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On higher dimensional Luecking's theorem

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Abstract. D. Luecking has recently proved that a Toeplitz operator with measure symbol on the Bergman space of unit disk has finite rank if and only if its symbols is a linear combination of point masses. In this note we extend this theorem to higher dimensional cases.

1. Introduction.

In a recent paper [**3**] D. Luecking has provided an elegant proof of a theorem asserting that a Toeplitz operator (interpreted in a more general way) with measure symbol on the Bergman space of unit disk has finite rank if and only if its symbol is a linear combination of point masses. This theorem was actually *proved* in [**2**] by D. Luecking himself, but there was a serious gap in its proof. So, he finally *reproved* his own "theorem" after twenty years.

Let D be the unit disk of the complex plane $\boldsymbol{C}$. The Bergman space $L^2_a(D)$ is defined to be space of all square-integrable holomorphic functions on D. Viewing $L^2_a(D)$ as the Hilbert subspace of $L^2(D) = L^2(D, dA)$ where dA denotes the normalized area measure on D, we see that there is a Hilbert space orthogonal projection $P : L^2(D) \to L^2_a(D)$, the Bergman projection, which is realized by the integral

$$Pf(z) = \int_D \frac{f(z)}{(1 - z\overline{w})^2}\, dA(w)$$

for $f \in L^2(D)$.

For $u \in L^\infty(D)$, the *Toeplitz* T_u *with symbol* u is defined by

$$T_u f = P(uf)$$

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for $f \in L_a^2(D)$. The symbol u being essentially bounded, T_u is clearly a bounded linear operator on $L_a^2(D)$. The integral realization of the Bergman projection allows one to extend the notion of Toeplitz operators to densely-defined Toeplitz operators with unbounded symbols, or even with measure symbols. That is, the Toeplitz operator T_μ with symbol μ, a complex Borel measure on D, is defined by

$$T_\mu f(z) = \int_D \frac{f(z)}{(1 - z\overline{w})^2}\, d\mu(w)$$

for holomorphic polynomials $f \in \mathscr{P}(\boldsymbol{C})$, the algebra of holomorphic polynomials on $\boldsymbol{C}$. Note that T_μ takes $\mathscr{P}(\boldsymbol{C})$ into the space of holomorphic functions on D.

Let $\mathscr{P}^*(\boldsymbol{C})$ denote the space of all conjugate-linear functionals on $\mathscr{P}(\boldsymbol{C})$. Given a compactly supported complex Borel measure μ on $\boldsymbol{C}$, let $L_\mu : \mathscr{P}(\boldsymbol{C}) \to \mathscr{P}^*(\boldsymbol{C})$ denote the linear operator defined by

$$L_\mu f(g) = \int_{\boldsymbol{C}} f\overline{g}\, d\mu$$

for $f, g \in \mathscr{P}(\boldsymbol{C})$. When μ is supported in D, we have, formally,

$$\int_D (T_\mu f)\overline{g}\, dA = L_\mu f(g) \qquad f, g \in \mathscr{P}(\boldsymbol{C}), \tag{1.1}$$

which reveals a close connection between operators T_μ and L_μ. Of course, this equality is not always true in the strict sense; justification of implicit exchange of integrals would be required. At this point of difficulty Luecking [**3**] noticed that there is a more natural way to connect operators T_μ and L_μ. Based on that, he has shown the following result which implies the analogue for Toeplitz operators; see Section 3.

THEOREM (Luecking). *L_μ has finite rank if and only if μ is a linear combination of point masses.*

Some steps of the proof of this theorem in [**3**] depend on certain properties restricted to one variable case and generalizations to several variable cases (though possible as in the current note) do not seem to be straightforward.

In this little note I follow the main scheme of Luecking's one variable proof and prove a higher dimensional analogue with hope that this might serve as a convenient reference for several variable theory. Some parts of Luecking's proofs are repeated for completeness, some more details are provided in some steps for reader's convenience, and, when necessary, appropriate adjustments are made for several variables.

The author thanks Hyungwoon Koo for his kind suggestion of the proof of Lemma 2.2.

2. Proofs.

Fix a positive integer n for the rest of the note. Let $\mathscr{P}(\boldsymbol{C}^n)$ denotes the algebra of all holomorphic polynomials on $\boldsymbol{C}^n$ and let $\mathscr{P}^*(\boldsymbol{C}^n)$ denote the space of all conjugate-linear functionals on $\mathscr{P}(\boldsymbol{C}^n)$.

