}^{r}(x^{\alpha}h(x,y)^{\beta} + \lambda_i)$ *where* $r \geq 2$

where $h(x,y) = y$ or $h(x,y) = x^l y + p(x)$ in (1) and $h(x,y) = x^l y + p(x)$ in (2), $p(x) \in \boldsymbol{C}[x]$, $p(0) \neq 0$; $\alpha, \beta, l \in \boldsymbol{Z}_{>0}$, $(\alpha, \beta) = 1$ and $\lambda_i \in \boldsymbol{C}^*$ are distinct.

Proof. We fix the notations first. Let $\phi : S \to \boldsymbol{A}^1$ be the $\boldsymbol{C}^*$-fibration on the surface S referred to in the statement of the proposition. The group $G = \boldsymbol{Z}/n\boldsymbol{Z}$ acts on the surface by n^{th}-roots of unity in the z-variable. Call ϕ' the quotient of ϕ such that $\phi' : \boldsymbol{A}^2 \to B$ is also a fibration where B is some algebraic curve. Let ψ be the quotient map from S to $\boldsymbol{A}^2$ and let $\xi : \boldsymbol{A}^1 \to B$ be the induced map on the base curves.

$$\begin{array}{ccc} S & \xrightarrow{\psi} & \boldsymbol{A}^2 \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ \boldsymbol{A}^1 & \xrightarrow{\xi} & B \end{array}$$

Note that if D is a fiber of ϕ with multiplicity μ such that its image in $\boldsymbol{A}^2$ is D' with multiplicity μ', ramification index of ξ at $\phi(D)$ is d and ramification index of ψ on D is d' then

$$\mu d = \mu' d'. \tag{4}$$

Step 1:

Claim: The curve C_0 is vertical.

Suppose that C_0 is horizontal to ϕ.

If ϕ is twisted then all its fibers are irreducible hence C_0 intersects all the fibers. So each fiber has a fixed point under G-action so it is stable for the action. Now the branch curve C_1 is also horizontal to ϕ as it is disjoint to C_0. This implies that a general fiber of ϕ has two fixed points under the G-action. But an automorphism of $\boldsymbol{C}^*$ with two fixed points is identity. Hence the general fibers of ϕ are pointwise fixed by G implying that they are branch curves for $S \to \boldsymbol{A}^2$, a contradiction.

Still continuing with the assumption that C_0 is horizontal, suppose that ϕ is untwisted. Then it has a reducible fiber containing an $\boldsymbol{A}^1$. Since a $\boldsymbol{Q}$-homology plane with $\bar{\kappa} = 1$ can have atmost two affine lines therefore the irreducible component of the reducible fiber of ϕ, other than the $\boldsymbol{A}^1$, is a $\boldsymbol{C}^*$ as C_0 is already an $\boldsymbol{A}^1$ present in the surface. Now C_0 will intersect atleast one of these two curves and under G-action both the irreducible components are stable since the other fibers of ϕ are stable as they have an intersection point with C_0. Now the quotient of a general fiber of ϕ by G is an $\boldsymbol{A}^1$. So on the quotient $S/G \cong \boldsymbol{A}^2$ we get an $\boldsymbol{A}^1$-fibration such that the image of the reducible fiber remains reducible. This is a contradiction as there is no $\boldsymbol{A}^1$-fibration on $\boldsymbol{A}^2$ with a reducible fiber. Therefore C_0 is in a fiber of ϕ.

Step 2:

Claim: C_i are also in fibers of ϕ for $i = 1, \ldots, r$.

