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K3 surfaces and sphere packings

Dedicated to Professor F. Hirzebruch for his 80th birthday

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Abstract. We determine the structure of the Mordell-Weil lattice, Néron-Severi lattice and the lattice of transcendental cycles for certain elliptic K3 surfaces. We find that such questions from algebraic geometry are closely related to the sphere packing problem, and a key ingredient is the use of the sphere packing bounds in establishing geometric results.

1. Introduction.

Let X be a complex smooth projective K3 surface. We consider three kinds of lattices attached to the surface X:

$$\mathrm{NS}(X), \quad \mathrm{T}(X), \quad \mathrm{MW}(X). \tag{1.1}$$

The Néron-Severi lattice $\mathrm{NS}(X)$ is the sublattice of $H^2(X, \boldsymbol{Z})$ consisting of algebraic cycles. The lattice of transcendental cycles $\mathrm{T}(X)$ is, by definition, its orthogonal complement in $H^2(X, \boldsymbol{Z})$.

The third lattice $\mathrm{MW}(X)$ is defined when $X = (X, f)$ is an elliptic surface given with an elliptic fibration $f : X \to \boldsymbol{P}^1$ with a section, and $\mathrm{MW}(X)$ denotes the group of sections of f, or equivalently, the group $E(k(t))$ of $k(t)$-rational points of the generic fibre E of f. (Throughout the paper, we let $k = \boldsymbol{C}$, the field of complex numbers, unless otherwise mentioned.) The height pairing $\langle , \rangle$ defines the structure of a positive-definite lattice on $\mathrm{MW}(X)$ modulo torsion, which is the Mordell-Weil lattice (MWL) of X. By abuse of language, we often call $\mathrm{MW}(X)$ itself the "Mordell-Weil lattice" of $X = (X, f)$, by which we mean the pair $(\mathrm{MW}(X), \langle , \rangle)$. We refer to [**12**] for the basic facts on MWL.

In this paper, we study the structure of three lattices (1.1) by taking X to be the elliptic K3 surface $F^{(n)}_{\alpha,\beta}$ defined by the Weierstrass equation:

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$$F^{(n)}_{\alpha,\beta}: \ y^2 = x^3 - 3\alpha x + \left(t^n + \frac{1}{t^n} - 2\beta\right) \tag{1.2}$$

where α, β are arbitrary complex numbers and $n = 1, 2, \ldots, 6$. The main results will be stated in the next section, which will settle the questions in our previous paper [**14**] in an improved form.

It is known (Inose [**3**]) that $F^{(2)}_{\alpha,\beta}$ is isomorphic to the Kummer surface of the product of two elliptic curves C_1, C_2:

$$F^{(2)}_{\alpha,\beta} \cong \mathrm{Km}(C_1 \times C_2) \tag{1.3}$$

where (α, β) and the absolute invariants j_1, j_2 of C_1, C_2 are related by

$$\alpha^3 = j_1 j_2, \quad \beta^2 = (1 - j_1)(1 - j_2). \tag{1.4}$$

[N.B. The absolute invariant j is normalized so that $j = 1$ for $y^2 = x^3 - x$ and $j = 0$ for $y^2 = x^3 - 1$.]

The rank $r^{(n)} = r^{(n)}_{\alpha,\beta}$ of the Mordell-Weil lattice $\mathrm{MW}(F^{(n)}_{\alpha,\beta}) = F^{(n)}_{\alpha,\beta}(k(t))$ is given by the following formula (see Section 4, cf. [**6**], [**14**]):

$$r^{(n)} = h + Min\{4(n-1), 16\} - \begin{cases} 0 & \text{if } j_1 \neq j_2 \\ n & \text{if } j_1 = j_2 \neq 0, 1 \\ 2n & \text{if } j_1 = j_2 = 0 \text{ or } 1 \end{cases} \tag{1.5}$$

where

$$h = \mathrm{rk}\,\mathrm{Hom}(C_1, C_2); \tag{1.6}$$

we have $h = 0, 1$ or 2 when $k = \boldsymbol{C}$.

Notation. Throughout the paper, we keep the above notation. Further, given a lattice $L = (L, \langle, \rangle)$ and a nonzero rational number m, $L[m]$ denotes the lattice $(L, m\langle, \rangle)$. We denote by $\det L$ the absolute value of the determinant of the Gram matrix for L; thus we have $\det L[m] = (\det L) m^r$ if r is the rank of L and $m > 0$. For general facts on lattices and sphere packings, we refer to the standard book of Conway-Sloane [**1**].

For the singular fibres of an elliptic surface, we use freely the results from [**5**] and [**18**].

2. The main results.

THEOREM 2.1. *For any α, β and any $n \leq 6$, we have a lattice isomorphism:*

$$T\big(F_{\alpha,\beta}^{(n)}\big) \cong T\big(F_{\alpha,\beta}^{(1)}\big)[n]. \tag{2.1}$$

In particular, we have

$$\det T\big(F_{\alpha,\beta}^{(n)}\big) = \det T\big(F_{\alpha,\beta}^{(1)}\big) \cdot n^{\lambda}, \quad \lambda = 4 - h. \tag{2.2}$$

THEOREM 2.2.

$$\det \mathrm{NS}\big(F_{\alpha,\beta}^{(n)}\big) = \det \mathrm{Hom}(C_1, C_2) \cdot n^{\lambda}. \tag{2.3}$$

In the above, $\mathrm{Hom}(C_1, C_2)$ is viewed as a positive-definite even integral lattice by defining the norm of $\varphi : C_1 \to C_2$ to be $2\deg(\varphi)$.