Some of notations, introduced for one variable case in the previous section, will also be used for several variable case. Namely, given a compactly supported complex Borel measure μ on $\boldsymbol{C}^n$, let $L_\mu : \mathscr{P}(\boldsymbol{C}^n) \to \mathscr{P}^*(\boldsymbol{C}^n)$ denote the linear operator defined by

$$L_\mu f(g) = \int_{\boldsymbol{C}^n} f\overline{g}\, d\mu$$

for $f, g \in \mathscr{P}(\boldsymbol{C}^n)$.

For the rest of this section we fix a complex Borel measure μ, supported in a compact subset of $\boldsymbol{C}^n$, and assume that L_μ has finite rank, say less than N. We will show that μ is supported on a finite set containing less than N points.

Given $f_1, \dots, f_N \in \mathscr{P}(\boldsymbol{C}^n)$, functionals $L_\mu f_1, \dots, L_\mu f_N$ are linearly dependent and thus

$$c_1 L_\mu f_1 + \cdots + c_N L_\mu f_N = 0$$

for some nontrivial choice of coefficients $c_1, \dots, c_N$. Let $g_1, \dots, g_N \in \mathscr{P}(\boldsymbol{C}^n)$ be given. Taking inner products and using the notation $\mu(f\overline{g}) := \int_{\boldsymbol{C}^n} f\overline{g}\, d\mu$ for simplicity, we see that the system of linear equations

$$\begin{pmatrix} \mu(f_1\overline{g}_1) & \dots & \mu(f_N\overline{g}_1) \\ \vdots & & \vdots \\ \mu(f_1\overline{g}_N) & \dots & \mu(f_N\overline{g}_N) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = 0$$

has a nontrivial solution, namely $(x_1, \dots, x_N) = (c_1, \dots, c_N)$. So, we have

$$\det \begin{pmatrix} \mu(f_1\overline{g}_1) & \dots & \mu(f_N\overline{g}_1) \\ \vdots & & \vdots \\ \mu(f_1\overline{g}_N) & \dots & \mu(f_N\overline{g}_N) \end{pmatrix} = 0. \tag{2.1}$$

Before proceeding, we introduce some more notation. Let $\boldsymbol{C}^{n\times N}$ stand for the product of N copies of $\boldsymbol{C}^n$. A general point in $\boldsymbol{C}^{n\times N}$ is denoted by $\boldsymbol{z} = (\boldsymbol{z}_1, \dots, \boldsymbol{z}_N)$ where $\boldsymbol{z}_j \in \boldsymbol{C}^n$ for each j. Denote by μ_N the product measure $\mu \times \cdots \times \mu$ on $\boldsymbol{C}^{n\times N}$ and denote by $\mathscr{P}(\boldsymbol{C}^{n\times N})$ the algebra of all holomorphic polynomials on $\boldsymbol{C}^{n\times N}$.

Now, since determinant is linear in each column, we may rephrase (2.1) (going back to the integral notation) as

$$\int_{\boldsymbol{C}^{n\times N}} f_1(\boldsymbol{z}_1)\cdots f_N(\boldsymbol{z}_N)\overline{\Delta_{(g_1,\dots,g_N)}(\boldsymbol{z})}\, d\mu_N(\boldsymbol{z}) = 0 \tag{2.2}$$

where

$$\Delta_{(g_1,\dots,g_N)}(\boldsymbol{z}) = \det \begin{pmatrix} g_1(\boldsymbol{z}_1) & \dots & g_1(\boldsymbol{z}_N) \\ \vdots & & \vdots \\ g_N(\boldsymbol{z}_1) & \dots & g_N(\boldsymbol{z}_N) \end{pmatrix}.$$

Inserting monomials f_j into (2.2) and then considering finite sums, we may replace the product $f_1(\boldsymbol{z}_1)\cdots f_N(\boldsymbol{z}_N)$ in (2.2) by any holomorphic polynomials on $\boldsymbol{C}^{n\times N}$. Namely, we have

$$\int_{\boldsymbol{C}^{n\times N}} \psi(\boldsymbol{z})\overline{\Delta_{(g_1,\dots,g_N)}(\boldsymbol{z})}\, d\mu_N(\boldsymbol{z}) = 0 \tag{2.3}$$

for $\psi \in \mathscr{P}(\boldsymbol{C}^{n\times N})$ and $g_j \in \mathscr{P}(\boldsymbol{C}^n)$ for each j.

