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Limits of characters of wreath products $\mathfrak{S}_n(T)$ of a compact group T with the symmetric groups and characters of $\mathfrak{S}_\infty(T)$, II
From a viewpoint of probability theory

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Abstract. This paper is the second part of our study on limiting behavior of characters of wreath products $\mathfrak{S}_n(T)$ of compact group T as $n \to \infty$ and its connection with characters of $\mathfrak{S}_\infty(T)$. Contrasted with the first part, which has a representation-theoretical flavor, the approach of this paper is based on probabilistic (or ergodic-theoretical) methods. We apply boundary theory for a fairly general branching graph of infinite valencies to wreath products of an arbitrary compact group T. We show that any character of $\mathfrak{S}_\infty(T)$ is captured as a limit of normalized irreducible characters of $\mathfrak{S}_n(T)$ as $n \to \infty$ along a path on the branching graph of $\mathfrak{S}_\infty(T)$. This yields reconstruction of an explicit character formula for $\mathfrak{S}_\infty(T)$.

Introduction.

In the present paper, we discuss the connection between limits of irreducible characters of wreath products of a compact group with symmetric groups and characters of its wreath product with the infinite symmetric group, taking an alternative route of [**8**] (Part I).

Wreath product group $\mathfrak{S}_n(T)$ of compact group T with the symmetric group $\mathfrak{S}_n$, where $n \in \boldsymbol{N} = \{1, 2, \dots\}$, is defined as $\mathfrak{S}_n(T) = D_n(T) \rtimes \mathfrak{S}_n$. Here $D_n(T) = T^n$ denotes the n-fold direct product of T. The action of $\sigma \in \mathfrak{S}_n$ on $D_n(T)$ is defined by

$$\sigma : d = (t_i) \longmapsto \sigma(d) = (t_{\sigma^{-1}(i)}).$$

Similarly, we consider $\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$ where

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$$D_\infty(T) = \{d = (t_i)_{i \in \boldsymbol{N}} \mid t_i \in T, t_i = e_T \text{ except for finite } i\text{'s}\},$$

$$\mathfrak{S}_\infty = \{\text{permutation } \sigma \text{ of } \boldsymbol{N} \mid \sigma(i) = i \text{ except for finite } i\text{'s}\},$$

e_T being the identity element of T. $\mathfrak{S}_\infty(T)$ is an inductive limit of $\mathfrak{S}_n(T)$. Equipped with its inductive limit topology, $\mathfrak{S}_\infty(T)$ is a topological group, which is no longer locally compact if T is continuous.

A probabilistic or ergodic method for describing the characters of $\mathfrak{S}_\infty$ was first developed by Vershik-Kerov [**17**]. The essential idea is to translate properties of the characters into those of probability measures on the path space of the Young graph, which is the branching graph of $\mathfrak{S}_\infty$. Developing this method due to Vershik-Kerov to the wreath product group $\mathfrak{S}_\infty(T)$ for any compact group T, we show the following results in this paper.

- Every character of $\mathfrak{S}_\infty(T)$ is described as a limit of normalized irreducible characters of $\mathfrak{S}_n(T)$ as $n \to \infty$.
- The classifying parameters for characters of $\mathfrak{S}_\infty(T)$ are expressed by rescaled limits of families of Young diagrams indexed by $\zeta \in \widehat{T}$.
- As a consequence we recapture the character formula for $\mathfrak{S}_\infty(T)$.

We fully use structure of the branching graph of $\mathfrak{S}_\infty(T)$. Reflecting the effect of compact group T, the graph naturally allows infinite valencies. We note that, for finite group T, such a character theory for wreath product groups was developed by Boyer [**2**].

This paper is organized as follows. In Section 1 we review fundamental facts on irreducible representations and their characters of the wreath product $\mathfrak{S}_n(T)$, including their branching rules. Section 2 and Section 3 are devoted to developing some materials in boundary theory of a general branching graph. In Section 4, applying these to our case of wreath product groups, we prove the above mentioned results.

1. Irreducible representations and the branching rule for $\mathfrak{S}_n(T)$.

In this section we briefly review the irreducible representations, the irreducible characters and the branching rule for $\mathfrak{S}_n(T)$.

1.1. Irreducible representations of $\mathfrak{S}_n(T)$.

Let T be an arbitrary compact group and $\widehat{T}$ denote the set of equivalence classes of *continuous irreducible unitary representations* (IURs). The equivalence class of IUR ζ of T is denoted by $[\zeta]$. For simplicity, however, we often use the notation like $\zeta \in \widehat{T}$ for IUR ζ. The equivalence classes of IURs of wreath product $G_n = \mathfrak{S}_n(T)$ are parametrized by

$$\boldsymbol{Y}_n(T) = \Big\{ \Lambda = (\lambda^\zeta)_{\zeta\in\widehat{T}} \;\Big|\; \lambda^\zeta \in \boldsymbol{Y},\ \sum_{\zeta\in\widehat{T}} |\lambda^\zeta| = n \Big\}. \tag{1.1}$$

Here $\boldsymbol{Y}$ denotes the set of all Young diagrams. The size (i.e. the number of boxes) of $\lambda \in \boldsymbol{Y}$ is denoted by $|\lambda|$. Thus $\Lambda \in \boldsymbol{Y}_n(T)$ is a map from $\widehat{T}$ to $\boldsymbol{Y}$ which assigns the empty diagram $\varnothing^\zeta$ to almost all ζ with finite exceptions. Construction of an IUR corresponding to $\Lambda \in \boldsymbol{Y}_n(T)$ was given in [**8**, Section 3] (Part I), which we recall below for the sake of convenience. For the case where T is a finite group, see e.g. [**11**, Chapter 4].

Let $\Lambda = (\lambda^\zeta)_{\zeta\in\widehat{T}} \in \boldsymbol{Y}_n(T)$ be arbitrarily given. Pick up a partition of $\{1, 2, \ldots, n\}$ whose block structure agrees with $\{|\lambda^\zeta|\}_{\zeta\in\widehat{T}}$:

$$\{1, 2, \ldots, n\} = \bigsqcup_{\zeta\in\widehat{T}} I_{n,\zeta}\ , \qquad |I_{n,\zeta}| = |\lambda^\zeta|.$$

$I_{n,\zeta}$ is empty except for finite numbers of ζ. According to this partition, we take IUR η of $D_n = D_n(T)$ given as

$$\eta = \boxtimes_{\zeta\in\widehat{T}} \big(\boxtimes_{i\in I_{n,\zeta}} \zeta_i \big) = \boxtimes_{\zeta\in\widehat{T}} \boxtimes_{i\in I_{n,\zeta}} \zeta_i, \qquad \text{where} \quad \zeta_i \equiv \zeta \quad (i \in I_{n,\zeta}).$$

Then the stationary subgroup $S_{[\eta]} = \{\sigma \in \mathfrak{S}_n \mid {}^\sigma\eta \cong \eta\}$ of $[\eta]$ coincides with $\prod_{\zeta\in\widehat{T}} \mathfrak{S}_{I_{n,\zeta}}$. Here $\sigma \in \mathfrak{S}_n$ acts on $\eta \in \widehat{D_n}$ as

$${}^\sigma\eta(d) = \eta(\sigma^{-1}(d)), \quad \sigma^{-1}(d) = (t_{\sigma(i)}) \quad (d = (t_i)_{i\in\{1,2,\ldots,n\}} \in D_n).$$

For $\zeta \in \widehat{T}$, let ρ_ζ be the IUR of $\mathfrak{S}_{I_{n,\zeta}}(T) = D_{I_{n,\zeta}}(T) \rtimes \mathfrak{S}_{I_{n,\zeta}}$ defined by

$$\rho_\zeta((d,\sigma)) = \big(\boxtimes_{i\in I_{n,\zeta}} \zeta_i \big)(d) I(\sigma) \qquad (d \in D_{I_{n,\zeta}}(T),\ \sigma \in \mathfrak{S}_{I_{n,\zeta}})$$

where we set $\zeta_i \equiv \zeta$ for $i \in I_{n,\zeta}$ and

$$I(\sigma) : \bigotimes_{i\in I_{n,\zeta}} v_i \longmapsto \bigotimes_{i\in I_{n,\zeta}} v_{\sigma^{-1}(i)}$$

on $\bigotimes_{i\in I_{n,\zeta}} V(\zeta_i)$, $V(\zeta_i) \equiv V(\zeta)$ being the representation space of IUR ζ of T for $i \in I_{n,\zeta}$. These ρ_ζ's yield an IUR of

$$H_n = D_n(T) \rtimes S_{[\eta]} = D_n(T) \rtimes \prod_{\zeta\in\widehat{T}} \mathfrak{S}_{I_{n,\zeta}} = \prod_{\zeta\in\widehat{T}} \mathfrak{S}_{I_{n,\zeta}}(T)$$

as the outer tensor product $\rho = \boxtimes_{\zeta\in\widehat{T}}\rho_\zeta$ on $V(\eta) = \bigotimes_{\zeta\in\widehat{T}} \bigotimes_{i\in I_{n,\zeta}} V(\zeta_i)$.

Let $\pi(\lambda^\zeta)$ be an IUR of $\mathfrak{S}_{I_{n,\zeta}}$ on $V(\pi(\lambda^\zeta))$ corresponding to Young diagram λ^ζ. Take IUR $\xi = \boxtimes_{\zeta\in\widehat{T}}\pi(\lambda^\zeta)$ of $S_{[\eta]} = \prod_{\zeta\in\widehat{T}} \mathfrak{S}_{I_{n,\zeta}}$ on $V(\xi) = \bigotimes_{\zeta\in\widehat{T}} V(\pi(\lambda^\zeta))$. The normal subgroup $D_n(T)$ acting trivially, ξ is regarded as a representation of the semi-direct product group $H_n = D_n(T) \rtimes S_{[\eta]}$.

Set $\eta\boxdot\xi = \rho\otimes\xi$, which is an IUR of H_n on $V(\eta)\otimes V(\xi)$. The desired IUR $\Pi(\Lambda)$ of G_n corresponding to $\Lambda = (\lambda^\zeta)_{\zeta\in\widehat{T}}$ is thus given by the induced representation

$$\Pi(\Lambda) = \mathrm{Ind}_{H_n}^{G_n} \eta \boxdot \xi.$$

1.2. Irreducible characters of $\mathfrak{S}_n(T)$.

We recall the description of the conjugacy classes of a wreath product group. See [**6**] and also [**8**] (Part I). Every element $g = (d, \sigma) \in G_n = \mathfrak{S}_n(T)$ admits a *standard decomposition*

$$g = \xi_{q_1} \cdots \xi_{q_r} g_1 \cdots g_m \tag{1.2}$$

uniquely determined modulo orders of ξ_q's and of g_j's. Here each ξ_{q_i} has the form $(t_{q_i}, (q_i))$ holding $t_{q_i} \in T$ at a certain position $q_i \in \{1, 2, \ldots, n\}$. The singleton $\{q_i\}$ is called the support of ξ_{q_i} and denoted by $\mathrm{supp}(\xi_{q_i})$. Each g_j has the form (d_j, σ_j) where σ_j is a cycle permutation in $\mathfrak{S}_n$ with length $\ell(\sigma_j) \geq 2$ and d_j holds an element of T at each position of $\mathrm{supp}(\sigma_j)$. Here the set of permuted letters by $\tau \in \mathfrak{S}_n$ is called the support of τ and denoted by $\mathrm{supp}(\tau)$. All the supports $\{q_1\}, \cdots, \{q_r\}, \mathrm{supp}(\sigma_1), \cdots, \mathrm{supp}(\sigma_m)$ are taken to be disjoint.

We use also $\mathrm{supp}(g_j)$ instead of $\mathrm{supp}(\sigma_j)$. Note that σ admits the cycle decomposition $\sigma_1 \cdots \sigma_m$. Each factor in (1.2), ξ_{q_i} or g_j, is called a basic element of G_n.