THEOREM 2.3. *The Mordell-Weil group $\mathrm{MW}(F_{\alpha,\beta}^{(n)})$ is torsion-free except for the cases (a) and (b) below, and then we have*

$$\det \mathrm{MW}\big(F_{\alpha,\beta}^{(n)}\big) = \frac{\det \mathrm{Hom}(C_1, C_2) \cdot n^{\lambda}}{c(n)^2 d^n} \tag{2.4}$$

where $c(n) \in \{1, 3, 4\}$ and $d \in \{1, 2, 3, 4\}$ are defined by (4.11) in Section 4.

The exception is: (a) $j_1 = j_2 = 0$ and $n = 2, 4, 6$, and (b) $j_1 = j_2 = 1$ and $n = 3, 6$. In case (a) [or (b)], the RHS of (2.4) is to be multiplied by 3^2 [or 4^2] which is the square of the order of the torsion part $\mathrm{MW}(F_{\alpha,\beta}^{(n)})_{tor}$.

THEOREM 2.4. *Assume that C_1, C_2 are not isogenous, i.e. $h = 0$. Then the lattice structure of $\mathrm{MW}(F_{\alpha,\beta}^{(n)})$ is independent of α, β and depends only on n; let us denote it by $\mathrm{MW}(F_{gen}^{(n)})$ or $M_{gen}^{(n)}$. This lattice $M_{gen}^{(n)}$ has the invariants in Table 1 (where μ and δ denote the minimal norm and the center density of a lattice.)*

n	1	2	3	4	5	6
rk	0	4	8	12	16	16
det	1	$2^4/3^2$	$3^4/4^2$	$4^4/3^2$	5^4	6^4
μ	-	$4/3$	2	$8/3$	4	4
δ	-	$1/12$	$1/6^2$	$2^2/3^5$	$1/5^2$	$1/6^2$

Table 1. Invariants of $M_{gen}^{(n)}$.

Theorem 2.5. *Assume that $h = 0$. Then, for any $n \le 6$, the Mordell-Weil lattice* $\mathrm{MW}(F_{\alpha,\beta}^{(n)}) \cong M_{gen}^{(n)}$ *is generated by the $k(t)$-rational points $P = (x, y)$ of the following form*:

$$
\begin{aligned}
x &= \frac{A_0 + A_1 t + A_2 t^2 + A_3 t^3 + A_4 t^4}{t^2}, \\
y &= \frac{B_0 + B_1 t + B_2 t^2 + \cdots + B_6 t^6}{t^3}.
\end{aligned}
\tag{2.5}
$$

More precisely, if $n = 2$, a set of generators is given by

$$
\left(\gamma, t + \frac{1}{t}\right), \quad \left(\gamma', t - \frac{1}{t}\right) \tag{2.6}
$$

where γ (or γ') runs over the 3 *roots of the cubic equation*:

$$
x^3 - 3\alpha x - 2 - 2\beta = 0 \quad (\textit{or } x^3 - 3\alpha x + 2 - 2\beta = 0). \tag{2.7}
$$

If $n > 2$, let $L = \mathrm{MW}(F_{\alpha,\beta}^{(n)+})[2]$ *be the sublattice of half rank in* $M = \mathrm{MW}(F_{\alpha,\beta}^{(n)})$, *obtained from a rational elliptic surface $F_{\alpha,\beta}^{(n)+}$ over the s-line where $s = t + 1/t$ (cf.* (5.2), (5.3), *Lemma* 5.1). *Then M is generated by L and L^σ for a suitable automorphism σ of M. In other words, M is generated by $\{P_i\}$ and $\{P_i^\sigma\}$, if we take a set of generators $\{P_i = (x, y)\}$ of L of the form*

$$
x = a_0 + a_1 s + a_2 s^2, \quad y = b_0 + b_1 s + b_2 s^2 + b_3 s^3. \tag{2.8}
$$

Theorem 2.6. *For any α, β such that $j_1 \ne j_2$ and for any $n \le 6$, the Mordell-Weil lattice* $M = \mathrm{MW}(F_{\alpha,\beta}^{(n)})$ *contains a direct sum $M_0 \oplus M_1$ as a sublattice of finite index where*

$$
M_0 := \mathrm{MW}\big(F_{\alpha,\beta}^{(1)}\big)[n] \cong \mathrm{Hom}(C_1, C_2)[n] \tag{2.9}
$$

and where $M_1 \cong M_{gen}^{(n)}$ has a set of generators induced from the sections of rational elliptic surfaces.

3. Preliminaries.

Let us recall some general results on the relationship of the lattices in (1.1).

