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Image of asymptotic Bers map

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Abstract. In this paper, we give a geometric characterization for developing mappings such that the asymptotic class of its Schwarzian derivative is in the image of the asymptotic Bers map from the asymptotic Teichmüller space of the unit disk $\boldsymbol{D}$. We also give a characterization of points in the closure of the image, and discuss the density problem for the asymptotic Teichmüller space.

1. Introduction.

1.1. Background.

In this paper, we discuss a geometric characterization of points in the *asymptotic Teichmüller space* $AT(\mathfrak{R})$ of a hyperbolic Riemann surface $\mathfrak{R}$ and its closure. The asymptotic Teichmüller space is the set of equivalence classes of quasiconformal mappings on $\mathfrak{R}$. Two quasiconformal mappings f and g on $\mathfrak{R}$ are equivalent if there is an asymptotically conformal mapping h of $f(\mathfrak{R})$ onto $g(\mathfrak{R})$ such that $h \circ f$ is homotopic to g rel the ideal boundary of $f(\mathfrak{R})$. We have another deformation space of $\mathfrak{R}$, the *Teichmüller space* $\mathrm{Teich}(\mathfrak{R})$ of $\mathfrak{R}$. The Teichmüller space has the same definition with one exception. The mapping h should be conformal. Since conformal mappings are asymptotically conformal, there is the canonical projection from $\mathrm{Teich}(\mathfrak{R})$ onto $AT(\mathfrak{R})$. Furthermore, $\mathrm{Teich}(\mathfrak{R})$ and $AT(\mathfrak{R})$ admit complex Banach manifold structures such that the canonical projection between them becomes holomorphic (cf. [**7**], [**9**] and [**12**]).

1.2. Motivation.

Let Γ be the Fuchsian group acting on the unit disk $\boldsymbol{D}$ uniformizing $\mathfrak{R}$. By virtue of a famous theorem due to L. Bers, the Teichmüller space $\mathrm{Teich}(\mathfrak{R})$ admits a holomorphic embedding, called the *Bers embedding*, into the Banach space $B(\Gamma)$ of bounded holomorphic automorphic forms on $\boldsymbol{D}^* := \widehat{\boldsymbol{C}} - \mathrm{cl}(\boldsymbol{D})$ of weight -4

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with respect to Γ, where $\mathrm{cl}(X)$ is the closure of a set X. The image of the Bers embedding is characterized as the set of automorphic forms $\varphi \in B(\Gamma)$ with the property that the developing mapping $f_\varphi : \boldsymbol{D}^* \to \widehat{\boldsymbol{C}}$ associated to φ admits a quasiconformal extension on the Riemann sphere, where the *developing mapping* f_φ is a locally univalent function whose Schwarzian derivative is equal to φ (cf. [**1**] and [**5**]). Since the developing mapping f_φ varies holomorphically with respect to φ, developing mappings are standard and powerful tools for studying complex-analytical variations of quasiconformal mappings and univalent functions, and the image of the Bers embedding is an ample and basic field which yields rich results on complex analytic structures of Teichmüller spaces and the set of univalent functions (cf. [**15**]).

Meanwhile, it is known that there is also a canonical holomorphic embedding of $AT(\mathfrak{R})$, called the *asymptotic Bers map*, which is induced from the Bers embedding and the canonical projection from $\mathrm{Teich}(\mathfrak{R})$ onto $AT(\mathfrak{R})$. The asymptotic Bers map embeds $AT(\mathfrak{R})$ into a quotient Banach space $\hat{B}(\Gamma)$ of $B(\Gamma)$ (cf. [**8**] and [**10**]).

However, because of the abundance and instability (non-rigidity) of asymptotically conformal mappings, the equivalence relation in defining the asymptotic Teichmüller space has quite a big flexibility. Hence, it is hard to imagine points in the asymptotic Teichmüller space as geometric objects. However, it seems to be important for studying the (complex analytic) theory of asymptotic Teichmüller spaces to correlate points in the image under the asymptotic Bers map with analytic and geometric objects which vary holomorphically, as well as the study of the complex analytic theory of Teichmüller space. This paper is motivated from this point of view.

1.3. Results.

This paper is devoted to the study of the image of asymptotic Bers maps and its closure. We mainly deal with the case of the asymptotic Teichmüller space $AT(\boldsymbol{D})$ of the unit disk $\boldsymbol{D}$.

This paper has two aims. One of the aims is to characterize $\varphi \in B(\mathbf{1})$ whose equivalence class $[\varphi]$ is in the image $\mathscr{A}\mathscr{T}_{\mathbf{1}}$ of $AT(\boldsymbol{D})$ under the asymptotic Bers map in terms of the geometric property of developing mappings, where $\mathbf{1}$ is the trivial Fuchsian group. For arbitrary Γ we will also give a necessary condition for $[\varphi] \in \hat{B}(\Gamma)$ to be contained in the image of the asymptotic Bers map. The author optimistically hopes the condition is sufficient. The second aim is to give a necessary and sufficient condition for $\varphi \in B(\mathbf{1})$ such that its asymptotic class $[\varphi]$ is contained in the closure of the image of asymptotic Bers map. We will also give a negative answer to a version of Bers density problem in the case of asymptotic Teichmüller space.

Characterization of the preimage.

We give notions to summarize our main theorem. An automorphic form $\varphi \in B(\Gamma)$ is said to have the *quasiconformal extension property at ends with respect to* Γ when for any $L > 1$, there exist a constant $\varepsilon(\varphi) \geq 1$ and a compact set $C(\varphi, L)$ of $\mathfrak{R}^* := \boldsymbol{D}^*/\Gamma$ such that for any L-quasidisk $D \subset \boldsymbol{D}^*$ avoiding the preimage of $C(\varphi, L)$, the restriction of the developing mapping f_φ to D admits an $\varepsilon(\varphi)$-quasiconformal extension on the Riemann sphere, where an *L-quasidisk* is the image of a round disk under an L-quasiconformal mapping on $\widehat{\boldsymbol{C}}$. When Γ is the trivial group $\mathbf{1}$, we simply say $\varphi \in B(\mathbf{1})$ *has the quasiconformal extension property at ends*. We note that when φ is in the image under the Bers embedding, φ has the quasiconformal extension property at ends since f_φ itself has a quasiconformal extension on $\widehat{\boldsymbol{C}}$.

A mapping f on a domain in $\widehat{\boldsymbol{C}}$ is called *locally univalent quasiregular* if any point in a domain has a neighborhood where f is K-quasiconformal with a constant K depending only on f. This definition is stronger than the definition of usual quasiregular mappings. Indeed, locally univalent quasiregular mappings are not allowed to have branch points. A *quasiloop* is the image of a mapping $\gamma : \boldsymbol{S}^1 := \partial \boldsymbol{D} \to \widehat{\boldsymbol{C}}$ such that for any $z_0 \in \boldsymbol{S}^1$, there is a neighborhood U_0 of z_0 in $\boldsymbol{C}$ such that $\gamma\,|_{\boldsymbol{S}^1 \cap U_0}$ is the restriction of a quasiconformal mapping on U_0. By definition, the image of $\boldsymbol{S}^1$ under a locally univalent quasiregular mapping is a quasiloop.

One of our main theorems in this paper is as follows.

Theorem 1 (Characterization of the preimage). *For $\varphi \in B(\mathbf{1})$, the following four conditions are equivalent.*

(a) *The equivalence class $[\varphi] \in \hat{B}(\mathbf{1})$ is contained in the image $\mathscr{A}\mathscr{T}_{\mathbf{1}}$ of $AT(\boldsymbol{D})$ under the asymptotic Bers map.*
(b) *φ has the quasiconformal extension property at ends.*
(c) *The developing mapping f_φ associated to φ is extended as a locally univalent quasiregular mapping on a neighborhood of* $\mathrm{cl}(\boldsymbol{D}^*)$.
(d) *f_φ has a continuous extension on* $\mathrm{cl}(\boldsymbol{D}^*)$ *with the property that f_φ is locally injective on* $\mathrm{cl}(\boldsymbol{D}^*)$ *and the image $f_\varphi(\boldsymbol{S}^1)$ is a quasiloop.*

The condition (d) in Theorem 1 is comparable to the condition for $\varphi \in B(\mathbf{1})$ to be in the universal Teichmüller space $\mathscr{T}_{\mathbf{1}}$. Indeed, $\varphi \in B(\mathbf{1})$ is in $\mathscr{T}_{\mathbf{1}}$ if and only if $f_\varphi(\boldsymbol{D}^*)$ is a quasidisk, that is, f_φ admits an extension as an injective mapping on $\mathrm{cl}(\boldsymbol{D}^*)$ such that $f_\varphi(\boldsymbol{S}^1)$ is a quasicircle. We here note that in the condition (d) we can not erase the local injectivity condition of the extension (see Remark in Section 5).