Given a permutation σ on $\{1, \dots, N\}$, let $\sigma\boldsymbol{z} = (\boldsymbol{z}_{\sigma(1)}, \dots, \boldsymbol{z}_{\sigma(N)})$ for $\boldsymbol{z} \in \boldsymbol{C}^{n\times N}$. Let φ be a function defined on a permutation-invariant set $U \subset \boldsymbol{C}^{n\times N}$. We say that φ is *n-symmetric* on U if

$$\varphi(\sigma\boldsymbol{z}) = \varphi(\boldsymbol{z}), \qquad \boldsymbol{z} \in U$$

for all permutations σ. Also, we say that φ is *n-antisymmetric* on U if

$$\varphi(\sigma\boldsymbol{z}) = (\operatorname{sgn}\sigma)\varphi(\boldsymbol{z}), \qquad \boldsymbol{z} \in U$$

for all permutations σ. Here, $\operatorname{sgn}\sigma = 1$ if σ is even and $\operatorname{sgn}\sigma = -1$ if σ is odd. We denote by $C_S(U)$ the space of all continuous n-symmetric functions on U.

Given a function φ on U that is not necessarily n-(anti)symmetric, we let φ_s and φ_a denote n-symmetrization and n-antisymmetrization of φ, respectively. That is,

$$\varphi_s(\boldsymbol{z}) = \frac{1}{N!}\sum_\sigma \varphi(\sigma\boldsymbol{z}) \quad \text{and} \quad \varphi_a(\boldsymbol{z}) = \frac{1}{N!}\sum_\sigma (\operatorname{sgn}\sigma)\varphi(\sigma\boldsymbol{z})$$

for $\boldsymbol{z} \in U$ where sums are taken over all permutations σ.

Let $\mathscr{P}_S(\boldsymbol{C}^{n\times N})$ and $\mathscr{P}_A(\boldsymbol{C}^{n\times N})$ denote the subspace of $\mathscr{P}(\boldsymbol{C}^{n\times N})$ consisting of all n-symmetric holomorphic polynomials and all n-antisymmetric holomorphic polynomials, respectively.

LEMMA 2.1. *The equality*

$$\int_{\boldsymbol{C}^{n\times N}} \psi\overline{\varphi}\, d\mu_N = 0$$

holds for all $\psi \in \mathscr{P}(\boldsymbol{C}^{n\times N})$ *and* $\varphi \in \mathscr{P}_A(\boldsymbol{C}^{n\times N})$.

PROOF. Note that $\Delta_{(p_1,\dots,p_N)} \in \mathscr{P}_A(\boldsymbol{C}^{n\times N})$ for all monomials $p_1,\dots,p_N \in \mathscr{P}(\boldsymbol{C}^n)$. Thus, such functions span a subspace, say $\mathscr{S}$, of $\mathscr{P}_A(\boldsymbol{C}^{n\times N})$. By (2.3) it is sufficient to show $\mathscr{S} = \mathscr{P}_A(\boldsymbol{C}^{n\times N})$.

We first observe a close connection with n-antisymmetrization and determinant. Given functions $g_1,\dots,g_N$ on $\boldsymbol{C}^n$, denote by $\otimes_{j=1}^N g_j$ the function defined by $\left[\otimes_{j=1}^N g_j\right](\boldsymbol{z}) = g_1(\boldsymbol{z}_1)\cdots g_N(\boldsymbol{z}_N)$. Note that we have

$$\left[\otimes_{j=1}^N g_j\right]_a(\boldsymbol{z}) = \frac{1}{N!}\sum_\sigma (\operatorname{sgn}\sigma) g_1(\boldsymbol{z}_{\sigma(1)})\cdots g_N(\boldsymbol{z}_{\sigma(N)}) = \frac{1}{N!}\Delta_{(g_1,\dots,g_N)}(\boldsymbol{z}).$$

Thus, taking arbitrary monomials g_j on $\boldsymbol{C}^n$ and considering linear combinations, we see that $\varphi_a \in \mathscr{S}$ for all $\varphi \in \mathscr{P}(\boldsymbol{C}^{n\times N})$. Accordingly, for $\varphi \in \mathscr{P}_A(\boldsymbol{C}^{n\times N})$, we have $\varphi = \varphi_a \in \mathscr{S}$, as required. $\square$

