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On a flexible class of continuous functions with uniform local structure

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Abstract. This paper considers a class of continuous functions constructed as a series of iterates of the "tent map" multiplied by variable signs. This class includes Takagi's nowhere-differentiable function, and contains the functions studied by Hata and Yamaguti [*Japan J. Appl. Math.*, **1** (1984), 183–199] and Kono [*Acta Math. Hungar.*, **49** (1987), 315–324] as a proper subclass. A complete description is given of the differentiability properties of the functions in this class, and several statements are proved concerning their uniform and local moduli of continuity. The results are applied to generation of random functions.

1. Introduction.

Various authors [**3**], [**7**], [**8**], [**11**], [**13**], [**17**] have studied continuous functions of the form

$$f(x) = \sum_{n=1}^{\infty} c_n \phi^{(n)}(x), \quad 0 \le x \le 1, \tag{1}$$

where $\{c_n\}$ is a sequence of real numbers; $\phi^{(1)} := \phi$ is the "tent map" defined by

$$\phi(x) := \begin{cases} 2x, & 0 \le x \le 1/2, \\ 2 - 2x, & 1/2 \le x \le 1; \end{cases}$$

and inductively, $\phi^{(n)} := \phi \circ \phi^{(n-1)}$ for $n \ge 2$.

For example, taking $c_n = b^n$, where $1/2 \le b < 1$, one obtains "fractal" functions analogous to the Weierstrass nowhere-differentiable but continuous function; see Ledrappier [**13**]. The borderline case $b = 1/2$ yields the Takagi function [**17**];

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see Figure 1(a). Other choices of c_n were considered by Faber [**3**], Kahane [**8**] and others, usually to demonstrate the existence of functions having a given modulus of continuity. Hata and Yamaguti [**7**] studied the functions (1) in full generality, and showed that (1) defines a continuous (and then a uniformly continuous) function if and only if $\{c_n\} \in \ell^1$. The first complete treatment of the differentiability properties and the modulus of continuity of these functions was given by Kono [**11**].

Observe, however, that f defined by (1) is always symmetric with respect to $x = 1/2$. To introduce more flexibility, this paper considers the functions

$$f(x) = \sum_{n=1}^{\infty} c_n r_{n-1}(x)\phi^{(n)}(x), \quad 0 \le x \le 1, \tag{2}$$

where for $n \in \boldsymbol{Z}_+$, r_n is a function from $[0,1]$ to $\{-1,1\}$ which is constant on each open subinterval $((j-1)/2^n, j/2^n)$, $j = 1, 2, \ldots, 2^n$. Assume without loss of generality that $c_n \ge 0$, $n \in \boldsymbol{N}$. To ensure that (2) defines a continuous function, assume furthermore that $\{c_n\} \in \ell^1$. (Observe that the summands in (2) are continuous, since the discontinuities of r_{n-1} coincide with zeros of $\phi^{(n)}$.)

Readers with a background in wavelet analysis will no doubt recognize (2) as a special kind of Schauder series. In fact, $2^{-(m+2)/2}\phi^{(m+1)}\chi_{[k/2^m,(k+1)/2^m]}$ is the Schauder-Ciesielski function of index $2^m + k$ (see [**16**]). Thus, for fixed m, the Schauder functions of index $2^m + k$ ($k = 0, 1, \ldots, 2^m - 1$) all occur with the same amplitude in (2), but the multiplication by $r_m(x)$ gives each Schauder function (or "tent") its own orientation (up or down), independently of the others. This means that on the one hand, the local properties of the graph of f will be fairly uniform throughout the domain, whereas on the other hand, the graph of f can have a wide variety of general shapes.

Some particular non-symmetric functions of the form (2) which have occurred in the literature are specified below. First, define the system of Rademacher functions $\{X_n\}_{n\in\boldsymbol{N}}$ by

$$X_n(x) := \begin{cases} 1, & \text{if } [2^n x] \text{ is even,} \\ -1, & \text{if } [2^n x] \text{ is odd,} \end{cases}$$

for $x \in \boldsymbol{R}$, where $[2^n x]$ denotes the greatest integer less than or equal to $2^n x$. It is clear that $r_n(x)$ may be assumed to be of the form

$$r_0(x) \equiv R_0, \qquad r_n(x) = R_n(X_1(x), \ldots, X_n(x)), \quad n \in \boldsymbol{N}, \tag{3}$$

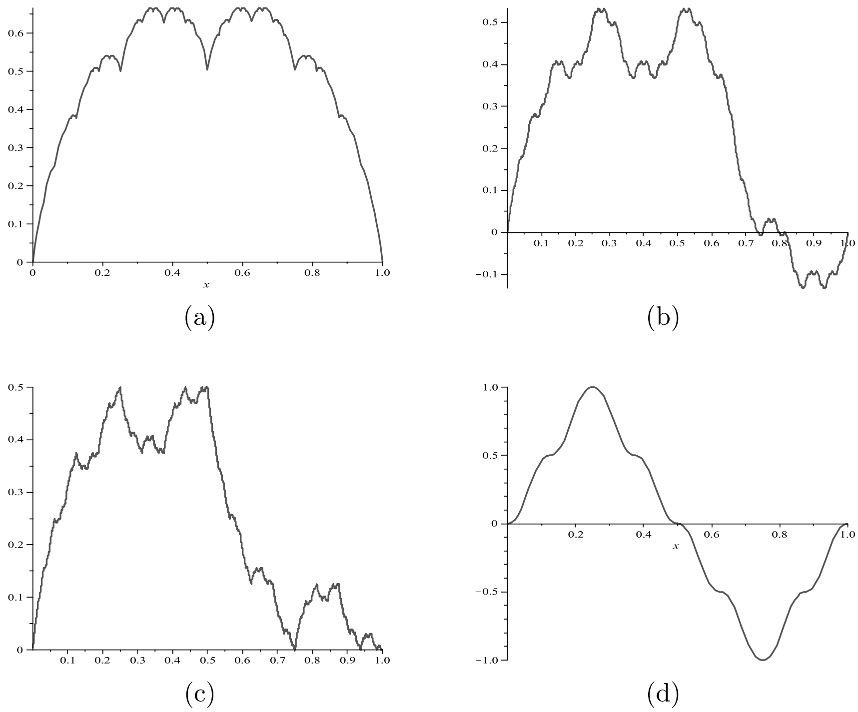

Figure 1. (a) the Takagi function; (b) the Gray Takagi function; (c) Kawamura's T^3; (d) a smooth function (see Section 4).

where $R_0 \in \{-1,1\}$ is constant, and for $n \geq 1$, R_n is a function from $\{-1,1\}^n$ to $\{-1,1\}$.

EXAMPLE 1.1. Taking $c_n = 2^{-n}$, $r_0(x) \equiv 1$, and $r_n(x) = X_n(x)$ for $n \in \boldsymbol{N}$ we obtain the function shown in Figure 1(b). Kobayashi [**10**] named it the *Gray Takagi function* because of its relationship to Gray codes. (See the last remark in Section 6.)

EXAMPLE 1.2. Let $c_n = 2^{-n}$, $r_0(x) \equiv 1$, and $r_n(x) = X_1(x)X_2(x)\cdots X_n(x)$ for $n \in \boldsymbol{N}$. This yields the function T^3 of Kawamura [**9**], which she used to study a family of self-similar sets in the plane. (See Figure 1(c).)

The objective of this paper is to study the differentiability properties and the modulus of continuity of the functions (2). In particular, it will be shown that the results of Kono [**11**] remain valid for this larger class of functions. The local modulus of continuity is explored in somewhat greater depth. The methods used are largely Kono's, and include the use of probabilistic techniques and the representation of ϕ in terms of the Rademacher system $\{X_n\}$. However, the general setting considered here presents several new challenges, and some of the proofs require some fundamentally new ideas. Note finally that related results on differentiability of Schauder series were proved by Pál and Schipp [**16**], but their results do not contain ours.