From Theorem 1, we conclude the following corollaries which describe a geometric and topological property of the developing mapping f_φ associated to $\varphi \in B(\mathbf{1})$ with $[\varphi] \in \mathscr{AT}_{\mathbf{1}}$ as follows.

Corollary 1 (Topology of developing mappings). *Let $\varphi \in B(\mathbf{1})$. When the equivalence class $[\varphi]$ of φ is in $\mathscr{AT}_{\mathbf{1}}$, the developing mapping f_φ admits a continuous extension on* $\mathrm{cl}(\boldsymbol{D}^*)$ *and is a finite covering map onto its image in the sense that there is an $N > 0$ such that the number of the preimage $f_\varphi^{-1}(w)$ is at most N for every point w in the image $f_\varphi(\boldsymbol{D}^*)$.*

Corollary 2 (Necessary condition for interior). *For $\varphi \in B(\mathbf{1})$, suppose that f_φ is univalent. Then, the complement $\widehat{\boldsymbol{C}} - f_\varphi(\boldsymbol{D})$ is locally connected, provided when the asymptotic class $[\varphi]$ is in the image $\mathscr{AT}_{\mathbf{1}}$.*

However, the local connectivity is not sufficient for the asymptotic class to be in the interior (cf. Section 7).

The conditions (a) and (b) in Theorem 1 are related even in the case of arbitrary Fuchsian groups. Indeed, we will conclude

Proposition 1.1 (Necessary condition for arbitrary Γ). *Let Γ be a Fuchsian group acting on $\boldsymbol{D}$. For $\varphi \in B(\Gamma)$, if the equivalence class $[\varphi] \in \hat{B}(\Gamma)$ is contained in the image $\mathscr{AT}_\Gamma$ of $AT(\boldsymbol{D}/\Gamma)$ under the asymptotic Bers map, then φ has the quasiconformal extension property at ends with respect to Γ.*

Closure of the image.

Teichmüller space has a standard closure and boundary by taking the closure of the image of the Bers embedding. The boundary is called the *Bers boundary.* In this paper, we treat the Bers boundary and the closure for asymptotic Teichmüller space of the unit disk.

1.3.1. Failure of density and density problem.

Let Γ be a Fuchsian group. Denote by $S(\Gamma) \subset B(\Gamma)$ the set of Schwarzian derivatives of univalent functions on $\boldsymbol{D}^*$ equivariant under the action of Γ. By definition, the image $\mathscr{T}_\Gamma$ of the Bers embedding of $\mathrm{Teich}(\boldsymbol{D}/\Gamma)$ is contained in $S(\Gamma)$. L. Bers raised a problem, called the *density problem*, which asks whether $\mathscr{T}_\Gamma$ is dense in $S(\Gamma)$. In his celebrated paper [**11**], F. Gehring solved the Bers density problem in the negative for the universal Teichmüller space $\mathscr{T}_{\mathbf{1}}$. Indeed, he explicitly gave a simply connected domain Ω_{Geh}, so-called a *Gehring's spiral domain*, and showed that the Schwarzian derivative φ_{Geh} of the Riemann mapping of Ω_{Geh} is not in the closure $\mathrm{cl}(\mathscr{T}_{\mathbf{1}})$ of the universal Teichmüller space.

In the case of asymptotic Teichmüller spaces, we will also observe a similar phenomenon as follows.

THEOREM 2 (Failure of density). *There is $\varphi_\infty \in S(\mathbf{1})$ with $[\varphi_\infty] \notin \mathrm{cl}(\mathscr{A}\mathscr{T}_{\mathbf{1}})$.*

Indeed, from the proof of this theorem, we can see that $[\varphi_\infty]$ can not be approximated from the asymptotic classes of Schwarzian derivatives whose developing mappings are Riemann mappings of Jordan domains (cf. Proposition 7.2 in Section 7.2). We remark that the equivalence class $[\varphi_{Geh}]$ of φ_{Geh} *is contained* in the Bers boundary $\partial_b \mathscr{A}\mathscr{T}_{\mathbf{1}}$ of $\mathscr{A}\mathscr{T}_{\mathbf{1}}$, and we will construct another simply connected domain to show the failure of the density.

1.3.2. Characterization of closure of $\mathscr{A}\mathscr{T}_{\mathbf{1}}$.

We will give a characterization of points in the closure of $\mathscr{A}\mathscr{T}_{\mathbf{1}}$ by using quasiconformal extensions. Our characterization follows from Astala and Gehring [**4**] for the case of universal Teichmüller space.

THEOREM 3 (Characterization of the preimage of the closure). *For $\varphi \in B(\mathbf{1})$, the equivalence class $[\varphi]$ of φ is in the closure* $\mathrm{cl}(\mathscr{A}\mathscr{T}_{\mathbf{1}})$ *of $\mathscr{A}\mathscr{T}_{\mathbf{1}}$ in $\hat{B}(\mathbf{1})$ if and only if for any $K, L > 1$, there are a K-quasiconformal mapping g on Σ_φ onto a quasidisk and a compact set C in Σ_φ such that for any univalent disk B in $\Sigma_\varphi - C$, $g \circ (\mathrm{pr}_\varphi|_B)^{-1} : \mathrm{pr}_\varphi(B) \to \widehat{\boldsymbol{C}}$ admits an L-quasiconformal extension on $\widehat{\boldsymbol{C}}$, where Σ_φ is the image of $\boldsymbol{D}^*$ under f_φ which is regarded as a Riemann surface spread over a domain in $\widehat{\boldsymbol{C}}$ and $\mathrm{pr}_\varphi : \Sigma_\varphi \to \widehat{\boldsymbol{C}}$ is the canonical projection.*

This paper is organized as follows. In the three sections Section 3, Section 4 and Section 5, we devote to the proof of Theorem 1, and we will check Corollaries 1 and 2 in Section 6. In Section 7 we will treat Theorem 2 and also show that the asymptotic class of the Schwarzian derivative associated to the Gehring's spiral domain is in the Bers boundary of the asymptotic Teichmüller space of the unit disk. In Section 8, we will prove Theorem 3 and give another (simpler) characterization for the closure. In Section 9, we discuss open problems related to the results in this paper.

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2. Notation.

In what follows, we fix a hyperbolic Riemann surface $\mathfrak{R}$ and the Fuchsian group Γ acting on $\boldsymbol{D}$ which uniformizes $\mathfrak{R}$. Though we mainly deal with $\mathfrak{R} = \boldsymbol{D}$

and $\Gamma = \mathbf{1}$, we define symbols in general situation. Throughout this paper we denote by $\boldsymbol{D}(z_0, r)$ the Euclidean open disk of radius r with center z_0 and let $\boldsymbol{D}(r) = \boldsymbol{D}(0, r)$.

2.1. Differentials, Teichmüller spaces and asymptotic Teichmüller spaces.

Let $M(\mathfrak{R})$ be the set of Beltrami differentials on $\mathfrak{R}$ with the essential supremum norm $\|\cdot\|_\infty$. A Beltrami differential μ on $\mathfrak{R}$ is said to *vanish at infinity* when for any $\epsilon > 0$, there is a compact set C of $\mathfrak{R}$ such that $|\mu| < \epsilon$ a.e. in $\mathfrak{R} - C$. An *asymptotically conformal* mapping on $\mathfrak{R}$ is a quasiconformal mapping whose Beltrami coefficient vanishes at infinity.

Let $\mathrm{Teich}(\mathfrak{R})$ and $AT(\mathfrak{R})$ be the Teichmüller space and the asymptotic Teichmüller space as in Section 1. For a quasiconformal mapping f on $\mathfrak{R}$, we denote by $[f]_T$ and $[f]_{AT}$ the Teichmüller and the asymptotic Teichmüller equivalence classes of f, respectively. As noted in Section 1, the mapping $\mathrm{Teich}(\mathfrak{R}) \ni [f]_T \to [f]_{AT} \in AT(\mathfrak{R})$ is a holomorphic submersion.

Let Ω be a hyperbolic domain in $\widehat{\boldsymbol{C}}$ and Γ a Kleinian group acting on Ω. We denote by $B(\Omega, \Gamma)$ the Banach space of holomorphic automorphic forms φ of weight -4 on Ω with respect to Γ which satisfies

$$\|\varphi\|_\Omega := \sup_{z \in \Omega} \lambda_\Omega(z)^{-2} |\varphi(z)| < \infty,$$

where $\lambda_\Omega = \lambda_\Omega(z)|dz|$ is the hyperbolic metric on Ω of curvature -4. An automorphic form $\varphi \in B(\Omega, \Gamma)$ is said to *vanish at infinity* with respect to Γ when the $(-1,1)$-form $\lambda_\Omega(z)^{-2}\overline{\varphi(z)}$ on Ω descends to a Beltrami differential on Ω/Γ which vanishes at infinity. Let $B_0(\Omega, \Gamma)$ denote the closed subspace of $B(\Omega, \Gamma)$ consisting of automorphic forms vanishing at infinity with respect to Γ. Set $\hat{B}(\Omega, \Gamma) := B(\Omega, \Gamma)/B_0(\Omega, \Gamma)$ and denote by $[\varphi]$ the equivalence class of $\varphi \in B(\Omega, \Gamma)$ and by $\|\cdot\|_\Omega^\wedge$ the quotient norm on $\hat{B}(\Omega, \Gamma)$.