Let Φ_S be the algebra consisting of all linear combinations of functions of the form $\psi\overline{\varphi}$ where $\psi, \varphi \in \mathscr{P}_S(\boldsymbol{C}^{n\times N})$. Clearly, Φ_S contains constant functions and is self-adjoint, i.e., closed under complex conjugation. Since an n-symmetric holomorphic polynomial has the same values at points that are permutations of one another, Φ_S does not separate points. However, Φ_S separates two points that are not permutations of each another, as in Lemma 2.2 below. Given a point $\boldsymbol{a} \in \boldsymbol{C}^{n\times N}$, we denote by $[\boldsymbol{a}]$ the set of all points obtained by permuting $\boldsymbol{a}_1,\dots,\boldsymbol{a}_N$. Note that if $[\boldsymbol{a}] \neq [\boldsymbol{b}]$, then $[\boldsymbol{a}] \cap [\boldsymbol{b}] = \emptyset$.

LEMMA 2.2. *If* $[\boldsymbol{a}] \neq [\boldsymbol{b}]$, *then there is some* $\psi \in \mathscr{P}_S(\boldsymbol{C}^{n\times N})$ *such that* $\psi(\boldsymbol{a}) \neq \psi(\boldsymbol{b})$.

The proof below is quite analytic. I wonder whether there is a simple algebraic proof.

Proof. We first introduce some temporary notation. For positive integers $d \le N$, denote by ϵ_d the d-th elementary symmetric polynomial in $\lambda = (\lambda_1, \dots, \lambda_N) \in \boldsymbol{C}^N$. This means that $\epsilon_d(\lambda)$'s are determined by the equation

$$\prod_{j=1}^{N}(t-\lambda_j) = \sum_{d=1}^{N}(-1)^d \epsilon_d(\lambda) t^{N-d}.$$

So, we have

$$\nu = (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(N)}) \text{ for some } \sigma \iff \epsilon_d(\lambda) = \epsilon_d(\nu) \text{ for all } d \tag{2.4}$$

for $\nu, \lambda \in \boldsymbol{C}^N$.

Assume $[\boldsymbol{a}] \ne [\boldsymbol{b}]$. Let s be a sufficiently large positive number. Given $j, k = 1, \dots, N$, denote by $\boldsymbol{c}_j^k(s)$ the point in $\boldsymbol{C}^n$ whose m-th component is $\log\{(\boldsymbol{a}_{km} + s)/(\boldsymbol{b}_{jm} + s)\}$ where log is the principal branch. Note $\boldsymbol{c}_j^k(s) = 0$ if and only if $\boldsymbol{a}_k = \boldsymbol{b}_j$. Since $[\boldsymbol{a}] \ne [\boldsymbol{b}]$, we have $\boldsymbol{c}_j^k(s) \ne 0$ for some j and k. Let H_s be the set of all points $z \in \boldsymbol{C}^n$ such that $\boldsymbol{c}_j^k(s) \cdot \overline{z} \ne 0$ whenever $\boldsymbol{a}_k \ne \boldsymbol{b}_j$. Here, $x \cdot \overline{y} = \sum_{j=1}^n x_j \overline{y}_j$ denotes the Hermitian inner product of $x, y \in \boldsymbol{C}^n$. Put $\eta_j^k(s) = \boldsymbol{c}_j^k(s)|\boldsymbol{c}_j^k(s)|^{-1}$ when $\boldsymbol{a}_k \ne \boldsymbol{b}_j$. Now we choose a sequence $s_\ell \to \infty$ such that $\eta_j^k(s_\ell)$ converges to some $\eta_j^k \in \boldsymbol{C}^n$ with $|\eta_j^k| = 1$ for each j, k. Let $H = \cap_\ell H_{s_\ell}$. Note $\boldsymbol{C}^n \setminus H$ is the union of countably many $(n-1)$-dimensional subspaces. In particular, H is dense in $\boldsymbol{C}^n$. Thus we can pick some $\zeta \in H$ such that $|\zeta| = 1$, $\zeta_j \ge 0$ for all j, and $\eta_j^k \cdot \overline{\zeta} \ne 0$ for all η_j^k. Put $\delta = \frac{1}{2}\min_{j,k} |\eta_j^k \cdot \overline{\zeta}| > 0$. Note δ is independent of ℓ.