Suppose C_1 is not in a fiber. Then C_1 intersects all fibers except perhaps one. All those fibers which intersect C_1 have a fixed point and hence are stable under G action and their quotient by G is an $\boldsymbol{A}^1$. Moreover, the induced map on the base $\boldsymbol{A}^1$ is identity. So ϕ' is an $\boldsymbol{A}^1$ fibration on $\boldsymbol{A}^2$. Therefore the fiber of ϕ containing C_0 is also irreducible. In other words, ϕ is twisted. We note that any of the branch curves $C_i,\ i \geq 2$, can't be a fiber since otherwise C_1 will intersect it which is not allowed since they are disjoint. Suppose F_1 is a singular fiber of ϕ other than C_0 and let μ_1 be its multiplicity in S and μ' be the multiplicity of its image in $\boldsymbol{A}^2$. By the equation (4) we have:

$$\mu \cdot 1 = \mu' \cdot 1$$

therefore $\mu = \mu'$. But we know that there can be no singular fiber for ϕ', so $\mu' = 1$, which implies $\mu = 1$. Therefore ϕ has exactly one singular fiber, namely C_0. This is a contradiction since a $\boldsymbol{C}^*$-fibration on a $\bar{\kappa} = 1$ surface has atleast two singular fibers by Lemma 3.14. The upshot is that C_1 is in a fiber. Note that it is possible that both C_0 and C_1 are in a single fiber.

For the same reason as above we see that C_2 etc. are also in a fiber. Moreover, since one of the branch curves might occur as an irreducible component of the reducible fiber, all except possibly one of the $\boldsymbol{C}^*$'s in the branch locus is a full fiber of ϕ.

Step 3:

Claim: ϕ is untwisted.

If ϕ were twisted then the fiber containing C_0 would be irreducible hence in the quotient we would get a twisted $\boldsymbol{C}^*$-fibration on $\boldsymbol{A}^2$ but this is not possible as shown by the equations of Lemma 2.14. Therefore ϕ is untwisted.

Step 4:

Claim: If C_0 and C_1 are the only branch curves then they cannot occur in the same fiber ϕ.

Suppose the contrary. Then since $\bar{\kappa}(S) = 1$, there is atleast one other singular fiber of ϕ, say F_1, which is not a branch curve and has multiplicity $\mu_1 \geq 2$. Suppose that the image of F_1 in $\boldsymbol{A}^2$ is F_1' with multiplicity μ_1'. The image F_1' is a $\boldsymbol{C}^*$. Let the ramification index of ξ at $\phi(F_1)$ be $d \geq 1$, and $d' = 1$ since F_1 is not a branch curve. So we have $1 \cdot \mu_1' = \mu_1 \cdot d$, i.e., $\mu_1' \geq 2$. But we know by Lemma 2.14 that any $\boldsymbol{C}^*$-fibration on $\boldsymbol{A}^2$ over $\boldsymbol{A}^1$ has exactly one singular fiber, which provides the contradiction. This implies that only the branch curves can be singular for ϕ. In particular, the said result holds by using Lemma 3.14.

Step 5: Suppose C_0 and C_1 are in the only reducible fiber and C_2 is present as a fiber. Then the induced map on the base is identity since it has two fixed points, namely the images of C_0 and C_2. We get a $\boldsymbol{C}^*$-fibration on $\boldsymbol{A}^2$, and since the reducible fiber is disconnected because of the disjointness of C_0 and C_1, the fibration is defined by the following polynomial due to Lemma 2.14 :

$$x^\alpha (x^l y + p(x))^\beta + 1.$$

Therefore, $C_1 := \{x^l y + p(x) = 0\}$ and $C_i := \{x^\alpha (x^l y + p(x))^\beta + \lambda_i\}$ for $i = 2, \ldots, r$ and for some $\lambda_i \in \boldsymbol{C}^*$. This gives rise to the second polynomial in the proposition.

Step 6: Suppose C_0 and C_1 are in different fibers. Then the $\boldsymbol{C}^*$-fibration on $\boldsymbol{A}^2$ is given by either of the following polynomials, again by Lemma 2.14:

$$x^\alpha y^\beta + 1,$$
$$x^\alpha (x^l y + p(x))^\beta + 1.$$

Therefore $C_i := \{x^\alpha h^\beta + \lambda_i\}$ where $\lambda_i \in \boldsymbol{C}^*$, for $i = 1, \ldots, r$, and $h(x, y) = y$ or $h(x, y) = (x^l y + p(x))$. This gives rise to the first polynomial in the proposition.