Let $[t]$ denote the conjugacy class of $t \in T$. When σ_j is expressed as $\sigma_j = (i_{j,1}\ i_{j,2}\ \cdots\ i_{j,\ell_j})$ with $\ell_j = \ell(\sigma_j)$, we set for $d_j = (t_i)_{i\in\mathrm{supp}(\sigma_j)}$

$$P_{\sigma_j}(d_j) = [t_{i_{j,\ell_j}} t_{i_{j,\ell_j-1}} \cdots t_{i_{j,1}}]. \tag{1.3}$$

The conjugacy class $P_{\sigma_j}(d_j)$ in T is well-defined since it does not depend on the cyclic orders of the product. Using these notations, we know that the conjugacy classes of $G_n = \mathfrak{S}_n(T)$ are parametrized by the data

$$[t_{q_i}] \quad (i = 1, \dots, r) \quad \text{and} \quad (P_{\sigma_j}(d_j), \ell(\sigma_j)) \quad (j = 1, \dots, m)$$

under the standard decomposition (1.2). To visualize this parametrization, we may assign a color to each conjugacy class of T, the identity element e_T being white (= non-colored). Then, a conjugacy class of $\mathfrak{S}_n(T)$ is indicated by a family of Young diagrams

$$\mathrm{P} = (\rho_\theta \mid \theta : \text{ color } (\leftrightarrow \text{ conjugacy class of } T)), \qquad \sum_\theta |\rho_\theta| = n,$$

by putting together the cycles of color θ to form ρ_θ.

Example 1.1. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 1 & 2 & 5 \end{pmatrix} = (1\ 4)(2\ 6\ 5)(3) \quad \in \mathfrak{S}_6,$$

$d = (a_1, b_1, c_1, a_2, b_2, b_3) \in D_6(T)$ and $g = (d, \sigma) \in \mathfrak{S}_6(T)$. Then we have $g = \xi_1 g_1 g_2$ where $\xi_1 = (c_1, (3)), g_1 = (d_1, \sigma_1), g_2 = (d_2, \sigma_2)$ with $d_1 = (a_1, a_2), d_2 = (b_1, b_2, b_3), \sigma_1 = (1\ 4), \sigma_2 = (2\ 6\ 5)$. Here $\ell(\sigma_1) = 2$, $\ell(\sigma_2) = 3$. We have $[c_1]$, $P_{\sigma_1}(d_1) = [a_2 a_1]$ and $P_{\sigma_2}(d_2) = [b_2 b_3 b_1]$ as colors.

Example 1.2. Consider $\mathfrak{S}_6(\boldsymbol{T})$ where $\boldsymbol{T} = \{z \in \boldsymbol{C} \mid |z| = 1\}$ is a one-dimensional torus. The set of colors is $\boldsymbol{T}$ itself. Moreover, the order of product in (1.3) is meaningless. For $g = (d, \sigma)$ in Example 1.1, let $a_1 a_2 = c_1 = 1$ and $b_1 b_2 b_3 = \sqrt{-1}$. Then, as its conjugacy class, we have a family of Young diagrams $\mathrm{P} = (\rho_\theta)$ where

$$\rho_1 = (1^1 2^1), \quad \rho_{\sqrt{-1}} = (3^1), \quad \text{and} \quad \rho_\theta = \varnothing\ (\theta \neq 1, \sqrt{-1}).$$

The character of an IUR of G_n corresponding to $\Lambda \in \boldsymbol{Y}_n(T)$ described in Subsection 1.1 were computed in [**8**, Section 4] (Part I) by using the induced character formula. We review the result below. See [**8**, Theorem 4.5].

Let $\Pi(\Lambda) = \mathrm{Ind}_{H_n}^{G_n} \eta \boxdot \xi$ be the IUR of G_n corresponding to $\Lambda = (\lambda^\zeta)_{\zeta \in \widehat{T}} \in \boldsymbol{Y}_n(T)$ as constructed in Subsection 1.1. The character of $\Pi(\Lambda)$ is denoted by $\chi_{\Pi(\Lambda)}$ or simply χ^Λ. Then the normalized character is

$$\widetilde{\chi}_{\Pi(\Lambda)} = \widetilde{\chi}^\Lambda = \frac{\chi_{\Pi(\Lambda)}}{\dim \Pi(\Lambda)}.$$

Since $\Pi(\Lambda)$ is induced from a representation of H_n, we see $\chi^\Lambda(g) = 0$ if $g \in G_n$

is not conjugate to an element of H_n. Let us write down a formula for $\chi^\Lambda(g)$ assuming that $g=(d,\sigma)\in G_n$ is conjugate to an element of H_n. Take a standard decomposition of g as in (1.2). Set $Q=\{q_1,\dots,q_r\}$ and $J=\{1,\dots,m\}$. We call $\mathscr{Q}=(Q_\zeta)_{\zeta\in\widehat{T}}$ and $\mathscr{J}=(J_\zeta)_{\zeta\in\widehat{T}}$ partitions of Q and J respectively if they yield disjoint unions

$$Q=\bigsqcup_{\zeta\in\widehat{T}}Q_\zeta\quad\text{and}\quad J=\bigsqcup_{\zeta\in\widehat{T}}J_\zeta.$$

The common value of the character of IUR $\pi(\lambda^\zeta)$ of $\mathfrak{S}_{I_{n,\zeta}}$ on the conjugacy class determined by partition $(\ell_j)=(\ell_1,\ell_2,\dots)$ is denoted by $\chi(\lambda^\zeta,(\ell_j))$. Similarly $\widetilde{\chi}(\lambda^\zeta,(\ell_j))$ is the normalized one. Recall that $\ell(\sigma_j)$ denotes the cardinality of $\mathrm{supp}(\sigma_j)$ (i.e. the length of σ_j) for each j. Under these notations, we have

$$\begin{aligned}\chi^\Lambda(g)=\sum_{\mathscr{Q},\mathscr{J}}&\frac{\big(n-\sum_{\zeta\in\widehat{T}}|Q_\zeta|-\sum_{j\in J}\ell(\sigma_j)\big)!}{\prod_{\zeta\in\widehat{T}}\big(|I_{n,\zeta}|-|Q_\zeta|-\sum_{j\in J_\zeta}\ell(\sigma_j)\big)!}\\&\times\prod_{\zeta\in\widehat{T}}\Bigg\{(\dim\zeta)^{|I_{n,\zeta}|-|Q_\zeta|-\sum_{j\in J_\zeta}\ell(\sigma_j)}\Bigg(\prod_{q\in Q_\zeta}\chi_\zeta(t_q)\Bigg)\Bigg(\prod_{j\in J_\zeta}\chi_\zeta(P_{\sigma_j}(d_j))\Bigg)\\&\qquad\times\chi\big(\lambda^\zeta,(\ell(\sigma_j))_{j\in J_\zeta}\big)\Bigg\}\end{aligned}\tag{1.4}$$

where $\mathscr{Q}=(Q_\zeta)_{\zeta\in\widehat{T}}$ and $\mathscr{J}=(J_\zeta)_{\zeta\in\widehat{T}}$ run over all the partitions of Q and J respectively. Note also that we adopt the notational convention of $1/(-k)!(=1/\Gamma(-k+1))=0$ for positive integer k. The normalized character $\widetilde{\chi}^\Lambda$ is obtained by dividing (1.4) by $\dim\Pi(\Lambda)$:

$$\chi^\Lambda(g)=\frac{n!}{\prod_{\zeta\in\widehat{T}}|I_{n,\zeta}|!}\prod_{\zeta\in\widehat{T}}\big\{(\dim\zeta)^{|I_{n,\zeta}|}\dim\lambda^\zeta\big\}\widetilde{\chi}^\Lambda(g).\tag{1.5}$$

Remark 1.3. For IUR $\pi(\lambda^\zeta)$ of $\mathfrak{S}_{I_{n,\zeta}}$ and partition $(\ell_j)_{j\in J_\zeta}$, $\chi(\lambda^\zeta,(\ell_j)_{j\in J_\zeta})$ may be expressed alternatively by $\chi^{\lambda^\zeta}_{(\tau,1^{n_\zeta-|\tau|})}$, where we set $n_\zeta=|I_{n,\zeta}|=|\lambda^\zeta|$ and τ is the Young diagram indicating (ℓ_j) such that $|\tau|=\sum_{j\in J_\zeta}\ell_j$. Equation (1.4) remains valid when Q or J is empty, in particular when g is the identity element. In the case of J_ζ is empty, we have

$$\chi\big(\lambda^\zeta,(\ell(\sigma_j))_{j\in\varnothing^\zeta}\big)=\chi^{\lambda^\zeta}_{(1^{n_\zeta})}=\dim\lambda^\zeta.$$

Here $\varnothing^\zeta$ is the empty diagram assigned to ζ.

1.3. Branching rule for $\mathfrak{S}_n(T)$'s.

$\mathfrak{S}_n$ is embedded into $\mathfrak{S}_{n+1}$ as the permutations fixing the letter $n+1$, while $D_n(T)$ is embedded into $D_{n+1}(T)$ with $e_T \in T$ as the last entry. This yields embedding $G_n = \mathfrak{S}_n(T) \subset G_{n+1} = \mathfrak{S}_{n+1}(T)$. For $\Lambda = (\lambda^\zeta)_{\zeta\in\widehat{T}} \in \boldsymbol{Y}_n(T)$ and $\mathrm{M} = (\mu^\zeta)_{\zeta\in\widehat{T}} \in \boldsymbol{Y}_{n+1}(T)$, we use the notation $\Lambda \nearrow \mathrm{M}$ if there exists $\zeta \in \widehat{T}$ such that $\lambda^\zeta \nearrow \mu^\zeta$. Here the latter NE-arrow means that Young diagram μ^ζ is obtained by adding one box to Young diagram λ^ζ. This ζ is uniquely determined for such a pair (Λ, M) and hence denoted by $\zeta_{\Lambda,\mathrm{M}}$.

PROPOSITION 1.4. *Let* $\mathrm{M} \in \boldsymbol{Y}_{n+1}(T)$*. Restricted on* G_n*,* IUR $\Pi(\mathrm{M})$ *of* G_{n+1} *has irreducible decomposition*

$$\Pi(\mathrm{M})\big|_{G_n} \;\cong \bigoplus_{\Lambda\in\boldsymbol{Y}_n(T);\Lambda\nearrow\mathrm{M}} [\dim\zeta_{\Lambda,\mathrm{M}}]\,\Pi(\Lambda).$$

PROOF. Instead of looking into detailed structure of the irreducible decomposition, we show the assertion by using the character formula in Subsection 1.2. In other words, we just verify

$$\chi^{\mathrm{M}}\big|_{G_n} = \sum_{\Lambda\in\boldsymbol{Y}_n(T);\Lambda\nearrow\mathrm{M}} (\dim\zeta_{\Lambda,\mathrm{M}})\;\chi^\Lambda. \tag{1.6}$$

Equation (1.4) together with an obvious identity for multinomial coefficients:

$$\frac{n!}{n_1!\cdots n_p!} = \sum_{k=1}^{p} \frac{(n-1)!}{n_1!\cdots(n_k-1)!\cdots n_p!} \qquad \text{for} \quad \sum_{k=1}^{p} n_k = n$$

yields the following. Let $g \in G_n$ have a standard decomposition as (1.2). We use the notations in (1.2) and (1.4), setting further $\ell(\sigma_j) = \ell_j$ and $P_{\sigma_j}(d_j) = P_j$ for simplicity. Let $\mathrm{M} = (\mu^\zeta)_{\zeta\in\widehat{T}} \in \boldsymbol{Y}_{n+1}(T)$. We have for $(\chi^{\mathrm{M}}\big|_{G_n})(g) = \chi^{\mathrm{M}}(g)$,