Note $\min_{j,k} |\eta_j^k(s_\ell) \cdot \overline{\zeta}| \to 2\delta$ and $\max_{j,k} |\boldsymbol{c}_j^k(s_\ell)| \to 0$ as $\ell \to \infty$. We now fix a large ℓ such that

$$\min_{j,k} \left|\eta_j^k(s_\ell) \cdot \overline{\zeta}\right| > \delta \quad \text{and} \quad \max_{j,k} \left|\boldsymbol{c}_j^k(s_\ell)\right| < \frac{1}{\sqrt{n}(M+1)} \tag{2.5}$$

where $M = 2\sqrt{n}/\delta$. For $w \in \boldsymbol{C}^n$ with $|w_j| \le 1$ for all j, we have by the first part of (2.5)

$$\left|\eta_j^k(s_\ell) \cdot \overline{(M\zeta + w)}\right| \ge M\left|\eta_j^k(s_\ell) \cdot \overline{\zeta}\right| - |w| \ge M\delta - \sqrt{n} > 0,$$

and thus $M\zeta + w \in H$.

Now, choose numbers $w_j \in [0,1]$ in such a way that components of $M\zeta + w$ are all positive integers. Put $\alpha = M\zeta + w \in H$. Let $\pi = \pi_{\alpha,\ell}$ denote the polynomial $(z_1 + s_\ell)^{\alpha_1} \cdots (z_n + s_\ell)^{\alpha_n}$ on $\boldsymbol{C}^n$. Also, for $d = 1, \dots, N$, let $E_d = E_{d,\pi}$ denote the function on $\boldsymbol{C}^{n\times N}$ defined by

$$E_d(\boldsymbol{z}) = \epsilon_d\big(\pi(\boldsymbol{z}_1), \dots, \pi(\boldsymbol{z}_N)\big).$$

Note $E_d \in \mathscr{P}_S(\boldsymbol{C}^{n\times N})$ for all d. We claim $E_d(\boldsymbol{a}) \neq E_d(\boldsymbol{b})$ for some d. To reach a contradiction assume $E_d(\boldsymbol{a}) = E_d(\boldsymbol{b})$ for all d. By (2.4) there is a permutation $\sigma = \sigma_\pi$ such that

$$\big(\pi(\boldsymbol{b}_1), \dots, \pi(\boldsymbol{b}_N)\big) = \big(\pi(\boldsymbol{a}_{\sigma(1)}), \dots, \pi(\boldsymbol{a}_{\sigma(N)})\big),$$

which yields

$$\frac{\pi(\boldsymbol{a}_{\sigma(1)})}{\pi(\boldsymbol{b}_1)} = \cdots = \frac{\pi(\boldsymbol{a}_{\sigma(N)})}{\pi(\boldsymbol{b}_N)} = 1.$$

Rephrasing this in terms of components and then taking logarithms, we have

$$\boldsymbol{c}_j^{\sigma(j)}(s_\ell) \cdot \overline{\alpha} = 0 \pmod{2\pi i} \tag{2.6}$$

for each j. Since $\alpha_m \leq M + 1$ for each m, we have by the second part of (2.5)

$$\big|\boldsymbol{c}_j^{\sigma(j)}(s_\ell) \cdot \overline{\alpha}\big| \leq 1$$

for each j. Thus (2.6) holds without (mod $2\pi i$) so that $\boldsymbol{c}_j^{\sigma(j)}(s_\ell) \cdot \overline{\alpha} = 0$ for all j. Since $\alpha \in H$, we obtain $\boldsymbol{c}_j^{\sigma(j)}(s_\ell) = 0$. This yields $\boldsymbol{b}_j = \boldsymbol{a}_{\sigma(j)}$ for all j. So, we have $\boldsymbol{b} = \sigma\boldsymbol{a}$ and thus $[\boldsymbol{a}] = [\boldsymbol{b}]$, which is a contradiction. $\square$

The following lemma shows that Φ_S is dense in $C_S(\boldsymbol{C}^{n\times N})$ in the topology of uniform convergence on compact sets.