The paper is organized as follows. Section 2 introduces the expression of $f(x)$ in terms of Rademacher series. Section 3 deals with the differentiability of f; as in Kono's setting there are three different cases, depending on the tail behavior of the sequence $\{c_n\}$. Section 4 investigates the question of smoothness in the sense of Zygmund [**18**]. While f defined by (1) is smooth if and only if $f(x) = ax(1-x)$ for some $a \in \boldsymbol{R}$ (see [**11**, Theorem 3]), the larger class (2) contains a variety of smooth functions, such as the one shown in Figure 1(d). However, it seems difficult to give a complete characterization.

The modulus of continuity of f is studied in Sections 5 and 6. First, Theorem 5.1 gives the uniform modulus of continuity. Using the law of the iterated logarithm, it is then shown (in Theorem 5.2) that the local modulus of continuity is strictly sharper at almost every point in $[0,1]$. This is done by extending a method of Gamkrelidze [**5**]. It is noted that Kono's original proof of this theorem contained a critical mistake (see Remark 5.4); thus, the proof given here also confirms the correctness of Kono's theorem.

While Theorem 5.2 gives the almost-everywhere modulus of continuity of f, Section 6 investigates behavior on the exceptional set. Under suitable restrictions, we demonstrate two different kinds of behavior of f on reasonably large sets; that is, on sets of strictly positive Hausdorff dimension. The proofs use two singular measures (including the well-known *binomial measure*), and exploit a connection with Gray codes. This section is new in the sense that Kono [**11**] did not study the exceptional set.

The results of the paper are applied in Section 7 to generation of random functions having specified differentiability or continuity properties. This method may be useful for simulating certain physical, biological or financial processes whose sample paths possess a known differentiability structure.

2. Representation by Rademacher series.

We need some notation and facts from [**11**]. Let x and h be real numbers such that $0 \leq x < x+h < 1$, and write

$$x = \sum_{k=1}^{\infty} 2^{-k}\varepsilon_k, \qquad x+h = \sum_{k=1}^{\infty} 2^{-k}\varepsilon'_k,$$

where $\varepsilon_k, \varepsilon'_k \in \{0,1\}$, and we make the convention that $\varepsilon_k = 0$ (resp. $\varepsilon'_k = 0$) eventually when x (resp. $x+h$) is dyadic rational. Note that

$$X_k(x) = 1 - 2\varepsilon_k, \qquad X_k(x+h) = 1 - 2\varepsilon'_k \qquad (k \in \boldsymbol{N}).$$

Let $p := p(h)$ be the unique integer such that

$$2^{-p-1} < h \leq 2^{-p}, \tag{4}$$

and let

$$k_0 := k_0(x,h) := \max\{k : \varepsilon_1 = \varepsilon'_1, \ldots, \varepsilon_k = \varepsilon'_k\}$$

(or $k_0 = 0$ if $\varepsilon_1 \neq \varepsilon'_1$). The following facts are easy to verify:

(a) $0 \leq k_0 \leq p$;
(b) $\varepsilon_{k_0+1} = 0$ and $\varepsilon'_{k_0+1} = 1$;
(c) If $k_0 + 2 \leq p$, then $\varepsilon'_k = 0$ and $\varepsilon_k = 1$ for $k_0 + 2 \leq k \leq p$.

Now we can write

$$\begin{aligned} f(x+h) - f(x) &= \sum_{n=1}^{\infty} c_n \big[r_{n-1}(x+h)\phi^{(n)}(x+h) - r_{n-1}(x)\phi^{(n)}(x) \big] \\ &= \sum_{n=1}^{k_0} + \sum_{n=k_0+1}^{p} + \sum_{n=p+1}^{\infty} =: \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned} \tag{5}$$

Since $r_{n-1}(t)\phi^{(n)}(t)$ is linear on $[x, x+h]$ with derivative $2^n r_{n-1}(x) X_n(x)$ for $n = 1, \ldots, k_0$, it is immediate from (2) that

$$\Sigma_1 = h \sum_{n=1}^{k_0} a_n r_{n-1}(x) X_n(x), \tag{6}$$

where we put

$$a_n := 2^n c_n.$$

In all cases considered in this paper, the magnitude of $f(x+h) - f(x)$ is controlled by the term Σ_1, and suitable estimates are required for the magnitudes of Σ_2 and Σ_3. In the case of Σ_3 this is easy, since

$$|\Sigma_3| \leq \sum_{n=p+1}^{\infty} 2c_n \leq 2 \sum_{n=p+1}^{\infty} 2^{-n} \cdot \sup_{n>p} a_n \leq 4h \sup_{n>p} a_n. \tag{7}$$

In order to be able to deal with Σ_2 we use Kono's expressions (see [**11**])

$$\phi(x) = \frac{1}{2} - X_1(x) \sum_{k=2}^{\infty} 2^{-k} X_k(x) = \sum_{k=2}^{\infty} 2^{-k}\{1 - X_1(x)X_k(x)\},$$

and

$$\phi^{(n)}(x) = \sum_{k=n+1}^{\infty} 2^{n-k-1}\{1 - X_n(x)X_k(x)\}. \tag{8}$$

Define

$$U_{n,k}(x,h) := r_{n-1}(x+h)\{1 - X_n(x+h)X_k(x+h)\} - r_{n-1}(x)\{1 - X_n(x)X_k(x)\}.$$

Substituting (8) into (5) yields

$$\Sigma_2 = \sum_{n=k_0+1}^{p} a_n \sum_{k=p+1}^{\infty} 2^{-k-1} U_{n,k}(x,h),$$

as the terms with $n < k \leq p$ cancel. Therefore (since $|U_{n,k}(x,h)| \leq 4$),

$$|\Sigma_2| \leq 4h \sum_{n=k_0+1}^{p} a_n. \tag{9}$$

Finally, note that if x and $x+h$ belong to the same (closed) dyadic interval of length 2^{-p}: $j/2^p \leq x < x+h \leq (j+1)/2^p$, we simply have

$$f(x+h) - f(x) = h \sum_{n=1}^{p} a_n r_{n-1}(x) X_n(x) + \Sigma_3. \tag{10}$$

3. Differentiability.

The first main result is a generalization of [**11**, Theorem 2].

THEOREM 3.1. *Let f be defined by* (2).

(i) *If $\{a_n\} \in \ell^2$, then f is absolutely continuous with derivative*

$$f'(x) = \sum_{n=1}^{\infty} a_n r_{n-1}(x) X_n(x) \quad a.s. \tag{11}$$

(ii) *If $\{a_n\} \notin \ell^2$ but $\lim_{n\to\infty} a_n = 0$, then f is nondifferentiable at almost every point of $[0,1]$, but f is differentiable on an uncountably large set, and the range of f' is $\boldsymbol{R}$.*

(iii) *If $\limsup_{n\to\infty} a_n > 0$, then f is nowhere differentiable.*

REMARK 3.2. Curiously, a quite analogous result has been observed before in a very different setting: Lax [**12**] showed that Pólya's space-filling curve, which maps the unit interval onto a solid right triangle, is either differentiable almost everywhere; nondifferentiable almost everywhere but differentiable on an uncountably large set; or nowhere differentiable; depending on the size of the smaller acute angle of the triangle. There does not, however, appear to be a direct relationship between our functions and the Pólya curve.

The following lemmas are needed in the proofs of parts (i) and (ii) of Theorem 3.1, respectively.