3.1. About $\mathrm{NS}(X)$ **and** $\mathrm{T}(X)$**.**

First suppose that X is a complex smooth projective surface with torsion-free $H^2(X, \boldsymbol{Z})$. Then it is well-known that the lattice $H^2(X, \boldsymbol{Z})$ (with respect to cup-product pairing) is unimodular by the Poincaré duality. $\mathrm{NS}(X)$ is a primitive sublattice of $H^2(X, \boldsymbol{Z})$, since the exponential sequence induces an injection of the quotient group $H^2(X, \boldsymbol{Z})/\mathrm{NS}(X)$ into $H^2(X, \mathscr{O}_X)$. Further $\mathrm{T}(X)$ is also a primitive sublattice of $H^2(X, \boldsymbol{Z})$ (as the orthogonal complement of $\mathrm{NS}(X)$ in $H^2(X, \boldsymbol{Z})$), and we have

$$\det \mathrm{T}(X) = \det \mathrm{NS}(X). \tag{3.1}$$

The rank of $\mathrm{T}(X)$ is called the Lefschetz number, denoted by $\lambda(X)$. Obviously we have

$$\lambda(X) = b_2(X) - \rho(X) \tag{3.2}$$

where $\rho(X) = \mathrm{rkNS}(X)$ is the Picard number of X.

Lemma 3.1.

(i) *The lattice* $\mathrm{T}(X)$ *is a birational invariant of a surface* X*.*

(ii) $\mathrm{T}(X) \otimes \boldsymbol{Q}$ *is the smallest* $\boldsymbol{Q}$*-subHodge structure of* $H^2(X, \boldsymbol{Q})$ *such that* $\mathrm{T}(X) \otimes \boldsymbol{C}$ *contains* $H^{2,0}(X) = H^0(X, \Omega^2)$*.*

Proof.

(i) It is easy to show that, if $\beta : X' \to X$ is a blowing-up of X at one point, then $\beta^* : H^2(X, \boldsymbol{Z}) \to H^2(X', \boldsymbol{Z})$ induces a lattice isomorphism of $\mathrm{T}(X)$ onto $\mathrm{T}(X')$. This implies the assertion (i), since any birational map is composed of finitely many blowing-up and blowing-down by surface theory. [N.B. It follows that one can speak of the lattice of transcendental cycles $\mathrm{T}(Z)$ on any irreducible surface Z which may not be smooth nor projective.]

(ii) $\mathrm{NS}(X) \otimes \boldsymbol{Q}$ is the largest $\boldsymbol{Q}$-subHodge structure of $H^2(X, \boldsymbol{Q})$ contained in $H^{1,1}$ by the Lefschetz-Hodge theorem. Taking the orthogonal complement, one gets the assertion (ii). □

Lemma 3.2. *Suppose* $\pi : X \to Y$ *is a rational map of finite degree between surfaces with the same geometric genus* $p_g(X) = p_g(Y) > 0$*. Then we have* $\lambda(X) = \lambda(Y)$*, and there exist natural homomorphisms* $\pi^* : \mathrm{T}(Y) \to \mathrm{T}(X)$ *and* $\pi_* : \mathrm{T}(X) \to \mathrm{T}(Y)$ *satisfying the two-way projection formulas*

$$\pi_* \circ \pi^* = \deg(\pi), \quad \pi^* \circ \pi_* = \deg(\pi). \tag{3.3}$$

Moreover we have

$$\pi^*\mathrm{T}(Y) \cong \mathrm{T}(Y)[\deg(\pi)], \quad \pi_*\mathrm{T}(X) \cong \mathrm{T}(X)[\deg(\pi)]. \tag{3.4}$$

Proof (cf. Inose [**2**, Proposition 1.1]). Given a rational map $\pi : X \to Y$, there is a birational morphism $\psi : X' \to X$ such that $\pi \circ \psi : X' \to Y$ is a morphism. By (i) of Lemma 3.1, we can identify $\mathrm{T}(X) = \mathrm{T}(X')$, and hence, by replacing X by X', we can assume $\pi : X \to Y$ is a morphism of finite degree. Then we consider the restriction of the standard homomorphisms π^*, π_* between $H^2(X, \boldsymbol{Z})$ and $H^2(Y, \boldsymbol{Z})$ to $\mathrm{T}(Y)$ or $\mathrm{T}(X)$. Observe that $\pi^*\mathrm{T}(Y) \otimes \boldsymbol{C}$ contains $\pi^* H^{2,0}(Y)$ which coincides with $H^{2,0}(X)$ by assumption. Hence, by (ii), we have $\pi^*\mathrm{T}(Y) \otimes \boldsymbol{Q} = \mathrm{T}(X) \otimes \boldsymbol{Q}$. It follows that $\lambda(X) = \lambda(Y)$ and $\pi^*\mathrm{T}(Y) \subset \mathrm{T}(X)$, $\pi_*\mathrm{T}(X) \subset \mathrm{T}(Y)$. Now the first projection formula of (3.3) is obvious (as restriction), while the second one easily follows from the first one since $\pi^*\mathrm{T}(Y)$ has finite index in $\mathrm{T}(X)$. (3.4) follows immediately from (3.3). □

Lemma 3.3. *Suppose $\{X_w\}$ is a smooth family of surfaces parametrized by an irreducible variety W. Let w be a generic point of W. Then, for any point $w_0 \in W$, there is an inclusion of lattices*:

$$\mathrm{NS}(X_w) \hookrightarrow \mathrm{NS}(X_{w_0}) \subset H^2(X_{w_0}, \boldsymbol{Z}) \cong H^2(X_w, \boldsymbol{Z}) \tag{3.5}$$

In particular, if $\rho(X_{w_0}) = \rho(X_w)$, then we have $\mathrm{NS}(X_w) \cong \mathrm{NS}(X_{w_0})$ *and* $\mathrm{T}(X_w) \cong \mathrm{T}(X_{w_0})$.

