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Poisson structures and generalized Kähler submanifolds

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Abstract. Let X be a compact Kähler manifolds with a non-trivial holomorphic Poisson structure β. Then there exist deformations $\{(\mathscr{J}_{\beta t}, \psi_t)\}$ of non-trivial generalized Kähler structures with one pure spinor on X. We prove that every Poisson submanifold of X is a generalized Kähler submanifold with respect to $(\mathscr{J}_{\beta t}, \psi_t)$ and provide non-trivial examples of generalized Kähler submanifolds arising as holomorphic Poisson submanifolds. We also obtain unobstructed deformations of bihermitian structures constructed from Poisson structures.

Introduction.

A generalized complex structure interpolates between symplectic and complex structures [**17**], where a generalized Kähler structure is obtained by extending the notion of ordinary Kähler structures from the viewpoint of generalized geometry [**14**]. Recently, Hitchin gave a construction of generalized Kähler structures on Del Pezzo surfaces by using holomorphic Poisson structures [**18**], [**19**], while we showed a stability theorem of generalized Kähler structures to obtain a family $\{(\mathscr{J}_{\beta t}, \psi_t)\}$ of non-trivial generalized Kähler structures on a compact Kähler manifold with a holomorphic Poisson structure [**13**]. The present paper is a sequel to [**13**] by discussing holomorphic Poisson submanifolds and generalized Kähler submanifolds from the viewpoint of deformations.

In Section 1.1, we give an exposition of generalized complex structures. In Section 1.2, we introduce a notion of $\mathscr{J}$-submanifolds of a generalized complex manifold $(X, \mathscr{J})$ which is due to Oren Ben-Bassart and Mitya Boyarchenko. A $\mathscr{J}$-submanifold M inherits the induced generalized complex structure $\mathscr{J}_M$ [**4**], [**31**][1]. Both complex submanifolds and symplectic submanifolds arise as special classes of $\mathscr{J}$-submanifolds. Let $\mathscr{J}_b$ be the generalized complex structure given by the action of d-closed b-fields. Then a $\mathscr{J}$-submanifold is also a $\mathscr{J}_b$-submanifold

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[1]Note that there is a different notion of generalized complex submanifolds due to Gualtieri. To avoid a confusion, we use the terminology of $\mathscr{J}$-submanifolds in this paper.

(see Example 1.8). We denote by N^* the conormal bundle to M in X. If a submanifold M admits a $\mathscr{J}$-invariant conormal bundle, i.e.,

$$\mathscr{J}(N^*) = N^*,$$

then M is a $\mathscr{J}$-submanifold. After a short explanation of generalized Kähler structures in Section 1.3, we prove in Section 1.4 that if a submanifold M of a generalized Kähler manifold $(X, \mathscr{J}_0, \mathscr{J}_1)$ admits a $\mathscr{J}_0$-invariant conormal bundle, then M is also a $\mathscr{J}_1$-submanifold and M inherits the induced generalized Kähler structure $(\mathscr{J}_{0,M}, \mathscr{J}_{1,M})$ (see Theorem 1.9). In Section 1.5, we introduce *a generalized Kähler structure with one pure spinor* which is a pair $(\mathscr{J}, \psi)$ consisting of a generalized complex structure $\mathscr{J}$ and a d-closed, non-degenerate, pure spinor ψ such that the induced pair $(\mathscr{J}, \mathscr{J}_\psi)$ is a generalized Kähler structure. On a Kähler manifold with a Kähler form ω, there is the ordinary generalized Kähler strucutre with one pure spinor, where the pure spinor is given by $\exp(\sqrt{-1}\omega)$.

Theorem 1.14. *Let $(X, \mathscr{J}_0, \psi)$ be a generalized Kähler manifold with one pure spinor. Let M be a submanifold with invariant conormal bundle with respect to $\mathscr{J}_0$. Then the pull back $i_M^*\psi$ is a d-closed, non-degenerate, pure spinor on M and the induced pair $(\mathscr{J}_{0,M}, i_M^*\psi)$ is a generalized Kähler structure with one pure spinor on M.*

This is analogous to the fact that the pull back of a Kähler form to a complex submanifold is also a Kähler form. In Section 1.6, let X be a compact Kähler manifold with a Poisson structure β and a Kähler form ω. (We always consider holomorphic Poisson structures in this paper.) A complex submanifold M is a Poisson submanifold if there exists the induced Poisson structure β_M on M (see Definition 1.17). Note that in algebraic geometry, Poisson schemes and Poisson subschemes are developed [**26**], [**27**]. A Poisson structure β on a compact Kähler manifold generates deformations of generalized complex structures $\{\mathscr{J}_{\beta t}\}$ parametrized by the complex number t. By applying the stability theorem [**13**], we obtain deformations of generalized Kähler structures $\{\mathscr{J}_{\beta t}, \psi_t\}_{t\in\triangle'}$ with one pure spinor, where $\triangle'$ is a one dimensional complex disk. Then it turns out that every Poisson submanifold M of X admits a $\mathscr{J}_{\beta t}$-invariant conormal bundle, which is a generalized Kähler submanifold of $(X, \mathscr{J}_{\beta t}, \psi_t)$ for $t \in \triangle'$. Thus every Poisson submanifold inherits the induced generalized Kähler structures.

Theorem 1.20. *Let M be a Poisson submanifold of a Poisson manifold X with a Kähler structure ω. Then M is a generalized Kähler manifold with the induced structure $(\mathscr{J}_{\beta t,M}, \psi_{t,M})$.*

In Section 1.7, we discuss examples of generalized Kähler manifolds arising as Poisson submanifolds. For instance, every hypersurface of degree $d \leq 3$ in $\boldsymbol{CP}^3$ is a Poisson submanifold which admits a non-trivial generalized Kähler structure. We also exhibit invariant Poisson submanifolds of Poisson manifolds given by the action of commutative complex group.

In Section 2, we will give a formal proof of the stability theorem in the case of a Kähler manifold with a Poisson structure β[2]. The construction in the special case is based on the Hodge decomposition and the Lefschetz decomposition, which is an application of the theorem showing unobstructed deformations of generalized Calabi-Yau (metrical) structures [**12**]. (Note that the proof in the general case is similar and depends on the generalized Hodge decomposition.) The method of our proof is a generalization of the one in unobstructed deformations of Calabi-Yau manifolds due to Bogomolov-Tian-Todorov [**30**].

In Section 3, we discuss an application of the stability theorem to deformations of bihermitian structures. By the one to one correspondence between generalized Kähler structures and bihermitian structures with a torsion condition, our deformations of generalized Kähler structures $\{\mathscr{J}_{\beta t}, \psi_t\}$ give rise to deformations of bihermitian structures $\{I^+(t), I^-(t), h_t\}$. In particular, we show that the infinitesimal deformations of $\{I^{\pm}(t)\}$ are respectively given by the class $[\pm\beta\cdot\omega] \in H^{0,1}(T^{1,0})$ defined by the contraction of the Poisson structure β and the Kähler from ω. In other words, it follows from the stability theorem that the class $[\beta\cdot\omega]$ gives rise to unobstructed deformations of complex structures. As an example, we discuss deformations of complex structures on the product of $\boldsymbol{CP}^1$ and a complex torus. We show that Poisson structures on the product generate the Kuranishi family in terms of the classes $[\beta\cdot\omega]$.

Recently Gualtieri extended Hithicn's construction of bihermitain structures to Poisson manifolds by the Hamiltonian diffeomorphisms [**16**]. It seems to be interesting to compare the construction by the stability theorem as in [**13**] and the one by Hamiltonian diffeomorphsims. It must be noted that the construction by Hamiltonian diffeomorphisms requires that the Kodaira-Spencer class $[\beta\cdot\omega]$ vanishes[3].

The author would like to thank Professor Fujiki for valuable comment about bihermitian structures. He would like to express his gratitude to Professor Namikawa for his meaningful suggestions about Poisson geometry. He also thanks to Professor Vaisman for a kind comment about definition of generalized com-

[2]This construction provides a formal family of deformations of generalized Kähler structures in the case of Poisson deformations (see Section 4 in [**13**] for the convergence of the formal power series).

[3]For instance, on a complex torus and a hyper Kähler manifold, there are Poisson structures with non-vanishing class $[\beta\cdot\omega]$.

plex submanifolds. He is grateful to Professor Gualtieri for his comment about deformations of bihermitian structures. Finally he would like to thank Professor Hitchin for explaining the generalized geometry.

1. Generalized Kähler submanifolds.

1.1. Generalized complex structures.

Let X be a real compact manifold of dimension $2n$. We denote by $\pi : TX \oplus T^*X \to TX$ the projection to the first component. The natural coupling between TX and T^*X defines the symmetric bilinear form $\langle\, ,\, \rangle$ on the direct sum $TX \oplus T^*X$. Then we have the fibre bundle $\mathrm{SO}(TX \oplus T^*X)$ over X with fibre the special orthogonal group with respect to $TX \oplus T^*X$. Note that $\mathrm{SO}(TX \oplus T^*X)$ is a subbundle of $\mathrm{End}(TX \oplus T^*X)$. *An almost generalized complex structure* $\mathscr{J}$ is a section of fibre bundle $\mathrm{SO}(TX \oplus T^*X)$ with $\mathscr{J}^2 = -\mathrm{id}$. Let $L_{\mathscr{J}}$ be the $(-\sqrt{-1})$-eigenspace with respect to $\mathscr{J}$ and $\overline{L}_{\mathscr{J}}$ its complex conjugate. Then an almost generalized complex structure $\mathscr{J}$ gives rise to the decomposition of the complexified $(TX \oplus T^*X)^{\boldsymbol{C}}$ into eigenspaces

$$(TX \oplus T^*X)^{\boldsymbol{C}} = L_{\mathscr{J}} \oplus \overline{L}_{\mathscr{J}}.$$

An almost generalized complex structure $\mathscr{J}$ is *integrable* if the space $L_{\mathscr{J}}$ is involutive with respect to the Courant bracket $[\, ,\,]_c$. An integrable $\mathscr{J}$ is called a *generalized complex structure.*

1.2. $\mathscr{J}$-submanifolds.