Step 7: In the first polynomial in the proposition, if $r = 1$ and $\beta = 1$ then x will give an $\boldsymbol{A}^1$-fibration on S which will imply $\bar{\kappa}(S) = -\infty$ which is false. Hence $\{r = 1\} \Rightarrow \{\beta > 1\}$ in polynomial (1) of the proposition. Similarly in the second polynomial, if $r = 1$ then x gives an $\boldsymbol{A}^1$-fibration on S. Therefore $r \geq 2$ for the second polynomial. □

PROPOSITION 4.8. *Suppose $S = \{z^n - f(x,y) = 0\} \subset \boldsymbol{A}^3$ is a smooth affine algebraic surface with branch locus $C = C_0 \amalg \dots \amalg C_r$, $C_0 \cong \boldsymbol{A}^1$ is defined by x, $C_i \cong \boldsymbol{C}^*$, $\bar{\kappa}(S) = 1$ and with a $\boldsymbol{C}^*$-fibration to $\boldsymbol{A}^1$. Then S is a $\boldsymbol{Q}$-homology plane if and only if*:

(1) $f(x,y) = x\prod_{i=1}^{r}(x^\alpha h^\beta + \lambda_i)$ *such that* $(n,\beta) = 1$ *and* $\beta > 1$ *if* $r = 1$; *or*

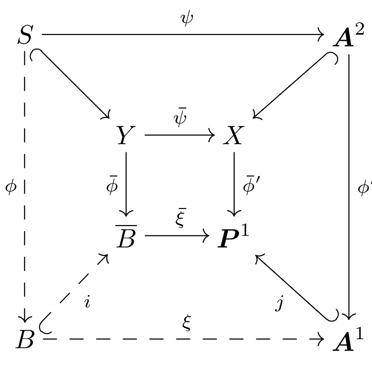
(2) $f(x,y) = xh(x,y)\prod_{i=2}^{r}(x^\alpha h^\beta + \lambda_i)$ *such that* $(n, |\alpha - \beta|) = 1$ *and* $r \geq 2$

where $h(x,y) = y$ *or* $x^l y + p(x)$ *in* (1) *and* $h(x,y) = x^l y + p(x)$ *in* (2), $p(x) \in \boldsymbol{C}[x]$, $p(0) \neq 0$; $\alpha, \beta, l \in \boldsymbol{Z}_{>0}$, $(\alpha, \beta) = 1$ *and* $\lambda_i \in \boldsymbol{C}^*$ *are distinct.*

PROOF. The strategy of the proof is to first show that S with the above equations has a $\boldsymbol{C}^*$-fibration over a curve B. Then by calculating Euler characteristic of B we show that it is not an $\boldsymbol{A}^1$ if $(n,\beta) > 1$ (resp. $(n, |\alpha - \beta|) > 1$) for the polynomial (1) (resp. (2)) in the proposition. And finally we show that $(n,\beta) = 1$ (resp. $(n, |\alpha - \beta|) = 1$) indeed implies that S is a $\boldsymbol{Q}$-homology plane.

For the 'if' case we already have a potential list of polynomials from Proposition 4.7. We will prune this list further. Let ϕ' be the fibration on $\boldsymbol{A}^2$ given by the polynomial $x^\alpha h^\beta + 1$. This is clearly a $\boldsymbol{C}^*$-fibration and the branch curves are in the fibers of ϕ'.

Step 1: Let $X \supset \boldsymbol{A}^2$ and $Y \supset S$ be smooth compactifications such that ϕ' extends to $\bar{\phi}' : X \to \boldsymbol{P}^1$ as a $\boldsymbol{P}^1$-fibration and ψ extends to $\bar{\psi} : Y \to X$. We can choose Y such that $Y \backslash S$ is a normal crossings divisor and G action extends to Y. The above notations are shown in the diagram below:



For the map $\bar{\phi}' \circ \bar{\psi} : Y \to \boldsymbol{P}^1$ let $\overline{B}$ be the normalization of $\boldsymbol{P}^1$ in the function field of Y, $B = \bar{\phi}(S)$, $\phi = \bar{\phi}|_S$, i and j the inclusion maps and ξ the induced map from $\bar{\xi}$.