LEMMA 3.3. *For every $n \in \boldsymbol{N}$ and $x \in [0,1]$,*

$$\int_0^x a_n r_{n-1}(t) X_n(t)\, dt = c_n r_{n-1}(x) \phi^{(n)}(x).$$

PROOF. Let j be the largest integer such that $x \geq j/2^{n-1}$. Then

$$\int_0^{j/2^{n-1}} a_n r_{n-1}(t) X_n(t)\, dt = 0$$

and, since $r_{n-1}(t)$ is constant on $j/2^{n-1} < t < (j+1)/2^{n-1}$,

$$\int_{j/2^{n-1}}^x a_n r_{n-1}(t) X_n(t)\, dt = r_{n-1}(x) \int_{j/2^{n-1}}^x a_n X_n(t)\, dt = c_n r_{n-1}(x) \phi^{(n)}(x).$$

The lemma follows. □

LEMMA 3.4. *Assume* $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$. *Then for every real number* L, *there exists a point* $x \in [0,1]$ *such that*

(*) *the binary expansion of* x *contains neither a string of four or more consecutive* 1*'s nor a string of four or more consecutive* 0*'s,*

and

$$\sum_{n=1}^{\infty} a_n r_{n-1}(x) X_n(x) = L.$$

PROOF. For notational simplicity, we write r_n, X_n rather than $r_n(x)$, $X_n(x)$. For $i = 0, 1, 2$, and $n \in \boldsymbol{N}$, define

$$t_n^{(i)} = \begin{cases} -1 & \text{if } n \equiv i \pmod 3 \\ 1 & \text{otherwise,} \end{cases}$$

and

$$s_n^{(i)} := \sum_{k=1}^{3n} a_k t_k^{(i)}.$$

Since

$$s_n^{(0)} + s_n^{(1)} + s_n^{(2)} = \sum_{k=1}^{3n} a_k \to \infty,$$

it follows that $\limsup_{n\to\infty} s_n^{(i)} = \infty$ for at least one i; say this holds for $i = 0$. Observe that for any sequence $\{\delta_k\} \in \{-1,1\}^{\boldsymbol{N}}$,

$$\begin{aligned} &\delta_k = 1 \text{ whenever } k \equiv 1 \text{ or } 2 \pmod 3 \\ &\quad\Rightarrow \limsup_{n\to\infty} \sum_{k=1}^{3n} a_k \delta_k \geq \limsup_{n\to\infty} \sum_{k=1}^{3n} a_k t_k^{(0)} = \infty. \end{aligned} \tag{12}$$

The proof now proceeds inductively. Let $m \geq 0$, and assume $X_1, \ldots, X_{3m}$ have been constructed without violating (*). (This condition is void if $m = 0$.) Let $A_m := \sum_{k=1}^{3m} a_k r_{k-1} X_k$, and choose X_{3m+1}, X_{3m+2} and X_{3m+3} so that

$$r_{3m}X_{3m+1} = r_{3m+1}X_{3m+2} = \begin{cases} 1 & \text{if } A_m \leq L \\ -1 & \text{if } A_m > L, \end{cases} \tag{13}$$

and

$$X_{3m+3} = -X_{3m+2}.$$

The last condition ensures that the sequence $X_1, \dots, X_{3m+3}$ still satisfies (*). Thus, by induction, this method yields a point x whose binary expansion is as required. The convergence to L follows since $a_n \to 0$, so $A_{m+1} - A_m \to 0$, and A_m and A_{m+1} lie on opposite sides of L infinitely often in view of (12) and (13). □

Proof of Theorem 3.1. (i) Assume $\{a_n\} \in \ell^2$. Note that $X_1, X_2, \dots$ are independent, identically distributed random variables with respect to Lebesgue measure on $[0,1]$, with mean 0 and variance 1. The same is true for the sequence $r_0X_1, r_1X_2, \dots$, since $r_{n-1}(x)$ depends only on $X_1(x), \dots, X_{n-1}(x)$. Thus, the series $\sum a_n X_n(x)$ and $\sum a_n r_{n-1}(x) X_n(x)$ both converge almost surely (see, for instance, Theorem 4.2.4 in [**14**]). In particular,

$$\lim_{h \downarrow 0} \sum_{n=k_0+1}^{p} a_n X_n(x) = 0 \quad \text{a.s.}$$

since $k_0 \to \infty$ as $h \downarrow 0$. Using (9), it follows that for $h > 0$,

$$|\Sigma_2| \leq 4h\biggl(- \sum_{n=k_0+1}^{p} a_n X_n(x) + 2a_{k_0+1} \biggr) = o(h) \quad \text{a.s.}$$

Similarly, $\Sigma_3 = o(h)$ by (7). Hence,

$$\lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{h} = \sum_{n=1}^{\infty} a_n r_{n-1}(x) X_n(x) \quad \text{a.s.}$$

by (6). The left derivative follows by applying the above argument to the function $\tilde{f}(x) := f(1-x)$, which is also of the type (2). Thus we have (11).

We now show that

$$f(x) = \int_0^x f'(t)\, dt, \tag{14}$$

and therefore f is absolutely continuous. Let

$$\zeta_k(t) := a_k r_{k-1}(t) X_k(t).$$

Since the random variables $\{\zeta_k\}$ are independent with mean zero,

$$\int_0^1 \left(\sum_{k=N}^{n} \zeta_k(t) \right)^2 dt = \sum_{k=N}^{n} \int_0^1 \zeta_k(t)^2 dt = \sum_{k=N}^{n} a_k^2.$$

Fatou's lemma therefore implies that

$$\int_0^1 \left(\sum_{k=N}^{\infty} \zeta_k(t) \right)^2 dt \le \sum_{k=N}^{\infty} a_k^2 \to 0, \quad \text{as } N \to \infty.$$

But then,

$$\int_0^x \left| \sum_{k=N}^{\infty} \zeta_k(t) \right| dt \to 0 \quad \text{as } N \to \infty,$$

by the Schwarz inequality. This, along with (11) and Lemma 3.3, yields (14).

(ii) Next, assume that $\{a_n\} \notin \ell^2$ but $a_n \to 0$. Fix $x = \sum_{k=1}^{\infty} 2^{-k} \varepsilon_k$, and put

$$x_m = \sum_{k=1}^{m} 2^{-k} \varepsilon_k, \qquad h_m = 2^{-m} \qquad (m \in \boldsymbol{N}). \tag{15}$$

A necessary condition for f to be differentiable at x is that

$$P_m := \frac{f(x_m + h_m) - f(x_m)}{h_m} \tag{16}$$

have a finite limit as $m \to \infty$. However, since x and x_m have their first m binary digits in common, (10) yields

$$f(x_m + h_m) - f(x_m) = h_m \sum_{n=1}^{m} a_n r_{n-1}(x) X_n(x). \tag{17}$$

Since $\{a_n\} \notin \ell^2$, the law of the iterated logarithm [**6**] therefore implies that

$$\limsup_{m\to\infty} P_m = \infty \qquad \text{and} \qquad \liminf_{m\to\infty} P_m = -\infty \quad \text{a.s.}$$

Hence, f is nondifferentiable almost everywhere.

Next, let $L \in \boldsymbol{R}$ be given, and choose $x \in [0,1]$ satisfying the conclusion of Lemma 3.4. Then $h > 0$ implies $p - k_0 \le 4$, so by (9)

$$|\Sigma_2| \le 4h \sum_{n=p-3}^{p} a_n = o(h),$$

since $a_n \to 0$. Thus, (6) and (7) yield

$$\lim_{h\downarrow 0} \frac{f(x+h)-f(x)}{h} = L.$$

The left derivative follows again by considering the function $\tilde{f}(x) = f(1-x)$, since $1-x$ also satisfies property (*) in Lemma 3.4. Thus, $f'(x) = L$.