Let $i_M : M \to X$ be a submanifold of dimension $2m$. We denote by $T^*X|_M$ the restricted bundle $i_M^{-1}T^*X$ over M. Let p be a bundle map defined by the pull back i_M^* and the identity map id_{TM} of TM,

$$p = \mathrm{id}_{TM} \oplus i_M^* : TM \oplus T^*X|_M \to TM \oplus T^*M.$$

We denote by $N^*(= N^*_{M|X})$ the conormal bundle to M in X. Then we have the short exact sequence,

$$0 \longrightarrow N^* \longrightarrow TM \oplus T^*X|_M \xrightarrow{p} TM \oplus T^*M \longrightarrow 0.$$

We define an intersection $L_{\mathscr{J}}(M)$ by

$$L_{\mathscr{J}}(M) = L_{\mathscr{J}} \cap (TM \oplus T^*X|_M)^{\boldsymbol{C}} = L_{\mathscr{J}} \cap (\pi^{-1}(TM))^{\boldsymbol{C}}$$

and denote by $\overline{L}_{\mathscr{J}}(M)$ its complex conjugate. For simplicity, we assume that $L_{\mathscr{J}}(M)$ is a subbundle of $(TM \oplus T^*X|_M)^{\boldsymbol{C}}$. Then the map p is restricted to the direct sum $L_{\mathscr{J}}(M) \oplus \overline{L}_{\mathscr{J}}(M)$ and we have the map $q : L_{\mathscr{J}}(M) \oplus \overline{L}_{\mathscr{J}}(M) \longrightarrow (TM \oplus T^*M)^{\boldsymbol{C}}$. Let $L_{\mathscr{J}}(N^*)$ denotes the intersection,

$$L_{\mathscr{J}}(N^*) = L_{\mathscr{J}} \cap (N^*)^{\boldsymbol{C}}.$$

We also assume that $L_{\mathscr{J}}(N^*)$ a subbundle of $(N^*)^{\boldsymbol{C}}$. Then $L_{\mathscr{J}}(N^*) \oplus \overline{L}_{\mathscr{J}}(N^*)$ is a bundle in the kernel of the map q and we have the following sequence,

$$L_{\mathscr{J}}(N^*) \oplus \overline{L}_{\mathscr{J}}(N^*) \longrightarrow L_{\mathscr{J}}(M) \oplus \overline{L}_{\mathscr{J}}(M) \xrightarrow{q} (TM \oplus T^*M)^{\boldsymbol{C}}. \tag{1.1}$$

The sequence is not exact in general. Note that

$$L_{\mathscr{J}}(N^*) \oplus \overline{L}_{\mathscr{J}}(N^*) \subset \big(L_{\mathscr{J}}(M) \oplus \overline{L}_{\mathscr{J}}(M)\big) \cap (N^*)^{\boldsymbol{C}} = \ker q.$$

The following definition is same as those in [**4**], which was given in terms of the pulback of Dirac structures [**8**].

Definition 1.1. A submanifold M is a $\mathscr{J}$-submanifold if the sequence (1.1) is exact,

$$0 \longrightarrow L_{\mathscr{J}}(N^*) \oplus \overline{L}_{\mathscr{J}}(N^*) \longrightarrow L_{\mathscr{J}}(M) \oplus \overline{L}_{\mathscr{J}}(M) \xrightarrow{q} (TM \oplus T^*M)^{\boldsymbol{C}} \longrightarrow 0. \tag{1.2}$$

The image $q(L_{\mathscr{J}}(M))$ is a maximally isotropic subbundle of $TM \oplus T^*M$ (see [**8**]). Hence rank $q(L_{\mathscr{J}}(M)) = \dim_{\boldsymbol{R}} M$ and we have

Lemma 1.2. *There are three equivalent conditions*:

(1) *M is a $\mathscr{J}$-submanifold.*
(2) *The map $q : L_{\mathscr{J}}(M) \oplus \overline{L}_{\mathscr{J}}(M) \longrightarrow (TM \oplus T^*M)^{\boldsymbol{C}}$ is surjective.*
(3) [4] *$q(L_{\mathscr{J}}(M)) \cap q(\overline{L_{\mathscr{J}}(M)}) = \{0\}$.*

Since both bundles $L_{\mathscr{J}}(N^*) \oplus \overline{L}_{\mathscr{J}}(N^*)$ and $L_{\mathscr{J}}(M) \oplus \overline{L}_{\mathscr{J}}(M)$ are $\mathscr{J}$-invariant, $TM \oplus T^*M$ inherits the almost generalized complex structure $\mathscr{J}_M$ induced in the quotient bundle. The almost generalized complex structure $\mathscr{J}_M$ gives the decomposition into eigenspaces,

[4] The condition (3) is the definition of a generalized complex submanifold in [**4**]. There is a presentation by classical tensor fields [**31**].

$$(TM \oplus T^*M)^{\boldsymbol{C}} = L_{\mathscr{J},M} \oplus \overline{L}_{\mathscr{J},M},$$

and we have the exact sequence,

$$0 \longrightarrow L_{\mathscr{J}}(N^*) \longrightarrow L_{\mathscr{J}}(M) \longrightarrow L_{\mathscr{J},M} \longrightarrow 0,$$

where $L_{\mathscr{J},M} = q(L_{\mathscr{J}}(M))$. Then from a viewpoint of the Dirac structure, it is shown in [**4**] that

Theorem 1.3 ([**4**]). *The induced structure $\mathscr{J}_M$ is integrable and M inherits a generalized complex structure.*

Proof. For the sake of reader, we will give a proof. We denote by $E|_M \in \Gamma(M, TX \oplus T^*X|_M)$ the restriction of a smooth section $E \in \Gamma(X, TX \oplus T^*X)$ to M. Let E_i be a section of $\Gamma(X, TX \oplus T^*X)$ with $E_i|_M \in \Gamma(M, TM \oplus TX^*|_M)$ for $i = 1, 2$. The Courant bracket is given by

$$[E_1, E_2]_c = [u_1, u_2] + \frac{1}{2}\big\{\mathscr{L}_{v_1}\theta^2 - \mathscr{L}_{v_2}\theta^1 - i_{v_2}d\theta_1 + i_{v_1}d\theta^2\big\}, \tag{1.3}$$

where $E_i = u_i + \theta^i$ for $u_i \in TM$ and $\theta^i \in T^*X$ $(i = 1, 2)$. Since the pull back i_M^* commutes with the exterior derivative d, the Lie derivative $\mathscr{L}_u$ and the interior product i_u for $u \in TM$, the Courant bracket satisfies the following,

$$\big[p(E_1|_M),\, p(E_2|_M)\big]_c = p\big([E_1, E_2]_c|_M\big). \tag{1.4}$$

Since M is a $\mathscr{J}$-submanifold, we have the exact sequence,

$$0 \longrightarrow L_{\mathscr{J}}(N^*) \oplus \overline{L}_{\mathscr{J}}(N^*) \longrightarrow L_{\mathscr{J}}(M) \oplus \overline{L}_{\mathscr{J}}(M) \longrightarrow (TM \oplus T^*M)^{\boldsymbol{C}} \longrightarrow 0,$$

where $(TM \oplus T^*M)^{\boldsymbol{C}} = L_{\mathscr{J},M} \oplus \overline{L}_{\mathscr{J},M}$. For every smooth section $\tilde{E} \in \Gamma(M, L_{\mathscr{J},M})$, there exists a smooth section $E \in \Gamma(X, TX \oplus T^*X)$ with $E|_M \in \Gamma(M, L_{\mathscr{J}}(M))$ satisfying $p(E|_M) = \tilde{E}$. It follows from (1.3) that for $\tilde{E}_1$, $\tilde{E}_2 \in \Gamma(M, L_{\mathscr{J},M})$ we have

$$\big[\tilde{E}_1, \tilde{E}_2\big]_c = p\big([E_1, E_2]_c|_M\big).$$

Since $[E_1, E_2]_c\,|_M \in L_{\mathscr{J}} \cap (\pi^{-1}(TM))^{\boldsymbol{C}} = L_{\mathscr{J}}(M)$ for $\tilde{E}_1$, $\tilde{E}_2 \in \Gamma(M, L_{\mathscr{J}}(M))$, we have

$$\left[\tilde{E}_1, \tilde{E}_2\right]_c \in p(L_{\mathscr{J}}(M)) = L_{\mathscr{J},M}.$$

Hence $\mathscr{J}_M$ is integrable. □

Note that Theorem 1.3 holds for sheaves $L_{\mathscr{J}}(M)$ and $L_{\mathscr{J}}(N^*)$ with a similar proof.