Step 2:

Claim: $\bar{\phi}$ is a $\boldsymbol{P}^1$-fibration and ϕ is a $\boldsymbol{C}^*$-fibration.

We find out the fibers of the map $\phi' \circ \psi$.

Case A: $f(x, y) = x \prod_{i=1}^{r} (x^\alpha h^\beta + \lambda_i)$. The general fiber of ϕ' is disjoint from C_0 so x is invertible. The inverse image of a fiber of ϕ' by ψ is given by the ring $A := \boldsymbol{C}[x, 1/x, y, z]/(z^n - f, x^\alpha h^\beta + \lambda)$. In A, h can be replaced by y since x is a unit and f can be replaced by cx in the ideal for some $c \neq 0$. Hence x can be eliminated to get $A \cong \boldsymbol{C}[z, 1/z, y, z]/(z^{n\alpha} y^\beta + \lambda)$ after a linear change of variables. The curve defined by A has $(n\alpha, \beta) = (n, \beta)$ irreducible and disjoint factors. Each of these curves is of the type $\boldsymbol{C}[z, 1/z, y, z]/(z^a y^b + \gamma)$ where $(a, b) = 1$ and $\gamma \neq 0$. By the parametrization $z = (-\gamma)^{1/a}/t^b$, $y = t^a$ it is easily seen that the last ring defines a $\boldsymbol{C}^*$.

Case B: $f(x, y) = xh(x, y) \prod_{i=2}^{r} (x^\alpha h^\beta + \lambda_i)$. In this case the inverse image of a general fiber of ϕ' is defined by $A := \boldsymbol{C}[x, 1/x, y, z]/(z^n - f, x^\alpha h^\beta + \lambda)$, h can again be replaced by y and in the ideal $f = cxy$ for some $c \neq 0$. So $A = \boldsymbol{C}[x, 1/x, y, z]/(z^n - cxy, x^\alpha y^\beta + \lambda)$. We can now eliminate y to get $A = \boldsymbol{C}[x, 1/x, z]/(x^\alpha (z^n/cx)^\beta + \lambda)$ which implies after a linear change of variables that $A \cong \boldsymbol{C}[x, 1/x, z]/(x^{\alpha-\beta} z^{n\beta} + \lambda)$. Depending on whether $\alpha > \beta$ or $\alpha < \beta$ the ideal above is either $(x^{\alpha-\beta} z^{n\beta} + \lambda)$ or $(z^{n\beta} + \lambda x^{\beta-\alpha})$. In both the cases the curve defined by the ring A has $(n\beta, |\alpha - \beta|) = (n, |\alpha - \beta|)$ irreducible factors. Each factor is of the type $x^a z^b + \gamma$ where $(|a|, b) = 1$, $a \in \boldsymbol{Z}$, $b \in \boldsymbol{Z}_{>0}$ and $\gamma \neq 0$. By a parametrization of the type $z = (-\gamma)^{1/a}/t^b$, $x = t^a$ this curve is isomorphic to $\boldsymbol{C}^*$. So this proves that ϕ is a $\boldsymbol{C}^*$-fibration. Also, any fiber of ϕ' other than the branch curves, has the same number of inverse images by ψ. Therefore the ramification locus of ξ is exactly the points corresponding to the branch curves.

Step 3:

Claim: If $(n, \beta) > 1$ for polynomial (1) or $(n, |\alpha - \beta|) > 1$ for polynomial (2) of the proposition, then S is not a $\boldsymbol{Q}$-homology plane.

Let $t = (n, \beta)$ in case of (1) and $t = (n, |\alpha - \beta|)$ in case of (2). By hypothesis $t > 1$.