Proof. By choosing a specialization $X_w \to X_{w_0}$ and considering the specialization of cycles, we have an injection $\mathrm{NS}(X_w) \hookrightarrow \mathrm{NS}(X_{w_0})$ preserving intersection pairing, hence the first assertion. The second assertion follows from the primitivity of $\mathrm{NS}(X_w)$ in $H^2(X_w, \boldsymbol{Z})$. □

3.2. Mordell-Weil lattices.

Next suppose that $X = (X, f)$ is an elliptic surface with a section. Then the Mordell-Weil group $\mathrm{MW}(X)$ is isomorphic to a quotient group of $\mathrm{NS}(X)$ (cf. [**12**, Theorem 1.3]):

$$\mathrm{MW}(X) \cong \mathrm{NS}(X)/\mathrm{Triv}(X) \tag{3.6}$$

where $\mathrm{Triv}(X)$ is the trivial sublattice of $\mathrm{NS}(X)$ which is generated by the zero-section (O) and all the irreducible components of fibres of f. In particular, the rank r of $\mathrm{MW}(X)$ is given by

$$r = \rho(X) - \mathrm{rk}\ \mathrm{Triv}(X). \tag{3.7}$$

The structure of the Mordell-Weil lattice is defined by sending $\mathrm{MW}(X)$ naturally into the orthogonal complement of $\mathrm{Triv}(X)$ in $\mathrm{NS}(X) \otimes \boldsymbol{Q}$, and the height pairing is defined by using the intersection theory on the elliptic surface X (see [**12**, Section 8]). In particular, we have

LEMMA 3.4. *The determinant of the Mordell-Weil lattice is given by*

$$\frac{\det(\mathrm{MW}(X)/\mathrm{MW}(X)_{tor})}{|\mathrm{MW}(X)_{tor}|^2} = \frac{\det \mathrm{NS}(X)}{\det \mathrm{Triv}(X)}. \tag{3.8}$$

If $\mathrm{MW}(X)$ *is torsion-free, this simplifies to*:

$$\det \mathrm{MW}(X) = \frac{\det \mathrm{NS}(X)}{\det \mathrm{Triv}(X)}. \tag{3.9}$$

PROOF. This follows easily from [**12**, Theorem 8.7]. □

LEMMA 3.5. *Suppose* $\{X_w\}$ *is a smooth family of* (*smooth*) *elliptic surfaces parametrized by an irreducible variety* W. *Let* w *be a generic point of* W. *Then, for any point* $w_0 \in W$ *such that the trivial lattice stays the same under the specialization* $w \to w_0$ *i.e.* $\mathrm{Triv}(X_w) \cong \mathrm{Triv}(X_{w_0})$ *compatibly with* $\mathrm{NS}(X_w) \hookrightarrow \mathrm{NS}(X_{w_0})$, *there is an inclusion of lattices*:

$$\mathrm{MW}(X_w) \hookrightarrow \mathrm{MW}(X_{w_0}) \tag{3.10}$$

If, in addition, the rank of $\mathrm{MW}(X_{w_0})$ *is equal to that of* $\mathrm{MW}(X_w)$ [*equivalently, if* $\rho(X_{w_0}) = \rho(X_w)$], *then we have* $\mathrm{MW}(X_w) \cong \mathrm{MW}(X_{w_0})$.

PROOF. This follows from Lemma 3.3 by using the relation (3.6) of NS and MW as groups and by noting that the structure of $\mathrm{MW}(X)$ as lattice is completely determined by the embedding of $\mathrm{Triv}(X)$ into $\mathrm{NS}(X)$ as recalled above. □

4. Review on the elliptic K3 surfaces $F^{(n)}_{\alpha,\beta}$.

For a K3 surface X, we have

$$H^2(X, \boldsymbol{Z}) \cong U^{\oplus 3} \oplus E_8[-1]^2, \quad b_2 = 22 \tag{4.1}$$

where U is an even unimodular lattice of rank 2 and E_8 is the root lattice of rank

8 (the unique positive-definite even unimodular lattice of rank 8) (cf. [**1**]). Thus the signature of $H^2(X, \boldsymbol{Z})$ is $(3, 19)$, while that of the Néron-Severi lattice $\mathrm{NS}(X)$ is $(1, \rho - 1)$ by the Hodge index theorem.

If $X = \mathrm{Km}(C_1 \times C_2)$ is a Kummer surface of the product of elliptic curves, then

$$\rho(X) = h + 18, \quad \lambda(X) = 4 - h \tag{4.2}$$

where h is the rank of $\mathrm{Hom}(C_1, C_2)$.

Now let us consider the elliptic K3 surfaces $F^{(n)} = F^{(n)}_{\alpha,\beta} (n \leq 6)$ defined by (1.2). Clearly $F^{(n)}$ is obtained from $F^{(1)}$ by the base change $t \to t^n$; in other words, there is a rational map from $F^{(n)}$ to $F^{(1)}$ of degree n

$$\pi = \pi_n : F^{(n)} \to F^{(1)}, \quad (x, y, t) \mapsto (x, y, t^n). \tag{4.3}$$

Similarly, for any divisor m of n, there is a rational map $\pi_{n,m}$ of degree n/m:

$$\pi_{n,m} : F^{(n)} \to F^{(m)}, \quad (x, y, t) \mapsto (x, y, t^{n/m}). \tag{4.4}$$