(iii) Finally, assume $\limsup_{n\to\infty} a_n > 0$. Fix $x = \sum_{k=1}^{\infty} 2^{-k}\varepsilon_k$. Define x_m and h_m by (15), and P_m by (16). From (17) it is clear that

$$|P_{m+1} - P_m| = a_{m+1},$$

and hence $\{P_m\}$ does not converge. Therefore, f is not differentiable at x. This completes the proof of (iii), and of the theorem. □

REMARK 3.5. The above proof shows that the range of f' is $\boldsymbol{R}$ also if $\{a_n\} \in \ell^2 \backslash \ell^1$. When $\{a_n\} \in \ell^1$, however, the range of f' can be much more complicated. For example, if $a_n = \beta^n$ where $0 < \beta < 1/2$, the range of f' is the set $\{\sum_{n=1}^{\infty} \xi_n \beta^n : \xi_n \in \{-1,1\} \text{ for } n \in \boldsymbol{N}\}$, a Cantor set of Hausdorff dimension $-\log 2/\log\beta$.

4. Smoothness.

In this section, say a continuous function f defined on $(0,1)$ is *smooth* if

$$f(x+h) + f(x-h) - 2f(x) = o(h) \tag{18}$$

for all $x \in (0,1)$. This concept of smoothness is due to Zygmund [**18**]. Kono [**11**, Theorem 3] showed that f defined by (1) is smooth if and only if $f(x) = ax(1-x)$ for some constant a; or equivalently, if $c_n = a4^{-n}$, $n \in \boldsymbol{N}$.

In the setting of this paper there are more possibilities (see the examples below). The closest the author has come to a characterization of smooth functions is the following:

THEOREM 4.1. *Let f be defined by* (2), *and let R_n $(n \in \boldsymbol{Z}_+)$ be as in* (3).

(i) *If f is smooth, then*

$$a_{l+1}R_l(\eta_1,\dots,\eta_l) = \frac{1}{2}\sum_{k=0}^{\infty} a_{l+k+2}\Big[R_{l+k+1}(\eta_1,\dots,\eta_l,-1,\underbrace{1,\dots,1}_{k}) + R_{l+k+1}(\eta_1,\dots,\eta_l,1,\underbrace{-1,\dots,-1}_{k})\Big] \tag{19}$$

for every $l \in \boldsymbol{Z}_+$ and $\eta_1,\dots,\eta_l \in \{-1,1\}$. (Here, $R_l(\eta_1,\dots,\eta_l)$ is to be read as R_0 if $l=0$.)

(ii) *If $\{a_n\} \in \ell^1$ and* (19) *holds, then f is smooth.*

REMARK 4.2. The author does not know whether there exist solutions to (19) with $\{a_n\} \notin \ell^1$.

PROOF OF THEOREM 4.1.

(i) Suppose f is smooth. Then f is differentiable on a set of cardinality of the continuum (see [**18**]). Hence $\lim_{n\to\infty} a_n = 0$ by Theorem 3.1.

Fix $m \in \boldsymbol{N}$, and consider a point $x = \sum_{k=1}^{m} 2^{-k}\varepsilon_k$ with $\varepsilon_m = 1$. Let $p > m$, and $2^{-p-1} < h < 2^{-p}$. We have $k_0(x,h) = p$, and $k_0(x-h,h) = m-1$. Applying (10) to both differences we obtain

$$f(x+h)-f(x) = h\Bigg[\sum_{n=1}^{m-1} a_n r_{n-1}(x)X_n(x) - a_m r_{m-1}(x) + \sum_{n=m+1}^{p} a_n r_{n-1}(x)\Bigg] + o(h)$$

and

$$f(x)-f(x-h) = h\Bigg[\sum_{n=1}^{m-1} a_n r_{n-1}(x)X_n(x) + a_m r_{m-1}(x) - \sum_{n=m+1}^{p} a_n r_{n-1}(x-h)\Bigg] + o(h),$$

and therefore,

$$f(x+h)+f(x-h)-2f(x)$$
$$= h\Bigg[\sum_{n=m+1}^{p} a_n\{r_{n-1}(x)+r_{n-1}(x-h)\} - 2a_m r_{m-1}(x)\Bigg] + o(h). \tag{20}$$

This expression can be seen to be correct also if $h = 2^{-p}$. Now from (3),

$$r_{m-1}(x) = R_{m-1}(X_1(x),\dots,X_{m-1}(x)),$$
$$r_{n-1}(x) = R_{n-1}(X_1(x),\dots,X_{m-1}(x),-1,\underbrace{1,\dots,1}_{n-m-1}), \quad n \ge m+1,$$
$$r_{n-1}(x-h) = R_{n-1}(X_1(x),\dots,X_{m-1}(x),1,\underbrace{-1,\dots,-1}_{n-m-1}), \quad n \ge m+1.$$

Substituting these expressions into (20), using the definition (18) of smoothness, and re-indexing yields (19).

(ii) Conversely, if (19) holds, then (18) is certainly satisfied at any dyadic rational point x. If moreover $\{a_n\} \in \ell^1$, then (18) holds for nondyadic points as well, since for such points, both $k_0(x,h)$ and $k_0(x-h,h)$ tend to ∞ as $h \downarrow 0$; thus, $\Sigma_2 = o(h)$ in the expansion of both $f(x+h)-f(x)$ and $f(x)-f(x-h)$. The terms contributed by Σ_1 will cancel, as they did in the dyadic case. □

Examples of functions satisfying (19) can be created from the "basic function" $H(x) := x(1-x)$. For example, 2^n copies of the graph of H can be placed side by side (every second one reflected in the x-axis) to create a wave-like pattern. Such a pattern may then be superimposed, appropriately scaled, on the graph of $\phi^{(m)}$ ($m \le n-1$) to create further examples. These ideas are now made precise.

Example 4.3. Choose $N \in \boldsymbol{N}$, and take $c_n = 0$ for $n \le N$, and $c_n = 4^{N+1-n}$ for $n > N$. Take $r_n = X_N$ for $n \ge N$. It is easy to verify (19). The graph of f is a piecewise quadratic curve resembling a "quadratic sine wave".

Example 4.4. Choose $K \in \boldsymbol{Z}_+$ and $N \ge 2$. Put $c_n = 0$ for $n \le K+N$, $n \ne K+1$; $c_{K+1} = 1$; and $c_n = 2^{N+1}4^{K-n}$ for $n > K+N$. Now choose $r_0,\dots,r_{K+N-1}$ arbitrarily, and put

$$r_n = -r_K X_{K+1} X_{K+N}, \quad n \ge K+N.$$

Again (19) is easily checked. This gives the "quadratic sine wave" from Example

4.3 traveling up and down the graph of $r_K\phi^{(K+1)}$. The simplest example, with $K=0$ and $N=2$, yields the "bell-shaped" curve given by

$$f(x)=\begin{cases}8x^2, & x\le 1/4\\ 8x(1-x)-1, & 1/4\le x\le 3/4\\ 8(1-x)^2, & x\ge 3/4.\end{cases}$$

Similarly, the graph in Figure 1(d) is obtained by taking $K=1$, $N=3$, $r_0\equiv 1$, and $r_1=X_1$.

The author does not know whether there exist other examples of functions satisfying (19).