The image of the map i is mapped by $\bar{\xi}$ to the image of the map j because by the properness of $\bar{\psi}$, the inverse image in Y of the fiber of $\bar{\phi}'$ over $\infty \in \boldsymbol{P}^1$ will not intersect S. Therefore we can define the map ξ by restriction of $\bar{\xi}$ to the image of i. Now the map ξ has degree t as noted in Step 2. Moreover ξ has $r + 1$ points of total ramification, namely the images of the branch curves $C_0, C_1, \ldots, C_r$ in B. We calculate the Euler characteristic of B.

$$\chi(B) = t(1 - r - 1) + r + 1$$
$$\Rightarrow \chi(B) = r(1 - t) + 1$$
$$\Rightarrow \chi(B) \leq 0.$$

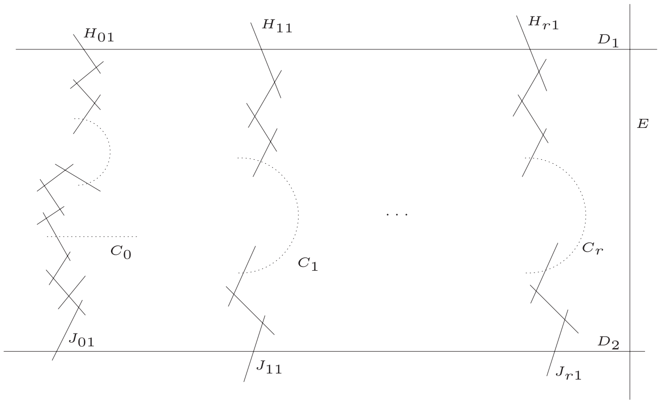
Therefore B is never $\boldsymbol{A}^1$ if $t \geq 2$. Since a $\boldsymbol{Q}$-homology plane has a $\boldsymbol{C}^*$-fibration only over $\boldsymbol{A}^1$ or $\boldsymbol{P}^1$ we conclude that S is not a $\boldsymbol{Q}$-homology plane whenever $t \geq 2$.

Step 4:

Claim: The polynomials of proposition define a $\boldsymbol{Q}$-homology plane provided $(n, \beta) = 1$ for polynomial (1) or $(n, |\alpha - \beta|) = 1$ for polynomial (2) of the proposition.

We have constructed the appropriate compactifications above. Consider now the boundary divisor $\overline{S} - S$ as shown in the figure below. The dotted curves shown are in the affine part S.

We first show that ϕ is untwisted. Suppose not. Then there is a 2-section of $\bar{\phi}$ at ∞ and the irreducible components of the divisor at infinity are automatically linearly independent (by almost the same reasoning as in Proposition 4.4). This implies that S is a $\boldsymbol{Q}$-homology plane. But we know that a twisted $\boldsymbol{C}^*$-fibration on a $\boldsymbol{Q}$-homology plane cannot have any reducible fiber. However, the fiber of ϕ containing C_0 is the inverse image of the reducible fiber of ϕ', hence is reducible, which provides a contradiction. Therefore ϕ must be an untwisted fibration.



The fibration $\bar{\phi}$ is a $\boldsymbol{P}^1$-fibration, the divisor containing C_i are linear chains for $i = 1, \ldots, r$ by Lemma 2.23 and the curves D_1, H_{ij}, J_{ij} are linearly independent. So along with D_2, the divisor $\overline{S} - S$ has atmost one relation. If there is no relation among these divisors, then S is a $\boldsymbol{Q}$-homology plane, but if there is a relation then $\Gamma(S, \mathscr{O}_S)/\boldsymbol{C}^* \cong \boldsymbol{Z}$. We work with the unit, say u, which generates this free group supposing that S is not a $\boldsymbol{Q}$-homology plane. Note that it is non-constant on S. If σ is the generator of $G = \boldsymbol{Z}/n\boldsymbol{Z}$ then we prove that $\sigma(u) \neq \omega u$ for some root of unity ω. For, if $\sigma(u) = \omega u$ then $\sigma(u^n) = u^n$ which implies that u^n is G-invariant and therefore descends to the quotient $\boldsymbol{A}^2$ of S as a unit. Since all the units on $\boldsymbol{A}^2$ are constants, it follows that u^n is a constant therefore u is a constant, a contradiction. Hence $\sigma(u) = c/u$ for $c \in \boldsymbol{C}^*$ which can be assumed to be 1 after substituting $u/\sqrt{c}$ for u.