Applying Lemma 3.1 to this situation, it follows from (1.3) and (4.2) that $F^{(n)}_{\alpha,\beta}$ have the same Lefschetz number $\lambda = 4 - h$ for all $n \leq 6$. Hence we have:

$$\rho(F^{(n)}_{\alpha,\beta}) = h + 18, \quad \lambda(F^{(n)}_{\alpha,\beta}) = 4 - h \quad (n \leq 6). \tag{4.5}$$

On the other hand, the singular fibres of the elliptic surface $F^{(n)}_{\alpha,\beta}$ are determined as follows. For $n = 1$, there are two reducible fibres of type II^* at $t = 0$ and $t = \infty$, and (i) no other reducible fibres if

$$D(\alpha, \beta) := (1 + \alpha^3 - \beta^2)^2 - 4\alpha^3 = (\alpha^3 - (1 - \beta)^2)(\alpha^3 - (1 + \beta)^2) \tag{4.6}$$

does not vanish. If $D(\alpha, \beta) = 0$, then (ii) a reducible fibre of type I_2 at $t = 1$ (or $t = -1$) if $\alpha \cdot \beta \neq 0$ and $\alpha^3 = (1 - \beta)^2$ (or $\alpha^3 = (1 + \beta)^2$); (iii) a reducible fibre of type IV at $t = \beta$ if $\alpha = 0$ and $\beta = 1$ or $\beta = -1$; (iv) two reducible fibres of type I_2 at $t = 1, -1$ if $\beta = 0$.

Note that the above conditions for (α, β) are respectively equivalent to the following conditions on j_1, j_2 in view of the relation (1.4):

$$(i)\ j_1 \neq j_2, \quad (ii)\ j_1 = j_2 \neq 0, 1, \quad (iii)\ j_1 = j_2 = 0, \text{ or } (iv)\ j_1 = j_2 = 1. \tag{4.7}$$

For $n > 1$, note that the base change $t \to t^n$ is ramified only at $t = 0$ and $t = \infty$, which implies that $F^{(n)}_{\alpha,\beta}$ has, at $t = 0$ and $t = \infty$, fibres of type IV^*, I_0^*, IV, II, I_0 according as $n = 2, \dots, 6$. At $t \neq 0, \infty$ where $t \to t^n$ is unramified, any fibre of $F^{(1)}$ induces n fibres of the same type on $F^{(n)}$. Hence the trivial lattice of $F^{(n)}_{\alpha,\beta}$ (cf. [**14**, Section 1]) is given as follows:

For any $n \leq 6$, we have

$$\operatorname{Triv}\bigl(F^{(n)}_{\alpha,\beta}\bigr) = U \oplus V^{(n)}[-1] \tag{4.8}$$

where $V^{(n)}$ is a sum of the root lattices: in case (i) (i.e. if $j_1 \neq j_2$), then

$$V^{(n)} = E_8^{\oplus 2}, E_6^{\oplus 2}, D_4^{\oplus 2}, A_2^{\oplus 2}, \{0\}, \{0\} \tag{4.9}$$

according as $n = 1, \dots, 6$. Further $V^{(n)}$ has an additional factor $A_1^{\oplus n}$ in case (ii), $A_2^{\oplus n}$ in case (iii), and $A_1^{\oplus 2n}$ in case (iv).

Therefore the determinant of the trivial lattice is given by the following formula:

$$\det \operatorname{Triv}\bigl(F^{(n)}_{\alpha,\beta}\bigr) = c(n)^2 d^n \tag{4.10}$$

where $c(n)$ and d are integers defined as follows according to the cases:

$$c(n) = \begin{cases} 1 & (n = 1, 5, 6) \\ 3 & (n = 2, 4) \\ 4 & (n = 3) \end{cases} \quad \text{and} \quad d = \begin{cases} 1 & \text{if } j_1 \neq j_2 \\ 2 & \text{if } j_1 = j_2 \neq 0, 1 \\ 3 & \text{if } j_1 = j_2 = 0 \\ 4 & \text{if } j_1 = j_2 = 1. \end{cases} \tag{4.11}$$

The rank formula (1.5) stated in Introduction follows immediately from (3.7), (4.5) and the above information about the trivial lattices.

5. Rational elliptic surfaces $F^{(n)+}_{\alpha,\beta}$.

Let us introduce auxiliary rational elliptic surfaces, to be denoted below by $F^{(n)+}_{\alpha,\beta} (n \leq 6)$. Since we know the MWL of rational elliptic surfaces quite well, it can be used for studying the MWL of elliptic K3 surfaces $F^{(n)}_{\alpha,\beta}$.

Letting $s = t + 1/t$, we have

$$t^n + \frac{1}{t^n} = \begin{cases} s^2 - 2 & (n = 2) \\ s^3 - 3s & (n = 3) \\ (s^2 - 2)^2 - 2 & (n = 4) \\ s^5 - 5s^3 + 5s & (n = 5) \\ (s^3 - 3s)^2 - 2 & (n = 6). \end{cases} \tag{5.1}$$

Let $E = F_{\alpha,\beta}^{(n)+}/k(s)$ denote the elliptic curve:

$$E = F_{\alpha,\beta}^{(n)+} : y^2 = x^3 - 3\alpha x + (s^n - ns^{n-2} + \cdots - 2\beta), \tag{5.2}$$

such that $E \otimes_{k(s)} k(t) \cong F_{\alpha,\beta}^{(n)}$. For all $n \leq 6$, this defines a rational elliptic surface, denoted by the same symbol $F_{\alpha,\beta}^{(n)+}$. We let

$$L := E(k(s)) = \mathrm{MW}\big(F_{\alpha,\beta}^{(n)+}\big) \tag{5.3}$$

be its Mordell-Weil lattice.