CONVENTION. For simplicity, we abbreviate $B(\boldsymbol{D}^*, \Gamma)$ and $B(\Omega, \mathbf{1})$ to $B(\Gamma)$ and $B(\Omega)$, respectively.

2.2. The Bers embedding and the asymptotic Bers map.

2.2.1. The Bers embedding.

Let $[f]_T \in \mathrm{Teich}(\mathfrak{R})$ and $\tilde{\mu}$ the lift of the Beltrami differential of f on $\boldsymbol{D}$. Let $W_f : \widehat{\boldsymbol{C}} \to \widehat{\boldsymbol{C}}$ be a quasiconformal mapping satisfying

$$\frac{\overline{\partial} W_f}{\partial W_f} = \begin{cases} \tilde{\mu} & \text{on } \boldsymbol{D} \\ 0 & \text{on } \mathrm{cl}(\boldsymbol{D}^*). \end{cases}$$

Then the *Bers embedding* of $\mathrm{Teich}(\mathfrak{R})$ is defined by

$$\beta_\Gamma : \mathrm{Teich}(\mathfrak{R}) \ni [f]_T \mapsto \mathscr{S}(W_f\ |_{\boldsymbol{D}^*}) \in B(\Gamma).$$

where $\mathscr{S}(g)$ is the Schwarzian derivative of g. It is known that $\beta_\Gamma([f]_T)$ depends only on the equivalence class of f. Conversely, for $\varphi \in B(\Gamma)$ one can define a locally univalent function f_φ, called the *developing mapping for* φ, on $\boldsymbol{D}^*$ with $\mathscr{S}(f_\varphi) = \varphi$. It is well-known that the Schwarzian derivative of the composition of locally univalent functions is calculated as

$$\mathscr{S}(f \circ g)(z) = \mathscr{S}(f)(g(z))g'(z)^2 + \mathscr{S}(g)(z) \tag{2.1}$$

(e.g. (1.3) of p.52 in [**15**]).

Let us denote by $\mathscr{T}_\Gamma \subset B(\Gamma)$ the image of the Bers embedding β_Γ. The image $\mathscr{T}_\Gamma \subset B(\Gamma)$ is characterized as the property of developing mappings: $\varphi \in B(\Gamma)$ is contained in $\mathscr{T}_\Gamma$ if and only if f_φ admits a quasiconformal extension on $\widehat{\boldsymbol{C}}$.

Proposition 2.1 (Ahlfors). *For $L \geq 1$, there is a constant $c(L) > 0$ such that any locally univalent function g on an L-quasidisk Ω satisfying $\|\mathscr{S}(g)\|_\Omega \leq c(L)$ admits a 2-quasiconformal extension on $\widehat{\boldsymbol{C}}$.*

2.2.2. The asymptotic Bers map.

In [**7**], C. Earle, F. Gardiner and N. Lakic established the existence of a holomorphic mapping $\widehat{\beta}_\Gamma : AT(\mathfrak{R}) \to \widehat{B}(\Gamma)$ satisfying the following commutative diagram

$$\begin{array}{ccc} \mathrm{Teich}(\mathfrak{R}) & \xrightarrow{\beta_\Gamma} & B(\Gamma) \\ \big\downarrow & & \big\downarrow \\ AT(\mathfrak{R}) & \xrightarrow{\widehat{\beta}_\Gamma} & \widehat{B}(\Gamma) \end{array} \tag{2.2}$$

where the vertical directions are canonical projections. Furthermore, in [**8**], C. Earle, V. Markovic and D. Saric obtained that the mapping $\widehat{\beta}_\Gamma$ is actually an embedding. The mapping $\widehat{\beta}_\Gamma$ is called the *asymptotic Bers map* and we denote by $\mathscr{AT}_\Gamma \subset \widehat{B}(\Gamma)$ the image of $AT(\mathfrak{R})$. Notice from the commutative diagram (2.2) that

$$\mathscr{AT}_\Gamma = \{[\varphi] \mid \varphi \in \mathscr{T}_\Gamma\}. \tag{2.3}$$

This implies that any point $[\varphi] \in \mathscr{AT}_\Gamma$ contains a representative $\varphi \in B(\Gamma)$ such that f_φ extends to a quasiconformal mapping on $\widehat{\boldsymbol{C}}$.

3. Necessary condition for arbitrary Γ.

In this section, we prove Proposition 1.1. Let Γ be a Fuchsian group acting on $\boldsymbol{D}$ and fix $\varphi \in B(\Gamma)$ whose equivalence class $[\varphi] \in \hat{B}(\Gamma)$ is in $\mathscr{A}\mathscr{T}_{\Gamma}$. By (2.3), there is $\psi \in \mathscr{T}_{\Gamma}$ such that $[\psi] = [\varphi]$ and $\varphi - \psi \in B_0(\Gamma)$. Then, f_ψ admits a K_1-quasiconformal extension on $\widehat{\boldsymbol{C}}$ for some $K_1 \geq 1$. By (2.1) $g := f_\varphi \circ f_\psi^{-1}$ is a locally univalent function on $\Omega^* := f_\psi(\boldsymbol{D}^*)$ with

$$(\mathscr{S}(g) \circ f_\psi)(z)(f_\psi')^2(z) = \varphi(z) - \psi(z) \in B_0(\Gamma).$$

Let $L \geq 1$ and take a constant $c(K_1 L)$ as in Proposition 2.1. By definition, there is a compact set C in $\mathfrak{R}^*$ such that

$$\lambda_{\boldsymbol{D}^*}(z)^{-2}|\varphi(z) - \psi(z)| \leq c(K_1 L)$$

for $z \in \boldsymbol{D}^* - \tilde{C}$ where $\tilde{C}$ is the preimage of C.

Let D be an L-quasidisk in $\boldsymbol{D}^* - \tilde{C}$. Then $f_\psi(D) \subset \Omega^*$ is a $K_1 L$-quasidisk and g satisfies

$$\begin{aligned}
\lambda_{f_\psi(D)}(w)^{-2}|\mathscr{S}(g)(w)| &\leq \lambda_{\Omega^*}(w)^{-2}|\mathscr{S}(g)(w)| \\
&= \lambda_{\boldsymbol{D}^*}(z)^{-2}|\varphi(z) - \psi(z)| \\
&\leq c(K_1 L)
\end{aligned}$$

for $w \in f_\psi(D)$ and $w = f_\psi(z)$. Hence, by Proposition 2.1, $g\mid_{f_\psi(D)}$ extends to a 2-quasiconformal mapping on the Riemann sphere and $f_\varphi\mid_D = g \circ f_\psi\mid_D$ is the restriction of $2K_1$-quasiconformal mapping to D. Finally, we note that K_1 can be taken to depend only on φ by definition, so is $2K_1$.

4. Quasiregular extension of developing mappings.

Throughout this section, we assume that $\Gamma = \mathbf{1}$ and $\mathfrak{R} = \boldsymbol{D}$. We will prove in this section that the three conditions (a), (b) and (c) in Theorem 1 are equivalent.

4.1. Fans.

For $a = e^{i\theta} \in \boldsymbol{S}^1$ and $r, R > 0$, we define

$$\begin{aligned}
S_1(a, r) &:= \{|\arg z - \theta| < r,\ 1 < |z| < e^r\}, \text{ and} \\
S_2(a, r, R) &:= S_1(a, r) \cup \{|\arg z - \theta| < r/3,\ e^{-R} < |z| < e^r\}.
\end{aligned}$$

LEMMA 4.1. *There exist universal constants $R_1 > 0$ and $K_3 > 1$ such that for $a, b \in \boldsymbol{S}^1$ with $|\arg a - \arg b| = 2r$, $S_1(a, r)$, $S_2(a, r, R)$, and the interior of* $\mathrm{cl}(S_1(a, r) \cup S_1(b, r))$ *are K_3-quasidisks for $0 < R \leq r \leq R_1$.*

PROOF. We can easily observe that the boundaries of polygons

$$\{-R < \mathrm{Re} z \leq 0,\ |\mathrm{Im} z| < r/3\} \cup \{0 < \mathrm{Re} z < r,\ |\mathrm{Im} z| < r\}$$

and

$$\{-R < \mathrm{Re} z \leq 0,\ 2r/3 < |\mathrm{Im} z| < 4r/3\} \cup \{0 < \mathrm{Re} z < r,\ |\mathrm{Im} z| < 2r\}$$

with $R \leq r$ are bi-Lipschitz equivalent to the boundaries of rectangles

$$\{0 < \mathrm{Re} z < r,\ |\mathrm{Im} z| < r\}$$

and

$$\{0 < \mathrm{Re} z < r,\ |\mathrm{Im} z| < 2r\}$$

with uniform bi-Lipschitz constant, respectively. Hence these are quasiconformally equivalent with uniform dilatation (cf [**23**]). Since the Schwarzian derivative of e^z satisfies $\|\mathscr{S}(e^z)\|_{\boldsymbol{D}(z_0, 2/\sqrt{3})} \leq 2 \cdot (1/3)$ for any $z_0 \in \boldsymbol{C}$, the restriction of e^z to $\boldsymbol{D}(z_0, 2/\sqrt{3})$ admits a 2-quasiconformal extension on $\widehat{\boldsymbol{C}}$ (cf. [**15**]). Therefore, $R_1 := 2/(3\sqrt{3})$ satisfies the desired property with suitable K_3. $\square$

4.2. Equivalence of conditions.

The condition (a) implies (b) in Theorem 1 from Proposition 1.1.