If we restrict u to the fibers of ϕ it is a non-constant unit on them. Since σ takes u to $1/u$, the points at ∞ of a general fiber of ϕ (which is a $\boldsymbol{C}^*$) are interchanged. Hence $\sigma(D_1) = D_2$ and vice-versa. But the points at ∞ of the branch curves, say C_2, remain fixed. This is a contradiction to the continuity of G-action. Hence S is a $\boldsymbol{Q}$-homology plane. □

5. More than one lines in the branch locus.

We prove the following

Proposition 5.1. *Suppose $S = \{z^n - f(x,y) = 0\} \subset \boldsymbol{A}^3$ is a $\boldsymbol{Q}$-homology plane such that the branch locus $f(x,y) = 0$ has at least two lines. Then in a suitable coordinate system on $\boldsymbol{C}^2$, $f(x,y) = \phi(x)(\alpha(x)y + \beta(x))$ where $\phi, \alpha, \beta \in \boldsymbol{C}[x]$, $\phi(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_r)$, λ_i's are distinct complex numbers, $\sqrt{(\alpha)} = (\phi)$ and $(\alpha(x), \beta(x)) = 1$.*

Proof. Let $C := \{f(x,y) = 0\} \subset \boldsymbol{A}^2$ be the branch curve. Let $C = C_1 \amalg \ldots \amalg C_r \amalg D$ be the irreducible decomposition of C such that for $i = 1, \ldots, r$, $C_i \cong \boldsymbol{A}^1$ are all the lines in C and D is some disjoint curve. By Lemma 2.9 we can assume $C_1 = \{x = 0\}$. Consider the map $\xi : S \xrightarrow{x} \boldsymbol{A}^1$. The zero fiber is $\xi^{-1}(0) = C_1$. By Lemma 3.5(a), C_i, $i = 2, \ldots, r$ are also fibers of ξ and in $\boldsymbol{C}^2$, $C_i = \{x - \lambda_i = 0\}$ for distinct $\lambda_i \in \boldsymbol{C}$ and $\lambda_1 = 0$. We will use the notation λ_1 throughout. Let $\phi(x) = \prod_i (x - \lambda_i)$. Then $f(x,y) = \phi(x)g(x,y)$ where $g(x,y) \in \boldsymbol{C}[x,y]$. The fiber of ξ at λ_1 and λ_2 are isomorphic to $\boldsymbol{A}^1$ with multiplicity n. Let F be a general fiber of ξ. By Suzuki's formula (Lemma 2.8):

$$
\begin{aligned}
&\chi(S) = \chi(\boldsymbol{A}^1)\chi(F) + (1 - \chi(F)) + (1 - \chi(F)) + (\text{non-neg terms}) \\
&\Rightarrow 1 = \chi(F) + (1 - \chi(F)) + (1 - \chi(F)) + (\text{non-neg terms}) \\
&\Rightarrow 0 = (1 - \chi(F)) + (\text{non-neg terms}) \\
&\Rightarrow \chi(F) = 1.
\end{aligned}
$$

Therefore $F \cong \boldsymbol{A}^1$ by Lemma 2.5. So ξ is an $\boldsymbol{A}^1$-fibration. At a general point c the fiber of ξ is $\{z^n - \phi(c)g(c,y) = 0\} \cong \boldsymbol{A}^1$. It follows by Lemma 3.6 that $g(x,y)$ is linear in y. Hence D is rational and irreducible.