Lemma 5.1. *Assume $j_1 \neq j_2$. Then*

$$L = \mathrm{MW}\big(F_{\alpha,\beta}^{(n)+}\big) \cong \{0\}, A_2^*, D_4^*, E_6^*, E_8, E_8 \tag{5.4}$$

according as $n = 1, 2, 3, 4, 5, 6$ (A_2^* indicates the dual lattice of root lattice A_2, etc.). The minimal norm of $L[2]$, i.e. L viewed as a sublattice of $\mathrm{MW}(F_{\alpha,\beta}^{(n)})$, is equal to 4/3, 2, 8/3, 4, 4 *according as $n = 2, 3, 4, 5, 6$.*

Proof. The MWL of a rational elliptic surface (with a section) is determined by the trivial lattice and its embedding into E_8 (see [**10**]). In the case under consideration, the trivial lattice is given by the "half" of $V^{(n)}$ in (4.9), which immediately implies (5.4). □

Lemma 5.2. *Assume $j_1 \neq j_2$. Then, for any non-zero $P \in \mathrm{MW}(F_{\alpha,\beta}^{(n)})$, we have*

$$\langle P, P \rangle \geq 4,\ \frac{4}{3},\ 2,\ \frac{8}{3},\ 4,\ 4 \tag{5.5}$$

according as $n = 1, 2, 3, 4, 5, 6$. In particular, $\mathrm{MW}(F_{\alpha,\beta}^{(n)})$ is torsion-free.

PROOF. By the height formula ([**12**, Theorem 8.6]), we have

$$\langle P, P\rangle = 2\chi + 2(PO) - \sum_v contr_v(P) \tag{5.6}$$

where $\chi = 2$ for a K3 surface and the summation runs over v such that the fibre over v is a reducible fibre. Under the assumption $j_1 \neq j_2$, we can have reducible fibres only at $t = 0$ and $t = \infty$, and they are both of type $II^*, IV^*, I_0^*, IV, II, I_0$ according as $n = 1, 2, \dots, 6$ as recalled in the previous section. The local contribution for these types, if non-zero, are respectively given by

$$contr_v(P) = 0,\ 4/3,\ 1,\ 2/3,\ 0,\ 0, \tag{5.7}$$

hence the assertion follows since the intersection number (PO) of the two sections $(P), (O)$ is a non-negative integer for $P \neq O$. $\square$

COROLLARY 5.3. *Assume $j_1 \neq j_2$. Then the minimal norm of* $\mathrm{MW}(F^{(n)}_{\alpha,\beta})$ *is given as follows*:

$$\mu\big(\mathrm{MW}(F^{(n)}_{\alpha,\beta})\big) = \frac{4}{3}(n=2),\ 2(n=3),\ \frac{8}{3}(n=4),\ 4(n=5,6). \tag{5.8}$$

6. Proof of Theorems 2.1, 2.2, 2.3, 2.4.

Fix $n (n \le 6)$, and let $X = F^{(n)} = F^{(n)}_{\alpha,\beta}$, $Y = F^{(1)} = F^{(1)}_{\alpha,\beta}$, and consider the rational map $\pi : X \to Y$ of degree n given by (4.3). By Lemma 3.2, we have

$$\mathrm{T}(X) \supset \pi^*\mathrm{T}(Y) \supset n\mathrm{T}(X), \quad \pi^*\mathrm{T}(Y) \simeq \mathrm{T}(Y)[n]. \tag{6.1}$$

Letting $\nu = \nu(n, \alpha, \beta)$ be the index of $\pi^*\mathrm{T}(Y)$ in $T(X)$, we have

$$\det \mathrm{T}(X) = \det \mathrm{T}(Y) \cdot n^\lambda/\nu^2, \quad \nu \mid n^\lambda. \tag{6.2}$$

Theorem 2.1 is equivalent to the claim:

$$\nu = \nu(n, \alpha, \beta) = 1 \tag{6.3}$$

for any n, α, β.

6.1. Step 1: Case $h = 0$.

First let us prove the claim (6.3) assuming $h = 0$. In this case, we have $\lambda = 4$ and $\mathrm{T}(F^{(1)}) \cong U^{\oplus 2}$. The formula (6.2) becomes

$$\det \mathrm{T}(F^{(n)}) = n^4/\nu^2. \tag{6.4}$$

By (3.1), (3.9) and (4.10), this is equivalent to

$$\det \mathrm{MW}(F^{(n)}) = (n^4/c(n)^2)/\nu^2 \tag{6.5}$$

where $c(n)$ is given by (4.11).

Therefore we see that Theorems 2.1, 2.2, 2.3 are mutually equivalent to each other and to the claim $\nu = 1$, under the assumption $h = 0$. (Actually the equivalence holds true without this assumption. See Section 6.3 below.)

Now we prove $\nu = 1$, separately for each $n \leq 6$. The key is to look at the density of the sphere packing by the Mordell-Weil lattice $\mathrm{MW}(F^{(n)}_{gen})$. This idea has been used in [**14**, Section 3, Remark] for the case $n = 5$, and we adapt it to other cases as follows.