EXAMPLE 1.4 (symplectic submanifolds). A symplectic manifold with a symplectic structure ω gives the generalized complex structure $\mathscr{J}_\omega$. Then a symplectic submanifold of a symplectic manifold is a $\mathscr{J}_\omega$-submanifold. In this case we have

$$L_{\mathscr{J}_\omega}(M) \oplus \overline{L}_{\mathscr{J}_\omega}(M) \cong (TM \oplus T^*M)^{\boldsymbol{C}},$$

with $L_{\mathscr{J}_\omega}(N^*) = \{0\}$.

EXAMPLE 1.5 (complex submanifolds). A complex manifold with a complex structure J admits the generalized complex structure $\mathscr{J}_J$. Then a complex submanifold of a complex manifold is a $\mathscr{J}_J$-submanifold with the exact sequence

$$0 \longrightarrow L_{\mathscr{J}_J}(N^*) \oplus \overline{L}_{\mathscr{J}_J}(N^*) \longrightarrow L_{\mathscr{J}_J}(M) \oplus \overline{L}_{\mathscr{J}_J}(M) \xrightarrow{q} (TM \oplus T^*M)^{\boldsymbol{C}} \longrightarrow 0,$$

where $L_{\mathscr{J}_J}(N^*) = (N^*)^{1,0}$ and $L_{\mathscr{J}_J}(M) = TM^{0,1} \oplus (T^*X)^{1,0}|_M$.

EXAMPLE 1.6. Let $X_1 \times X_2$ be the product of generalized complex manifolds $(X_1, \mathscr{J}_1)$ and $(X_2, \mathscr{J}_2)$ with the generalized complex structure $\mathscr{J}_1 \times \mathscr{J}_2$. Let M_i be a $\mathscr{J}_i$-submanifold of $(X_i, \mathscr{J}_i)$ for $i = 1, 2$. Then the product $M_1 \times M_2$ is a $(\mathscr{J}_1 \times \mathscr{J}_2)$-submanifold of $X_1 \times X_2$.

EXAMPLE 1.7 ($\mathscr{J}$-invariant conormal bundles). Let $(X, \mathscr{J})$ be a generalized complex manifold and $i_M : M \to X$ a submanifold of X whose conormal bundle N^* is $\mathscr{J}$-invariant,

$$\mathscr{J}(N^*) = N^*.$$

Then M is a $\mathscr{J}$-submanifold.

PROOF OF EXAMPLE 1.7. The bundle $TM \oplus T^*X|_M$ is also defined in terms of N^*,

$$TM \oplus T^*X|_M = \{E \in TX \oplus T^*X|_M ; \langle E, \theta \rangle = 0, \forall \theta \in N^*\}.$$

Since $\mathscr{J}$ is a section of $\mathrm{SO}(TX \oplus T^*X)$, we have

$$\langle \mathscr{J} E, \theta\rangle = -\langle E, \mathscr{J}\theta\rangle = 0,$$

for $E \in TM \oplus T^*X|_M$ and $\theta \in N^*$. Thus the bundle $TM \oplus T^*X|_M$ is $\mathscr{J}$-invariant. Hence we have the exact sequence by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (N^*)^{\boldsymbol{C}} & \longrightarrow & (TM \oplus T^*X|_M)^{\boldsymbol{C}} & \longrightarrow & (TM \oplus T^*M)^{\boldsymbol{C}} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & L_{\mathscr{J}}(N^*) \oplus \overline{L}_{\mathscr{J}}(N^*) & \longrightarrow & L_{\mathscr{J}}(M) \oplus \overline{L}_{\mathscr{J}}(M) & \longrightarrow & (TM \oplus T^*M)^{\boldsymbol{C}} & \longrightarrow & 0 \end{array}$$

□

Example 1.8 (b-fields). Let M be a $\mathscr{J}$-submanifold of a generalized complex manifold $(X, \mathscr{J})$. For a real d-closed 2-form b, the exponential e^b acts on $\mathscr{J}$ by the adjoint action to obtain a generalized complex structure $\mathscr{J}_b = \mathrm{Ad}_{e^b}\mathscr{J}$. Then M is also a $\mathscr{J}_b$-submanifold.

Proof of Example 1.8. The bundles $L_{\mathscr{J}_b}(M)$ and $L_{\mathscr{J}_b}(N^*)$ are respectively given as the images by the adjoint action Ad_{e^b},

$$L_{\mathscr{J}_b}(M) = \mathrm{Ad}_{e^b}(L_{\mathscr{J}}(M)), \qquad L_{\mathscr{J}_b}(N^*) = \mathrm{Ad}_{e^b}\big(L_{\mathscr{J}}(N^*)\big),$$

and the adjoint action Ad_{e^b} preserves the bundles $TM \oplus T^*X|_M$ and N^*. Hence we have the exact sequence by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_{\mathscr{J}}(N^*) \oplus \overline{L}_{\mathscr{J}}(N^*) & \longrightarrow & (L_{\mathscr{J}}(M) \oplus \overline{L}_{\mathscr{J}}(M)) & \longrightarrow & (TM \oplus T^*M)^{\boldsymbol{C}} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \mathrm{Ad}_{e^b}} & & \downarrow{\scriptstyle \mathrm{Ad}_{e^{i^*b}}} & & \\ 0 & \longrightarrow & L_{\mathscr{J}_b}(N^*) \oplus \overline{L}_{\mathscr{J}_b}(N^*) & \longrightarrow & (L_{\mathscr{J}_b}(M) \oplus \overline{L}_{\mathscr{J}_b}(M)) & \longrightarrow & (TM \oplus T^*M)^{\boldsymbol{C}} & \longrightarrow & 0, \end{array}$$

where $\mathrm{Ad}_{e^{i^*b}}$ denotes the adjoint action by the exponential of the pull back i^*b.

□

1.3. Generalized metrics and generalized Kähler structures.

A generalized metric $\hat{G}$ is a section of $\mathrm{SO}(TX \oplus T^*X)$ with $\hat{G}^2 = $id satisfying the condition: a bilinear form G defined by $G(E_1, E_2) := \langle \hat{G}E_1, E_2\rangle$ is a positive-definite metric on $TX \oplus T^*X$, where $E_i = v_i + \eta_i$ for $v_i \in TX$ and $\eta_i \in T^*X$ ($i = 1, 2$). A generalized metric gives the decomposition of $TX \oplus T^*X$ into eigenspaces

$$TX \oplus T^*X = C^+ \oplus C^-,$$

where C^+ and C^- denotes the $(+1)$-eigenspace and the (-1)-eigenspace respectively. Then there are a Riemannian metric g and a 2-form b such that C^+ and C^- are respectively written as

$$\begin{aligned} C^+ &= \{v + g(v,\ ) + b(v,\ ) \mid v \in T\} \\ C^- &= \{v - g(v,\ ) + b(v,\ ) \mid v \in T\}, \end{aligned} \tag{1.5}$$

where $g(v,\ )$ and $b(v,\ )$ denote the 1-forms given by the interior product by $v \in TX$ respectively. Hence the restriction of the projection π to C^+ and C^- gives isomorphisms respectively,

$$\pi|_{C^+} : C^+ \cong TX, \quad \pi|_{C^-} : C^- \cong TX. \tag{1.6}$$

A generalized Kähler structure is a pair $(\mathscr{J}_0, \mathscr{J}_1)$ consisting of two commuting generalized complex structures with the generalized metric $G = -\mathscr{J}_0\mathscr{J}_1 = -\mathscr{J}_1\mathscr{J}_0$. Then $(TX \oplus T^*X)^{\boldsymbol{C}}$ is simultaneously decomposed into four eigenspaces by $\mathscr{J}_0$ and $\mathscr{J}_1$,

$$(TX \oplus T^*X)^{\boldsymbol{C}} = (L_{\mathscr{J}_0} \cap L_{\mathscr{J}_1}) \oplus (L_{\mathscr{J}_0} \cap \overline{L}_{\mathscr{J}_1}) \tag{1.7}$$

$$\oplus (\overline{L}_{\mathscr{J}_0} \cap L_{\mathscr{J}_1}) \oplus (\overline{L}_{\mathscr{J}_0} \cap \overline{L}_{\mathscr{J}_1}). \tag{1.8}$$

Then eigenspaces C^+ and C^- of the generalized metric $G = -\mathscr{J}_0\mathscr{J}_1 = -\mathscr{J}_1\mathscr{J}_0$ are respectively given by

$$(C^+)^{\boldsymbol{C}} = (L_{\mathscr{J}_0} \cap L_{\mathscr{J}_1}) \oplus (\overline{L}_{\mathscr{J}_0} \cap \overline{L}_{\mathscr{J}_1}), \tag{1.9}$$

$$(C^-)^{\boldsymbol{C}} = (L_{\mathscr{J}_0} \cap \overline{L}_{\mathscr{J}_1}) \oplus (\overline{L}_{\mathscr{J}_0} \cap L_{\mathscr{J}_1}). \tag{1.10}$$

1.4. Generalized Kähler submanifolds.

As in example 1.7, if M has a $\mathscr{J}$-invariant conormal bundle, then M admits the induced generalized complex structure $\mathscr{J}_M$. Then we shall show the following in this section 1.4.

THEOREM 1.9. *Let $(\mathscr{J}_0, \mathscr{J}_1)$ be a generalized Kähler structure on X. If a submanifold M of X admits a $\mathscr{J}_0$-invariant conormal bundle, then M is also a $\mathscr{J}_1$-submanifold and M inherits the induced generalized Kähler structure $(\mathscr{J}_{0,M}, \mathscr{J}_{1,M})$.*

At first we define a generalized Kähler submanifold which is due to Barton and Stiénon [**3**].