4.2.1. (b) implies (c).

We begin with the following proposition which deduces (b) $\Rightarrow$ (c) in Theorem 1.

PROPOSITION 4.1. *Let f be a locally univalent function on $\boldsymbol{D}^*$ whose Schwarzian derivative has the quasiconformal extension property at ends. Then f extends to a locally univalent quasiregular mapping on a neighborhood of* $\mathrm{cl}(\boldsymbol{D}^*)$.

PROOF. Take constants K_3 and R_1 as in Lemma 4.1. Since f has the quasiconformal extension property at ends, there is an $R_2 > 0$ such that for any K_3-quasidisk D in $\{1 < |z| < e^{R_2}\}$, the restriction $f\,|_D$ is extended as an $\varepsilon_1 := \varepsilon(\mathscr{S}(f))$-quasiconformal mapping on the Riemann sphere.

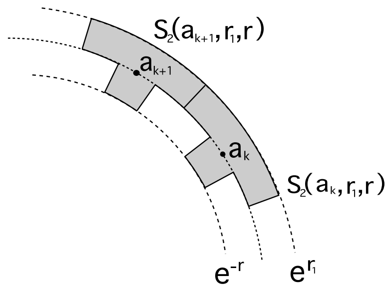

Figure 1. Union $S_3(k, r)$ of fans.

Take $n_1 \in \boldsymbol{N}$ so large that $r_1 := \pi/(n_1+1) \leq \min\{R_1, R_2\}/2$. Set $a_k := e^{2r_1 k i}$ for $k = 0, 1, \cdots, n_1$ and $a_{n_1+1} = 1 = a_0$. Since $S_1(a_k, r_1)$ is a K_3-quasidisk in $\{1 < |z| < e^{R_2}\}$, there is an ε_1-quasiconformal extension g_k on $\widehat{\boldsymbol{C}}$ of the restriction $f\,|_{S_1(a_k, r_1)}$. Thus, the mapping

$$G_1(z) := \begin{cases} f(z) & z \in \boldsymbol{D}^* \\ g_k(z) & z \in S_2(a_k, r_1, r_1) \end{cases}$$

is a locally univalent quasiregular mapping on $D_1 := \boldsymbol{D}^* \cup \cup_{k=0}^{n_1} S_2(a_k, r_1, r_1)$.

We check that G_1 extends to a locally univalent quasiregular mapping on a neighborhood of $\mathrm{cl}(\boldsymbol{D}^*)$. Set $S_3(k, r) := S_2(a_k, r_1, r) \cup S_2(a_{k+1}, r_1, r)$ (Figure 1). As we will observe later, there is an $r_2 > 0$ such that for $r < r_2$, G_1 is injective on the closure $\mathrm{cl}(S_3(k, r))$ of $S_3(k, r)$, and the image $G_1(S_3(k, r))$ is a quasidisk in $\widehat{\boldsymbol{C}}$ (cf. Lemma 4.2 below). Since $S_3(k, r_2)$ is also a quasidisk and G_1 is quasiconformal, the restriction $G_1\,|_{S_3(k, r_2)}$ extends to a quasiconformal mapping h_k on $\widehat{\boldsymbol{C}}$. Thus, the mapping

$$G_2(z) := \begin{cases} f(z) & z \in \boldsymbol{D}^* \\ h_k(z) & z \in \{e^{-r_2} < |z| \leq 1,\ r_1/3 < \arg z - 2r_1 k < 7r_1/3\} \end{cases}$$

is a locally univalent quasiregular extension of f on $\{e^{-r_2} < |z|\} \cup \{\infty\}$. □

To complete the proof of Proposition 4.1, we shall show the following lemma.

Lemma 4.2. *There is an $r_2 > 0$ such that G_1 is injective on the closure $\mathrm{cl}(S_3(k, r))$ of $S_3(k, r)$ for $r < r_2$. Furthermore, the image $G_1(S_3(k, r))$ is a quasidisk in $\widehat{\boldsymbol{C}}$ for $r < r_2$.*

Proof. First we show the injectivity of G_1. Indeed, since $G_1 = f$ on

$\mathrm{cl}(S_1(a_k, r_1) \cup S_1(a_{k+1}, r_1))$, by Proposition 4.1 and our assumption on f, G_1 is quasiconformal on the closure of a neighborhood of $\mathrm{cl}(S_1(a_k, r_1) \cup S_1(a_{k+1}, r_1))$, G_1 satisfies the inverse Hölder condition, that is

$$d_S(G_1(z), G_1(w)) \geq M|z - w|^\alpha$$

holds for $z, w \in \mathrm{cl}(S_1(a_k, r_1) \cup S_1(a_{k+1}, r_1))$ for some $M, \alpha > 0$, where d_S is the spherical distance on $\widehat{\boldsymbol{C}}$. Furthermore, G_1 is also quasiconformal on $\mathrm{cl}(S_2(a_s, r_1, r))$ for $r \leq r_1$ and $s = k, k+1$. G_1 satisfies the same Hölder condition above on $\mathrm{cl}(S_2(a_s, r_1, r))$ for $r \leq r_1$ and $s = k, k+1$. Since $\mathrm{cl}(S_3(k, 0)) = \mathrm{cl}(S_2(a_k, r_1) \cup S_2(a_{k+1}, r_1))$ and the (Euclidean) distance between $\mathrm{cl}(S_2(a_k, r_1, r) - S_1(a_k, r_1))$ and $\mathrm{cl}(S_2(a_{k+1}, r_1, r) - S_1(a_{k+1}, r_1))$ is positive, their images under G_1 cannot intersect for sufficiently small r, which implies that G_1 is injective on the closure $\mathrm{cl}(S_3(k, r_2))$ of $S_3(k, r_2)$ for some $r_2 > 0$.

We next check that the image $G_1(S_3(k, r))$ is a quasidisk for $r < r_2$. By Proposition 4.1 and the definitions of r_1 and R_1, $S_3(k, r) \cap \boldsymbol{D}^* = S_1(a_k, r_1) \cup S_1(a_{k+1}, r_1)$ is a K_3-quasidisk contained in $\{1 < |z| < e^{R_2}\}$. Therefore, the restriction of f to $S_3(k, r) \cap \boldsymbol{D}^*$ admits a quasiconformal extension on $\widehat{\boldsymbol{C}}$. In particular, f is injective on the closure of $S_3(k, r) \cap \boldsymbol{D}^*$ and any point in $\partial G_1(S_3(k, r) \cap \boldsymbol{D}^*)$ is in a open quasiarc (a quasiconformal image of an open interval), in $\partial(S_3(k, r) \cap \boldsymbol{D}^*)$.

Since g_k is quasiconformal on the closure of $S_2(a_k, r_1, r)$ for $r \leq r_1$, any point in $\partial G_1(S_3(k, r))$ is contained in an open quasiarc. Thus, the image of $S_3(k, r)$ is bounded by a quasicircle by Theorem 8.7 in p. 103 of [**16**] and $G_1(S_3(k, r))$ is a quasidisk for such r. □

4.2.2. (c) implies (a).

Next, we show the following proposition which implies (c) ⇒ (a) in Theorem 1.