The center density of a positive-definite lattice L is defined by

$$\delta(L) = \left(\frac{1}{2}\sqrt{\mu}\right)^r / \sqrt{\det L} \tag{6.6}$$

where r is the rank, μ is the minimal norm of L (see [**1**]). Hence, if L' is a sublattice of L of finite index (say ν) such that $\mu(L') = \mu(L)$, then $\delta(L)$ is equal to $\nu \cdot \delta(L')$, since $\det L = \det L'/\nu^2$. Thus the value of δ in Table 1 (Theorem 2.4) is to be multiplied by ν at the moment, since the value of minimal norm μ given there is correct by Corollary 5.3. Thus we have the modified Table 2 where β_r in the last row denotes the (lattice) sphere packing bound in dimension r copied from [**1**, Tables 1.1 and 1.2].

n	1	2	3	4	5	6
rk	0	4	8	12	16	16
δ	-	$\nu/12$	$\nu/6^2$	$\nu \cdot 2^2/3^5$	$\nu/5^2$	$\nu/6^2$
β_r	-	1/8	1/16	0.06559	0.11774	0.11774

Table 2. Center density of $M^{(n)}_{gen}$.

Now, for each n, we should have

$$\delta \le \beta_r \quad \text{and} \quad \nu \mid n^\lambda. \tag{6.7}$$

$n = 2, 3, 5$: In these cases, if $\nu > 1$, then the center density δ would exceed the bound β_r as Table 2 shows. This gives the required conclusion $\nu = 1$. (N.B. The case $n = 2$ is classically well-known from the theory of Kummer surfaces, cf. [**2**], [**16**].)

The above argument is not sufficient in the remaining cases $n = 4$ and $n = 6$. In fact, it shows only that $\nu \le 2$ in case $n = 4$, and $\nu \le 3$ in case $n = 6$. Thus we need a different approach.

$n = 6$: In this case, the rational map $\pi : F^{(6)} \to F^{(1)}$ factors through $F^{(3)}$ on one hand, and through $F^{(2)}$ on the other hand. We apply Lemma 3.2 here. Via the first factorization, we have the inclusion of lattices:

$$T(F^{(6)}) \supset T(F^{(3)})[2] = T(F^{(1)})[3][2] = T(F^{(1)})[6] \tag{6.8}$$

of index a power of 2 (= the degree of the rational map $\pi_{6,3} : F^{(6)} \to F^{(3)}$), where the first equality follows from the case $n = 3$ proven above. Similarly, via the second factorization, we have the inclusion of lattices:

$$T(F^{(6)}) \supset T(F^{(2)})[3] = T(F^{(1)})[2][3] = T(F^{(1)})[6] \tag{6.9}$$

of index a power of 3 (= the degree of the rational map $\pi_{6,2} : F^{(6)} \to F^{(2)}$), with the first equality following from the case $n = 2$. Since 2 and 3 are relatively prime, we conclude that

$$T(F^{(6)}) \cong T(F^{(1)})[6], \tag{6.10}$$

which proves the assertion for $n = 6$.

$n = 4$: To exclude the possibility $\nu = 2$, we directly show that the lattice $M_{gen}^{(4)}$ is "2-primitive", i.e. it has no overlattice of index 2 preserving the minimal norm. We omit the computation here, and refer to Usui [**19**, IV] where this is treated in the special case $\alpha = \beta = 0$. In view of the independence result in Theorem 2.4, this is in fact sufficient. (The independence will be shown below in Step 2.)

This completes the proof of Theorems 2.1, 2.2, 2.3 under the assumption $h = 0$.

6.2. Step 2: General case.

Now we prove Theorem 2.1 for any α, β and any $n \leq 6$. For this, we use a specialization (or deformation) argument. Take and fix any α_0, β_0.

We can specialize (or deform) a general $F^{(n)}_{\alpha,\beta}$ to $F^{(n)}_{\alpha_0,\beta_0}$ in a smooth family. In fact, we have a smooth family $\{X_w \mid w \in W\}$ of smooth elliptic surfaces such that $X_w = F^{(n)}_{\alpha,\beta}$ and $X_{w_0} = F^{(n)}_{\alpha_0,\beta_0}$ where W is a smooth irreducible variety. This follows from the existence of simultaneous resolution of rational double points, due to Brieskorn and others (cf. [**13**] and the references therein). Namely, if an elliptic surface is given by a Weierstrass equation, then we can consider two models of the surface: the Weierstrass model and the Kodaira-Néron model (a smooth elliptic surface). The former is the surface defined naturally by the given Weierstrass equation, which is a normal surface with rational double points (corresponding to the reducible fibres), and the latter is obtained as the minimal resolution of the former. The theory of simultaneous resolution assures that, given a family of Weierstrass models, a suitable finite base change of the parameter space gives a smooth family of Kodaira-Néron models.

Applying Lemma 3.3 to the specialization $w \to w_0$, we obtain

$$\mathrm{NS}(X_w) \hookrightarrow \mathrm{NS}(X_{w_0}) \subset H^2(X_{w_0}, \boldsymbol{Z}) \cong H^2(X_w, \boldsymbol{Z}). \tag{6.11}$$

A digression: we insert here the proof of Theorem 2.4. In the above situation, assume for a moment that $\mathrm{Triv}(X_w) \cong \mathrm{Triv}(X_{w_0})$. Then Lemma 3.5 gives an inclusion of lattices:

$$\mathrm{MW}(X_w) \hookrightarrow \mathrm{MW}(X_{w_0}). \tag{6.12}$$

In particular, if the rank of $\mathrm{MW}(X_{w_0})$ is equal to that of $\mathrm{MW}(X_w)$, then we have $\mathrm{MW}(X_w) \cong \mathrm{MW}(X_{w_0})$, which proves the independence statement in Theorem 2.4. The rest of Theorem 2.4 has been already shown in Step 1. This completes the proof of Theorems 2.1–2.4.