Definition 1.10. Let $(X, \mathscr{J}_0, \mathscr{J}_1)$ be a generalized Kähler manifold and M a submanifold of X. A submanifold M is a generalized Kähler submanifold if M is a $\mathscr{J}_0$-submanifold and M is also a $\mathscr{J}_1$-submanifold.

Then it is shown that a generalized Kähler submanifold M inherits a generalized Kähler structure $(\mathscr{J}_{0,M}, \mathscr{J}_{1,M})$.

Let $(X, \mathscr{J}_0, \mathscr{J}_1)$ be a $2n$-dimensional generalized Kähler manifold with the generalized metric G and M a submanifold of dimension $2m$. As in Section 1.3, we have subbundles C^+ and C^-. We define $C^+(M)$ and $C^-(M)$ respectively by the intersections

$$C^+(M) = C^+ \cap \pi^{-1}(TM), \quad C^-(M) = C^- \cap \pi^{-1}(TM). \tag{1.11}$$

Then from (1.6), we see that $C^+(M)$ and $C^-(M)$ are bundles with $\operatorname{rank} C^+(M) = \operatorname{rank} C^-(M) = \dim M = 2m$. Let p be the bundle map in section 1.2,

$$p : TM \oplus T^*X|_M \longrightarrow TM \oplus T^*M.$$

We denote by γ the bundle map given by the restriction of p to the subbundle $(C^+(M) \oplus C^-(M))$,

Lemma 1.11. *The map $\gamma : (C^+(M) \oplus C^-(M)) \longrightarrow (TM \oplus T^*M)$ is an isomorphism.*

Proof. The kernel of the map γ is the intersection $N^* \cap (C^+(M) \oplus C^-(M))$. From (1.5), $C^+(M)$ and $C^-(M)$ are respectively written as

$$C^+(M) = \{u_1 + g(u_1, \ ) + b(u_1, \ ) \mid u_1 \in TM\}, \tag{1.12}$$

$$C^-(M) = \{u_2 - g(u_2, \ ) + b(u_2, \ ) \mid u_2 \in TM\}. \tag{1.13}$$

Since g is positive-definite, it follows that $N^* \cap (C^+(M) \oplus C^-(M)) = \{0\}$. Hence γ is injective. Since $\operatorname{rank}(C^+(M) \oplus C^-(M)) = \operatorname{rank}(TM \oplus T^*M) = 4m$, it follows that γ is an isomorphism. $\square$

Lemma 1.12. *If M admits a $\mathscr{J}_0$-invariant conormal bundle, then $C^+(M)$ and $C^-(M)$ are respectively invariant under both actions of $\mathscr{J}_0$ and $\mathscr{J}_1$.*

Proof. If M admits a $\mathscr{J}_0$-invariant conormal bundle, as in Example 1.7, $\pi^{-1}(TM) = TM \oplus T^*X|_M$ is also $\mathscr{J}_0$-invariant. Since C^+ and C^- are respectively eigenspaces of G and $\mathscr{J}_0$ commutes with G, $C^+(M)$ and $C^-(M)$ are respectively invariant under both action $\mathscr{J}_0$ and G. It follows from $\mathscr{J}_1 = G\mathscr{J}_0$ that $\mathscr{J}_1$ is preserving $C^+(M)$ and $C^-(M)$. □

Proof of Theorem 1.9. We shall show the sequence is exact,

$$L_{\mathscr{J}_1}(N^*) \oplus \overline{L}_{\mathscr{J}_1}(N^*) \longrightarrow L_{\mathscr{J}_1}(M) \oplus \overline{L}_{\mathscr{J}_1}(M) \xrightarrow{q_1} (TM \oplus T^*M)^{\boldsymbol{C}}.$$

Since M is a $\mathscr{J}_0$-submanifold, it follows from Theorem 1.3 that we have the exact sequence with respect to $\mathscr{J}_0$,

$$0 \longrightarrow L_{\mathscr{J}_0}(N^*) \oplus \overline{L}_{\mathscr{J}_0}(N^*) \longrightarrow L_{\mathscr{J}_0}(M) \oplus \overline{L}_{\mathscr{J}_0}(M) \xrightarrow{q_0} (TM \oplus T^*M)^{\boldsymbol{C}} \longrightarrow 0,$$

and we have the induced generalized complex structure $\mathscr{J}_{o,M}$. From Lemma 1.12, $(C^+(M) \oplus C^-(M))^{\boldsymbol{C}}$ is a subbundle of both $L_{\mathscr{J}_0}(M) \oplus \overline{L}_{\mathscr{J}_0}(M)$ and $L_{\mathscr{J}_1}(M) \oplus \overline{L}_{\mathscr{J}_1}(M)$. It follows from Lemma 1.11 that we have the following commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & L_{\mathscr{J}_0}(N^*) \oplus \overline{L}_{\mathscr{J}_0}(N^*) \arrow[r] & L_{\mathscr{J}_0}(M) \oplus \overline{L}_{\mathscr{J}_0}(M) \arrow[r, "q_0"] & (TM \oplus T^*M)^{\boldsymbol{C}} \arrow[r] \arrow[d, equal] & 0 \\
& & (C^+(M) \oplus C^-(M))^{\boldsymbol{C}} \arrow[u] \arrow[d] \arrow[r, "\cong", "\gamma"'] & (TM \oplus T^*M)^{\boldsymbol{C}} \arrow[d, equal] & \\
& L_{\mathscr{J}_1}(N^*) \oplus \overline{L}_{\mathscr{J}_1}(N^*) \arrow[r] & L_{\mathscr{J}_1}(M) \oplus \overline{L}_{\mathscr{J}_1}(M) \arrow[r, "q_1"'] & (TM \oplus T^*M)^{\boldsymbol{C}}. &
\end{tikzcd}$$

Hence the map $q_1 : L_{\mathscr{J}_1}(M) \oplus \overline{L}_{\mathscr{J}_1}(M) \to (TM \oplus T^*M)^{\boldsymbol{C}}$ is surjective. Hence it follows from Lemma 1.2 (2) that M is a $\mathscr{J}_1$-submanifold and a generalized Kähler submanifold. □

Our Theorem 1.9 can be generalized. For instance, as in our proof, if $C^+(M) \oplus C^-(M)$ is invariant under the action of $\mathscr{J}_0$, then M is a generalized Kähler submanifold.

1.5. Generalized Kähler manifolds with one pure spinor.

A pure spinor of X is a complex differential form ψ with $\dim_{\boldsymbol{C}} \ker \psi = 2\dim_{\boldsymbol{C}} X$, where $\ker\psi = \{E \in (TX \oplus T^*X) \otimes \boldsymbol{C} \mid E \cdot \psi = 0\}$. A pure spinor is *non-degenerate* if we have a decomposition,

$$(TX \oplus T^*X)^{\boldsymbol{C}} = \ker\psi \oplus \overline{\ker\psi}.$$

Thus a non-degenerate, pure spinor induces the almost generalized complex structure $\mathscr{J}_\psi$ such that $\ker\psi$ is the $(-\sqrt{-1})$ eigenspace L_ψ of $\mathscr{J}_\psi$. If a non-degenerate, complex pure spinor ψ is d-closed, then the induced $\mathscr{J}_\psi$ is integrable.

Definition 1.13. A pair $(\mathscr{J}_0, \psi)$ consisting of a generalized complex structure and a d-closed, non-degenerate, complex pure spinor is a generalized Kähler structure with one pure spinor if the induced pair $(\mathscr{J}_0, \mathscr{J}_\psi)$ is a generalized Kähler structure.

For a point $x \in M$, a non-degenerate pure spinor ψ is written as

$$\psi_x = \psi_{l,x} e^{b+\sqrt{-1}\omega}, \tag{1.14}$$

where $\psi_{l,x}$ is a complex l-form which is given by $\psi_{l,x} = \theta^1 \wedge \cdots \wedge \theta^l$ in terms of 1-forms $\{\theta^i\}_{i=1}^l$ and ω and b are real 2-forms. The degree of $\psi_{l,x}$ is called Type of the pure spinor ψ at x. If Type $\psi_x = 0$, it follows that the pullback $i_M^*\psi_x$ of ψ to a submanifold M does not vanish. In general the pullback $i_M^*\psi$ may vanish which is not a pure spinor on M. However, we have

Theorem 1.14. *Let $(X, \mathscr{J}_0, \psi)$ be a generalized Kähler manifold with one pure spinor. Let M be a submanifold with invariant conormal bundle with respect to $\mathscr{J}_0$. Then the pull back $i_M^*\psi$ is a d-closed, non-degenerate, pure spinor on M and the induced pair $(\mathscr{J}_{0,M}, i_M^*\psi)$ is a generalized Kähler structure with one pure spinor on M.*

We shall show the following lemma for the proof of Theorem 1.14.