Proposition 4.2. *Let f be a locally univalent function on $\boldsymbol{D}^*$. Suppose that f extends to a locally univalent quasiregular mapping on a neighborhood of $\mathrm{cl}(\boldsymbol{D}^*)$. Then $\mathscr{S}(f) \in B(\mathbf{1})$ and there is $\psi \in \mathscr{T}_{\mathbf{1}}$ such that $\mathscr{S}(f) - \psi \in B_0(\mathbf{1})$.*

Proof. By assumption, there is a locally univalent quasiregular mapping F on a neighborhood N of $\mathrm{cl}(\boldsymbol{D}^*)$ such that $F\,|_{\boldsymbol{D}^*} = f$. Let μ_F be the complex dilatation of F on N and consider a quasiconformal mapping H on $\widehat{\boldsymbol{C}}$ with Beltrami differential

$$\frac{H_{\overline{z}}}{H_z} = \begin{cases} \mu_F & \text{on } N, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that, since F is holomorphic on $\boldsymbol{D}^*$, so is H. Then $F \circ H^{-1}$ is locally univalent on $\Omega_0 := H(N)$ and its Schwarzian derivative $\mathscr{S}(F \circ H^{-1})$ is holomorphic on Ω_0, which contains the closure of $\Omega := H(\boldsymbol{D}^*)$. This means that $\mathscr{S}(f \circ H^{-1}) \in B_0(\Omega)$, that is, for any $\epsilon > 0$ there is a compact set C of Ω such that

$$\lambda_\Omega(w)^{-2} |\mathscr{S}(f \circ H^{-1})(w)| < \epsilon$$

for every $w \in \Omega - C$, since $\lambda_\Omega(w)$ is comparable with the reciprocal of the distance from w to $\partial\Omega$ (e.g. Corollary 1.4 of [**20**]).

Let $\varphi_0 := \mathscr{S}(f \circ H^{-1}) \circ H \cdot (H')^2 \in B_0(\mathbf{1})$ and $\psi := \mathscr{S}(H) \in \mathscr{T}_{\mathbf{1}}$. Then

$$\begin{aligned} \mathscr{S}(f) &= \mathscr{S}(f \circ H^{-1} \circ H) \\ &= \mathscr{S}(f \circ H^{-1}) \circ H \cdot (H')^2 + \mathscr{S}(H) \\ &= \varphi_0 + \psi, \end{aligned}$$

and $\mathscr{S}(f) - \psi = \varphi_0 \in B_0(\mathbf{1})$. □

5. Quasi-loops.

In this section, we complete the proof of Theorem 1. Indeed, we prove the equivalence of the conditions (c) and (d) in Theorem 1, which is a characterization of interior points in terms of the boundary values of corresponding developing mappings. To show this, since the condition (c) clearly implies (d), it suffices to show the converse.

Assume that φ satisfies the condition (d). Then, for $z_0 \in \boldsymbol{S}^1$, there is a neighborhood U_0 of z_0 (in $\boldsymbol{C}$) such that f_φ is injective on $U_0 \cap \mathrm{cl}(\boldsymbol{D}^*)$ and the image $f_\varphi(\boldsymbol{S}^1 \cap U_0)$ is a quasiarc. By the standard argument, we can see that z_0 admits a neighborhood U_{z_0} ($\subset U_0$) where f_φ extends to a quasiconformal mapping. Since $\boldsymbol{S}^1$ is compact, $\boldsymbol{S}^1$ can be covered by finitely many such neighborhoods $U_{z_1}, \ldots, U_{z_m}$ of $z_1, \ldots, z_m \in \boldsymbol{S}^1$.

By the Lebesgue number lemma, there is an r_0 such that when $r < r_0$, fans $S_1(z, r)$ and $S_2(z, r)$ and the union of adjacent fans are in some U_{z_i}. This means that the restriction of f_φ to any such sufficiently small fans or to the union of adjacent fans can be extended as a quasiconformal mapping on $\widehat{\boldsymbol{C}}$ (we take r_0 small, if necessary). Thus, by the same argument as that in the proof of Proposition 4.1, we conclude that f_φ is allowed to have a locally quasiregular extension on a neighborhood of $\mathrm{cl}(\boldsymbol{D}^*)$.

Remark. In the condition (d), we can not ignore the property that f_φ is locally injective on $\mathrm{cl}(\boldsymbol{D}^*)$. Indeed, consider a cubic map $f(z) = z^3$ on the upper half-plane $\boldsymbol{H}$ instead of $\boldsymbol{D}^*$. Then, f is not locally injective on the closure $\mathrm{cl}(\boldsymbol{H})$, but $f\,|_{\boldsymbol{R}}$ is a quasiloop since $x \mapsto x^3$ ($x \in \boldsymbol{R}$) extends to a quasiconformal mapping on whole $\widehat{\boldsymbol{C}}$. For instance, we can easily check that the cubic map $x \mapsto x^3$ satsifies the M-condition with $M = 7 + 4\sqrt{3}$. See Chapter IV of [**2**].

On the other hand, since the Schwarzian derivative $\mathscr{S}(f)$ is $-4/z^2$, we have

$$\|\mathscr{S}(f)\|^{\wedge}_{\boldsymbol{H}} = 16,$$

which implies that the quotient norm is larger than the outer radius 6 of $\mathscr{A}\mathscr{T}_{\boldsymbol{1}}$ (see [**18**] and [**19**]). Therefore, the asymptotic class $[\mathscr{S}(f)]$ is not in the image $\mathscr{A}\mathscr{T}_{\boldsymbol{1}}$.

6. Proofs of Corollaries.

In this section, we discuss corollaries of our main theorem. However, Corollary 2 immediately follows from the topological result which tells that the continuous image of a locally connected compact set is again locally connected and compact (cf. Lemma 9.7 of [**20**]). Hence we only check Corollary 1.

A continuous mapping $g : V \to \widehat{\boldsymbol{C}}$ on an open set V is said to be a *finite covering map* if g is locally injective and there is an $N > 0$ such that for all $w \in \hat{\boldsymbol{C}}$, the number of the preimage $g^{-1}(w)$ is at most N (possibly $g^{-1}(w)$ is empty, when w is not in the image).

Let $\varphi \in B(\boldsymbol{1})$ with $[\varphi] \in \mathscr{A}\mathscr{T}_{\boldsymbol{1}}$. Take $w \in \widehat{\boldsymbol{C}}$ in the image of f_φ. By (d) of Theorem 1, f_φ is locally injective on $\mathrm{cl}(\boldsymbol{D}^*)$. Hence, there is an open covering $\{U_k\}_{k=1}^N$ of $\mathrm{cl}(\boldsymbol{D}^*)$ such that $f_\varphi\,|_{U_k}$ is injective on U_k for all $k = 1, \ldots, N$. Therefore, any two points in the preimage $f_\varphi^{-1}(w)$ are contained in different components of $\{U_k\}_{k=1}^N$, and the number of the preimage $f_\varphi^{-1}(w)$ is at most N. Thus, we complete the proof of Corollary 1.

7. Failure of Density.

In this section, we shall prove Theorem 2. The idea of our proof is based on Gehring's result in [**11**]. Figure 2 is a schematic picture which gives locations of Schwarzian derivatives which will be discussed in this section.

7.1. Gehring's spiral domain.

Fix a constant $a \in (0, 1/8\pi)$ and consider the following spiral curves:

$$\beta = \{z = \pm i e^{(-a+i)t} \mid t \in \boldsymbol{R}\} \cup \{0, \infty\}$$
$$\gamma = \beta \cap \{|z| \le 1\}.$$

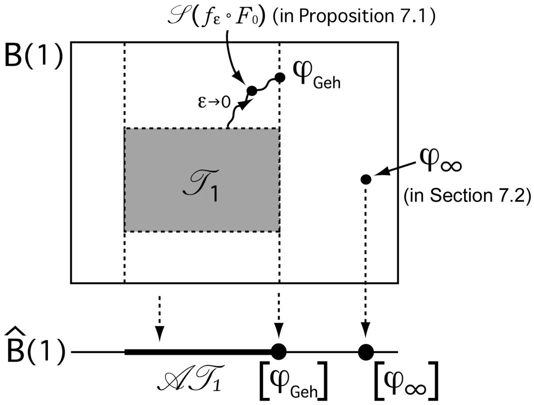

Figure 2. Locations of Schwarzian derivatives in Section 7.

We define the *spiral domain* Ω_{Geh} by $\Omega_{Geh} = \widehat{\boldsymbol{C}} - \gamma$. In [**11**], F. Gehring showed that there is a constant $\delta = \delta(a) > 0$ such that when the norm of the Schwarzian derivative of a conformal mapping h on Ω_{Geh} is less than δ, the image of h is not a Jordan domain, which means that the Schwarzian derivative φ_{Geh} of the Riemann mapping of Ω_{Geh} is not in the closure of the universal Teichmüller space. Meanwhile, we first show the following.

Proposition 7.1. *The asymptotic class* $[\varphi_{Geh}]$ *of* φ_{Geh} *is in the Bers boundary* $\partial_b \mathscr{A}\mathscr{T}_{\boldsymbol{1}}$ *of* $\mathscr{A}\mathscr{T}_{\boldsymbol{1}}$.