$$\begin{aligned}
&\chi^{\mathrm{M}}(g)\\
&= \sum_{\mathscr{Q},\mathscr{J}} \frac{\left(n-\sum_{\zeta\in\widehat{T}}|Q_\zeta| - \sum_{j\in J}\ell_j\right)!}{\prod_{\zeta\in\widehat{T}}\left(|I_{n,\zeta}| - |Q_\zeta| - \sum_{j\in J_\zeta}\ell_j\right)!}\\
&\quad\times \prod_{\zeta\in\widehat{T}} \left\{ (\dim\zeta)^{|I_{n,\zeta}|-|Q_\zeta|-\sum_{j\in J_\zeta}\ell_j} \prod_{q\in Q_\zeta}\chi_\zeta(t_q) \prod_{j\in J_\zeta}\chi_\zeta(P_j)\chi\big(\mu^\zeta,(\ell_j)_{j\in J_\zeta}\big)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mathscr{Q},\mathscr{J}} \Biggl\{ \sum_{\zeta\in\widehat{T}} \frac{\bigl(n-1-\sum_{\kappa\in\widehat{T}}|Q_\kappa|-\sum_{j\in J}\ell_j\bigr)!}{\bigl(|I_{n,\zeta}|-1-|Q_\zeta|-\sum_{j\in J_\zeta}\ell_j\bigr)!\prod_{\theta\neq\zeta}\bigl(|I_{n,\theta}|-|Q_\theta|-\sum_{j\in J_\theta}\ell_j\bigr)!} \Biggr\} \\
&\quad \times \prod_{\theta\in\widehat{T}} \Biggl\{ (\dim\theta)^{|I_{n,\theta}|-|Q_\theta|-\sum_{j\in J_\theta}\ell_j} \prod_{q\in Q_\theta}\chi_\theta(t_q) \prod_{j\in J_\theta}\chi_\theta(P_j)\chi\bigl(\mu^\theta,(\ell_j)_{j\in J_\theta}\bigr) \Biggr\} \\
&= \sum_{\mathscr{Q},\mathscr{J}} \Biggl[\sum_{\zeta\in\widehat{T}} \frac{\bigl(n-1-\sum_{\kappa\in\widehat{T}}|Q_\kappa|-\sum_{j\in J}\ell_j\bigr)!}{\bigl(|I_{n,\zeta}|-1-|Q_\zeta|-\sum_{j\in J_\zeta}\ell_j\bigr)!\prod_{\theta\neq\zeta}\bigl(|I_{n,\theta}|-|Q_\theta|-\sum_{j\in J_\theta}\ell_j\bigr)!} \\
&\quad \times \prod_{\theta\in\widehat{T}} \Biggl(\prod_{q\in Q_\theta}\chi_\theta(t_q) \prod_{j\in J_\theta}\chi_\theta(P_j) \Biggr) (\dim\zeta)^{|I_{n,\zeta}|-|Q_\zeta|-\sum_{j\in J_\zeta}\ell_j}\chi\bigl(\mu^\zeta,(\ell_j)_{j\in J_\zeta}\bigr) \\
&\quad \times \prod_{\theta\neq\zeta} \bigl\{ (\dim\theta)^{|I_{n,\theta}|-|Q_\theta|-\sum_{j\in J_\theta}\ell_j}\chi\bigl(\mu^\theta,(\ell_j)_{j\in J_\theta}\bigr) \bigr\} \Biggr] \\
&= \sum_{\zeta\in\widehat{T}} (\dim\zeta) \sum_{\mathscr{Q},\mathscr{J}} \Biggl[\frac{\bigl(n-1-\sum_{\kappa\in\widehat{T}}|Q_\kappa|-\sum_{j\in J}\ell_j\bigr)!}{\bigl(|I_{n,\zeta}|-1-|Q_\zeta|-\sum_{j\in J_\zeta}\ell_j\bigr)!\prod_{\theta\neq\zeta}\bigl(|I_{n,\theta}|-|Q_\theta|-\sum_{j\in J_\theta}\ell_j\bigr)!} \\
&\quad \times \prod_{\theta\in\widehat{T}} \Biggl(\prod_{q\in Q_\theta}\chi_\theta(t_q) \prod_{j\in J_\theta}\chi_\theta(P_j) \Biggr) \prod_{\theta\neq\zeta} (\dim\theta)^{|I_{n,\theta}|-|Q_\theta|-\sum_{j\in J_\theta}\ell_j} \\
&\quad \times (\dim\zeta)^{|I_{n,\zeta}|-1-|Q_\zeta|-\sum_{j\in J_\zeta}\ell_j} \Biggl\{ \sum_{\lambda^\zeta:\lambda^\zeta\nearrow\mu^\zeta} \chi\bigl(\lambda^\zeta,(\ell_j)_{j\in J_\zeta}\bigr) \prod_{\theta\neq\zeta}\chi\bigl(\mu^\theta,(\ell_j)_{j\in J_\theta}\bigr) \Biggr\} \Biggr] \\
&= \sum_{\zeta\in\widehat{T}} \sum_{\lambda^\zeta:\lambda^\zeta\nearrow\mu^\zeta} (\dim\zeta) \\
&\quad \times \sum_{\mathscr{Q},\mathscr{J}} \Biggl[\frac{\bigl(n-1-\sum_{\kappa\in\widehat{T}}|Q_\kappa|-\sum_{j\in J}\ell_j\bigr)!}{\bigl(|I_{n,\zeta}|-1-|Q_\zeta|-\sum_{j\in J_\zeta}\ell_j\bigr)!\prod_{\theta\neq\zeta}\bigl(|I_{n,\theta}|-|Q_\theta|-\sum_{j\in J_\theta}\ell_j\bigr)!} \\
&\quad \times (\dim\zeta)^{|I_{n,\zeta}|-1-|Q_\zeta|-\sum_{j\in J_\zeta}\ell_j} \prod_{\theta\neq\zeta} (\dim\theta)^{|I_{n,\theta}|-|Q_\theta|-\sum_{j\in J_\theta}\ell_j} \\
&\quad \times \prod_{\theta\in\widehat{T}} \Biggl(\prod_{q\in Q_\theta}\chi_\theta(t_q) \prod_{j\in J_\theta}\chi_\theta(P_j) \Biggr) \chi\bigl(\lambda^\zeta,(\ell_j)_{j\in J_\zeta}\bigr) \prod_{\theta\neq\zeta}\chi\bigl(\mu^\theta,(\ell_j)_{j\in J_\theta}\bigr) \Biggr] \\
&= \sum_{\Lambda:\Lambda\nearrow\mathrm{M}} (\dim\zeta_{\Lambda,\mathrm{M}})\chi^\Lambda(g),
\end{aligned}
$$

which completes the proof of (1.6). □

2. Branching graph and central measures.

In this section, we prepare some notions of harmonic analysis on a general branching graph along the lines of [**12**], [**14**], [**1**] and [**13**] in order to translate analysis on groups into that on their dual objects. For our purpose, we cannot help allowing infinite (even uncountable) valencies of the graph.

2.1. Branching graph.

Definition 2.1. A *branching graph* consists of the stratified vertex sets

$$\boldsymbol{G} = \bigsqcup_{n=0}^{\infty} \boldsymbol{G}_n \qquad \text{(disjoint union)}$$

and the edges satisfying the following conditions. We call $\boldsymbol{G}_n$ the vertices of the nth level.

(1) Two vertices $\alpha, \beta \in \boldsymbol{G}$ can be adjacent only if they belong to two consecutive levels. If $\alpha \in \boldsymbol{G}_n$ and $\beta \in \boldsymbol{G}_{n+1}$ are adjacent, we express them as $\alpha \nearrow \beta$ and call (α, β) the ingoing [resp. outgoing] edge of β [resp. α].
(2) $\boldsymbol{G}_0$ consists of the unique element $\varnothing$ that has no ingoing edges.
(3) For any vertex except $\varnothing$, its ingoing [resp. outgoing] edges form a nonempty finite [resp. nonempty (possibly infinite)] set.
(4) If $\alpha \nearrow \beta$ holds, the edge (α, β) carries multiplicity $\kappa(\alpha, \beta) > 0$.

For the sake of convenience we set $\kappa(\alpha, \beta) = 0$ if α and β belong to two consecutive levels but are not adjacent. The branching graph itself is also denoted by $\boldsymbol{G}$ for simplicity of the notation.

Remark 2.2. What is primarily in our mind is the branching graph for the wreath product groups $\mathfrak{S}_n(T)$, namely

$$\boldsymbol{G}_n = \mathfrak{S}_n(T)\widehat{\ } = \boldsymbol{Y}_n(T) \quad \text{and} \quad \kappa(\Lambda, \mathrm{M}) = \dim \zeta_{\Lambda,\mathrm{M}}$$

for $\Lambda \in \boldsymbol{G}_n$, $\mathrm{M} \in \boldsymbol{G}_{n+1}$. The unique element of $\boldsymbol{G}_0 = \boldsymbol{Y}_0(T)$ is $\varnothing = (\varnothing^{\zeta})_{\zeta \in \widehat{T}}$, where each $\varnothing^{\zeta}$ is the empty Young diagram. If T is a continuous compact group, the number of outgoing edges of a vertex is necessarily infinite.

Definition 2.3. A complex-valued function φ on $\boldsymbol{G}$ is usually said to be harmonic if it satisfies

$$\varphi(\alpha) = \sum_{\beta : \alpha \nearrow \beta} \kappa(\alpha, \beta) \varphi(\beta), \qquad \alpha \in \boldsymbol{G}. \tag{2.1}$$

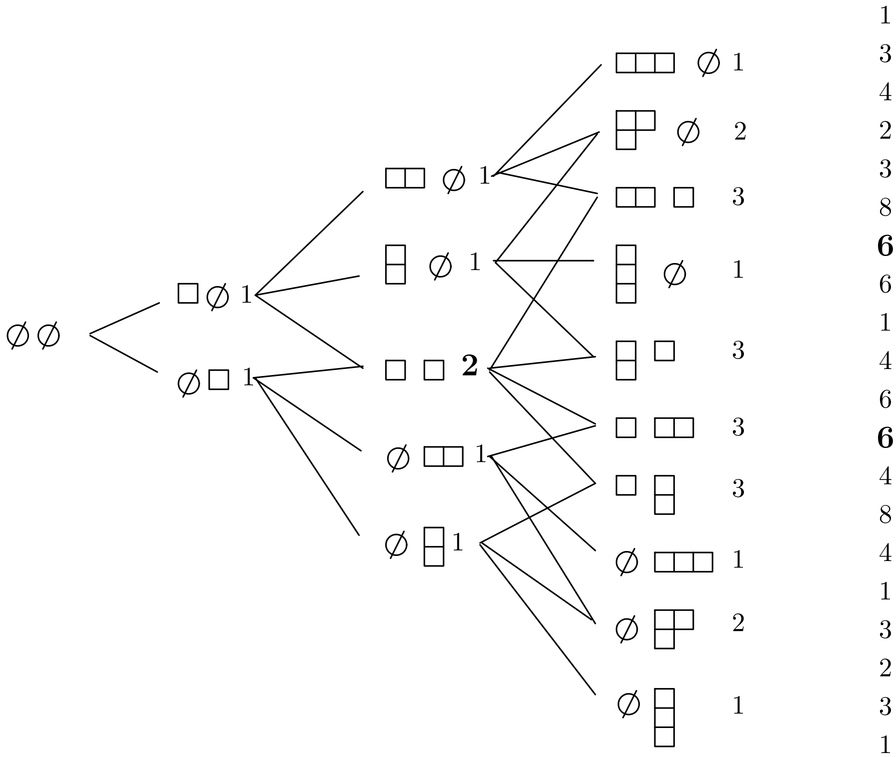

Figure 1. Branching for $\mathfrak{S}_n(\boldsymbol{Z}_2)$, Weyl group of type B_n/C_n; $\Lambda = (\lambda^{\zeta_0}, \lambda^{\zeta_1})$, $\zeta_0 = \mathbf{1}$: The integer associated with a pair indicates the dimension of the corresponding IUR. The meaning of a boldface integer concerns restriction to the Weyl group of type D_n. See Remark 4.10.

In this paper, however, we call φ a *harmonic function on a branching graph* $\boldsymbol{G}$ if it is

nonnegative :	$\varphi(\alpha) \geq 0, \quad \alpha \in \boldsymbol{G}$,	(2.2)
normalized :	$\varphi(\varnothing) = 1$,	(2.3)
countably supported :	$\operatorname{supp} \varphi$ is an at most countable set,	(2.4)

and satisfies (2.1). The meaning of the sum in (2.1) is now clear since $\operatorname{supp} \varphi$ is at most countable.

Note that (2.1) and (2.2) imply that if $\alpha \notin \operatorname{supp} \varphi$ and $\alpha \nearrow \beta$, then $\beta \notin \operatorname{supp} \varphi$, in other words that:

$$\text{If} \quad \beta \in \operatorname{supp}\varphi \ \text{ and } \alpha \text{ lies on a path terminating at } \beta, \text{ then } \quad \alpha \in \operatorname{supp}\varphi. \tag{2.5}$$

Let $\mathfrak{T} = \mathfrak{T}(\boldsymbol{G})$ denote the set of all infinite paths on branching graph $\boldsymbol{G}$ starting at $\varnothing$. A path $t \in \mathfrak{T}$ is expressed as

$$t = (t(0) \nearrow t(1) \nearrow \cdots \nearrow t(n) \nearrow \cdots)$$

where $t(n) \in \boldsymbol{G}_n$ is the nth level vertex of t. For any path $t \in \mathfrak{T}$, $t(0)$ is always $\varnothing$. Its truncated path up to the nth level is denoted by

$$t_n = (t(0) \nearrow t(1) \nearrow \cdots \nearrow t(n)).$$

$\mathfrak{T}_n = \mathfrak{T}_n(\boldsymbol{G})$ denotes the set of all finite paths up to the nth level. For finite path u connecting $\alpha \in \boldsymbol{G}_m$ and $\beta \in \boldsymbol{G}_n$: $\alpha = u(m) \nearrow \cdots \nearrow u(n) = \beta$, its *weight* w_u is defined by

$$w_u = \prod_{i=m}^{n-1} \kappa(u(i), u(i+1)). \tag{2.6}$$

Summing up the weights over all paths connecting α to β as

$$d(\alpha, \beta) = \sum_{\text{path } u\,:\,\alpha \nearrow \cdots \nearrow \beta} w_u, \tag{2.7}$$

we define the (*combinatorial*) *dimension function* d on branching graph $\boldsymbol{G}$. If there are no paths connecting α to β, our convention yields that some edge multiplicity in (2.6) vanishes and hence $d(\alpha, \beta) = 0$.