5. The uniform and local moduli of continuity.

This section investigates the continuity properties of f. Note first that, if $\{a_n\}\in\ell^1$, then f is Lipschitz continuous. This follows from Theorem 3.1, since $\{a_n\}\in\ell^1$ implies $\{a_n\}\in\ell^2$, and so (14) and (11) imply

$$|f(x+h)-f(x)|=\left|\int_x^{x+h}\sum_{n=1}^{\infty}a_n r_{n-1}(t)X_n(t)dt\right|\le\left(\sum_{n=1}^{\infty}a_n\right)|h|.$$

The theorems below generalize Theorems 4 and 5 of [**11**], respectively. In this section and the next, assume f is defined by (2), and let σ_u and σ_l denote nonincreasing continuous functions satisfying

$$\sigma_u(2^{-p})=\sum_{n=1}^{p}a_n,\qquad \sigma_l(2^{-p})=\left(\sum_{n=1}^{p}a_n^2\right)^{1/2}\quad(p=1,2,\dots).$$

THEOREM 5.1 (The uniform modulus of continuity). *If $\{a_n\}\in\ell^\infty\backslash\ell^1$, then*

$$\limsup_{|x-y|\downarrow 0}\frac{f(x)-f(y)}{(x-y)\sigma_u(|x-y|)}=1,\quad \textit{and}\quad \liminf_{|x-y|\downarrow 0}\frac{f(x)-f(y)}{(x-y)\sigma_u(|x-y|)}=-1.$$

While the above theorem illustrates "worst-case" behavior, the next theorem shows that at a "typical" point, the function f exhibits a stronger form of continuity.

THEOREM 5.2 (The local modulus of continuity). *If $\{a_n\}\in\ell^\infty\backslash\ell^2$, then*

$$\limsup_{h\to 0} \frac{f(x+h)-f(x)}{h\sigma_l(|h|)\sqrt{2\log\log\sigma_l(|h|)}} = 1 \quad a.s.,$$

and the corresponding lim inf *equals* -1 *a.s.*

PROOF OF THEOREM 5.1. Assume $a_n \le K$. Then for all x and h,

$$|f(x+h)-f(x)| \le \sum_{n=1}^{p} a_n|h| + 2\sum_{n=p+1}^{\infty} c_n,$$

where p is defined as in (4). The first term on the right is equal to $|h|\sigma_u(2^{-p})$, and the second term is bounded above by $4K|h|$. Since $\{a_n\} \notin \ell^1$, it follows that

$$-1 \le \liminf_{|x-y|\downarrow 0} \frac{f(x)-f(y)}{(x-y)\sigma_u(|x-y|)} \le \limsup_{|x-y|\downarrow 0} \frac{f(x)-f(y)}{(x-y)\sigma_u(|x-y|)} \le 1.$$

Conversely, given $m \in \boldsymbol{N}$, it is always possible to construct a point $x_m = \sum_{k=1}^{m} 2^{-k}\varepsilon_k$ such that $r_{n-1}(x_m)X_n(x_m) = 1$ for $n = 1, \dots, m$. Put $h_m = 2^{-(m+1)}$. Then

$$f(x_m+h_m) - f(x_m) = h_m\sigma_u(h_m) + O(h_m),$$

and letting $m \to \infty$ yields the lim sup in the statement of the theorem. The lim inf follows similarly. □

The idea of the proof of Theorem 5.2 is to apply the law of the iterated logarithm to the sum Σ_1 in (5). The following lemma is needed to ensure that Σ_2 remains small compared to Σ_1. Define the notation

$$b_n := a_n^2, \qquad s_n := \sum_{j=1}^{n} b_j \qquad (n \in \boldsymbol{N}).$$

LEMMA 5.3. *Under the hypotheses of Theorem* 5.2, *we have*
(a) $s_{k_0}/s_p \to 1$ *a.s.*; *and*
(b) $s_p^{-1/2} \sum_{n=k_0+1}^{p} a_n \to 0$ *a.s.*

PROOF. Without loss of generality, assume $a_n \le 1$ for all n. We adapt a method of Gamkrelidze [**5**], who considered the case $a_n \equiv 1$.

For any fixed x, k_0 is smallest when $h = 2^{-p}$, so it is sufficient to consider this special case. Thus, we have

$$\mathrm{P}(k_0 = p - j) = 2^{-j}, \quad 1 \le j < p; \qquad \mathrm{P}(k_0 = 0) = 2^{1-p},$$

and therefore,

$$\mathrm{E}(s_p - s_{k_0})^2 = \sum_{j=1}^{p-1} 2^{-j}(s_p - s_{p-j})^2 + 2^{1-p}s_p^2 \le 2\sum_{j=1}^{p} 2^{-j}(s_p - s_{p-j})^2. \tag{21}$$

Now fix $\varepsilon > 0$. By Chebyshev's inequality and (21),

$$\begin{aligned}\sum_{p=1}^{\infty} \mathrm{P}\left(\frac{s_p - s_{k_0}}{s_p} \ge \varepsilon\right) &\le \sum_{p=1}^{\infty} \frac{\mathrm{E}(s_p - s_{k_0})^2}{\varepsilon^2 s_p^2} \\ &\le \frac{2}{\varepsilon^2}\sum_{j=1}^{\infty} 2^{-j} \sum_{p=j}^{\infty}\left(\frac{s_p - s_{p-j}}{s_p}\right)^2. \end{aligned}\tag{22}$$

Let

$$p_k := \inf\{p : s_p > k\}, \quad k \in \boldsymbol{Z}_+.$$

Then, for each $j < p_k - 1$,

$$\sum_{n=p_k-j+1}^{p_{k+1}-1} b_n \le j + \sum_{n=p_k+1}^{p_{k+1}-1} b_n \le j + 1.$$

For $k \ge 1$, this yields the estimate

$$\begin{aligned}\sum_{p=p_k}^{p_{k+1}-1}\left(\frac{s_p - s_{p-j}}{s_p}\right)^2 &\le k^{-2}\sum_{p=p_k}^{p_{k+1}-1}\left(\sum_{n=p-j+1}^{p} b_n\right)^2 \\ &\le k^{-2}\left(\sum_{p=p_k}^{p_{k+1}-1}\sum_{n=p-j+1}^{p} b_n\right)^2 \\ &\le k^{-2}\left(j\sum_{n=p_k-j+1}^{p_{k+1}-1} b_n\right)^2 \le k^{-2}j^2(j+1)^2,\end{aligned}$$

since each b_n occurs at most j times in the double sum above. The case $k = 0$ must be handled separately in a similar manner. We obtain

$$\sum_{p=j}^{\infty}\left(\frac{s_p - s_{p-j}}{s_p}\right)^2 = \sum_{k=0}^{\infty}\sum_{p=p_k}^{p_{k+1}-1}\left(\frac{s_p - s_{p-j}}{s_p}\right)^2 \leq 4\Big(b_{p_0}^{-2} + \sum_{k=1}^{\infty} k^{-2}\Big) j^4.$$

Thus, the series in (22) converges, and the Borel-Cantelli lemma implies that

$$\frac{s_p - s_{k_0}}{s_p} \to 0 \quad \text{a.s.},$$

which is equivalent to (a). To prove (b), note that in view of the Schwarz inequality, it suffices to show that

$$\frac{(p-k_0)(s_p - s_{k_0})}{s_p} \to 0 \quad \text{a.s.}$$

This can be done in essentially the same way as the proof of part (a). □

Proof of Theorem 5.2. Assume first that $h > 0$. Then $\Sigma_3 = O(h)$ by (7), and Lemma 5.3(b) implies that

$$\frac{\Sigma_2}{h\sqrt{s_p}} \to 0 \quad \text{a.s.} \tag{23}$$

Since the random variables $r_{n-1}(x)X_n(x)$ are independent with mean 0 and variance 1, the law of the iterated logarithm (LIL) implies

$$\limsup_{k_0\to\infty} \frac{\Sigma_1}{h\sqrt{2s_{k_0}\log\log s_{k_0}}} = 1 \quad \text{a.s.}$$

Using part (a) of Lemma 5.3 we obtain the statement of the theorem for the case $h \downarrow 0$. The corresponding result for $h \uparrow 0$ follows by considering the function $\tilde{f}(x) := f(1-x)$. Thus, the proof is complete. □

Remark 5.4. It is noted here that Kono's proof of Theorem 5.2 (for the functions (1)) contains a gap. Kono claims that (with the notation of Section 2)

$$\sum_{k=p+1}^{\infty} 2^{-k}(1 - \varepsilon_k - \varepsilon_k') \geq 0, \tag{24}$$

an inequality which in his proof is of critical importance for dealing with the term

Σ_2. However, if $h = 2^{-p}$ and $\varepsilon_{p+1} = \varepsilon_{p+2} = 1$, it is easy to see that the inequality fails. In fact, the left hand side of (24) can be arbitrarily close to $-h$. The proof given above confirms that the statement of Kono's theorem was nonetheless correct.