Proof. Let $F_0 : \boldsymbol{D}^* \to \Omega_{Geh}$ be the Riemann mapping of Ω_{Geh}. Since $\gamma = \partial\Omega_{Geh}$ is locally connected, F_0 extends continuously to the boundary $\boldsymbol{S}^1 = \partial\boldsymbol{D}^*$. Since $F_0(\boldsymbol{S}^1) = \gamma$, $F_0 \mid_{\boldsymbol{S}^1}$ is not a quasiloop. Therefore, the equivalence class $[\varphi_{Geh}]$ is not contained in $\mathscr{A}\mathscr{T}_1$ by Theorem 1.

Next, we show that the asymptotic class is in the closure. Let $T_0(z) = (z - i)/(z + i)$. For a positive number ϵ with $\epsilon \leq 1/2$, consider a locally univalent function f_ϵ on Ω_{Geh} defined by

$$f_\epsilon(z) = (T_0(z))^{1-\epsilon}$$

where the branch is taken so that $1^{1-\epsilon} = 1$. Around $z = \pm i$, γ is mapped by f_ϵ to a piecewise smooth simple curve, which is a quasiarc. Hence, we have that the restriction of $f_\epsilon \circ F_0$ to $\boldsymbol{S}^1$ is a quasiloop, since γ is a quasiarc connecting $\pm i$. Since F_0 is a conformal map, one can check that $f_\epsilon \circ F_0$ is locally injective on $\mathrm{cl}(\boldsymbol{D}^*)$. Thus, by Theorem 1, the asymptotic class of the Schwarzian derivative $\mathscr{S}(f_\epsilon \circ F_0)$ of $f_\epsilon \circ F_0$ is in the image $\mathscr{A}\mathscr{T}_1$.

By a simple calculation, we have

$$\mathscr{S}(f_\epsilon)(z) = \frac{-i\epsilon(2-\epsilon)}{(z^2+1)^2} = \frac{4\epsilon(2-\epsilon)}{3}\mathscr{S}(f_{1/2})(z)$$

for $z \in \Omega_{Geh}$. Since $T_0(\gamma)$ is contained in the left half-plane, $f_{1/2}(\Omega_{Geh})$ is a Jordan domain, and hence $f_{1/2}$ is univalent. Therefore, the norm of $\mathscr{S}(f_\epsilon)$ on Ω_{Geh} is $O(\epsilon)$. Thus, we conclude that

$$\begin{aligned}\|[\mathscr{S}(f_\epsilon \circ F_0)] - [\varphi_{Geh}]\|^{\wedge}_{\boldsymbol{D}^*} &\leq \|\mathscr{S}(f_\epsilon \circ F_0) - \mathscr{S}(F_0)\|_{\boldsymbol{D}^*} \\ &= \|\mathscr{S}(f_\epsilon)\|_{\Omega_{Geh}} = O(\epsilon),\end{aligned}$$

which indicates what we wanted. □

7.2. Point not in the closure.

We continue to use symbols in the previous section. Let RF be a discrete group of similarities generated by two reflections $z \mapsto 4 - \bar{z}$ and $z \mapsto -4 - \bar{z}$. Define

$$\begin{aligned}\boldsymbol{H}_{-1} &:= \{z \in \boldsymbol{C} \mid \mathrm{Im} z > -1\}, \quad \text{and} \\ \Omega_\infty &:= \boldsymbol{H}_{-1} - \cup_{T \in RF} T(\gamma)\end{aligned}$$

(cf. Figure 3).

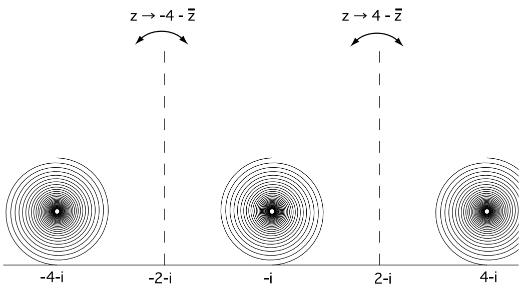

Figure 3. Domain Ω_∞.

Let φ_∞ be the Schwarzian derivative of the Riemann mapping of Ω_∞ from $\boldsymbol{D}^*$. The rest of this section is devoted to the proof of the following proposition, which implies Theorem 2.

PROPOSITION 7.2. *When $a \in (0, 1/8\pi)$, the asymptotic class $[\varphi_\infty]$ of φ_∞ is not in the closure* $\mathrm{cl}(\mathscr{A}\mathscr{T}_{\mathbf{1}})$ *of* $\mathscr{A}\mathscr{T}_{\mathbf{1}}$.

To show Proposition 7.2, we consider two domains D_0 and Ω_0 defined by

$$D_0 = \{z \in \boldsymbol{C} \mid |\mathrm{Re}z| < 2, \mathrm{Im}z > -1\}$$
$$\Omega_0 = D_0 - \gamma.$$

By the similar argument as that by Gehring in [**11**] we can check the following.

Lemma 7.1. *There is a constant $\delta_0 = \delta_0(a) > 0$ such that when the norm of the Schwarzian derivative of a conformal mapping h on Ω_0 is less than δ_0, the image $h(\Omega_0)$ is not a Jordan domain.*

Since D_0 and Ω_0 are simply connected domains and the impression of the prime end at each $\pm 2 - i$ and ∞ consists of one point for both domains D_0 and Ω_0, there is a conformal mapping g_0 from D_0 onto Ω_0 which fixes $\pm 2 - i$ and ∞. By Schwarz's reflection principle, we can extend g_0 as a Riemann mapping from $\boldsymbol{H}_{-1}$ onto Ω_∞. We denote the extension by the same symbol g_0. By definition, g_0 satisfies $T \circ g_0 = g_0 \circ T$ for all $T \in RF$.

Let $T_1(z) = 2i/(z-1)$ and

$$h_\infty(z) = g_0 \circ T_1(z).$$

Then, $T_1(\boldsymbol{D}^*) = \boldsymbol{H}_{-1}$ and h_∞ is a Riemann mapping from $\boldsymbol{D}^*$ onto Ω_∞ with $h_\infty(1) = \infty$. By definition, $\mathscr{S}(h_\infty) = \varphi_\infty$.

Proof of Proposition 7.2. Suppose to the contrary that the asymptotic class $[\varphi_\infty]$ is contained in the closure of $\mathscr{A}\mathscr{T}_{\boldsymbol{1}}$. Then there are sequences $\{E_n\}_{n=1}^\infty$ of quasidisks and $\{h_n\}_{n=1}^\infty$ of conformal mappings $h_n : \boldsymbol{D}^* \to E_n$ such that

$$\|[\mathscr{S}(h_n)] - [\varphi_\infty]\|^\wedge_{\boldsymbol{D}^*} \leq 1/n.$$

By definition, there is a compact set C_n in $\boldsymbol{D}^*$ such that

$$\lambda_{\boldsymbol{D}^*}(z)^{-2} |\mathscr{S}(h_n)(z) - \varphi_\infty(z)| \leq 2/n \tag{7.1}$$

for all $z \in \boldsymbol{D}^* - C_n$. Take $m = m(n) \in \boldsymbol{Z}$ such that $T_1(C_n) \cap T_2(D_0) = \emptyset$ where $T_2(w) = w + 8m \in RF$. Since $T_1^{-1}(T_2(D_0)) \subset \boldsymbol{D}^*$ and $g_0 \circ T_2 = T_2 \circ g_0$, by (7.1) we have

$$\begin{aligned}\|\mathscr{S}(h_n \circ h_\infty^{-1} \circ T_2)\|_{\Omega_0} &= \|\mathscr{S}(h_n) - \mathscr{S}(h_\infty)\|_{T_1^{-1} \circ T_2(D_0)} \\ &\leq \sup_{z \in \boldsymbol{D}^* - C_n} \lambda_{\boldsymbol{D}^*}(z)^{-2} |\mathscr{S}(h_n)(z) - \varphi_\infty(z)| \\ &\leq 2/n.\end{aligned}$$

On the other hand, since $h_n(\boldsymbol{D}^*) = E_n$ is a quasidisk (a Jordan domain), $h_n \circ h_\infty^{-1} \circ T_2(\Omega_0) = h_n(T_1^{-1}(T_2(D_0)))$ is a Jordan domain. This contradicts Lemma 7.1 when n is sufficiently large. □

8. Points in the closure of $\mathscr{A}\mathscr{T}_{\mathbf{1}}$.

In this section, we prove Theorem 3. Throughout this section, for $\varphi \in B(\mathbf{1})$, we regard the image $\Sigma_\varphi := f_\varphi(\boldsymbol{D}^*)$ of the developing mapping f_φ associated to φ as a Riemann surface spread over $\widehat{\boldsymbol{C}}$ with a canonical projection $\mathrm{pr}_\varphi : \Sigma_\varphi \to \widehat{\boldsymbol{C}}$. Let us denote by $\widehat{f}_\varphi$ the biholomorphic mapping from $\boldsymbol{D}^*$ to Σ_φ with $f_\varphi = \mathrm{pr}_\varphi \circ \widehat{f}_\varphi$.