Remark 2.4. In the case of $\boldsymbol{G}_n = \boldsymbol{Y}_n(T)$, the value $d(\varnothing, \Lambda)$ agrees with the dimension of IUR $\Pi(\Lambda)$ of $\mathfrak{S}_n(T)$ associated with $\Lambda \in \boldsymbol{Y}_n(T)$, which is readily seen from Proposition 1.4.

Definition 2.5. Consider a subset $\boldsymbol{G}^0 \subset \boldsymbol{G}$ as a new vertex set and the edges inherited from $\boldsymbol{G}$. Let $\boldsymbol{G}^0$ become a branching graph in the sense of Definition 2.1. Furthermore assume that, for any $\beta \in \boldsymbol{G}^0$ and any finite path in $\boldsymbol{G}$ connecting $\varnothing$ to β, all the vertices lying on the path belong to $\boldsymbol{G}^0$. Then we call $\boldsymbol{G}^0$ a *subgraph* of branching graph $\boldsymbol{G}$. If $\boldsymbol{G}^0$ is an at most countable set, we refer to it as a countable subgraph.

Remark 2.6. If $\boldsymbol{G}^0$ is a subgraph of $\boldsymbol{G}$, then we have for any $\alpha \in \boldsymbol{G}^0 \cap \boldsymbol{G}_n$,

$n = 0, 1, 2, \ldots$ that

$$\{u \in \mathfrak{T}_n(\boldsymbol{G}) \mid u(n) = \alpha\} = \{u \in \mathfrak{T}_n(\boldsymbol{G}^0) \mid u(n) = \alpha\}.$$

Equation (2.5) shows that $\operatorname{supp}\varphi$ of a harmonic function on $\boldsymbol{G}$ is a countable subgraph of $\boldsymbol{G}$.

Lemma 2.7. *Harmonic function φ on $\boldsymbol{G}$ satisfies*

$$\varphi(\alpha) = \sum_{\beta \in \boldsymbol{G}_n} d(\alpha, \beta)\varphi(\beta) \tag{2.8}$$

for any $m < n$ and $\alpha \in \boldsymbol{G}_m$.

Proof. Set $\boldsymbol{G}^0 = \operatorname{supp}\varphi$, which is a countable subgraph of $\boldsymbol{G}$. If $\beta \in \boldsymbol{G}_n^0$ and $d(\alpha, \beta) > 0$, then $\alpha \in \boldsymbol{G}_m^0$. Hence in the case of $\alpha \notin \boldsymbol{G}_m^0$, (2.8) holds trivially as 0.

Let $\alpha \in \boldsymbol{G}_m^0$ be taken. For $\beta_2 \in \boldsymbol{G}_{m+2}^0$ we have

$$d(\alpha, \beta_2) = \sum_{\beta_1 \in \boldsymbol{G}_{m+1}^0} d(\alpha, \beta_1)d(\beta_1, \beta_2)$$

since $\beta_2 \in \boldsymbol{G}_{m+2}^0$ and $\beta_1 \nearrow \beta_2$ imply $\beta_1 \in \boldsymbol{G}_{m+1}^0$. Then (2.8) is shown inductively by iterating (2.1). □

2.2. Central measures.

For each $u = (u(0) \nearrow \cdots \nearrow u(n)) \in \mathfrak{T}_n = \mathfrak{T}_n(\boldsymbol{G})$, we set

$$C_u = \{t \in \mathfrak{T} \mid t(k) = u(k),\ k = 0, 1, \ldots, n\}.$$

$\mathfrak{T} = \mathfrak{T}(\boldsymbol{G})$ is equipped with the topology in which each $t \in \mathfrak{T}$ has $\{C_{t_n}\}_{n=0,1,2\ldots}$ as its neighborhoods. Definition 2.1 yields that $\mathfrak{T}$ is totally disconnected under this topology. For the branching graph of $\mathfrak{S}_\infty(T)$, the set $\widehat{T}$ can be identified with the set $\mathfrak{T}_1$ of all paths of the first level, and it is equipped with the discrete topology. The Borel field of $\mathfrak{T}$ is denoted by $\mathfrak{B}(\mathfrak{T})$.

Definition 2.8. Probability M on measurable space $(\mathfrak{T}, \mathfrak{B}(\mathfrak{T}))$ is usually said to be central if it satisfies

$$\frac{M(C_u)}{w_u} = \frac{M(C_v)}{w_v} \tag{2.9}$$

for all n and $u, v \in \mathfrak{T}_n$ which share a common terminating vertex. In this paper, however, we call probability M on $(\mathfrak{T}, \mathfrak{B}(\mathfrak{T}))$ to be *central* if M is supported by the path space $\mathfrak{T}(\boldsymbol{G}^0)$ of some countable subgraph $\boldsymbol{G}^0$ of $\boldsymbol{G}$ in addition that it satisfies (2.9).

LEMMA 2.9. *There exists a bijective correspondence between the central probabilities M on $\mathfrak{T} = \mathfrak{T}(\boldsymbol{G})$ and the harmonic functions φ on $\boldsymbol{G}$ through*

$$\frac{M(C_u)}{w_u} = \varphi(\alpha) \tag{2.10}$$

for any $\alpha \in \boldsymbol{G}_n$ and $u \in \mathfrak{T}_n = \mathfrak{T}_n(\boldsymbol{G})$ such that $u(n) = \alpha$ $(n = 0, 1, 2, \dots)$.

PROOF. If $\boldsymbol{G}^0$ is a subgraph of $\boldsymbol{G}$, we have

$$\mathfrak{T}(\boldsymbol{G}^0) = \bigcap_{n=0}^{\infty} \big\{t \in \mathfrak{T}(\boldsymbol{G}) \mid t(0), \cdots, t(n) \in \boldsymbol{G}^0\big\}. \tag{2.11}$$

In fact, the inclusion $\subset$ is obvious. To show the converse inclusion $\supset$, note that $\mathfrak{T}(\boldsymbol{G})$ [resp. $\mathfrak{T}(\boldsymbol{G}^0)$] is identified with the projective limit of $(\mathfrak{T}_n(\boldsymbol{G}))_{n=0,1,\dots}$ [resp. $(\mathfrak{T}_n(\boldsymbol{G}^0))_{n=0,1,\dots}$]. Projection p_{mn} is defined by $p_{mn}(t_n) = t_m$ for $m < n$ for $t \in \mathfrak{T}(\boldsymbol{G})$ [resp. $t \in \mathfrak{T}(\boldsymbol{G}^0)$]. The projective sequence corresponding to $t \in \mathfrak{T}(\boldsymbol{G})$ is $(t_0, t_1, t_2, \cdots)$. If t belongs to the right hand side of (2.11), we have $t_n \in \mathfrak{T}_n(\boldsymbol{G}^0)$ for any n. This means that $(t_n)_{n=0,1,\dots}$ belongs to the projective limit of $(\mathfrak{T}_n(\boldsymbol{G}^0))_{n=0,1,\dots}$.

Let M be a central probability on $\mathfrak{T}$ and $\boldsymbol{G}^0$ an associated countable subgraph of $\boldsymbol{G}$ such that M is supported by $\mathfrak{T}(\boldsymbol{G}^0)$. Equation (2.9) for M assures that (2.10) determines the function φ well. Then $\operatorname{supp}\varphi$ is included in $\boldsymbol{G}^0$, which is at most countable. Harmonicity of φ follows from countable additivity of M.

Conversely, let φ be a harmonic function on $\boldsymbol{G}$ and set $\boldsymbol{G}^0 = \operatorname{supp}\varphi$. As noted in Remark 2.6, $\boldsymbol{G}^0$ is a countable subgraph of $\boldsymbol{G}$. Equation (2.10) defines atomic probability M_n on $\mathfrak{T}_n = \mathfrak{T}_n(\boldsymbol{G})$ which is supported by an at most countable set. Harmonicity of φ yields that $((\mathfrak{T}_n, M_n), (p_{mn}))$ is a consistent projective system. This means that we have $(p_{mn})_* M_n = M_m$ for $m < n$ where $_*$ indicates a push-forward. Then we obtain the unique probability M on $\mathfrak{T}$, which is the projective limit of $\mathfrak{T}_n$, such that $(p_n)_* M = M_n$ holds for any n where $p_n : \mathfrak{T} \longrightarrow \mathfrak{T}_n$ is the canonical projection. (See e.g. [**18**, Volume 1, Chapter 2] for a comprehensive account on extension theorems of measures. Our measure space $(\mathfrak{T}_n, M_n)$ is almost countably separated since M is supported by a countable set.) Centrality of M is obvious from the definition of (2.10). Furthermore (2.11) implies

$$M(\mathfrak{T}(\boldsymbol{G}^0)) = \lim_{n\to\infty} M(\{t \in \mathfrak{T} \mid t(0), \cdots, t(n) \in \boldsymbol{G}^0\}) = 1.$$

It is obvious that the above correspondences are mutually inverse. □

The centrality of a probability on the path space $\mathfrak{T}$ is rephrased as quasi-invariance with respect to groups. For $\alpha \in \boldsymbol{G}_n$ set

$$\mathfrak{T}(\alpha) = \{u \in \mathfrak{T}_n(\boldsymbol{G}) \mid u(n) = \alpha\}.$$

$\mathfrak{T}(\alpha)$ consists of all paths terminating at α. It is a finite set by virtue of Definition 2.1 (3). The set of all permutations of $\mathfrak{T}(\alpha)$ is denoted by $\mathfrak{S}_{\mathfrak{T}(\alpha)}$. We regard any element $\tau \in \mathfrak{S}_{\mathfrak{T}(\alpha)}$ as a permutation of $\mathfrak{T}$ by

$$t \longmapsto \tau(t) = \begin{cases} \tau\bigl(t(0) \nearrow \cdots \nearrow t(n)\bigr) \nearrow t(n+1) \nearrow \cdots, & t(n) = \alpha, \\ t, & t(n) \neq \alpha. \end{cases}$$

We have then canonical inclusion

$$\mathfrak{S}_{\mathfrak{T}(\alpha)} \subset \mathfrak{S}_{\mathfrak{T}(\beta)} \qquad \text{if} \quad \alpha \nearrow \cdots \nearrow \beta. \tag{2.12}$$

If $\boldsymbol{G}^0$ is a subgraph of $\boldsymbol{G}$, $\mathfrak{T}(\boldsymbol{G}^0)$ is invariant under any $\mathfrak{S}_{\mathfrak{T}(\alpha)}$.

Lemma 2.10. *Let $\boldsymbol{G}^0$ be a countable subgraph of $\boldsymbol{G}$. Probability M supported by $\mathfrak{T}(\boldsymbol{G}^0)$ satisfies* (2.9) *if and only if*

$$M(\tau^{-1}B) = \int_B \frac{w_{\tau^{-1}(t_n)}}{w_{t_n}} M(dt), \qquad B \in \mathfrak{B}(\mathfrak{T}(\boldsymbol{G})) \tag{2.13}$$

holds for any $\alpha \in \boldsymbol{G}$ and any $\tau \in \mathfrak{S}_{\mathfrak{T}(\alpha)}$.

Proof. Note that the definition of a function f_τ on $\mathfrak{T}$

$$t \in \mathfrak{T} \longmapsto f_\tau(t) = \frac{w_{\tau^{-1}(t_n)}}{w_{t_n}}, \qquad t_n = (t(0) \nearrow \cdots \nearrow t(n)) \tag{2.14}$$

for $\tau \in \mathfrak{S}_{\mathfrak{T}(\alpha)}$ is consistent with the inclusion (2.12).