LEMMA 2.3. *Let K be a permutation-invariant compact subset of $\boldsymbol{C}^{n\times N}$. Then Φ_S is uniformly dense in $C_S(K)$.*

PROOF. Define an equivalence relation $\sim$ on K by that $\boldsymbol{a} \sim \boldsymbol{b}$ if and only if $\boldsymbol{a} = \sigma\boldsymbol{b}$ for some permutation σ. So, the equivalence class containing $\boldsymbol{a} \in K$ is precisely the set $[\boldsymbol{a}]$. Equip $X := K/\sim$ with the standard quotient topology.

Namely, a set $E \subset X$ is open if and only if $\cup_{[\boldsymbol{a}]\in E}[\boldsymbol{a}]$ is open in K. It is not hard to see that X is compact and Hausdorff.

There is a canonical isomorphism between $C_S(K)$ and $C(X)$, which is isometric when both spaces are endowed with sup norm topologies. That is, for $\psi \in C_S(K)$, define $\widetilde{\psi}([\boldsymbol{z}]) = \psi(\boldsymbol{z})$. Clearly, $\widetilde{\psi}$ is well defined, $\widetilde{\psi} \in C(X)$ and $\|\widetilde{\psi}\|_{C(X)} = \|\psi\|_{C_S(K)}$. Conversely, given $h \in C(X)$, define $\widehat{h}(\boldsymbol{z}) = h([\boldsymbol{z}])$. Clearly, $\widehat{h} \in C_S(K)$ and $\|h\|_{C(X)} = \|\widehat{h}\|_{C_S(K)}$. Furthermore, we have $\widehat{\widetilde{\psi}} = \psi$ for $\psi \in C_S(K)$ and $\widetilde{\widehat{h}} = h$ for $h \in C(X)$.

Let $\widetilde{\Phi}_S$ be the set of all functions $\widetilde{\psi}$ induced by functions $\psi \in \Phi_S$. Note that $\widetilde{\Phi}_S$ is a self-adjoint subalgebra of $C(X)$ and contains constant functions. Moreover, $\widetilde{\Phi}_S$ separates points by Lemma 2.2. Thus $\widetilde{\Phi}_S$ is uniformly dense in $C(X)$ by the Stone-Weierstrass theorem. So, using the canonical isometric isomorphism mentioned above, we conclude that Φ_S is uniformly dense in $C_S(K)$, as required. □

Let V be the N-th Vandermonde polynomial defined by

$$V(\lambda) = \det \begin{pmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_N \\ \vdots & & \vdots \\ \lambda_1^{N-1} & \dots & \lambda_N^{N-1} \end{pmatrix} = \prod_{i>j}(\lambda_i - \lambda_j)$$

for $\lambda = (\lambda_1, \dots, \lambda_N) \in \boldsymbol{C}^N$.

Fix a compact set $E \subset \boldsymbol{C}^n$ containing the support of μ. Suppose the support of μ contains N distinct points $\boldsymbol{a}_1, \dots, \boldsymbol{a}_N$. Choose $p \in \mathscr{P}(\boldsymbol{C}^n)$ such that $p(\boldsymbol{a}_j) \neq p(\boldsymbol{a}_k)$ for all j, k with $j \neq k$. Note $V_p(\boldsymbol{a}) \neq 0$ where $\boldsymbol{a} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_N)$. Associated with p is the polynomial V_p on $\boldsymbol{C}^{n\times N}$ defined by

$$V_p(\boldsymbol{z}_1, \dots, \boldsymbol{z}_N) = V(p(\boldsymbol{z}_1), \dots, p(\boldsymbol{z}_N)).$$

Note that V_p is n-antisymmetric.

Since the product of an n-symmetric function and an n-antisymmetric function is n-antisymmetric, an application of Lemma 2.1 to functions $\psi = \psi_1 V_p$ and $\varphi = \psi_2 V_p$ with $\psi_1, \psi_2 \in \mathscr{P}_S(\boldsymbol{C}^{n\times N})$ yields

$$\int_{\boldsymbol{C}^{n\times N}} F|V_p|^2 \, d\mu_N = 0 \tag{2.7}$$

for all $F \in \Phi_S$. Since E^N is permutation-invariant, it follows from Lemma 2.3 that (2.7) is valid even for all $F \in C_S(E^N)$.