A *univalent disk* B in Σ_φ is a subset such that pr_φ is injective on B and $\mathrm{pr}_\varphi(B)$ is a round disk in $\widehat{\boldsymbol{C}}$. Then, Σ_φ admits the canonical projective structure containing a coordinate system $\{(B, \mathrm{pr}_\varphi|_B)\}_B$, where B runs over all univalent disks in Σ_φ.

For any map g on Σ_φ and a univalent disk B, the composition $g \circ (\mathrm{pr}_\varphi|_B)^{-1}$ is well-defined on the round disk $\mathrm{pr}_\varphi(B) \subset \widehat{\boldsymbol{C}}$. We denote this composition by $g|_B$ to simplify the notation.

As noted in Introduction, our characterization follows from Astala and Gehring [**4**]. Indeed, the proof of necessity is also done by a very similar way.

8.1. Proof of necessity.

Suppose that $[\varphi] \in \mathrm{cl}(\mathscr{A}\mathscr{T}_{\mathbf{1}})$. For any $L > 1$, there is a conformal mapping h on $\boldsymbol{D}^*$ onto a quasidisk such that $\|[\varphi] - [\mathscr{S}(h)]\|^{\wedge}_{\boldsymbol{D}^*} \leq (L-1)/(L+1)$. Then, $g = h \circ (f_\varphi)^{-1}$ is conformal (1-quasiconformal) on Σ_φ and satisfies

$$\|[\mathscr{S}(g)]\|^{\wedge}_{\Sigma_\varphi} = \|[\varphi] - [\mathscr{S}(h)]\|^{\wedge}_{\boldsymbol{D}^*} \leq (L-1)/(L+1), \tag{8.1}$$

where $\mathscr{S}(g)$ is the Schwarzian derivative of g with respect to the canonical projective structure on Σ_φ given above, and $\|\cdot\|^{\wedge}_{\Sigma_\varphi}$ means the quotient norm of holomorphic quadratic differential on Σ_φ. By definition, there is a compact set C in Σ_φ such that

$$\lambda_{\Sigma_\varphi}(p)^{-2} |\mathscr{S}(g)(p)| \leq 2(L-1)/(L+1)$$

for all $p \in \Sigma_\varphi - C$, where λ_{Σ_φ} is the hyperbolic metric on Σ_φ.

Let B be a univalent disk in $\Sigma_\varphi - C$. Then, by Schwarz lemma, we have

$$\lambda_B(p)^{-2}|\mathscr{S}(g)(p)| \leq \lambda_{\Sigma_\varphi}(p)^{-2}|\mathscr{S}(g)(p)| \leq 2(L-1)/(L+1)$$

for all $p \in B$. Since the Schwarzian derivative $\mathscr{S}(g)$ is defined under the canonical projective structure above,

$$\lambda_{\mathrm{pr}_\varphi(B)}(z)^{-2}|\mathscr{S}(g\mid_B)(z)| = \lambda_B(p)^{-2}|\mathscr{S}(g)(p)|$$

for all $z \in \mathrm{pr}_\varphi(B)$ and $p = (\mathrm{pr}_\varphi\mid_B)^{-1}(z)$. Hence, by Ahlfors-Weill theorem (cf. Theorem II.5.1 of [**15**]), $g\mid_B\colon \mathrm{pr}_\varphi(B) \to \widehat{\boldsymbol{C}}$ admits an L-quasiconformal extension on $\widehat{\boldsymbol{C}}$, which implies the necessity.

8.2. Proof of sufficiency.

8.2.1. A lemma.

To simplify notation and calculations, we consider f_φ as a locally univalent function of $\boldsymbol{D}$ by composing f_φ and $z \mapsto 1/z$. Hence we recognize φ as a holomorphic function on $\boldsymbol{D}$.

To show the sufficiency, we begin with the following lemma.

Lemma 8.1. *For $\varphi \in B(\boldsymbol{D})$ and $0 < r < 1$, the following inequality holds*:

$$\|[\varphi]\|^{\wedge}_{\boldsymbol{D}} \leq 2 \limsup_{|z|\to 1}(1-|z|^2)^2|\varphi(z)|.$$

Proof. Fix $R < 1$ and let $\varphi_R(z) = \varphi(Rz)$. Notice that $\varphi_R \in B_0(\boldsymbol{D})$ since $\varphi_R(z)$ is holomorphic on $\mathrm{cl}(\boldsymbol{D})$. The maximal principle tells us that

$$\max_{|z|=r}|\varphi_R(z)| = \max_{|z|=Rr}|\varphi(z)| \leq \max_{|z|=r}|\varphi(z)|$$

holds for all $r < 1$. Thus we have

$$\begin{aligned}
\|[\varphi]\|^{\wedge}_{\boldsymbol{D}} &= \|[\varphi - \varphi_R]\|^{\wedge}_{\boldsymbol{D}} \leq \|\varphi - \varphi_R\|_{\boldsymbol{D}} \\
&\leq \sup_{|z|\leq 1-\delta}(1-|z|^2)^2|\varphi(z)-\varphi_R(z)| + \sup_{1-\delta<|z|<1}(1-|z|^2)^2|\varphi(z)-\varphi_R(z)| \\
&\leq \sup_{|z|\leq 1-\delta}(1-|z|^2)^2|\varphi(z)-\varphi_R(z)| + \sup_{1-\delta<|z|<1}(1-|z|^2)^2(|\varphi(z)|+|\varphi_R(z)|) \\
&\leq \sup_{|z|\leq 1-\delta}(1-|z|^2)^2|\varphi(z)-\varphi_R(z)| + 2\sup_{1-\delta<|z|<1}(1-|z|^2)^2|\varphi(z)|. \qquad (8.2)
\end{aligned}$$

When δ is fixed and $R \to 1$, the first term of (8.2) tends to zero. Therefore, we get the desired estimate. $\square$

8.2.2. Proof of sufficiency.

Fix $\varphi \in B(\mathbf{1})$ with the sufficient condition in the theorem, and take any $\epsilon > 0$. Let $K, L > 1$ be constants satisfying that

$$12288(KL-1)/(KL+1) < \epsilon. \tag{8.3}$$

Then, there are a K-quasiconformal mapping g_1 on Σ_φ onto a quasidisk and a compact set C with the conditions in Theorem 3. By the measurable Riemann mapping theorem, there is a K-quasiconformal mapping g_2 on $\widehat{\boldsymbol{C}}$ such that $h := g_2 \circ g_1$ is conformal on Σ_φ. Notice that for any univalent disk B in the complement $\Sigma_\varphi - C$, $h\mid_B$ admits a KL-quasiconformal extension on $\widehat{\boldsymbol{C}}$. Furthermore, since $h(\Sigma_\varphi) = h \circ f_\varphi(\boldsymbol{D})$ is a quasidisk, $\psi := \mathscr{S}(h \circ f_\varphi) \in \mathscr{T}_{\mathbf{1}}$.

By Nehari's theorem, f_φ is univalent on any hyperbolic disks in $\boldsymbol{D}$ with radius $\tanh^{-1}(\rho)$, where $\rho > 0$ depends only on $\|\varphi\|_{\boldsymbol{D}}$ (cf. [**14**]). Take $r_0 > 0$ such that C does not intersect the hyperbolic $\tanh^{-1}(\rho)$-neighborhood of $f_\varphi(\{r_0 < |z| < 1\})$. Let $z_0 \in \boldsymbol{D}$ with $|z| > r_0$. Set $\Delta(z_0) = \{z \in \boldsymbol{D} \mid |z - z_0| < \rho(1 - |z_0|)\}$. We claim

Claim. *$\Delta(z_0)$ is contained in the hyperbolic disk of center z_0 and radius $\tanh^{-1}(\rho)$.*

Proof of claim. Indeed, let $z \in \Delta(z_0)$. Then, $|z - z_0| < \rho(1 - |z_0|) < \rho|1 - \overline{z_0}z|$ and hence the hyperbolic distance $\tanh^{-1}(|z - z_0|/|1 - \overline{z_0}z|)$ between z_0 and z is less than $\tanh^{-1}(\rho)$. $\square$