Assume that M satisfies (2.9). Let $\tau \in \mathfrak{S}_{\mathfrak{T}(\alpha)}$ be given for $\alpha \in \boldsymbol{G}_n$. Take a finite path $u = (u(0) \nearrow \cdots \nearrow u(m)) \in \mathfrak{T}_m$ and subset C_u from $\mathfrak{B}(\mathfrak{T})$.

(i) Case of $m = n$. If $u(m) = \alpha$, we have

$$\int_{C_u} f_\tau(t)M(dt) = \int_{C_u} \frac{w_{\tau^{-1}(u)}}{w_u} M(dt) = \frac{w_{\tau^{-1}(u)}}{w_u} M(C_u) = M(\tau^{-1}(C_u)).$$

Otherwise the left side is $M(C_u) = M(\tau^{-1}(C_u))$.

(ii) CASE OF $m < n$. A path extending u to $\beta \in \boldsymbol{G}^0$ is denoted by $u \nearrow \cdots \nearrow \beta \in \mathfrak{T}_n$. Since

$$C_u = \bigsqcup_{\beta \in \boldsymbol{G}^0} \bigsqcup_{\text{path: } u \nearrow \cdots \nearrow \beta} C_{u \nearrow \cdots \nearrow \beta} \quad \bigsqcup (M\text{-null set})$$

holds, where the first is a countable disjoint union and the second is a finite one, we have

$$\begin{aligned}
\int_{C_u} f_\tau(t)M(dt) &= \sum_{\beta \in \boldsymbol{G}^0} \sum_{u(m) \nearrow \cdots \nearrow \beta} \int_{C_{u \nearrow \cdots \nearrow \beta}} f_\tau(t)M(dt) \\
&= \sum_{u \nearrow \cdots \nearrow \alpha} \frac{w_{\tau^{-1}(u \nearrow \cdots \nearrow \alpha)}}{w_{u \nearrow \cdots \nearrow \alpha}} M(C_{u \nearrow \cdots \nearrow \alpha}) \\
&\quad + \sum_{\beta \in \boldsymbol{G}^0 : \beta \neq \alpha} \sum_{u \nearrow \cdots \nearrow \beta} M(C_{u \nearrow \cdots \nearrow \alpha}) \\
&= M(\tau^{-1}(C_u)).
\end{aligned}$$

(iii) CASE OF $m > n$. Independent of whether α lies in u or not, we have

$$\int_{C_u} f_\tau(t)M(dt) = \frac{w_{\tau^{-1}(u)}}{w_u} M(C_u) = M(\tau^{-1}(C_u)).$$

All cases summed up, (2.9) implies (2.13).

Conversely, following the above argument of (i), we see (2.13) implies (2.9). □

Consider a random variable $X_n : \mathfrak{T} \longrightarrow \boldsymbol{G}_n$ defined by $X_n(t) = t(n)$. Here any subset $B \subset \boldsymbol{G}_n$ is measurable by definition. Then $\mathfrak{B}(\mathfrak{T})$ is generated by random variables $X_1, X_2, \cdots$. Let $\mathfrak{B}_n$ be the sub-σ-field generated by the $X_n, X_{n+1}, \cdots$ and set the tail σ-field as $\mathfrak{B}_\infty = \bigcap_{n=0}^\infty \mathfrak{B}_n$. Lemma 2.10 says that centrality of M is equivalent to $\bigcup_{\alpha \in \boldsymbol{G}} \mathfrak{S}_{\mathfrak{T}(\alpha)}$-quasi-invariance. Among such probabilities, an extremal one is often said to be $\bigcup_{\alpha \in \boldsymbol{G}} \mathfrak{S}_{\mathfrak{T}(\alpha)}$-*ergodic*.

LEMMA 2.11. *Let M be an extremal central probability on $\mathfrak{T}$. Then M is* *trivial on* $\mathfrak{B}_\infty$, *namely* $M(B) = 0$ *or* 1 *for* $B \in \mathfrak{B}_\infty$, *and hence a* $\mathfrak{B}_\infty$*-measurable function is constant* M*-a.s.*

Proof. The following argument is standard, as is seen in e.g. [**18**, Volume 2, Chapter 2]. Let $E \in \mathfrak{B}_\infty$ satisfy $M(E) \neq 0,\ 1$. Set

$$M_1(B) = \frac{1}{M(E)} M(B \cap E), \qquad M_2(B) = \frac{1}{M(E^c)} M(B \cap E^c), \qquad B \in \mathfrak{B}(\mathfrak{T}).$$

Then M_1 and M_2 are central probabilities. In fact, let $\alpha \in \boldsymbol{G}$ and $\tau \in \mathfrak{S}_{\mathfrak{T}(\alpha)}$ be taken arbitrarily. Noting that $E \in \mathfrak{B}_\infty$ satisfies $\tau^{-1}(E) = E$, we have for $B \in \mathfrak{B}(\mathfrak{T})$

$$\begin{aligned} M_1(\tau^{-1}(B)) &= \frac{1}{M(E)} M(\tau^{-1}(B \cap E)) \\ &= \frac{1}{M(E)} \int_B f_\tau(t) 1_E(t) M(dt) = \int_B f_\tau(t) M_1(dt), \end{aligned}$$

and similarly for M_2. Thus, using disjoint central probabilities M_1 and M_2, we have a convex decomposition

$$M = M(E) M_1 + M(E^c) M_2,$$

which contradicts extremality of M. This completes the proof. □

3. Limit of Martin kernels on a branching graph.

In this section we continue working on a general branching graph to prove a limit theorem for Martin kernels.

3.1. Martingales and convergence theorem.

We briefly summarize necessary notions of martingales and a convergence theorem for them. See e.g. [**3**, Chapter 4].

As usual let $(\Omega, \mathfrak{F}, P)$ be a probability space and $\boldsymbol{E}[X] = \int_\Omega X(\omega) P(d\omega)$ denote the expectation of real-valued random variable X on Ω. For sub-σ-field $\mathfrak{E} \subset \mathfrak{F}$ the conditional expectation of X with respect to $\mathfrak{E}$ is denoted by $\boldsymbol{E}[X|\mathfrak{E}]$, which is characterized as the $\mathfrak{E}$-measurable function such that

$$\int_A \boldsymbol{E}[X|\mathfrak{E}](\omega) P(d\omega) = \int_A X(\omega) P(d\omega), \qquad A \in \mathfrak{E}.$$

Let $(\mathfrak{F}_n)_{n=0,1,2,\dots}$ be a decreasing sequence of sub-σ-fields of $\mathfrak{F}$, i.e. $\mathfrak{F}_n \supset \mathfrak{F}_{n+1}$. A sequence of integrable random variables $(X_n)_{n=0,1,2,\ldots}$ is called a *backward* $(\mathfrak{F}_n)$-*martingale* if it satisfies

$$\boldsymbol{E}[X_n|\mathfrak{F}_{n+1}] = X_{n+1} \quad \text{a.s.}, \qquad n = 0, 1, 2, \ldots.$$

PROPOSITION 3.1. *Let $(X_n)_{n=0,1,2,\ldots}$ be a backward martingale with respect to decreasing sub-σ-fields $(\mathfrak{F}_n)$ as above. Then*

$$X_\infty = \lim_{n\to\infty} X_n$$

exists a.s. The convergence holds also in L^1-topology. Clearly X_∞ is $(\bigcap_{n=0}^{\infty} \mathfrak{F}_n)$-measurable.

3.2. Martin kernels.

According to the common terminology of Markov chains, the ratio of Green kernels (or potential kernels) $G(x,y)/G(x_0,y)$ is referred to as a Martin kernel, where $G(x,y)$ denotes the expected number for the chain starting at x to visit y. Here x_0 is a fixed reference vertex. When we consider the simple random walk on the Young graph, whose transitions are made from a vertex to another lying in the adjacent upper level, and its long-time limiting behaviour, the ratio of dimension functions plays the role of a Martin kernel. In our case where the set $\boldsymbol{G}_n$ of the nth level vertices may be infinite, we can no longer associate a simple random walk with the branching graph $\boldsymbol{G}$. Nevertheless, since the combinatorial dimension function $d(\alpha,\beta)$ is well-defined by virtue of Definition 2.1 (3), we regard the ratio

$$\frac{d(\alpha,\beta)}{d(\varnothing,\beta)}, \qquad \alpha,\beta \in \boldsymbol{G}$$

as a *Martin kernel* on the branching graph $\boldsymbol{G}$.

Let a central probability M be given on $\mathfrak{T} = \mathfrak{T}(\boldsymbol{G})$. Take an associated countable subgraph $\boldsymbol{G}^0$ of $\boldsymbol{G}$ such that M is supported by $\mathfrak{T}(\boldsymbol{G}^0)$. Then M can be traced to probability M^0 on sub-σ-field

$$\mathfrak{B}^0 = \mathfrak{B}(\mathfrak{T}) \cap \mathfrak{T}(\boldsymbol{G}^0) = \{B \cap \mathfrak{T}(\boldsymbol{G}^0) \mid B \in \mathfrak{B}(\mathfrak{T})\}$$

which is defined well by

$$M^0(B \cap \mathfrak{T}(\boldsymbol{G}^0)) = M(B), \qquad B \in \mathfrak{B}(\mathfrak{T}). \tag{3.1}$$

Theorem 3.2. *Assume that M is an extremal central probability on $(\mathfrak{T}, \mathfrak{B}(\mathfrak{T}))$. Let φ be an extremal harmonic function on $\boldsymbol{G}$ associated with M which is determined in Lemma* 2.9. *Then, for M-a.s. $t \in \mathfrak{T}$,*

$$\lim_{n\to\infty} \frac{d(\alpha, t(n))}{d(\varnothing, t(n))} = \varphi(\alpha), \qquad \alpha \in \boldsymbol{G}^0 \tag{3.2}$$

holds.

Proof.

Step 1: Recall the notations X_n, $\mathfrak{B}_n$ and $\mathfrak{B}_\infty$ in Subsection 2.2. For each $\alpha \in \boldsymbol{G}^0_m$ and $n > m$, we consider random variables defined by

$$Z_n^{(\alpha)}(t) = \frac{d(\alpha, t(n))}{d(\varnothing, t(n))} = \frac{d(\alpha, X_n(t))}{d(\varnothing, X_n(t))}, \qquad t \in \mathfrak{T}(\boldsymbol{G}^0) \tag{3.3}$$

on probability space $(\mathfrak{T}(\boldsymbol{G}^0), \mathfrak{B}^0, M^0)$ where M^0 comes from (3.1). Set $\mathfrak{B}^0_n = \mathfrak{B}_n \cap \mathfrak{T}(\boldsymbol{G}^0)$ for $n = 0, 1, 2, \ldots, \infty$. $(\mathfrak{B}^0_n)_{n=0,1,2,\ldots}$ is a sequence of decreasing sub-σ-field of $\mathfrak{B}^0$.