Note that the measure $|V_p|^2\, d\mu_N$ is permutation-invariant. Thus, for a function $F \in C(E^N)$ that is not necessarily n-symmetric, the integral in (2.7) remains the same if F is replaced with its n-symmetrization $F_s \in C_S(E^N)$. Hence we see that (2.7) is valid even for all functions F continuous on E^N. Accordingly, $|V_p|^2\, d\mu_N$ is the zero measure and thus μ_N is supported in the zero variety of V_p. Note that $\boldsymbol{a}$ is contained in the support of μ_N. So, $V_p(\boldsymbol{a}) = 0$, which is a contradiction. This contradiction shows that μ is supported on a finite set containing less than N points, as desired.

Thus we conclude higher dimensional Luecking's theorem as follows. Note that the sufficiency is clear.

THEOREM 2.4. *Let μ be complex Borel measure on $\boldsymbol{C}^n$ supported in a compact set. Then L_μ has finite rank if and only if μ is a linear combination of point masses.*

3. Some consequences.

As in [**3**], one may apply Theorem 2.4 to Toeplitz operators on a bounded domain in $\boldsymbol{C}^n$ that might be quite general. For example, consider a complete circular bounded domain $\Omega \subset \boldsymbol{C}^n$ with normalized volume measure dV on it. By "complete circular" we mean that if $z \in \Omega$ and γ is a complex number with $|\gamma| \le 1$, then $\gamma z \in \Omega$.

Let $L^2_a(\Omega)$ and $L^2_{ph}(\Omega)$ be the Bergman space and the pluriharmonic Bergman space on Ω, respectively. The space $L^2_{ph}(\Omega)$ consists of all pluriharmonic functions in $L^2(\Omega) = L^2(\Omega, dV)$. Let $K_a(z,w)$ and $K_{ph}(z,w)$ be the Bergman kernel for $L^2_a(\Omega)$ and the pluriharmonic Bergman kernel for $L^2_{ph}(\Omega)$, respectively. These are the kernels uniquely determined by the reproducing properties

$$f(z) = \int_\Omega f(w) K_a(z,w)\; dV(w), \qquad f \in L^2_a(\Omega)$$

and

$$f(z) = \int_\Omega f(w) K_{ph}(z,w)\; dV(w), \qquad f \in L^2_{ph}(\Omega).$$

The domain being a complete circular domain, any pluriharmonic function on Ω can be uniquely written in the form $f + \overline{g}$ where f, g are holomorphic functions on

Ω with $g(0) = 0$. So, the Bergman kernel and the pluriharmonic Bergman kernel are closely related by

$$K_{ph}(z, w) = K_a(z, w) + K_a(w, z) - 1; \tag{3.1}$$

see [**1**].

Let μ be a complex Borel measure on Ω. We define T_μ^a, the Toeplitz operator associated with $L_a^2(\Omega)$, by

$$T_\mu^a f(z) = \int_\Omega f(w) K_a(z, w)\, d\mu(w)$$

for $f \in \mathscr{P}(\boldsymbol{C}^n)$. Similarly, we define T_μ^{ph}, the Toeplitz operator associated with $L_{ph}^2(\Omega)$, by replacing the kernel $K_a(z, w)$ with $K_{ph}(z, w)$. The operator T_μ^a (T_μ^{ph}) takes $\mathscr{P}(\boldsymbol{C}^n)$ into $H(\Omega)$ ($ph(\Omega)$), the space of holomorphic (pluriharmonic) functions on Ω.

Since the Bergman kernel is conjugate symmetric and the pluriharmonic Bergman kernel is symmetric, we again have the formal analogues of (1.1) in this context:

$$\int_\Omega (T_\mu^a f)\overline{g}\, dV = L_\mu f(g) = \int_\Omega (T_\mu^{ph} f)\overline{g}\, dV$$

for $f, g \in \mathscr{P}(\boldsymbol{C}^n)$. For either equality to be true in the strict sense, justification of implicit exchange of integrals would also be required. So, we follow below Luecking's interpretation of Toeplitz operators.