REMARK 5.5. The boundedness condition in Theorems 5.1 and 5.2 is sufficient but not necessary. For example, straightforward calculations show that the conclusions of both theorems hold if $a_n = n^\beta$ for some $\beta > 0$. It is not clear, however, to what extent the boundedness condition can be weakened.

REMARK 5.6. With no additional effort, we can obtain the following generalization of Gamkrelidze's first theorem [**5**, Theorem 1]. Let $a_n \equiv 1$, so

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} r_{n-1}(x) X_n(x).$$

Then

$$\lim_{h \downarrow 0} \lambda \left\{ x \in (0,1) : \frac{f(x+h) - f(x)}{h\sqrt{\log_2(1/h)}} < y \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-u^2/2} du,$$

for every $y \in \boldsymbol{R}$, where λ denotes Lebesgue measure on $[0,1]$. To see this, simply apply the Central Limit Theorem to Σ_1 in (5), and observe that the terms Σ_2 and Σ_3 are controlled by (23) and (7), respectively.

6. More on the local modulus of continuity.

While Theorem 5.2 gives the almost-everywhere modulus of continuity of f, it is interesting to consider what happens on the exceptional set. The following two theorems specify behavior on a reasonably "large" set, that is, on a set of strictly positive Hausdorff dimension.

THEOREM 6.1. *Assume r_n is constant for each n. Let $\delta \in (-1,1)$ be given. Under the hypotheses of Theorem* 5.2*,*

$$\limsup_{h \to 0} \frac{f(x+h) - f(x) - \delta h \tau(|h|)}{h \sigma_l(|h|) \sqrt{2(1-\delta^2) \log\log \sigma_l(|h|)}} = 1,$$

and the corresponding lim inf *equals -1, for all x in a set of strictly positive Hausdorff dimension. Here, τ is the continuous function satisfying $\tau(2^{-p}) = \sum_{n=1}^{p} r_{n-1} a_n$ for $p = 0, 1, 2, \ldots,$ and extended to $(0,1)$ by linear interpolation.*

Theorem 6.2. *Consider the Takagi function*

$$T(x) := \sum_{n=1}^{\infty} 2^{-n} \phi^{(n)}(x).$$

Let $A > 0$ be given. Then

$$\limsup_{h \to 0} \frac{T(x+h) - T(x)}{h\sigma_l(|h|)\sqrt{2 \log \log \sigma_l(|h|)}} = A,$$

and the corresponding $\liminf$ *equals $-A$, for all x in a set of strictly positive Hausdorff dimension.*

The proofs of these theorems use two singular measures of positive dimension. Recall that for a probability measure μ on $[0, 1]$ equipped with the Borel sigma algebra $\mathscr{B}([0, 1])$, the Hausdorff dimension of μ is defined by

$$\dim_H(\mu) = \inf\{\dim_H E : E \in \mathscr{B}([0, 1]) \text{ and } \mu(E) = 1\},$$

where $\dim_H E$ denotes the Hausdorff dimension of the set E. (See Falconer [**4**] for a definition and properties of Hausdorff dimension.)

Define the intervals $I := I_{0,0} := [0, 1]$, and

$$I_{n,j} := \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right), \quad j = 0, 1, \ldots, 2^n - 2; \qquad I_{n,2^n-1} := \left[\frac{2^n - 1}{2^n}, 1\right]$$

for $n \in \boldsymbol{N}$. For $0 < \alpha < 1$, denote by μ_α the unique probability measure on I determined by the conditions

$$\mu_\alpha(I_{n+1,2j}) = \alpha \mu_\alpha(I_{n,j}), \qquad \mu_\alpha(I_{n+1,2j+1}) = (1 - \alpha)\mu_\alpha(I_{n,j})$$

for $n = 0, 1, 2, \ldots$ and $j = 0, 1, \ldots, 2^n - 1$. The measure μ_α is called a *binomial measure*; it has been used in applications ranging from digital sum problems in number theory [**15**] to the mathematical theory of gambling [**2**]. Note that μ_α coincides with Lebesgue measure when $\alpha = 1/2$, but is singular otherwise. The dimension of μ_α, due to Besicovitch, is given by

$$\dim_H(\mu_\alpha) = -\alpha \log_2 \alpha - (1 - \alpha) \log_2(1 - \alpha) \tag{25}$$

(see [**4**, Proposition 10.4]).

PROOF OF THEOREM 6.1. Choose $0 < \alpha < 1$ so that $2\alpha - 1 = \delta$. Observe that under μ_α, the binary digits of a number $x \in [0,1]$ are independent, taking on the values 0 and 1 with probabilities α and $1-\alpha$, respectively. Hence

$$\mathrm{E}[X_n(x)] = \delta, \qquad \mathrm{Var}[X_n(x)] = 4\alpha(1-\alpha) = 1 - \delta^2,$$

where E and Var denote expectation and variance operators with respect to μ_α. The LIL applied to the random variables $a_n r_{n-1}\{X_n(x) - \delta\}$ yields

$$\limsup_{k\to\infty} \frac{\sum_{n=1}^k a_k r_{k-1} X_k(x) - \delta \sum_{n=1}^k a_n r_{n-1}}{\sigma_l(2^{-k})\sqrt{2(1-\delta^2)\log\log\sigma_l(2^{-k})}} = 1 \quad \text{a.s. } (\mu_\alpha).$$

Since Lemma 5.3 is still valid (and its proof easily modified), it follows as in the proof of Theorem 5.2 that

$$\limsup_{h\to 0} \frac{f(x+h) - f(x) - \delta h \tau(|h|)}{h\sigma_l(|h|)\sqrt{2(1-\delta^2)\log\log\sigma_l(|h|)}} = 1 \quad \text{a.s. } (\mu_\alpha),$$

and similarly for the lim inf. This clearly implies the statement of the theorem, since $\dim_H(\mu_\alpha) > 0$. $\square$

The second singular measure is defined as follows. For $0 < \alpha < 1$, let $\tilde\mu_\alpha$ denote the unique probability measure on I determined by the conditions

$$\tilde\mu_\alpha(I_{n+1,2j}) = \begin{cases} \alpha\tilde\mu_\alpha(I_{n,j}), & j \text{ even}, \\ (1-\alpha)\tilde\mu_\alpha(I_{n,j}), & j \text{ odd}, \end{cases}$$

$$\tilde\mu_\alpha(I_{n+1,2j+1}) = \begin{cases} (1-\alpha)\tilde\mu_\alpha(I_{n,j}), & j \text{ even}, \\ \alpha\tilde\mu_\alpha(I_{n,j}), & j \text{ odd} \end{cases}$$

for $n = 0,1,2,\dots$ and $j = 0,1,\dots,2^n - 1$. This measure was introduced by Kobayashi [**10**] and studied further by Cristea and Prodinger [**1**], who named it the *Gray code measure*. Again, $\tilde\mu_\alpha$ reduces to Lebesgue measure when $\alpha = 1/2$, but is singular otherwise.