$(Z_n^{(\alpha)})_{n=m+1,m+2,\ldots}$ is a backward $(\mathfrak{B}^0_n)$-martingale. In fact, we verify

$$\int_A Z_n^{(\alpha)} dM^0 = \int_A Z_{n+1}^{(\alpha)} dM^0, \qquad A \in \mathfrak{B}^0_{n+1}. \tag{3.4}$$

Since

$$\mathfrak{B}^0_{n+1} = \sigma[X_{n+1}, X_{n+2}, \cdots] = \sigma\Big[\bigcup_{r=1}^{\infty} \sigma[X_{n+1}, \cdots, X_{n+r}]\Big]$$

(where all X_i's are restricted on $\mathfrak{T}(\boldsymbol{G}^0)$) holds, it suffices to show (3.4) for any set having the form of

$$A = \{t \in \mathfrak{T}(\boldsymbol{G}^0) \mid t(n+1) = \beta_1, \cdots, t(n+r) = \beta_r\}, \qquad \beta_i \in \boldsymbol{G}^0_i.$$

We have

$$\begin{aligned} M^0(A) &= \sum_{u \in \mathfrak{T}_n(\boldsymbol{G}^0)\,:\, u(n) \nearrow \beta_1} M^0(C_{u \nearrow \beta_1 \nearrow \cdots \nearrow \beta_r}) \\ &= \sum_{u \in \mathfrak{T}_n(\boldsymbol{G}^0)\,:\, u(n) \nearrow \beta_1} w_{u \nearrow \beta_1 \nearrow \cdots \nearrow \beta_r}\, \varphi(\beta_r) \\ &= \varphi(\beta_r)\kappa(\beta_1, \beta_2) \cdots \kappa(\beta_{r-1}, \beta_r) d(\varnothing, \beta_1). \end{aligned} \tag{3.5}$$

Using this we have

$$\int_A Z_{n+1}^{(\alpha)} dM^0 = \frac{d(\alpha,\beta_1)}{d(\varnothing,\beta_1)} M^0(A) = d(\alpha,\beta_1)\ w_{\beta_1\nearrow\cdots\nearrow\beta_r}\ \varphi(\beta_r).$$

On the other hand, we have

$$\int_A Z_n^{(\alpha)} dM^0 = \sum_{\beta\,:\,\beta\nearrow\beta_1} \frac{d(\alpha,\beta)}{d(\varnothing,\beta)} M^0(A_\beta),$$

where A is decomposed as

$$A = \bigsqcup_{\beta\in \boldsymbol{G}_n\,:\,\beta\nearrow\beta_1} A_\beta,$$
$$A_\beta = \{t \in \mathfrak{T}(\boldsymbol{G}^0) \mid t(n)=\beta, t(n+1)=\beta_1,\cdots, t(n+r)=\beta_r\}.$$

Computing $M^0(A_\beta)$ similarly as (3.5), we have

$$\begin{aligned}\int_A Z_n^{(\alpha)} dM^0 &= \sum_{\beta\,:\,\beta\nearrow\beta_1} \varphi(\beta_r) d(\alpha,\beta)\kappa(\beta,\beta_1)\cdots\kappa(\beta_{r-1},\beta_r)\\ &= \varphi(\beta_r)\ w_{\beta_1\nearrow\cdots\nearrow\beta_r}\ d(\alpha,\beta_1).\end{aligned}$$

This completes the proof of (3.4).

Step 2: The mean of $Z_n^{(\alpha)}$ is computed as follows. Set

$$B_\beta = \{t(n) \in \mathfrak{T}(\boldsymbol{G}^0) \mid t(n)=\beta\}, \qquad \beta\in \boldsymbol{G}_n^0.$$

Using

$$M^0(B_\beta) = \sum_{u\in\mathfrak{T}_n(\boldsymbol{G}^0)\,:\,u(n)=\beta} w_u\varphi(\beta) = d(\varnothing,\beta)\varphi(\beta),$$

and decomposing the whole space into B_β's, we have

$$\begin{aligned}\int_{\mathfrak{T}(\boldsymbol{G}^0)} Z_n^{(\alpha)} dM^0 &= \sum_{\beta\in\boldsymbol{G}_n^0} \int_{B_\beta} \frac{d(\alpha,t(n))}{d(\varnothing,t(n))} M^0(dt) = \sum_{\beta\in\boldsymbol{G}_n^0} \frac{d(\alpha,\beta)}{d(\varnothing,\beta)} M^0(B_\beta)\\ &= \sum_{\beta\in\boldsymbol{G}_n^0} d(\alpha,\beta)\varphi(\beta) = \varphi(\alpha)\end{aligned}$$

by virtue of Lemma 2.7.

Step 3: Applying Proposition 3.1, from the backward martingale convergence theorem, we conclude that

$$\lim_{n\to\infty} Z_n^{(\alpha)} = Z_\infty^{(\alpha)} \tag{3.6}$$

exists M-a.s. as a $\mathfrak{B}_\infty$-measurable function. Since M is extremal, $Z_\infty^{(\alpha)}$ is M-a.s. constant as is seen from Lemma 2.11. The convergence of (3.6) is valid also in L^1-topology. Hence the constant agrees with

$$\boldsymbol{E}\big[Z_\infty^{(\alpha)}\big] = \lim_{n\to\infty} \boldsymbol{E}\big[Z_n^{(\alpha)}\big] = \varphi(\alpha).$$

Finally we note that α just runs over countable set $\boldsymbol{G}^0$ and hence that the exceptional subset of $\mathfrak{T}$ can be taken commonly. □

4. Limit of irreducible characters of $\mathfrak{S}_n(T)$.

4.1. Branching graph and characters of $\mathfrak{S}_\infty(T)$.

In what follows, we consider the branching graph of a wreath product group. Recalling notations, let T be an arbitrary compact group, $G_n = \mathfrak{S}_n(T)$ its wreath product with the symmetric group $\mathfrak{S}_n$, and $\boldsymbol{Y}_n(T)$ as defined in (1.1), where $n = 1, 2, \ldots$. Set

$$\boldsymbol{Y}(T) = \bigsqcup_{n=0}^{\infty} \boldsymbol{Y}_n(T).$$

Here $\boldsymbol{Y}_0(T)$ consists of the unique element $\varnothing = (\varnothing^\zeta)_{\zeta\in\widehat{T}}$, in which each $\varnothing^\zeta$ is the empty Young diagram. We equip $\boldsymbol{Y}(T)$ with the structure of a branching graph induced by the branching rule for $\mathfrak{S}_n(T)$'s in Proposition 1.4. We use $\Lambda, \mathrm{M}, \cdots$ to indicate vertices instead of $\alpha, \beta, \cdots$ and put

$$\kappa(\Lambda, \mathrm{M}) = \dim \zeta_{\Lambda,\mathrm{M}},$$

with $\zeta_{\Lambda,\mathrm{M}}$ in Subsection 1.3. It is obvious that $\boldsymbol{Y}(T)$ satisfies the conditions in Definition 2.1.

Set $G = \mathfrak{S}_\infty(T)$ for simplicity. $E(G)$ denotes the set of extremal elements among the continuous, positive definite, central and normalized functions on G. An element of $E(G)$ is also called a *character* of G since it is essentially a normalized trace of a factor representation of finite type of G. Using the machinery of Sections 2 and 3, we can transfer to $\boldsymbol{Y}(T)$ in investigating $E(G)$ as below (Theorem 4.2).

We begin with referring to a Bochner type theorem on a compact group.

PROPOSITION 4.1. *Let K be a compact group and g a complex-valued function on K. The following two statements for g are equivalent.*

- *g is a linear combination of continuous and positive definite functions.*
- *g belongs to $L^1(K)$ and admits an absolutely convergent Fourier series expansion.*

In particular, g is continuous, positive definite and central if and only if $g \in L^1(K)$ and

$$g = \sum_{\alpha \in \widehat{K}} c_\alpha \chi_\alpha, \qquad c_\alpha \geq 0, \quad \sum_{\alpha \in \widehat{K}} c_\alpha \dim \alpha < \infty \tag{4.1}$$

hold. Here χ_α denotes the (non-normalized) irreducible character associated with $\alpha \in \widehat{K}$.

PROOF. See [**4**, Section 34], especially Equations (34.13) and (34.37). □

THEOREM 4.2. *For $G = \mathfrak{S}_\infty(T)$, we have bijective correspondences between the following three objects*:

(1) *$E(G)$,*
(2) *the set of extremal harmonic functions on $\boldsymbol{Y}(T)$,*
(3) *the set of extremal central probabilities on $\mathfrak{T}(\boldsymbol{Y}(T))$.*

To be precise, f in (1) *and φ in* (2) *are connected as*

$$f|_{\mathfrak{S}_n(T)} = \sum_{\Lambda \in \boldsymbol{Y}_n(T)} \varphi(\Lambda) \chi^\Lambda \tag{4.2}$$

while the bijection between (2) *and* (3) *is described in Lemma* 2.9.

PROOF. Let $f \in E(G)$ be given. Restricted onto $G_n = \mathfrak{S}_n(T)$, f specifies countable subset $\boldsymbol{Y}_n^0$ of $\boldsymbol{Y}_n(T)$ for each n according to (4.2) as

$$f\big|_{G_n} = \sum_{\Lambda \in \boldsymbol{Y}_n^0} \varphi(\Lambda) \chi^\Lambda \tag{4.3}$$

with Fourier coefficients $\varphi(\Lambda) > 0$. Applying (4.3) for $n+1$ together with (1.6), we have

$$f|_{G_n} = \sum_{\mathrm{M}\in \boldsymbol{Y}^0_{n+1}} \varphi(\mathrm{M})\chi^{\mathrm{M}}|_{G_n} = \sum_{\mathrm{M}\in \boldsymbol{Y}^0_{n+1}} \varphi(\mathrm{M}) \sum_{\Lambda\in \boldsymbol{Y}_n(T):\,\Lambda\nearrow\mathrm{M}} (\dim \zeta_{\Lambda,\mathrm{M}})\,\chi^{\Lambda}$$
$$= \sum_{\Lambda\in \boldsymbol{Y}^{00}_n} \Bigg(\sum_{\mathrm{M}\in \boldsymbol{Y}^0_{n+1}:\,\Lambda\nearrow\mathrm{M}} (\dim \zeta_{\Lambda,\mathrm{M}})\,\varphi(\mathrm{M})\Bigg)\chi^{\Lambda}, \tag{4.4}$$

where we set $\boldsymbol{Y}^{00}_n = \{\Lambda \in \boldsymbol{Y}_n(T) \mid \Lambda \nearrow \mathrm{M}$ for some $\mathrm{M} \in \boldsymbol{Y}^0_{n+1}\}$. Each coefficient of the rightmost hand is strictly positive for $\Lambda \in \boldsymbol{Y}^{00}_n$. Hence comparing this with (4.3), we have $\boldsymbol{Y}^0_n = \boldsymbol{Y}^{00}_n$ and

$$\varphi(\Lambda) = \sum_{\mathrm{M}\in \boldsymbol{Y}^0_{n+1}:\,\Lambda\nearrow\mathrm{M}} (\dim \zeta_{\Lambda,\mathrm{M}})\,\varphi(\mathrm{M}), \qquad \Lambda \in \boldsymbol{Y}^0_n.$$

Accordingly we see that $\boldsymbol{Y}^0 = \bigsqcup_{n=0}^{\infty} \boldsymbol{Y}^0_n$ is a subgraph of $\boldsymbol{Y}(T)$ and that φ is a harmonic function with $\operatorname{supp}\varphi = \boldsymbol{Y}^0$.

Conversely, let φ in (2) be given. Set $\boldsymbol{Y}^0_n = (\operatorname{supp}\varphi)\cap \boldsymbol{Y}_n(T)$. Then $\boldsymbol{Y}^0_n = \boldsymbol{Y}^{00}_n$ holds. The same computation with (4.4) yields that (4.3) defines $f \in E(G)$ well, namely $f|_{G_n} = (f|_{G_{n+1}})|_{G_n}$ is valid.

The above correspondences clearly give mutual inverses. □

4.2. Limit of irreducible characters of $\mathfrak{S}_n(T)$.

Theorem 4.3. *Let $f \in E(\mathfrak{S}_\infty(T))$ be given and M the corresponding extremal central probability in Theorem 4.2. For M-a.s. path $t \in \mathfrak{T}$, the convergence*

$$\lim_{n\to\infty} \widetilde{\chi}^{t(n)} = f \tag{4.5}$$

holds uniformly on each $G_k = \mathfrak{S}_k(T)$, $k \in \boldsymbol{N}$.

Proof.