Lemma 1.15. *Let $(X, \mathscr{J}_0, \psi)$ be a generalized Kähler manifold with one pure spinor and M a submanifold with invariant conormal bundle with respect to $\mathscr{J}_0$. Then the pull back $i_M^*\psi$ does not vanish.*

Proof of Lemma 1.15. In the case $l = 0$, then $i_M^*\psi_x = i_M^* e^{b+\sqrt{-1}\omega} \neq 0$. Thus it suffices to consider the case $l > 1$. From (1.14), if $i_M^*\psi_x = 0$, then we have $i_M^*\psi_{l,x} = 0$. Thus $i_M^*\psi_{l,x}$ is generated by N^* and at least one element of $\{\theta^i\}_{i=1}^l$ belongs to N^*. We can assume that $\theta^i \neq 0 \in N^*$. It follows from $\theta^i \cdot \psi_x = 0$ that we see $\theta^i \in L_\psi$. Then we have

$$G(\theta^i, \overline{\theta}^i) = \langle G\theta^i, \overline{\theta}^i\rangle = \langle -\mathscr{J}_0\mathscr{J}_\psi\theta^i, \overline{\theta}^i\rangle \tag{1.15}$$

$$= \sqrt{-1}\langle \mathscr{J}_0\theta^i, \overline{\theta}^i\rangle. \tag{1.16}$$

Since $\mathscr{J}_0 N^* = N^*$, we see that $\mathscr{J}_0\theta^i$ is a 1-form and $\langle \mathscr{J}\theta^i, \overline{\theta}^i\rangle = 0$. Hence

$G(\theta^i, \overline{\theta}^i) = 0$. Since G is positive-definite, it implies $\theta^i = 0$, which is a contradiction. Hence we conclude that $i_M^*\psi_x \neq 0$ for all $x \in M$. $\square$

Proof of Theorem 1.14. Let $\mathscr{J}_\psi$ be the induced generalized complex structure by ψ with the $(-\sqrt{-1})$-eigenspace L_ψ. As in Section 1.4, $C^+(M) \oplus C^-(M)$ is $\mathscr{J}_\psi$-invariant and under the isomorphism $q : C^+(M) \oplus C^-(M) \cong TM \oplus T^*M$ we have the decomposition,

$$(TM \oplus T^*M)^{\boldsymbol{C}} = q(L_\psi(M)) \oplus q(\overline{L}_\psi(M)).$$

For $E = u + \eta \in L_\psi^{(M)}$, we see that

$$q(E) \cdot i_M^*\psi = (u + i_M^*\eta) \cdot i_M^*\psi \tag{1.17}$$

$$= i_M^*(u + \eta) \cdot \psi = i_M^*(E \cdot \psi) = 0. \tag{1.18}$$

It implies that $q(L_\psi(M)) \subset \ker i_M^*\psi$. Since $\dim q(L_\psi(M)) = 2m$ and $q(L_\psi(M)) \cap q(\overline{L}_\psi(M)) = \{0\}$, it follows from Lemma 1.15 that $q(L_\psi(M)) = \ker i_M^*\psi$ is maximally isotropic. Thus $i_M^*\psi$ is a non-degenerate, pure spinor on M with the induced structure $\mathscr{J}_{M,\psi}$. Since the pull back $i_M^*\psi$ is d-closed, the pair $(\mathscr{J}_{M,0}, \mathscr{J}_{M,\psi})$ is a generalized Kähler structure. Hence the pair $(\mathscr{J}_{M,0}, i_M^*\psi)$ is a generalized Kähler structure with one pure spinor. $\square$

1.6. Poisson submanifolds.

Definition 1.16. Let X be a complex manifold with a holomorphic 2-vector β. If the Schouten bracket vanishes, i.e., $[\beta, \beta]_{Sch} = 0$, we call β a (holomorphic) Poisson structure on X and the Poisson bracket is defined by

$$\{f, g\} = \beta(df \wedge dg).$$

Definition 1.17. Let X be a complex manifold with a Poisson structure β and M a complex submanifold with the defining ideal sheaf I_M. A submanifold M is a Poisson submanifold if we have $\{f, g\} \in I_M$ for all $f \in I_M$ and $g \in \mathscr{O}_X$.

A Poisson submanifold admits the induced Poisson structure.
Let $\mathscr{J}_J$ be the generalized complex structure defined by the usual complex structure J. By using a Poisson structure β, we obtain a family of generalized complex structures $\mathscr{J}_{\beta t}$ parameterized by the complex numbers t

$$\mathscr{J}_{\beta t} = e^{\beta t} \circ \mathscr{J}_J \circ e^{-\beta t},$$

The structure $\mathscr{J}_{\beta t}$ is written in the form of a matrix,

$$\mathscr{J}_{\beta t} = \begin{pmatrix} J & -2\sqrt{-1}(\beta t - \overline{\beta t}) \\ 0 & -J^* \end{pmatrix}. \tag{1.19}$$

Then we have

Lemma 1.18. *Every Poisson submanifold M admits a $\mathscr{J}_{\beta t}$-invariant conormal bundle for all t.*

Proof. Since $\beta(df, dg) \in I_M$ for $f \in I_M$ and $(N^*)^{1,0}$ is generated by the set $\{df \mid f \in I_M\}$, we have the restriction $\beta(df, \,)|_M = 0$. It follows from (1.19) that $\mathscr{J}_\beta(N^*) = N^*$. $\square$

In [**13**] we obtain a stability theorem of generalized Kähler structures with one pure spinor. It implies that a generalized Kähler structure with one pure spinor is stable under small deformations of generalized complex structures. By applying the stability theorem to small deformations of generalized complex structures $\{\mathscr{J}_{\beta t}\}$ starting from $\mathscr{J}_J$, we have deformations of generalized Kähler structures with one pure spinor $\{\mathscr{J}_{\beta t}, \psi_t\}$. The type of $\mathscr{J}_{\beta t}$ is given by

$$\operatorname{Type} \mathscr{J}_{\beta t} = n - 2 \operatorname{rank} \beta.$$

It implies that if $\beta \neq 0$, then deformations of generalized Kähler structures with one pure spinor $\{\mathscr{J}_{\beta t}, \psi_t\}$ can not be obtained from ordinary Kähler structures by the action of b-fields. Hence we have,

Theorem 1.19. *Let X be a Kähler manifold with non-trivial Poisson structure β. Then there exists an analytic family of non-trivial generalized Kähler structures with one pure spinor $\{\mathscr{J}_{\beta t}, \psi_t\}$.*

Hence it follows from 1.18 and 1.14 that

Theorem 1.20. *Let M be a Poisson submanifold of a Poisson manifold X with a Kähler structure ω. Then M is a generalized Kähler manifold with the induced structure $(\mathscr{J}_{\beta t, M}, \psi_{t,M})$.*

1.7. Examples of generalized Kähler submanifolds arising as Poisson submanifolds.

Let X be a compact Kähler manifold on which an l-dimensional commutative complex group G acts holomorphically. The Lie algebra $\mathfrak{g}$ of G generates holomorphic vector fields $\{V_i\}_{i=1}^l$ on X. Since $[V_i, V_j] = 0$, it follows that a linear combination of 2-vectors $V_i \wedge V_j$'s gives a holomorphic Poisson structure β,

$$\beta = \sum_{i,j} \lambda_{i,j} V_i \wedge V_j, \tag{1.20}$$

where $\lambda_{i,j}$ is a constant. Note that $[V_i, V_j] = 0$ implies $[\beta, \beta]_{Sch} = 0$. If $\beta \neq 0$, from the stability theorem we have deformations of generalized Kähler structures $\{\mathscr{J}_{\beta t}, \psi_t\}$.

LEMMA 1.21. *Let M be a complex submanifold with defining ideal I_M. If the ideal I_M is invariant under the action of G, then M is a Poisson submanifold of (X, β).*

PROOF. Since we have $V_i f \in I_M$ for $f \in I_M$ and $i = 1, \ldots, l$. Hence $\beta(df, \ ) \in I_M \otimes T^{1,0}X$ for $f \in I_M$. It implies that M is a Poisson submanifold. □

EXAMPLE 1.22 (toric submanifolds). Let X be a compact toric manifold of dimension n. Then there is the action of n-dimensional complex torus G on X. Then a toric submanifold M which is invariant under the action of G is a Poisson submanifold with respect to β as in (1.20).

EXAMPLE 1.23. Let $\boldsymbol{C}P^4$ be the complex projective space with the homogeneous coordinates $[z_0, z_1, z_2, z_3, z_4]$ on which the commutative group $\boldsymbol{C}^\times \times \boldsymbol{C}^\times$ acts by a homomorphism $\rho : \boldsymbol{C}^\times \times \boldsymbol{C}^\times \to \mathrm{GL}(5, \boldsymbol{C})$,

$$\rho(\lambda_1, \lambda_2) = \mathrm{diag}(1, \lambda_1, \lambda_1, \lambda_2, \lambda_2).$$

Then we have a Poisson structure $\beta = V_1 \wedge V_2$ as in (1.20). We take a following quadratic function F of $\boldsymbol{C}P^4$

$$F(z) = \sum_{\substack{i=1,2\\ j=3,4}} a_{ij} z_i z_j,$$

where a_{ij} are constants. The hypersurface M defined as the zero of F becomes a smooth manifold of complex dimension 3 for suitably chosen constants a_{ij}. Since $\rho^*(\lambda_1, \lambda_2) F(z) = \lambda_1 \lambda_2 F(z)$, the hypersurface M is a Poisson submanifold in $\boldsymbol{C}P^4$ which admits the deformations generalized Kähler structure with one pure spinor $(\mathscr{J}_{\beta t}, \psi_t)$ from Theorem 1.20. Since the Type of $\mathscr{J}_{\beta t}$ is 2 at generic points of X and the type of the induced $\mathscr{J}_{\beta t, M}$ is 1 at generic points of M, the generalized Kähler structure $(\mathscr{J}_{\beta t}, \psi_t)$ and the induced generalized Kähler structure $(\mathscr{J}_{\beta t M}, \psi_t)$ are not obtained from Kähle structures by the action of b-fields.