It is precisely the connection of the Gray code measure to Gray codes which allows us to prove Theorem 6.2. The Gray code is an encoding of the nonnegative integers by sequences of 0's and 1's, with the property that representations of adjacent integers differ in exactly one position. Specifically, if the nonnegative integer n is written as $n = \sum_{k=0}^{\infty} 2^k \varepsilon_k$ with $\varepsilon_k \in \{0,1\}$, then the nth *Gray code* is the infinite sequence

$$(\ldots \varepsilon_{k+1} \oplus \varepsilon_k, \ldots, \varepsilon_2 \oplus \varepsilon_1, \varepsilon_1 \oplus \varepsilon_0),$$

where $\oplus$ denotes addition modulo 2. The definition extends naturally to points in $[0,1)$: if $x = \sum_{k=1}^{\infty} 2^{-k} \varepsilon_k$, the *Gray code expansion* of x is the sequence

$$(\varepsilon_1, \varepsilon_1 \oplus \varepsilon_2, \varepsilon_2 \oplus \varepsilon_3, \ldots, \varepsilon_k \oplus \varepsilon_{k+1}, \ldots).$$

The important point to observe is that, under $\tilde{\mu}_\alpha$, the digits in the Gray code expansion of a point $x \in [0,1)$ are independent, taking on 0 and 1 with probabilities α and $1-\alpha$, respectively. With this in mind, one can show that

$$\dim_H(\tilde{\mu}_\alpha) = -\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha).$$

The proof of (25), as given in [**4**, Proposition 10.4], can be copied with only one change: the quantities $n_0(x|_k)$ and $n_1(x|_k)$ must denote the number of 0's (resp. 1's) in the first k Gray code digits, rather than binary digits, of x.

While $\tilde{\mu}_\alpha$ makes the Gray code digits of x independent, it makes the binary digits of x Markovian, with transition probabilities

$$p(x,y) = \begin{cases} \alpha, & x = y, \\ 1-\alpha, & x \neq y, \end{cases}$$

for $x, y \in \{0,1\}$. Thus, under $\tilde{\mu}_\alpha$, the partial sums defined by

$$S_0 := 0, \qquad S_n := \sum_{j=1}^{n} X_j(x), \quad n \in \boldsymbol{N}$$

follow a *correlated random walk*; that is, a random walk which at each stage continues in its present direction with probability α, and reverses its direction with probability $1-\alpha$. For such a random walk, the LIL takes the following form. (This result is probably known, but since no reference is known to the author, a proof is included for completeness.)

Proposition 6.3. *We have*

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = \sqrt{\frac{\alpha}{1-\alpha}} \quad a.s.\ (\tilde{\mu}_\alpha).$$

PROOF. Without loss of generality, assume the process $\{X_n\}$ starts at time 0 with $X_0 \equiv -1$. (The initial condition clearly does not affect the long-run behavior.) Define $\tau_0 \equiv 0$, and inductively,

$$\tau_j := \inf\{n > \tau_{j-1} : X_n \neq X_{n-1}\}, \quad j = 1, 2, \ldots.$$

Let $T_j := \tau_j - \tau_{j-1}$, $j \in \boldsymbol{N}$. Then $T_1, T_2, \ldots$ are independent, each having a geometric distribution with parameter $1-\alpha$, so

$$\mathrm{E}(T_j) = \frac{1}{1-\alpha}, \qquad \mathrm{Var}(T_j) = \frac{\alpha}{(1-\alpha)^2},$$

where E and Var now denote expectation and variance with respect to $\tilde{\mu}_\alpha$.

Next, let $Z_j := S_{\tau_{2j}} - S_{\tau_{2j-2}}$, $j \in \boldsymbol{N}$. Then $Z_j = T_{2j} - T_{2j-1}$, and hence

$$\mathrm{E}(Z_j) = 0, \qquad \mathrm{Var}(Z_j) = \frac{2\alpha}{(1-\alpha)^2} \qquad (j \in \boldsymbol{N}).$$

Thus, with $b_n := 2\alpha n/(1-\alpha)^2$, the LIL implies that

$$\limsup_{j\to\infty} \frac{S_{\tau_{2j}}}{\sqrt{2b_j \log\log b_j}} = 1 \quad \text{a.s. } (\tilde{\mu}_\alpha). \tag{26}$$

Let $u(n) := \sqrt{2n\log\log n}$. By the strong law of large numbers,

$$\frac{u(\tau_{2j})}{u(2j)} \to \frac{1}{\sqrt{1-\alpha}} \quad \text{a.s. } (\tilde{\mu}_\alpha). \tag{27}$$

Finally, note that $\{S_n\}$ takes on its local maxima at the times $\tau_{2j} - 1$ $(j \in \boldsymbol{N})$. It follows from (26) and (27) that

$$\limsup_{n\to\infty} \frac{S_n}{u(n)} = \limsup_{j\to\infty} \frac{S_{\tau_{2j}}}{u(\tau_{2j})} = \sqrt{\frac{\alpha}{1-\alpha}} \quad \text{a.s. } (\tilde{\mu}_\alpha),$$

and the proof is complete. $\square$

Proof of Theorem 6.2. Observe that the statement of Lemma 5.3 is valid even with respect to $\tilde{\mu}_\alpha$. (The details of the proof are left to the interested reader.) Hence, Proposition 6.3 and the discussion preceding it imply that

$$\limsup_{h\to 0} \frac{T(x+h)-T(x)}{h\sigma_l(|h|)\sqrt{2\log\log\sigma_l(|h|)}} = \sqrt{\frac{\alpha}{1-\alpha}} \quad \text{a.s. } (\tilde{\mu}_\alpha),$$

and similarly for the lim inf. Since $\dim_H(\tilde{\mu}_\alpha) > 0$ and α can be chosen so that $\sqrt{\alpha/(1-\alpha)} = A$, the theorem follows. □

Remark 6.4. Let $\tilde{T}$ denote the Gray Takagi function (see Example 1.1). Kobayashi [**10**] showed that the measure $\tilde{\mu}_\alpha$ is related to $\tilde{T}$ by the formula

$$\frac{1}{2}\frac{\partial\tilde{\mu}_\alpha([0,x])}{\partial\alpha}\bigg|_{\alpha=1/2} = \tilde{T}(x), \qquad x\in[0,1].$$

The same relationship holds between the Takagi function $T(x)$ and the measure μ_α; see Hata and Yamaguti [**7**, Theorem 4.6].

7. Application to random functions.

The above results can be used to generate random functions having specified differentiability properties or a specified modulus of continuity. First, it is obvious that the functions R_n may be chosen completely at random: Given fixed coefficients $\{c_n\}$, all of the theorems in this paper (except those in Section 6) hold for *every* realization of the R_n's. For each fixed n, the signs $R_n(\eta_1,\dots,\eta_n)$ may be chosen completely independently for all 2^n vectors $(\eta_1,\dots,\eta_n)$ if a maximum degree of randomness is desired; or they may be chosen dependently if more symmetry in the graph is desired. To introduce even more randomness, one may choose the coefficients $\{c_n\}$ at random as well. The apparent limitation that all functions of the form (2) vanish at 0 and 1 is easily overcome by adding a random linear function $Cx+D$ to $f(x)$, where C and D are appropriate random variables.