Step 1: For $t \in \mathfrak{T}$ and $k < n$, we have

$$\widetilde{\chi}^{t(n)}|_{G_k} = \sum_{\Lambda\in \boldsymbol{Y}_k(T)} \frac{d(\Lambda, t(n))}{d(\varnothing, t(n))}\,\chi^{\Lambda} \tag{4.6}$$

by iterating (1.6). Indeed,

$$\begin{aligned}\chi^{t(n)}\big|_{G_k} &= \sum_{\mathrm{M}\in \boldsymbol{Y}_{n-1}(T):\, \mathrm{M}\nearrow t(n)} (\dim \zeta_{\mathrm{M},t(n)})\, \chi^{\mathrm{M}}|_{G_k} \\ &= \sum_{\mathrm{M}\in \boldsymbol{Y}_{n-1}(T):\, \mathrm{M}\nearrow t(n)} \sum_{\mathrm{N}\in \boldsymbol{Y}_{n-2}(T):\, \mathrm{N}\nearrow \mathrm{M}} (\dim \zeta_{\mathrm{M},t(n)} \dim \zeta_{\mathrm{N},\mathrm{M}})\, \chi^{\mathrm{N}}|_{G_k} \\ &= \sum_{\mathrm{N}\in \boldsymbol{Y}_{n-2}(T)} d(\mathrm{N}, t(n))\chi^{\mathrm{N}}|_{G_k} = \cdots = \sum_{\Lambda\in \boldsymbol{Y}_k(T)} d(\Lambda, t(n))\chi^{\Lambda}.\end{aligned}$$

Step 2: Under the correspondences of $f \leftrightarrow \varphi \leftrightarrow M$ in Theorem 4.2, set $\boldsymbol{Y}^0 = \operatorname{supp}\varphi$. Then, M is supported by $\mathfrak{T}(\boldsymbol{Y}^0)$. Theorem 3.2 tells us that we have, for M-a.s. path t,

$$\lim_{n\to\infty} \frac{d(\Lambda, t(n))}{d(\varnothing, t(n))} = \varphi(\Lambda), \qquad \Lambda \in \boldsymbol{Y}^0. \tag{4.7}$$

Take a path $t \in \mathfrak{T}(\boldsymbol{Y}^0)$ satisfying (4.7). We see

$$\Lambda \in \boldsymbol{Y}_k(T) \text{ and } d(\Lambda, t(n)) > 0 \;\text{ imply }\; \Lambda \in \boldsymbol{Y}_k(T)^0 = \boldsymbol{Y}_k(T) \cap \boldsymbol{Y}^0 \tag{4.8}$$

since $\boldsymbol{Y}^0$ is a subgraph. Set

$$\begin{aligned} Q(\Lambda) &= \varphi(\Lambda) d(\varnothing, \Lambda), \\ Q_{t(n)}(\Lambda) &= \frac{d(\Lambda, t(n))}{d(\varnothing, t(n))} d(\varnothing, \Lambda) \end{aligned} \tag{4.9}$$

for $\Lambda \in \boldsymbol{Y}_k(T)$. Clearly $\operatorname{supp} Q \subset \boldsymbol{Y}_k(T)^0$ is countable. Also (4.8) yields $\operatorname{supp} Q_{t(n)} \subset \boldsymbol{Y}_k(T)^0$. Furthermore, both are probabilities. In fact, it follows from

$$\sum_{\Lambda\in \boldsymbol{Y}_k(T)^0} d(\Lambda, t(n)) d(\varnothing, \Lambda) = \sum_{u\in \mathfrak{T}_n(\boldsymbol{Y}^0):\, u(n)=t(n)} w_u = d(\varnothing, t(n)),$$

$$\begin{aligned}\sum_{\Lambda\in \boldsymbol{Y}_k(T)^0} \varphi(\Lambda) d(\varnothing, \Lambda) &= \sum_{\Lambda\in \boldsymbol{Y}_k(T)^0} \varphi(\Lambda) \sum_{\mathrm{M}\in \boldsymbol{Y}_{k-1}(T)^0:\, \mathrm{M}\nearrow\Lambda} d(\varnothing, \mathrm{M}) \dim \zeta_{\mathrm{M},\Lambda} \\ &= \sum_{\mathrm{M}\in \boldsymbol{Y}_{k-1}(T)^0} \Bigg(\sum_{\Lambda\in \boldsymbol{Y}_k(T)^0:\, \mathrm{M}\nearrow\Lambda} (\dim \zeta_{\mathrm{M},\Lambda})\, \varphi(\Lambda) \Bigg) d(\varnothing, \mathrm{M}) \\ &= \sum_{\mathrm{M}\in \boldsymbol{Y}_{k-1}(T)^0} \varphi(\mathrm{M}) d(\varnothing, \mathrm{M}) = \cdots = \varphi(\varnothing) = 1.\end{aligned}$$

Step 3: We estimate the difference of the following:

$$\begin{aligned}\widetilde{\chi}^{t(n)}|_{G_k} &= \sum_{\Lambda\in \boldsymbol{Y}_k(T)^0} Q_{t(n)}(\Lambda)\,\widetilde{\chi}^{\Lambda},\\ f|_{G_k} &= \sum_{\Lambda\in \boldsymbol{Y}_k(T)^0} \varphi(\Lambda)\chi^{\Lambda} = \sum_{\Lambda\in \boldsymbol{Y}_k(T)^0} Q(\Lambda)\,\widetilde{\chi}^{\Lambda}\end{aligned} \tag{4.10}$$

where the first equality follows from (4.6) and (4.9). Take $\epsilon > 0$ arbitrarily. There exists finite set $F \subset \boldsymbol{Y}_k(T)^0$ such that $Q(F) > 1-\epsilon$. Equation (4.7) shows that, for M-a.s. path $t \in \mathfrak{T}(\boldsymbol{Y}^0)$, sufficiently large n allows

$$\begin{aligned}&|Q_{t(n)}(F) - Q(F)| < \epsilon, \qquad \text{and also}\\ &Q_{t(n)}(F^c) \le 1 - Q(F) + |Q_{t(n)}(F) - Q(F)| < 2\epsilon.\end{aligned}$$

Putting these into (4.10), we have for $g \in G_k$

$$\begin{aligned}&|\widetilde{\chi}^{t(n)}(g) - f(g)|\\ &\quad\le \Big|\sum_{\Lambda\in F}(Q_{t(n)}(\Lambda) - Q(\Lambda))\widetilde{\chi}^{\Lambda}(g)\Big| + \Big|\sum_{\Lambda\in \boldsymbol{Y}_k(T)^0\setminus F} Q_{t(n)}(\Lambda)\widetilde{\chi}^{\Lambda}(g)\Big|\\ &\qquad + \Big|\sum_{\Lambda\in \boldsymbol{Y}_k(T)^0\setminus F} Q(\Lambda)\widetilde{\chi}^{\Lambda}(g)\Big|\\ &\quad\le \sum_{\Lambda\in F}|Q_{t(n)}(\Lambda) - Q(\Lambda)| + Q_{t(n)}(\boldsymbol{Y}_k(T)^0\setminus F) + Q(\boldsymbol{Y}_k(T)^0\setminus F) \;\le\; 4\epsilon.\end{aligned}$$

We have thus obtained, for M-a.s. path t,

$$\lim_{n\to\infty}\sup_{g\in G_k}\big|\widetilde{\chi}^{t(n)}(g) - f(g)\big| = 0. \qquad \square$$

Theorem 4.3 enables us to determine an explicit form of character f in terms of two sorts of parameters, one being the Fourier coefficients of $f|_T$ and the other being families of asymptotic frequencies of Young diagrams. In this procedure, asymptotics for irreducible characters of $\mathfrak{S}_n$ play an essential role. Given Young diagram $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots)$ as a sequence of row lengths, we set

$$a_i(\lambda) = \lambda_i - i, \quad b_i(\lambda) = \lambda'_i - i, \qquad i = 1, 2, \ldots, d$$

where λ' is the transposed diagram and $d = d_\lambda$ denotes the main diagonal length of λ. These are called the Frobenius coordinates of λ.

PROPOSITION 4.4. *The value of the irreducible character corresponding to Young diagram λ at k-cycle has an asymptotic expression*

$$\begin{aligned} \widetilde{\chi}^{\lambda}_{(k,1^{|\lambda|-k})} &= \frac{1}{|\lambda|^k} p_k(\lambda) + O\Big(\frac{1}{|\lambda|}\Big), \\ p_k(\lambda) &= \sum_{i=1}^{d_\lambda} \big(a_i(\lambda)^k + (-1)^{k-1} b_i(\lambda)^k\big) \end{aligned} \tag{4.11}$$

as the size of diagram $|\lambda|$ grows to infinity. Actually, the O-term in (4.11) *is a polynomial of $p_j(\lambda)$, $j = 1, \dots, k-1$, of total degree $\leq k-1$ divided by $|\lambda|^k$.*

PROOF. We refer to [**15**, Chapter Five, Section 1] , [**17**] and [**10**]. □

THEOREM 4.5. *Let $f \in E(\mathfrak{S}_\infty(T))$ be given and M the corresponding extremal central probability in Theorem* 4.2*. Along M-a.s. path $t = (t(0) \nearrow \cdots \nearrow t(n) \nearrow \cdots)$ in Theorem* 4.3 *where $t(n) = (t(n)^\zeta)_{\zeta\in\widehat{T}} \in \boldsymbol{Y}_n(T)^0$, the following limits exist*:

$$B_\zeta = \lim_{n\to\infty} \frac{|t(n)^\zeta|}{n}, \quad \zeta \in \widehat{T}, \qquad \textit{moreover} \qquad \sum_{\zeta\in\widehat{T}} B_\zeta = 1, \tag{4.12}$$

$$\alpha_{\zeta,0,i} = \lim_{n\to\infty} \frac{a_i(t(n)^\zeta)}{n}, \quad \alpha_{\zeta,1,i} = \lim_{n\to\infty} \frac{b_i(t(n)^\zeta)}{n}, \qquad \zeta \in \widehat{T},\ i \in \boldsymbol{N}. \tag{4.13}$$

Since $B_\zeta = 0$ implies $\alpha_{\zeta,0,i} = \alpha_{\zeta,1,i} = 0$ for any $i \in \boldsymbol{N}$, these are 0 *except for at most countable ζ's.*

PROOF.

Step 1: Recall that every element of a wreath product group is factorized into basic elements as (1.2). We write down the values of irreducible characters of G_n at two kinds of basic elements $(s, (q))$ and (d, σ).

Let $\Lambda = (\lambda^\zeta)_{\zeta\in\widehat{T}} \in \boldsymbol{Y}_n(T)$, $n^\zeta = |\lambda^\zeta|$, $s \in T$, σ a k-cycle and $d \in D(T)$ such that $\operatorname{supp} d \subset \operatorname{supp} \sigma$. Then (1.4) yields

$$\widetilde{\chi}^{\Lambda}(s, (q)) = \sum_{\zeta\in\widehat{T}} \frac{n^\zeta}{n} \widetilde{\chi}_\zeta(s), \tag{4.14}$$

$$\widetilde{\chi}^{\Lambda}(d,\sigma) = \sum_{\zeta\in\widehat{T}} \frac{n^{\zeta}(n^{\zeta}-1)\cdots(n^{\zeta}-k+1)}{n(n-1)\cdots(n-k+1)} \frac{1}{(\dim\zeta)^k}\chi_\zeta(P_\sigma(d))\,\widetilde{\chi}^{\lambda^\zeta}_{(k,1^{n^\zeta-k})}. \tag{4.15}$$

Here we regard $\chi^{\lambda^\zeta}_{(k,1^{n^\zeta-k})}$ to be 0 if $n^\zeta < k$.

Step 2: We show (4.12). Proposition 4.1 ensures that

$$f(s,(q)) = \sum_{\zeta\in\widehat{T}} B_\zeta \widetilde{\chi}_\zeta(s) \qquad \text{with} \quad B_\zeta \geq 0, \quad \sum_{\zeta\in\widehat{T}} B_\zeta = 1 \tag{4.16}$$

since $\sum_{\zeta\in\widehat{T}} B_\zeta = f(e,(q)) = 1$. Theorem 4.3 tells us that $\widetilde{\chi}^{t(n)}(s,(q))$ converges to $f(s,(q))$ uniformly in $s\in T$. Combining these with (4.14) for $\lambda^\zeta = t(n)^\zeta$, we obtain convergence of their Fourier coefficients, namely (4.12).

Step 3: We consider (4.13). Putting $\Lambda = t(n)$ and $d = (s,e,\cdots,e)$ ($k-1$ times repetition of the identity element e of T) in (4.15), we have

$$\begin{aligned}&\widetilde{\chi}^{t(n)}((s,e,\cdots,e),\sigma)\\ &= \sum_{\zeta\in\widehat{T}} \frac{|t(n)^\zeta|(|t(n)^\zeta|-1)\cdots(|t(n)^\zeta|-k+1)}{n(n-1)\cdots(n-k+1)} \frac{1}{(\dim\zeta)^{k-1}} \widetilde{\chi}^{t(n)^\zeta}_{(k,1^{|t(n)^\zeta|-k})}\,\widetilde{\chi}_\zeta(s)\end{aligned} \tag{4.17}$$

as a function on T. See Remark 1.3 for the notation of an irreducible character. The k-cycles in $\mathfrak{S}_p$ is denoted by $(k,1^{p-k})$. The left side converges to $f((s,e,\cdots,e),\sigma)$ uniformly on T by virtue of Theorem 4.3. Hence the convergence of the Fourier coefficients implies that

$$\lim_{n\to\infty} \frac{|t(n)^\zeta|(|t(n)^\zeta|-1)\cdots(|t(n)^\zeta|-k+1)}{n(n-1)\cdots(n-k+1)} \widetilde{\chi}^{t(n)^\zeta}_{(k,1^{|t(n)^\zeta|-k})} \tag{4.18}$$

exists for any $\zeta\in\widehat{T}$.