Example 1.24. Let $\boldsymbol{C}P^3$ be the complex projective space with the homogeneous coordinates $[z_0, z_1, z_2, z_3]$. On the open set $\{z_0 \neq 0\}$, we have the inhomogeneous coordinates $\{\zeta_1, \zeta_2, \zeta_3\}$ given by $\zeta_i = \frac{z_i}{z_0}$. Let $f = f(\zeta_1, \zeta_2, \zeta_3)$ be a polynomial of degree $d \leq 3$ and we assume that $df \neq 0$. Then we define a 2-vector field β_f by

$$\beta_f = f_1 \frac{\partial}{\partial \zeta_2} \wedge \frac{\partial}{\partial \zeta_3} + f_2 \frac{\partial}{\partial \zeta_3} \wedge \frac{\partial}{\partial \zeta_1} + f_3 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \zeta_2},$$

where $f_i = \frac{\partial}{\partial z_i} f$. Then it turns out that $[\beta, \beta]_{\mathrm{Sch}} = 0$. Thus β_f is a Poisson structure, which is called the exact quadratic Poisson structure [**24**], [**27**]. We also see that $\beta(df, \ ) = 0$. Thus the zero of f is a Poisson submanifold with respect to the Poisson structure β_f on $\boldsymbol{C}^3$. Let $F = F(z_0, \dots, z_3)$ be the homogeneous polynomial defined by

$$F = z_0^d f\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \frac{z_3}{z_0}\right).$$

Since each f_i is a quadratic polynomial, β_f can be extended as a holomorphic Poisson structure β_F on $\boldsymbol{C}P^3$. Then a complex surface M given by the zero of F is a Poisson submanifold.

Theorem 1.25. *Let M be a complex smooth hypersurface of the projective space $\boldsymbol{C}P^3$ defined by a homogeneous polynomial F of degree $d \leq 3$. Then M is a non-trivial generalized Kähler manifold arising as Poisson submanifold of $\boldsymbol{C}P^3$ with respect to the Poisson structure β_F.*

Proof. It suffices to show that the induced Poisson structure β_M is nontrivial. On $\{z_0 \neq 0\}$, β_M is the induced structure from β_f. Since M is smooth, we can assume that there exists an open set defined by $\{f_3 \neq 0\}$ with coordinates (η_1, η_2, η_3),

$$\begin{aligned}\eta_1 &= \zeta_1\\ \eta_2 &= \zeta_2\\ \eta_3 &= f(\zeta_1, \zeta_2, \zeta_3).\end{aligned}$$

Then β_f is written as

$$\beta_f = f_3 \frac{\partial}{\partial \eta_1} \wedge \frac{\partial}{\partial \eta_2}.$$

Since M is defined by $\eta_3 = 0$, it follows that β_M is non-trivial. Hence the type of the induced generalized complex structure $\mathscr{J}_M$ is 0 on the complement of the zero of β_M. Hence the induced generalized Kähler structure on M is not obtained from ordinary Kähler structures by b-field action. □

2. Deformations of generalized Kähler structures via Poisson structures.

Let X be a compact Kähler manifold with a Kähler form ω. Then we have the generalized Kähler structure with one pure spinor $(\mathscr{J}_J, \psi)$, where $\mathscr{J}_J$ denotes the generalized complex structure induced from the complex structure J on X and ψ is the pure spinor defined by

$$\psi = e^{\sqrt{-1}\omega}.$$

We assume that there exists a Poisson structure β on X. Then we have deformations of generalized complex structures $\{\mathscr{J}_{\beta t}\}_{t\in\triangle}$ as in Section 1. Applying the stability Theorem in [**13**] to deformations of generalized complex structures $\{\mathscr{J}_{\beta t}\}_{t\in\triangle}$, we obtain

Theorem 1.19. Let X be a Kähler manifold with non-trivial Poisson structure β. Then there exists an analytic family of non-trivial generalized Kähler structure with one pure spinor $\{\mathscr{J}_{\beta t}, \psi_t\}$.

In the case of deformations starting from ordinary Kähler manifolds, the proof of stability theorem becomes simple which is based on the ordinary Hodge decomposition and the Lefschetz decomposition. Note that in general case, we used the generalized Hodge decomposition. We shall give an exposition of our proof in the special cases. We use the same notation as in [**13**].

Let CL be the real Clifford algebra of $TX \oplus T^*X$ with respect to $\langle\, ,\, \rangle$. Then CL acts on differential forms by the spin representation. The Clifford group G_{cl} is defined in terms of the twisted adjoint $\widetilde{\mathrm{Ad}}_g$,

$$G_{cl} := \{g \in \mathrm{CL}^\times \mid \widetilde{\mathrm{Ad}}_g(TX \oplus T^*X) \subset TX \oplus T^*X\}.$$

Let CL^2 be the Lie algebra which consists of elements of the Clifford algebra of degree less than or equal to 2. It turns out that CL^2 is the Lie algebra of the Clifford group G_{cl}. The set of almost generalized complex structures forms an orbit of the adjoint action of the Clifford group and the set of almost generalized Kähler structures with one pure spinor is also an orbit of the diagonal action of the Clifford group. Thus it follows that small deformations of almost generalized

Kähler structures with one pure spinor are given by the action of exponential of CL^2 on $(\mathscr{J}_J, \psi)$,

$$\big(\mathrm{Ad}_{e^{z(t)}}\mathscr{J}_J,\ e^{z(t)}\cdot\psi\big),$$

where $z(t)\in\mathrm{CL}^2[[t]]$. Let $\{\mathscr{J}_{\beta t}\}$ be deformations of generalized complex structures by a Poisson structure β as before. The deformations $\{\mathscr{J}_{\beta t}\}$ are given by the adjoint action of real 2-vector $a=\beta+\overline{\beta}$,

$$\mathscr{J}_{\beta t}=\mathrm{Ad}_{e^{at}}\mathscr{J}_J.$$

Let $b(t)$ be an analytic family of $b(t)$ of $\mathrm{CL}^2[[t]]$,

$$b(t)=b_1t+b_2\frac{t^2}{2!}+\cdots=\sum_{i=1}^{\infty}b_i\frac{t^i}{i!}.$$

We denote by $\wedge^{n,0}$ the canonical line bundle of (X,J). We assume that there exists a family $\{b(t)\}$ with the following conditions (2.1) and (2,2),

$$b_i\cdot\wedge^{n,0}\subset\wedge^{n,0} \tag{2.1}$$

$$d(e^{at}\,e^{b(t)}\cdot\psi)=0. \tag{2.2}$$

Then it follows from the Campbel-Hausdorff formula that there is the $z(t)\in\mathrm{CL}^2[[t]]$ with

$$e^{z(t)}=e^{at}e^{b(t)}.$$

From (2.1), we see that the action by $b(t)$ is preserving $\mathscr{J}_J$,

$$\mathrm{Ad}_{e^b(t)}\mathscr{J}_J=\mathscr{J}_J.$$

Thus we have

$$\mathrm{Ad}_{e^{z(t)}}\mathscr{J}_J=\mathrm{Ad}_{e^{at}}\circ\mathrm{Ad}_{e^{b(t)}}\mathscr{J}_J \tag{2.3}$$

$$=\mathrm{Ad}_{e^{at}}\mathscr{J}_J=\mathscr{J}_{\beta t}. \tag{2.4}$$

From (2.2), the non-degenerate pure spinor $\psi_t=e^{z(t)}\cdot\psi$ is d-closed. Hence the pair $(\mathrm{Ad}_{z(t)}\mathscr{J}_J,\ e^{z(t)}\cdot\psi) = (\mathscr{J}_{\beta t}, \psi_t)$ is a generalized Kähler structure with one pure spinor. We shall construct $b(t)$ which satisfies the (2.1) and (2.2). Let $\mathrm{CL}^{[i]}$ be the subspace of the CL of degree i. We define CL^i for $i = 0, \dots, 3$ by

$$\mathrm{CL}^0 = C^\infty(X), \quad \mathrm{CL}^1 = TX \oplus T^*X, \tag{2.5}$$

$$\mathrm{CL}^2 = \mathrm{CL}^0 \oplus \mathrm{CL}^{[2]}, \quad \mathrm{CL}^3 = \mathrm{CL}^1 \oplus \mathrm{CL}^{[3]}. \tag{2.6}$$

Then we define bundles $\widetilde{\ker}^1$ and $\widetilde{\ker}^2$ respectively by

$$\widetilde{\ker}^1 = \{b \in \mathrm{CL}^2 \mid b \cdot \wedge^{n,0} \subset \mathrm{CL}^0 \cdot \wedge^{n,0}\}, \tag{2.7}$$

$$\widetilde{\ker}^2 = \{b \in \mathrm{CL}^3 \mid b \cdot \wedge^{n,0} \subset \mathrm{CL}^1 \cdot \wedge^{n,0}\}, \tag{2.8}$$

where $\mathrm{CL}^i \cdot \wedge^{n,0}$ denotes the image by the action of CL^i on the canonical line bundle $\wedge^{n,0}$. Then a section $b \in \widetilde{\ker}^i$ $(i = 1, 2)$ acts on $\psi = e^{\sqrt{-1}\omega}$ by the spin representation and we obtain bundles $\widetilde{K}^1$ and $\widetilde{K}^2$,

$$\widetilde{K}^i = \left\{b \cdot \psi \mid b \in \widetilde{\ker}^i\right\}.$$

The bundle $\widetilde{K}^1$ is the direct sum of $U^{0,-n}$ and $U^{0,-n+2}$,

$$\widetilde{K}^1 = U^{0,-n} \oplus U^{0,-n+2},$$

where $U^{0,-n} = \mathrm{CL}^0 \cdot \psi = \{f\psi \mid f \in C^\infty(X)\}$ and $U^{0,-n+2}$ is given by the contraction $\wedge_\omega$ by the Kähler form ω,

$$U^{0,-n+2} = \{h\psi + p \wedge \psi \mid h \in C^\infty(M),\ p \in \wedge^{1,1},\ \ \wedge_\omega p + 2h = 0\}, \tag{2.9}$$

where $\wedge^{1,1}$ denotes forms of type $(1,1)$ with respect to the complex structure J. We define K^1 to be $U^{0,-n+2}$ and write $\widetilde{K}^2$ as K^2. Then K^2 is written as

$$K^2 = \widetilde{K}^2 = \{\eta \wedge \psi \mid \eta \in \wedge^1 \oplus \wedge^{2,1} \oplus \wedge^{1,2}\}.$$