Let's consider the pluriharmonic case; the holomorphic case is similar and simpler. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, let e_α be the normalized monomial on $\boldsymbol{C}^n$ of multi-degree α, i.e., $e_\alpha(z) = z^\alpha \|z^\alpha\|_{L^2(\Omega)}^{-1}$. Given a function $F \in ph(\Omega)$ with Taylor series $F = \sum_\alpha a_\alpha(F) e_\alpha + \sum_{\beta \neq 0} b_\beta(F)\overline{e_\beta}$, define $\Lambda_F(e_\alpha) = a_\alpha(F)$ for each α and then extend Λ_F to all of $\mathscr{P}(\boldsymbol{C}^n)$ by conjugate-linearity. Each $F \in ph(\Omega)$ thus naturally induces $\Lambda_F \in \mathscr{P}^*(\boldsymbol{C}^n)$. Since $K_a(z, w) = \sum_\alpha e_\alpha(z)\overline{e_\alpha(w)}$ (as is well-known), it is easily seen from (3.1) that

$$\Lambda_{T_\mu^{ph} f}(g) = \int_\Omega f\overline{g}\, d\mu$$

for all $f, g \in \mathscr{P}(\boldsymbol{C}^n)$. So, $\Lambda_{T_\mu^{ph} f} = L_\mu f$ for $f \in \mathscr{P}(\boldsymbol{C}^n)$. This implies that if the range of T_μ^{ph} is spanned by finitely many functions F_j's in $ph(\Omega)$, then the range of L_μ is spanned by Λ_{F_j}'s.

Thus we have the following consequence.

COROLLARY 3.1. *Let μ be a complex Borel measure on Ω. Then the following conditions are equivalent*:

(a) $T^a_\mu : \mathscr{P}(\boldsymbol{C}^n) \to H(\Omega)$ *has finite rank*;
(b) $T^{ph}_\mu : \mathscr{P}(\boldsymbol{C}^n) \to ph(\Omega)$ *has finite rank*;
(c) μ *is a linear combination of point masses.*

Following [**3**], we observe another application on certain approximation properties associated with zero sets. We first introduce some notation.

For a subset Q of $C(\Omega)$, we denote by $Z(Q)$ the zero set of Q. That is, $Z(Q)$ is the set of all points in Ω where functions in Q vanish simultaneously. Conversely, for a subset E of Ω, we denote by $J(E)$ the ideal in $C(\Omega)$ consisting of all functions in $C(\Omega)$ vanishing on E.

Given a subspace W of $L^2_a(\Omega)$, denote by $\widehat{W}$ the closure in $C(\Omega)$ of the span of $L^2_a\overline{W} := \{f\overline{g} : f \in L^2_a(\Omega), g \in W\}$ in the topology of uniform convergence on compact sets.

COROLLARY 3.2. *Let W be a subspace of $L^2_a(\Omega)$ with finite codimension. Then $Z(W)$ is a finite set and $\widehat{W} = J(Z(W))$. In particular, if $Z(W) = \emptyset$, then $\widehat{W} = C(\Omega)$.*

PROOF. Endowed with the topology of uniform convergence on compact sets, the space $C(\Omega)$ is locally convex and its continuous linear functionals are identified with complex Borel measures supported on compact sets in Ω; see, for example, [**4**, p. 88]. Let Y be the annihilator space of $L^2_a\overline{W}$. Assume first $Y \neq \{0\}$. Then there is a complex Borel measure $\mu \neq 0$ supported on a compact set in Ω such that

$$0 = \int_\Omega f\overline{g}\,d\mu = \int_\Omega (T_\mu f)\overline{g}\,dV$$

for all $f \in L^2_a(\Omega)$ and $g \in W$. This shows that $T_\mu L^2_a(\Omega)$ is contained in $W^\perp$, which is finite dimensional. By Corollary 3.1 μ is supported on some finite set in Ω, say E_μ. It follows that $T_\mu L^2_a(\Omega)$ is spanned by finitely many kernel functions $K_a(\cdot, b)$ for $b \in E_\mu$. Note $K_a(\cdot, b) \in W^\perp$ for $b \in E_\mu$. Set $E = \cup_{\mu \in Y} E_\mu$.

Note that functions $K_a(\cdot, b)$ with $b \in E$ are linearly independent and contained in $W^\perp$. Hence E is a finite set and is contained in $Z(W)$ by the reproducing property. In fact we have $E = Z(W)$, because point masses (or point evaluations) at each point of $Z(W)$ belong to Y. Now, since each $\mu \in Y$ is supported in E, $J(E) = J(Z(W))$ is annihilated by all $\mu \in Y$. Thus $J(Z(W)) \subset \widehat{W}$ and the converse containment is clear. Next, the case $Y = \{0\}$ is easily treated by the fact that $Y = \{0\}$ if and only if $Z(W) = \emptyset$, which one may see from the proof above. $\square$

The pluriharmonic analogue of Corollary 3.2 also holds by the same proof.

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