Let $\{\alpha_n\}_{n\in\boldsymbol{N}}$ be a sequence of nonnegative random variables defined on an exterior probability space $(\Omega,\mathscr{F},\mathrm{P})$, and satisfying

$$\sum_{n=1}^{\infty} 2^{-n}\,\mathrm{E}(\alpha_n) < \infty, \tag{28}$$

where E denotes expectation with respect to P. Assume further that $(\Omega,\mathscr{F},\mathrm{P})$ is large enough to support a collection of $\{-1,1\}$-valued random variables

$\{R_0\} \cup \{R_n(\eta_1, \ldots, \eta_n) : n \in \boldsymbol{N}, (\eta_1, \ldots, \eta_n) \in \{-1, 1\}^n\}$, having any desired joint distribution. Put

$$\rho_0(x) \equiv R_0, \qquad \rho_n(x) = R_n(X_1(x), \ldots, X_n(x)), \quad n \in \boldsymbol{N}.$$

Put $\gamma_n := 2^{-n}\alpha_n$ for $n \in \boldsymbol{N}$, and define the random function

$$f(x) = \sum_{n=1}^{\infty} \gamma_n \rho_{n-1}(x) \phi^{(n)}(x), \quad 0 \leq x \leq 1.$$

The assumption (28) implies that $\{\gamma_n\} \in \ell^1$ almost surely, and hence f is a continuous function with probability one. By imposing additional restrictions on the distribution of the sequence $\{\alpha_n\}$, it is possible to control various aspects of the graph of f. We focus here on differentiability and modulus of continuity. The first result is a direct consequence of Theorem 3.1 and the Kolmogorov three series theorem [**14**, Theorem 4.2.6].

THEOREM 7.1.

(i) *If* $\sum_{n=1}^{\infty} \mathrm{E}(\alpha_n^2) < \infty$, *then f is absolutely continuous almost surely, with derivative*

$$f'(x) = \sum_{n=1}^{\infty} \alpha_n \rho_{n-1}(x) X_n(x) \quad \textit{a.s.}\ (\lambda \times \mathrm{P}).$$

(ii) *If $\{\alpha_n\}$ is independent, $\alpha_n \to 0$ almost surely, and there exists a positive constant c such that either*

$$\sum_{n=1}^{\infty} \mathrm{P}(\alpha_n \geq c) = \infty, \qquad \textit{or} \qquad \sum_{n=1}^{\infty} \mathrm{E}(\alpha_n^2; \alpha_n < c) = \infty,$$

then with probability one f is nondifferentiable at almost every point of $[0, 1]$, *but f is differentiable on an uncountably large set, and the range of f' is $\boldsymbol{R}$.*

(iii) *If* $\limsup_{n\to\infty} \alpha_n > 0$ *almost surely, then f is nowhere differentiable with probability one.*

It is easy to give examples fulfilling the hypothesis of each statement. For instance, the condition of (ii) is satisfied when α_n is uniformly distributed on the interval $(0, n^{-1/2})$ for each n. The condition in (iii) clearly holds when the α_n's are independent and identically distributed, with $\mathrm{P}(\alpha_n > 0) > 0$.

Similarly, the above scheme can be used to generate random functions having a given modulus of continuity. The last two theorems illustrate this.

THEOREM 7.2. *If* $\sum_{n=1}^{\infty} \mathrm{E}(\alpha_n) < \infty$, *then* f *is Lipschitz continuous with probability one.*

PROOF. Immediate, from the remark at the beginning of Section 5.

THEOREM 7.3. *Fix* $d \geq 0$, *and let the sequence* $\{\alpha_n\}$ *be independent, with* $\mathrm{P}(\alpha_n \leq 2n^{d-1}) = 1$ *and* $\mathrm{E}(\alpha_n) = n^{d-1}$, *for* $n \in \boldsymbol{N}$.

(i) *If* $d > 0$, *then with probability one,*

$$\limsup_{|x-y|\downarrow 0} \frac{f(x) - f(y)}{(x-y)(-\log_2 |x-y|)^d} = \frac{1}{d}.$$

(ii) *If* $d = 0$, *then with probability one,*

$$\limsup_{|x-y|\downarrow 0} \frac{f(x) - f(y)}{(x-y)\log(-\log_2 |x-y|)} = 1.$$

PROOF. Recall from the proof of Theorem 5.1 that

$$|f(x+h) - f(x)| \leq |h| \sum_{n=1}^{p} \alpha_n + 2 \sum_{n=p+1}^{\infty} 2^{-n} \alpha_n. \tag{29}$$

The second term is bounded above by $4 \sum_{n=p+1}^{\infty} 2^{-n} n^{d-1}$, which is asymptotically of order $2^{-p} p^{d-1}$, and therefore of the order of $|h| p^{d-1}$. As for the first term, notice that by the strong law of large numbers [**14**, Theorem 4.3.1],

$$\frac{\alpha_1 + \cdots + \alpha_n}{b_n} \to 1 \quad \text{a.s.},$$

where

$$b_n = \sum_{k=1}^{n} \mathrm{E}(\alpha_k) \sim \begin{cases} n^d/d & \text{if } d > 0 \\ \log n & \text{if } d = 0 \end{cases} \qquad \text{as } n \to \infty.$$

Thus the first term in (29) dominates the second with probability one as $p \to \infty$. Since

$$p \sim \log_2(1/|h|) \quad \text{as } h \to 0,$$

the rest of the proof follows as in the proof of Theorem 5.1. □

References

[1] L. L. Cristea and H. Prodinger, Moments of distributions related to digital expansions, J. Math. Anal. Appl., **315** (2006), 606–625.

[2] L. E. Dubins and L. J. Savage, How to gamble if you must. Inequalities for stochastic processes, McGraw-Hill, New York, 1965.

[3] G. Faber, Einfaches Beispiel einer stetigen nirgends differenzierbaren Funktion, Jahresber. Deutschen Math.-Verein, **16** (1907), 538–540.

[4] K. Falconer, Techniques in Fractal Geometry, Wiley, Chichester, 1997.

[5] N. G. Gamkrelidze, On a probabilistic properties of Takagi's function (sic), J. Math. Kyoto Univ., **30** (1990), 227–229.

[6] P. Hartman and A. Wintner, On the law of the iterated logarithm, Amer. J. Math., **63** (1941), 169–176.

[7] M. Hata and M. Yamaguti, Takagi function and its generalization, Japan J. Appl. Math., **1** (1984), 183–199.

[8] J.-P. Kahane, Sur l'exemple, donné par M. de Rham, d'une fonction continue sans dérivée, Enseignement Math., **5** (1959), 53–57.

[9] K. Kawamura, On the classification of self-similar sets determined by two contractions on the plane, J. Math. Kyoto Univ., **42** (2002), 255–286.

[10] Z. Kobayashi, Digital sum problems for the Gray code representation of natural numbers, Interdiscip. Inform. Sci., **8** (2002), 167–175.

[11] N. Kono, On generalized Takagi functions, Acta Math. Hungar., **49** (1987), 315–324.

[12] P. D. Lax, The differentiability of Pólya's function, Adv. Math., **10** (1973), 456–464.

[13] F. Ledrappier, On the dimension of some graphs, Contemp. Math., **135** (1992), 285–293.

[14] E. Lukacs, Stochastic Convergence (2nd ed.), Academic Press, New York, 1975.

[15] T. Okada, T. Sekiguchi and Y. Shiota, Applications of binomial measures to power sums of digital sums, J. Number Theory, **52** (1995), 256–266.

[16] L. G. Pál and F. Schipp, On Haar and Schauder series, Acta Sci. Math. (Szeged), **31** (1970), 53–58.

[17] T. Takagi, A simple example of the continuous function without derivative, Phys. Math. Soc. Japan, **1** (1903), 176–177; The Collected Papers of Teiji Takagi, (ed. S. Kuroda), Iwanami, 1973, pp. 5–6.

[18] A. Zygmund, Smooth functions, Duke Math. J., **12** (1945), 47–76.

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