Then we have a differential complex $\{K^i, d\}$ by the exterior derivative d,

$$0 \longrightarrow K^1 \xrightarrow{d} K^2 \xrightarrow{d} \cdots.$$

It turns out that the complex (K^i, d) is elliptic since we have the following elliptic complex,

$$0 \longrightarrow P^{1,1} \xrightarrow{d} \wedge^{2,1} \oplus \wedge^{1,2} \xrightarrow{d} \cdots ,$$

where $P^{1,1}$ denotes the primitive $(1,1)$-forms on the Kähler manifold X (*cf*, Proposition 4.7 in [**11**]). The complex (K^*, d) is a subcomplex of the full de Rham complex,

$$\begin{tikzcd}
0 \arrow[r] & K^1 \arrow[r, "d"] \arrow[d] & K^2 \arrow[r, "d"] \arrow[d] & \cdots \\
\oplus_{i=0}^{2n}\wedge^i \arrow[r, "d"] & \oplus_{i=0}^{2n}\wedge^i \arrow[r, "d"] & \oplus_{i=0}^{2n}\wedge^i \arrow[r] & \cdots
\end{tikzcd}$$

We denote by $H^i(K^*)$ the cohomology group of the complex (K^*, d). It follows from the Hodge decomposition and the Lefschetz decomposition that the map $p^i : H^i(K^*) \to \oplus_{j=0}^{2n} H^j_{dR}(X)$ is injective for $i = 1, 2$.

Let $(de^{z(t)}\psi)_{[k]}$ denotes the term of $de^{z(t)}\psi$ of degree k in t. The first term is given by

$$(de^{z(t)}\psi)_{[1]} = da\psi + db_1\psi,$$

where $da\psi = (d(\beta + \overline{\beta})\omega^2) \wedge \psi \in (\wedge^{2,1} \oplus \wedge^{2,1}) \wedge \psi$. Thus $da\psi \in K^2$ is d-exact. Since the map p^2 is injective, the class $[da\psi] \in H^2(K^*)$ vanishes and we have a solution $b_1 \in K^1$ of the first equation $da\psi + db_1\psi = 0$.

Next we consider an operator $e^{-z(t)}de^{z(t)}$ acting on differential forms, where $z(t) = \log e^{at}e^{b(t)}$. It follows that the operator $e^{-z(t)}de^{z(t)}$ is a Clifford-Lie operator of order 3 which is locally written in terms of the Clifford algebra valued Lie derivative,

$$e^{-z(t)}de^{z(t)} = \sum_i E_i \mathscr{L}_{v_i} + N_i, \tag{2.10}$$

where $\mathscr{L}_{v_i}$ denotes the Lie derivative by a vector v_i and $E_i \in \mathrm{CL}^1$, $N_i \in \mathrm{CL}^3$ (*cf* Definition 2.2 in [**12**]). We find an open covering $\{U_\alpha\}$ of X with a non-vanishing holomorphic n-form Ω_α on each U_α. We denote by Φ_α the pair (Ω_α, ψ). Since the set of almost generalized Kähler structures is invariant under the action of diffeomorphisms, the Lie derivative of Φ_a by a vector field v is given by

$$\mathscr{L}_v \Phi_\alpha = a_\alpha \cdot \Phi_\alpha = (a_\alpha \cdot \Omega_\alpha, a_\alpha \cdot \psi),$$

for a section $a_\alpha \in \mathrm{CL}^2$ on U_α. It follows from (2.10) that there is a $h_\alpha \in \mathrm{CL}^3$ such that

$$e^{-z(t)} d e^{z(t)} \cdot \Phi_\alpha = h_\alpha \cdot \Phi_\alpha = (h_\alpha \cdot \Omega_\alpha, h_\alpha \cdot \psi).$$

Since $b(t) \in \widetilde{\ker}^1$ and $\mathrm{Ad}_{e^{z(t)}} \mathscr{J}_J = \mathscr{J}_{\beta t}$ is integrable, we have

$$d e^{z(t)} \Omega_\alpha = E_\alpha \cdot e^{z(t)} \Omega_\alpha,$$

for a $E_\alpha \in \mathrm{CL}^1$, which is the integrablity condition of $\mathscr{J}_{\beta t}$ in terms of pure spinors. Thus

$$e^{-z(t)} d e^{z(t)} \cdot \Omega_\alpha = \widetilde{E}_\alpha \cdot \Omega_a,$$

where $\widetilde{E}_a = e^{-z(t)} E_a e^{z(t)} \in \mathrm{CL}^1$. It follows from $h_a \cdot \Omega_\alpha = \tilde{E}_\alpha \cdot \Omega_\alpha$ that $h_\alpha \in \widetilde{\ker}^2$. It implies that $h_\alpha \cdot \psi \in K^2$.

We shall find a solution $b(t)$ of the equation $(d e^{z(t)} \psi) = 0$ by the induction on the degree k. We assume that there exists a solution $b_j \in \widetilde{\ker}^1$ for $0 \le j < k$ of the equation $(d e^{z(t)} \psi)_{[i]} = 0$, for all $0 \le i < k$. Then we have

$$(e^{-z(t)} d e^{z(t)} \psi)_{[k]} = \sum_{\substack{i+j=k \\ 0 \le i,j \le k}} (e^{-z(t)})_{[j]} (d\, e^{z(t)} \psi)_{[i]} \tag{2.11}$$

$$= (d\, e^{z(t)} \psi)_{[k]}. \tag{2.12}$$

Since $(e^{-z(t)} d e^{z(t)} \psi)|_{U_\alpha} = h_\alpha \cdot \psi|_{U_\alpha} \in K^2$ for $h_\alpha \in \widetilde{\ker}^2$ on each U_α, it follows that $(d\, e^{z(t)} \psi)_{[k]} = (h_\alpha \cdot \psi)_{[k]} \in K^2$. The d-exact form $(d\, e^{z(t)} \psi)_{[k]}$ is written as

$$(d\, e^{z(t)} \psi)_{[k]} = \frac{1}{k!} (d b_k \cdot \psi) + \mathrm{Ob}_k,$$

where Ob_k is also a d-exact form in K^2 which defined in terms of a and b_j for $1 \le j < k$. Since the map p^2 is injective, it follows that the class $[\mathrm{Ob}_k] \in H^2(K^*)$ vanishes and we have a solution b_k of the equation $(d\, e^{z(t)} \psi)_{[k]} = 0$. By our assumption of the induction, we have a solution $b(t)$ in the form of formal power series, which can be shown to be a convergent series.

The cohomology group $H^1(K^*)$ is given by $H^{1,1}(X)$. Then by applying Theorem 3.2 in [**13**], we obtain a 2-parameter family of deformations of generalized Kähler structures $(\mathscr{J}_{\beta t}, \psi_{t,s})$,

Theorem 2.1. *There exists a family of solutions $b_s(t)$ parameterized by $s \in H^{1,1}(M)$, which gives rise to deformations of generalized Kähler structures $(\mathscr{J}_{\beta t}, \psi_{t,s})$.*

3. Deformations of bihermitian structures.

Let (X, ω) be an n-dimensional compact Kähler manifold with a Poisson structure β and a complex structure J. Then we have deformations of generalized Kähler structures $\{\mathscr{J}_{\beta t}, \psi_t\}$ as in Section 2. According to theorem by Gualtieri, there is the one to one correspondence between generalized Kähler structures and bihermitian structures with a torsion condition. A bihermitian structure is a triple (I^+, I^-, h) consisting of two complex structures I^+ and I^- and a Hermitian structure h with respect to both I^+ and I^-. Let $\omega^\pm$ be the Hermitian 2-form and $\overline{\partial}^\pm$ the $\overline{\partial}$-operator with respect to $I^\pm$ respectively. Then the torsion condition is given by

$$d_c^+\omega^+ = -d_c^-\omega^- = H,$$

where H is a d-exact 3-form and $d_c^\pm = \sqrt{-1}(\partial^\pm - \overline{\partial}^\pm)$. Let $z(t)$ be a solution of the equation $de^{z(t)}\psi = 0$ as in section 2, which gives rise to deformations of generalized Kähler structures $\{\mathscr{J}_{\beta t}, \psi_t\}$, where $e^{z(t)} = e^{at}\, e^{b(t)}$ and $a = \beta + \overline{\beta}$. Then we have the corresponding deformations of bihermitian structures $\{(I^+(t), I^-(t), h_t)\}$, where $I^+(0) = I^-(0) = J$. Let b_1 be the first term of power series $b(t)$ in t. On an open set U, we find a basis $\{Z_i\}_{i=1}^n$ of vector fields of type $(1,0)$ with respect to the complex structure J. We denote by $\overline{\theta}^i$ the 1-form of type $(0,1)$ defined by the interior product of $-\sqrt{-1}\omega$ by Z_i,