Suppose $g(x,y) = \alpha_1(x)y + \beta_1(x)$. Let $h(x) = (\alpha_1, \beta_1)$ be the g.c.d., $\alpha_1(x) = h(x)\alpha(x)$ and $\beta_1(x) = h(x)\beta(x)$. Then $f(x,y) = \phi(x)h(x)(\alpha y + \beta)$ and $(\alpha y + \beta)$ is irreducible. If h has a different linear factor than those which appear in ϕ then we would have found a new line in the branch locus. This is impossible as we have already counted all of the lines in the branch locus. We observe that h cannot have a factor common with ϕ otherwise one of the branch curves will appear with multiplicity and hence cannot be smooth. Therefore we conclude that h is a constant and $g(x,y) = \alpha(x)y + \beta(x)$ is irreducible. It follows that α and β have no common factor.

We prove the rest of the proposition in the following steps:

Step 1: $\alpha = 0$ implies that $f(x,y) \in \boldsymbol{C}[x]$ and is linear.

If $\alpha = 0$ then $f(x,y) = \phi(x)\beta(x)$, $S = \mathrm{Spec}(\boldsymbol{C}[x,y,z]/(z^n - f(x,y)) \cong \boldsymbol{A}^1 \times X$ where X is the curve $\mathrm{Spec}(\boldsymbol{C}[x,z]/(z^n - \phi\beta))$. Since S is a $\boldsymbol{Q}$-homology plane, its first betti number is zero, so the first betti number of the curve X is also zero, hence it is an $\boldsymbol{A}^1$. It follows that $S \cong \boldsymbol{A}^2$. So we need to find out when $X \cong \boldsymbol{A}^1$. By Lemma 3.6 it follows that $X \cong \boldsymbol{A}^1$ implies $f = \phi\beta$ is linear. Our claim is proved.

We assume that $\alpha \neq 0$ for the rest of the proof.

Step 2: $D \cong \boldsymbol{C}^{r*}$.

Follows from Lemma 3.5.

Step 3: $\sqrt{(\alpha)} = (\phi)$.

We know from Step 2 that the zero locus of $g(x,y) = \alpha(x)y + \beta(x)$ is $\boldsymbol{C}^{r*}$, in other words $\mathrm{Spec}\ \boldsymbol{C}[x,y]/(\alpha y + \beta) \cong \mathrm{Spec}\ \boldsymbol{C}[x, -\beta(x)/\alpha(x)] \cong \boldsymbol{C}^{r*}$. It follows that α has exactly r different linear factors. Suppose if possible that $x - \mu$ is a factor of α not dividing ϕ. Then $\xi^{-1}(\mu) = \mathrm{Spec}\ \boldsymbol{C}[x,y]/(z^n - \phi(\mu)\beta(\mu))$. Note that $\phi(\mu)\beta(\mu) \neq 0$. Therefore $\xi^{-1}(\mu)$ is a disjoint union of n copies of $\boldsymbol{A}^1$. Such a fiber cannot occur in an $\boldsymbol{A}^1$-fibration on a $\boldsymbol{Q}$-homology plane hence any linear factor of α must divide ϕ. But α has precisely r different linear factors therefore $\sqrt{(\alpha)} = (\phi)$ as required. □

PROPOSITION 5.2. *The polynomials as found in the Proposition* 5.1 *indeed give rise to a* $\boldsymbol{Q}$*-homology plane.*

PROOF. Any polynomial in our list gives rise to an $\boldsymbol{A}^1$-fibration with irreducible fibers given by $x : S \to \boldsymbol{A}^1$. We use the exact sequence from Suzuki's paper [**Su77**, Lemme 7]

$$H_1(F) \to H_1(S) \to H_1(B) \to 0$$

where a smooth surface S has an F-fibration over a smooth curve B and F is an irreducible general fiber. In the present context $F \cong \boldsymbol{A}^1$ and $B \cong \boldsymbol{A}^1$ so H_1 of both of them is zero. Hence $H_1(S) = (0)$ proving that S is indeed a $\boldsymbol{Q}$-homology plane. □

This finishes the proof of the Theorem in the introduction.

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