Step 4: Equation (4.13) is deduced by using (4.18) through a compactness argument, which is a repetition of the argument in [**17**, Section 5]. We state the procedure, however, for reader's convenience below.

It is obvious that (4.13) holds as totally 0 if $B_\zeta = 0$.

Let $\zeta\in\widehat{T}$ be such that $B_\zeta > 0$. It suffices to show that, for every $i\in\boldsymbol{N}$, two sequences $\{a_i(t(n)^\zeta)/n\}_n$ and $\{b_i(t(n)^\zeta)/n\}_n$ have the unique limit points respectively. Combining (4.18) with (4.11), we have the existence of

$$\lim_{n\to\infty}\sum_{i=1}^{\infty}\left\{\left(\frac{a_i(t(n)^\zeta)}{n}\right)^k+(-1)^{k-1}\left(\frac{b_i(t(n)^\zeta)}{n}\right)^k\right\}. \tag{4.19}$$

Let $\alpha_i = \alpha_i^\zeta$ [resp. $\beta_i = \beta_i^\zeta$] be a limit point of $\{a_i(t(n)^\zeta)/n\}_n$ [resp. $\{b_i(t(n)^\zeta)/n\}_n$]. Then, Lemma 4.6 below tells us that (4.19) agrees with

$$\sum_{i=1}^{\infty}\left(\alpha_i^k+(-1)^{k-1}\beta_i^k\right) \tag{4.20}$$

if $k \geq 2$. Hence (4.20) does not depend on the choice of limit points α_i and β_i. However, (4.20) determines α_i and β_i uniquely since it holds that

$$\exp\left\{\sum_{k=2}^{\infty}\sum_{i=1}^{\infty}\left(\alpha_i^k+(-1)^{k-1}\beta_i^k\right)\frac{z^k}{k}\right\}=\exp\left\{-z\sum_{i=1}^{\infty}(\alpha_i+\beta_i)\right\}\prod_{i=1}^{\infty}\frac{1+\beta_i z}{1-\alpha_i z}, \quad z\in\boldsymbol{C}.$$

(Note that $\sum_{i=1}^{\infty}(\alpha_i+\beta_i)\leq 1$ follows from Fatou's lemma.) These unique limit points give (4.13). □

Lemma 4.6. *Let $\{c_i(n)\}_{(i,n)\in\boldsymbol{N}^2}$ satisfy*

$$c_1(n)\geq c_2(n)\geq\cdots\geq 0 \qquad \textit{for any } n,$$

$$\sum_{i=1}^{\infty}c_i(n)\leq n \qquad \textit{for any } n,$$

$$\lim_{n\to\infty}\frac{c_i(n)}{n}=c_i \qquad \textit{for any } i.$$

Then we have

$$\lim_{n\to\infty}\sum_{i=1}^{\infty}\left(\frac{c_i(n)}{n}\right)^k=\sum_{i=1}^{\infty}c_i^k, \qquad k\in\{2,3,\dots\}.$$

The proof is elementary and omitted. We note, however, that it can fail to hold for $k=1$.

Remark 4.7. Along a path chosen in Theorem 4.5, we saw that $\sum_{\zeta\in\widehat{T}}B_\zeta = 1$ holds for B_ζ defined in (4.12). It is possible to have the situation that $\sum_{\zeta\in\widehat{T}}B_\zeta < 1$ along other paths. In fact, this is the case where normalized irreducible characters of $\mathfrak{S}_n(T)$ converge to a discontinuous function on $\mathfrak{S}_\infty(T)$. See [**8**, Section 6] for more details.

Theorem 4.8 (Recapturing the character formula for $\mathfrak{S}_\infty(T)$). *Let a character $f \in E(\mathfrak{S}_\infty(T))$ be given. Take the corresponding extremal central probability M on $\mathfrak{T}(\boldsymbol{Y}(T))$ in Theorem* 4.2 *and parameters $\alpha_{\zeta,\epsilon,i}$, B_ζ in Theorem* 4.5. *Set*

$$\mu_\zeta = B_\zeta - \sum_{i=1}^{\infty} \sum_{\epsilon\in\{0,1\}} \alpha_{\zeta,\epsilon,i}, \qquad \zeta \in \widehat{T}. \tag{4.21}$$

Then f is completely characterized by these parameters

$$\alpha_{\zeta,\epsilon,i}, \quad \mu_\zeta; \qquad \zeta \in \widehat{T},\ \epsilon \in \{0,1\},\ i \in \boldsymbol{N}$$

so that its values on the basic elements of $\mathfrak{S}_\infty(T)$ are given by

$$f(s,(q)) = \sum_{\zeta\in\widehat{T}} \left(\sum_{i=1}^{\infty} \sum_{\epsilon\in\{0,1\}} \frac{\alpha_{\zeta,\epsilon,i}}{\dim\zeta} + \frac{\mu_\zeta}{\dim\zeta} \right) \chi_\zeta(s), \qquad s \in T, \tag{4.22}$$

$$f(d,\sigma) = \sum_{\zeta\in\widehat{T}} \left\{ \sum_{i=1}^{\infty} \sum_{\epsilon\in\{0,1\}} (-1)^{\epsilon(k-1)} \left(\frac{\alpha_{\zeta,\epsilon,i}}{\dim\zeta} \right)^k \right\} \chi_\zeta(P_\sigma(d)), \tag{4.23}$$

where $\sigma \in \mathfrak{S}_\infty$ is a k-cycle, $k \geq 2$, and $d \in D(T)$ satisfies $\operatorname{supp} d \subset \operatorname{supp}\sigma$. *($P_\sigma(d)$ is defined in* (1.3).*)*

Proof. Equation (4.22) immediately follows from (4.16) and (4.21). Consider the Fourier expansion

$$f((s,e,\cdots,e),\sigma) = \sum_{\zeta\in\widehat{T}} C_\zeta \widetilde{\chi}_\zeta(s), \qquad s \in T.$$

Since (4.17) converges uniformly to this, (4.11) yields

$$\begin{aligned} C_\zeta &= \lim_{n\to\infty} \frac{|t(n)^\zeta|(|t(n)^\zeta|-1)\cdots(|t(n)^\zeta|-k+1)}{n(n-1)\cdots(n-k+1)} \frac{1}{(\dim\zeta)^{k-1}} \widetilde{\chi}^{t(n)^\zeta}_{(k,1^{|t(n)^\zeta|-k})} \\ &= \frac{1}{(\dim\zeta)^{k-1}} \sum_{i=1}^{\infty} \sum_{\epsilon\in\{0,1\}} (-1)^{\epsilon(k-1)} \alpha^k_{\zeta,\epsilon,i}. \end{aligned}$$

We hence obtain (4.23) for $d = (s, e, \cdots, e)$. Since f is a central function, we see it is enough to take $[s] = P_\sigma(d)$, recalling structure of the conjugacy classes of $\mathfrak{S}_\infty(T)$ described in Subsection 1.2.

Finally, we know that $f \in E(G)$ is completely determined by the values on the basic elements since it is factorizable (see [**6**, Section 4]). □

REMARK 4.9. Let us consider a special situation where all $\alpha_{\zeta,\epsilon,i}$'s are 0. In the case of $\mathfrak{S}_\infty$, this condition means that we treat the regular character (= the delta function at the identity element) of $\mathfrak{S}_\infty$ and the Plancherel measure on the path space $\mathfrak{T}$ of the Young graph. It is well known that typical Young diagrams in the Plancherel ensemble are balanced, i.e. row and column lengths of $\lambda \in \boldsymbol{Y}_n$ are proportional to $\sqrt{n}$. Then, the quantities of (4.13) obviously vanish along growing typical Young diagrams. The Plancherel measure is no longer captured as a probability if T is a continuous group. For general T, the situation of all $\alpha_{\zeta,\epsilon,i}$'s being 0 and an associated growth process on the branching graph $\boldsymbol{Y}(T)$ are described as follows. Let $(B_\zeta)_{\zeta\in\widehat{T}}$ satisfy $B_\zeta \geq 0$ and $\sum_{\zeta\in\widehat{T}} B_\zeta = 1$ so that it gives a probability on $\widehat{T}$ with an at most countable support. Let ψ be the continuous positive-definite central normalized function on T which has Fourier coefficients B_ζ:

$$\psi(t) = \sum_{\zeta\in\widehat{T}} \frac{B_\zeta}{\dim\zeta}\chi_\zeta(t), \qquad t \in T \tag{4.24}$$

(see Proposition 4.1). We consider $f \in E(\mathfrak{S}_\infty(T))$ determined by

$$\begin{aligned} f(t,(q)) &= \psi(t), && t \in T, \\ f(d,\sigma) &= 0, && \text{if } \sigma \text{ is a nontrivial cycle of } \mathfrak{S}_\infty \end{aligned} \tag{4.25}$$

at basic elements $(t,(q))$ and (d,σ) respectively, and multiplicatively extended to the whole $\mathfrak{S}_\infty(T)$. Then the extremal harmonic function φ on $\boldsymbol{Y}(T)$ corresponding to f in (4.25) (see Theorem 4.2) is given by

$$\varphi(\Lambda) = \prod_{\zeta\in\widehat{T}} \frac{B_\zeta^{|\lambda^\zeta|}\dim\lambda^\zeta}{|\lambda^\zeta|!\,(\dim\zeta)^{|\lambda^\zeta|}}, \qquad \Lambda = (\lambda^\zeta) \in \boldsymbol{Y}(T).$$

It can be seen that the corresponding central probability on the path space $\mathfrak{T}(\boldsymbol{Y}(T))$ induces a system of parallel Plancherel growth processes parametrized by $\zeta \in \widehat{T}$ for which the chain switches from one to another according to the probability $(B_\zeta)_{\zeta\in\widehat{T}}$. This growth process canonically associated with the wreath product group $\mathfrak{S}_\infty(T)$ seems to be interesting and will be treated in separate papers.

Remark 4.10. In this section we treated the branching graph $\boldsymbol{Y}(T)$ to obtain the characters of $G=\mathfrak{S}_\infty(T)$. Let T be a compact abelian group and S its subgroup. Set

$$G^S = D_\infty(T)^S \rtimes \mathfrak{S}_\infty, \quad D_\infty(T)^S = \left\{ d=(t_i)_{i\in\boldsymbol{N}} \in D_\infty(T) \,\middle|\, \prod_{i\in\boldsymbol{N}} t_i \in S \right\},$$

and call it a canonical subgroup of G. It is the inductive limit of

$$G_n^S = D_n(T)^S \rtimes \mathfrak{S}_n, \quad D_n(T)^S = \left\{ d=(t_i)_{i=1,\dots,n} \in D_n(T) \,\middle|\, \prod_{i=1}^{n} t_i \in S \right\}$$

as $n\to\infty$. The character formula for G^S is studied in [**5**], [**6**] and [**8**]. For IUR Π of G_{n+1}^S, the branching rule of $\Pi\big|_{G_n^S}$ is described in [**8**, Section 8]. We thus obtain the branching graph $\boldsymbol{Y}(T)^S$ for G^S by modifying $\boldsymbol{Y}(T)$. For example, let T be $\boldsymbol{Z}_2$ and S its trivial subgroup. This describes the case of Weyl groups of type B/C and D. An IUR of $W_{B_n/C_n}=\mathfrak{S}_n(\boldsymbol{Z}_2)$ corresponding to a pair (λ^0,λ^1), where $|\lambda^0|+|\lambda^1|=n$, splits into two IURs of $W_{D_n}=\mathfrak{S}_n(\boldsymbol{Z}_2)^{\{e\}}$ if and only if λ^0 coincides with λ^1. Moreover, (λ^0,λ^1) and (λ^1,λ^0) correspond to equivalent IURs of W_{D_n} if $\lambda^0\neq\lambda^1$. In Figure 1, an IUR of W_{B_n/C_n} which splits into two IURs of W_{D_n} is specified by using boldface for its dimension. Applying the general theory in Section 2 and Section 3 to $\boldsymbol{Y}(T)^S$, we have a similar result to Theorem 4.3, namely, any character of G^S is obtained as a limit of normalized irreducible characters of G_n^S as $n\to\infty$ along some path on the branching graph $\boldsymbol{Y}(T)^S$. This fact was proved in [**8**, Theorem 8.6] while we see here its probabilistic aspect.

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