$$\overline{\theta}^i = -\sqrt{-1}\, i_{Z_i}\, \omega.$$

We define $\overline{E}_i^\pm$ to be $Z_i \pm \overline{\theta}^i \in (TX \oplus T^*X) \otimes \boldsymbol{C}$. Then $b_1 \in \mathrm{CL}^2$ acts on $\overline{E}_i^\pm$ by the adjoint action,

$$[b_1, \overline{E}_i^\pm] \in (TX \oplus T^*X) \otimes \boldsymbol{C}.$$

We denote by $\overline{\beta}(\overline{\theta}^i)$ the vector filed given by the contraction of 2-vector $\overline{\beta}$ by 1-

from $\overline{\theta}^i$. Then we have a deformed basis $\{Z_i^{\pm}(t)\}$ of vectors of type $(1,0)$ with respect to $I^{\pm}(t)$ on U which is written by the followings up to degree 1 in t,

$$Z_i^+(t) \equiv Z_i + \big(\overline{\beta}(\overline{\theta}^i) + \pi_{TX}[b_1, \overline{E}_i^+]\big)t, \qquad (\mathrm{mod}\ t^2) \tag{3.1}$$

$$Z_i^-(t) \equiv Z_i + \big(-\overline{\beta}(\overline{\theta}^i) + \pi_{TX}[b_1, \overline{E}_i^-]\big)t, \qquad (\mathrm{mod}\ t^2), \tag{3.2}$$

where $\pi_{TX} : TX \oplus T^*X \to TX$ denotes the projection.

LEMMA 3.1. *For $a = \beta + \overline{\beta}$, there exists a solution $z(t)$ of the equation $de^{z(t)} \cdot \psi = 0$ such that the first term b_1 is a real 2-form.*

PROOF. The first term of the equation $de^{z(t)}\psi = 0$ is given by

$$da \cdot \psi + db_1 \cdot \psi = 0.$$

Then we have

$$da \cdot \psi = d(\beta + \overline{\beta}) \cdot \psi = -\frac{1}{2}(\beta \cdot \omega^2 + \overline{\beta} \cdot \omega^2)\psi,$$

where $\beta \cdot \omega^2$ denotes the interior product of the 4-form ω^2 by the 2-vector β. The d-exact form $-\frac{1}{2}d(\beta\cdot\omega^2 + \overline{\beta}\cdot\omega^2)$ is a real form of type $\wedge^{2,1} \oplus \wedge^{1,2}$. As in Proposition 4.7 in [**11**], we have a real elliptic complex,

$$0 \longrightarrow P_{\boldsymbol{R}}^{1,1} \xrightarrow{d} (\wedge^{2,1} \oplus \wedge^{1,2})_{\boldsymbol{R}} \xrightarrow{d} \cdots,$$

whose cohomology groups are respectively given by the harmonic real primitive form $\boldsymbol{P}_{\boldsymbol{R}}^{1,1}$ of type $(1,1)$ and the real part of the Dolbeault cohomology $(H^{2,1}(X) \oplus H^{1,2}(X))_{\boldsymbol{R}}$. Hence we obtain a real $b_1 \in P_{\boldsymbol{R}}^{1,1}$ with $da \cdot \psi + db_1 \cdot \psi = 0$. Hence the result follows. □

Hence for $b_1 \in \wedge^2 T^*X$, it follows from $\pi_{TX}[b_1, \overline{E}_i^+] = 0$ that the $Z_i^{\pm}(t)$ is given by

$$Z_i^{\pm}(t) = Z_i \pm \overline{\beta} \cdot \overline{\theta}^i t, \qquad (\mathrm{mod}\ t^2). \tag{3.3}$$

The contraction between β and $\sqrt{-1}\omega$ is written as

$$\sqrt{-1}\beta \cdot \omega = \sum_i (\beta \cdot \theta)\overline{\theta}^i \in T^{1,0} \otimes \wedge^{0,1}.$$

Since $\sqrt{-1}\beta\cdot\omega$ is $\overline{\partial}$-closed, we have a class $\sqrt{-1}[\beta\cdot\omega] \in H^{0,1}(X, T^{1,0})$. Then it follows from (3.3) that the infinitesimal tangents of deformations $\{I^+(t)\}$ and $\{I^-(t)\}$ are respectively given by the classes $\sqrt{-1}[\beta\cdot\omega]$ and $-\sqrt{-1}[\beta\cdot\omega] \in H^{0,1}(X, T^{1,0})$. Hence we have

Theorem 3.2. *Let X be a compact Kälher manifold with a Poisson structure β. The class $[\beta\cdot\omega] \in H^{0,1}(X, T^{1,0})$ defined by the contraction of β by a Kähler form ω gives rise to unobstructed deformations. In other words, we have a vanishing of the obstruction class, $[\beta\cdot\omega, \beta\cdot\omega] = 0 \in H^{0,2}(X, T^{1,0})$.*

Proof. Let $\{\mathscr{J}_{\beta t}, \psi_t\}$ be deformations of generalized Kähler structures as in section 2 with the corresponding deformations of bihermitian structures $\{I^+(t), I^-(t), h_t\}$. Then the class $\sqrt{-1}[\beta\cdot\omega] \in H^{0,1}(X, T^{1,0})$ is the infinitesimal tangent of the deformations of $I^+(t)$. Hence we obtain a vanishing of the obstruction class, $[\beta\cdot\omega, \beta\cdot\omega] = 0 \in H^{0,2}(X, T^{1,0})$. □

Example 3.3. Let M be a complex torus of dimension n and X the product of M and the projective space $\boldsymbol{C}P^1$. Deformations of X were explicitly studied in [**22**]. A holomorphic vector field on $\boldsymbol{C}P^1$ is written as a linear combination,

$$a\frac{\partial}{\partial\zeta} + b\zeta\frac{\partial}{\partial\zeta} + c\zeta^2\frac{\partial}{\partial\zeta},$$

where ζ is the affine coordinates of $\boldsymbol{C}P^1$ and a, b, c are constants. Let $\{z_1, \dots, z_n\}$ be the coordinates of complex torus M. Then every representative p of $H^{0,1}(X, T^{1,0})$ is given in the from,

$$p = \sum_i \left(a_i\frac{\partial}{\partial\zeta} + b_i\zeta\frac{\partial}{\partial\zeta} + c_i\zeta^2\frac{\partial}{\partial\zeta}\right)d\overline{z}_i + \sum_{j,k}\lambda_{jk}\frac{\partial}{\partial z_j}d\overline{z}_k, \tag{3.4}$$

where λ_{jk} are constants. We define an $n\times 3$ matrix P by

$$P = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \vdots & \vdots & \vdots \\ a_n & b_n & c_n \end{pmatrix}.$$

Then we see that the class of obstruction $[p, p]$ vanishes if and only if the rank of the matrix P is less than or equal to 1. On the other hand, every holomorphic 2-vector on X is given by

$$\beta=\sum_i\left(a_i\frac{\partial}{\partial\zeta}+b_i\zeta\frac{\partial}{\partial\zeta}+c_i\zeta^2\frac{\partial}{\partial\zeta}\right)\wedge\frac{\partial}{\partial z_i}+\sum_{j,k}\lambda_{jk}\frac{\partial}{\partial z_j}\wedge\frac{\partial}{\partial z_k}.$$

Then the Schouten bracket $[\beta,\beta]$ also vanishes if and only if the rank of P is less than or equal to 1. Let ω be the Kähler form $\omega_{FS}+\omega_M$, where ω_{FS} denotes the Fubini-Study form of $\boldsymbol{C}P^1$ and ω_M is the standard Kähler form of M. Then the contraction $\beta\cdot\omega$ is the representative p and we have a surjective map

$$H^0(X,\wedge^2T^{1,0})\to H^{0,1}(X,T^{1,0}).$$

Let $\Lambda=\{\omega_\alpha\}_{\alpha=1}^{2n}$ be the discrete lattice of maximal rank $2n$ in $\boldsymbol{C}^n$ with $M=\boldsymbol{C}^n/\Lambda$, where $\omega_\alpha=(\omega_{\alpha1},\dots\omega_{\alpha n})$. We denote by V_i the holomorphic vector field $a_i(\partial/\partial\zeta)+b_i\zeta(\partial/\partial\zeta)+c_i\zeta^2(\partial/\partial\zeta)$. Then V_i generates the automorphism $\exp(V_i)$ of $\boldsymbol{C}P^1$. For each α, we define an automorphism $\rho_t(\omega_\alpha)$ by

$$\rho_t(\omega_\alpha)=\sum_i\exp(\omega_{\alpha i}V_it).$$

In the case of the rank of $P=1$, $V_1,\dots,V_n$ are commuting vector fields. Hence ρ_t gives rise to a representation of $\Lambda=\pi_1(M)$ on automorphisms of $\boldsymbol{C}P^1$. By the family of representations $\{\rho_t\}$, we obtain deformations of complex fibre bundles $\{X_t\}$ over the torus M starting from the trivial bundle $X_0=M\times\boldsymbol{C}P^1$,

$$X_t=M\times_{\rho_t}\boldsymbol{C}P^1\to M.$$

In [**22**], it is shown that the infinitesimal deformation of $\{X_t\}$ is the class $[p]\in H^{0,1}(X,T^{1,0})$ in (3.4), where $\lambda_{jk}=0$.

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