Without loss of generality, we may assume that $f_\varphi(\Delta(z_0)) \subset \boldsymbol{C}$. By Koebe 1/4-theorem, $f_\varphi(\Delta(z_0))$ contains a disk $B' := \{|w - w_0| < \rho|f_\varphi'(z_0)|(1 - |z_0|)/4\}$, where $w_0 = f_\varphi(z_0)$. Furthermore, from the claim above, B' can be regarded as a univalent disk on Σ_φ with $B' \cap C = \emptyset$. Hence, $h\mid_{B'}$ extends to a KL-quasiconformal mapping on $\widehat{\boldsymbol{C}}$ as noted before. Therefore, we have

$$\|\mathscr{S}(h\mid_{B'})\|_{B'} \le 6\frac{KL-1}{KL+1}. \tag{8.4}$$

Let $B = (f_\varphi\mid_{\Delta(z_0)})^{-1}(B') \subset \boldsymbol{D}$ and δ denote the Euclidean distance from z_0 to ∂B. Then, $\lambda_B(z_0) \le 1/\delta$ by Schwarz lemma. By applying the Koebe 1/4 theorem for $(f_\varphi^{-1})\mid_{B'}$, $B = (f_\varphi\mid_{\Delta(z_0)})^{-1}(B')$ contains a Euclidean disk of center z_0 and radius

$$\big(|(f_\varphi^{-1})'(w_0)| \cdot \rho|f_\varphi'(z_0)|(1 - |z_0|)/4\big)/4 = \rho(1 - |z_0|)/16 \ge \rho(1 - |z_0|^2)/32.$$

Therefore, we obtain

$$\lambda_B(z_0)^{-1} \geq \delta \geq \rho(1-|z_0|^2)/32$$

and

$$\|\varphi-\psi\|_B \geq \lambda_B(z_0)^{-2}|\varphi(z_0)-\psi(z_0)| \geq \frac{\rho^2}{1024}(1-|z_0|^2)^2|\varphi(z_0)-\psi(z_0)|. \quad (8.5)$$

Since z_0 is chosen arbitrarily in $\{r_0 < |z| < 1\}$, from (8.3), (8.4) and (8.5) and Lemma 8.1, we conclude that

$$\begin{aligned}
\|[\varphi]-[\psi]\|^{\wedge}_{\boldsymbol{D}} &\leq 2 \limsup_{|z|\to 1}(1-|z|^2)^2|\varphi(z)-\psi(z)| \\
&\leq 2 \sup_{r_0<|z|<1}(1-|z|^2)^2|\varphi(z)-\psi(z)| \\
&\leq 2\cdot\frac{1024}{\rho^2}\|\varphi-\mathscr{S}(h\circ f_\varphi)\|_B \\
&= \frac{2048}{\rho^2}\|\mathscr{S}(h)\|_{B'} \leq \frac{12288}{\rho^2}\frac{KL-1}{KL+1} < \epsilon/\rho^2.
\end{aligned}$$

Since $\psi = \mathscr{S}(h\circ f_\varphi) \in \mathscr{T}_{\mathbf{1}}$ and ρ depends only on $\|\varphi\|_{\boldsymbol{D}}$, the asymptotic class $[\varphi]$ is in the closure $\mathrm{cl}(\mathscr{A}\mathscr{T}_{\mathbf{1}})$ of $\mathscr{A}\mathscr{T}_{\mathbf{1}}$ in $\hat{B}(\mathbf{1})$. □

By the same argument as in the proof of Theorem 3, we also have the following characterization, which is intuitively comprehensible.

Theorem 4. *For $\varphi \in B(\mathbf{1})$, the equivalence class $[\varphi]$ of φ is in the closure $\mathrm{cl}(\mathscr{A}\mathscr{T}_{\mathbf{1}})$ of $\mathscr{A}\mathscr{T}_{\mathbf{1}}$ in $\hat{B}(\mathbf{1})$ if and only if for any $K, L > 1$, there are a K-quasiconformal mapping g on $\boldsymbol{D}^*$ onto a quasidisk Ω and a compact set C in Ω such that for any round disk B in $\Omega - C$, the restriction $(f_\varphi\circ g^{-1})\,|_B\colon B \to \widehat{\boldsymbol{C}}$ admits an L-quasiconformal extension on $\widehat{\boldsymbol{C}}$.*

9. Open problems.

As the conclusion, we give open problems concerning results in this paper.

9.1. Characterization of interior.

Proposition 1.1 gives a necessary condition of the Schwarzian derivative such that the asymptotic class is in the interior of the image of asymptotic Bers map for arbitrary Fuchsian group Γ. However, it is open whether *the quasiconformal* *extension property at ends is sufficient.*

9.2. Density problem.

In this paper we disproved the density conjecture for the asymptotic Teichmüller space of the unit disk. We note that the density problem is recently modified as follows.

Modified Density Problem. Characterize Fuchsian groups Γ with the property that $\mathscr{A}\mathscr{T}_\Gamma$ (resp. $\mathscr{T}_\Gamma$) is dense in $\hat{S}(\Gamma)$ (resp. $S(\Gamma)$).

Indeed, T. Sugawa [**21**] extended Gehring's result to the case of Teichmüller spaces of Fuchsian groups of the second kind. K. Matsuzaki [**17**] also solved the density problem in the negative for some (infinitely generated) Fuchsian group of the first kind. On the contrary, by virtue of the ending lamination theorem by J. Brock, R. Canary and Y. Minsky [**6**], the density problem is solved in the affirmative for the Teichmüller spaces of finitely generated Fuchsian groups of the first kind.

However, a basic problem also remains. Indeed, it is open *whether* $\hat{S}(\Gamma)$ *is closed or not in* $\hat{B}(\Gamma)$ (*even in the case where* $\Gamma = \mathbf{1}$), while $S(\Gamma)$ is closed. This basic problem will be important for studying the degenerations of asymptotic classes and the boundaries of the asymptotic Teichmüller spaces.

References

[1] L. V. Ahlfors, Quasiconformal reflections, Acta Math., **109** (1963), 291–301.

[2] L. V. Ahlfors, Lectures on quasiconformal mappings, Second edition, With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard, University Lecture Series, **38**, American Mathematical Society, 2006.

[3] K. Astala, Selfsimilar zippers, In Holomorphic Functions and Moduli I, **10**, Springer-Verlag: MSRI publications, 1988, pp. 61–73.

[4] K. Astala and F. Gehring, Crickets, Zippers, and the Bers universal Teichmüller space, Proc. Amer. Math. Soc., **110** (1990), 675–687.

[5] L. Bers, A non-standard integral equation with applications to quasiconformal mappings, Acta Math., **116** (1966), 113–134.

[6] J. Brock, D. Canary and Y. Minsky, The classification of Kleinian surface groups II, The Ending Lamination Conjecture, preprint (2005).

[7] C. J. Earle, F. Gardiner and N. Lakic, Asymptotic Teichmüller space, Part I, The complex structure, Comtemp. Math., **256** (2000), 17–38.

[8] C. J. Earle, V. Markovic and D. Saric, Barycentric extension and the Bers embedding for asymptotic Teichmüller space, Contemp. Math., **311** (2002), 87–105.

[9] F. Gardiner and N. Lakic, Quasiconformal Teichmüller theory, Mathematical Surveys and Monograph, **76**, American Mathematical Society, Providence, RI, 2000.

[10] F. Gardiner and D. Sullivan, Symmetric structures on a closed curve, Amer. J. Math., **114** (1992), 683–736.

[11] F. W. Gehring, Spirals and the universal Teichmüller space, Acta Math., **141** (1978),

99–113.

[12] Y. Imayoshi and M. Taniguchi, An introduction to Teichmüller spaces, Springer-Verlag, Tokyo, 1992.

[13] I. Kra, Automorphic forms and Kleinian groups, Mathematics Lecture Note Series, W. A. Benjamin, Inc., Reading, Mass., 1972.

[14] I. Kra and B. Maskit, Remarks on projective structures. Riemann surfaces and related topics, Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), Ann. of Math. Stud., **97**, Princeton Univ. Press, Princeton, N.J., 1981, pp. 343–359.

[15] O. Lehto, Univalent functions and Teichmüller spaces, Graduate Texts in Mathematics, **109**, Springer-Verlag, New York-Heidelberg, 1986.

[16] O. Lehto and K. I. Virtanen, Quasiconformal mappings in the plane, Second edition, Springer-Verlag, New York-Heidelberg, 1973.

[17] K. Matsuzaki, Simply connected invariant domains of Kleinian groups not in the closures of Teichmüller spaces, Complex Variables Theory Appl., **22** (1993), 93–100.

[18] H. Miyachi, On invariant distances on asymptotic Teichmüller spaces, Proc. Amer. Math. Soc., **134** (2006), 1917–1925.

[19] H. Miyachi, The inner and outer radii of asymptotic Teichmüller spaces, Complex Var. Elliptic Equ., **53** (2008), 139–158.

[20] Ch. Pommerenke, Univalent functions, Studia Mathematica/Mathematische Lehrbücher, BandXXV, Vandenhoeck & Ruprecht, Göttingen, 1975.

[21] T. Sugawa, On the Bers conjecture for Fuchsian groups of the second kind, J. Math. Kyoto Univ., **32** (1992), 45–52.

[22] W. P. Thurston, Zippers and univalent functions, The Bieberbach conjecture, Math. Surveys Monogr., **21** (1986), 185–197.

[23] P. Tukia, The planar Schönflies theorem for Lipschitz maps, Ann. Acad. Sci. Fenn. A.I. Math., **5** (1980), 49–72.

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