Going back to (6.11) and considering the orthogonal complement, we have

$$\mathrm{T}(X_{w_0}) \subset \mathrm{T}(X_w) \subset H^2(X_w, \boldsymbol{Z}) \tag{6.13}$$

where both inclusions are obviously primitive.

In particular, letting $n = 1$, we have a primitive embedding:

$$\mathrm{T}(F^{(1)}_{\alpha_0,\beta_0}) \subset \mathrm{T}(F^{(1)}_{\alpha,\beta}). \tag{6.14}$$

This implies that, for any $n \le 6$, the inclusion in the first row of the following diagram is primitive:

$$\begin{array}{ccc} \mathrm{T}\big(F^{(1)}_{\alpha_0,\beta_0}\big)[n] & \subset & \mathrm{T}\big(F^{(1)}_{\alpha,\beta}\big)[n] \\ \bigcap & & \| \\ \mathrm{T}\big(F^{(n)}_{\alpha_0,\beta_0}\big) & \subset & \mathrm{T}\big(F^{(n)}_{\alpha,\beta}\big) \end{array} \tag{6.15}$$

The equality on the right is true by Step 1. Therefore the inclusion on the left

$$\mathrm{T}\big(F^{(1)}_{\alpha_0,\beta_0}\big)[n] \subset \mathrm{T}\big(F^{(n)}_{\alpha_0,\beta_0}\big) \tag{6.16}$$

is also primitive. Since this inclusion is of finite index for any α_0, β_0 (see (6.1)), it must be an equality. This proves Theorem 2.1 for $F^{(n)}_{\alpha_0,\beta_0}$. $\square$

6.3. Proof of Theorems 2.2 and 2.3 in general.

The equivalence of Theorem 2.1 and 2.2 in general is a consequence of the following:

LEMMA 6.1.

$$\det \mathrm{T}\big(F^{(1)}_{\alpha,\beta}\big) = \det \mathrm{Hom}(C_1, C_2). \tag{6.17}$$

PROOF. Recall that $F^{(1)}_{\alpha,\beta}$ is a double cover of the Kummer surface $\mathrm{Km}(A)$ ($A = C_1 \times C_2$), not only a degree two quotient of $F^{(2)}_{\alpha,\beta} \cong \mathrm{Km}(A)$ (see [**15**]). By [**4**] (cf. [**8**]), this implies that there are isomorphisms of lattices (and also of Hodge structures):

$$\mathrm{T}(F^{(1)}_{\alpha,\beta}) \cong \mathrm{T}(A), \quad \mathrm{T}(\mathrm{Km}(A)) \cong \mathrm{T}(A)[2]. \tag{6.18}$$

Hence we have

$$\det \mathrm{T}(F^{(1)}_{\alpha,\beta}) = \det \mathrm{T}(A) = \det \mathrm{NS}(A) = \det \mathrm{Hom}(C_1, C_2), \tag{6.19}$$

where the second equality results from (3.1) and the last one from the well-known fact that $\mathrm{Hom}(C_1, C_2)$ is the orthogonal complement of $T_0 = \{C_1 \times pt, pt \times C_2\}$ in $\mathrm{NS}(A)[-1]$ (cf. [**16**, (14.22)]. Note that, in [**16**], the norm of $\phi \in \mathrm{Hom}(C_1, C_2)$ was defined by $\deg(\phi)$ instead of $2\deg(\phi)$ as in the present paper.) $\square$

The equivalence of Theorem 2.2 and 2.3 easily follows from Section 3 and Section 4, except for the statement about torsion.

Lemma 6.2. *The Mordell-Weil group* $\mathrm{MW}(F^{(n)}_{\alpha,\beta})$ *is torsion-free, except for the following cases*: (*a*) $\alpha = 0, \beta = \pm 1$ (*i.e.* $j_1 = j_2 = 0$) *and* $n = 2, 4, 6$, *and* (*b*) $\alpha^3 = 1, \beta = 0$ (*i.e.* $j_1 = j_2 = 1$) *and* $n = 3, 6$. *The torsion subgroup* $\mathrm{MW}(F^{(n)}_{\alpha,\beta})_{tor}$ *is* $\boldsymbol{Z}/3\boldsymbol{Z}$ *in case* (*a*), *and* $(\boldsymbol{Z}/2\boldsymbol{Z})^{\oplus 2}$ *in case* (*b*).

Proof. In Lemma 5.2, the torsion-freeness is proven in case (i) $j_1 \neq j_2$ by means of the height formula (5.6). In case (ii) $j_1 = j_2 \neq 0, 1$, it is similarly proven, so we omit it.

Let us consider the remaining cases (iii) $j_1 = j_2 = 0$ and (iv) $j_1 = j_2 = 1$.

In case (iii), the elliptic surface $F^{(n)}_{\alpha,\beta}$ has n singular fibres of type IV at $t \neq 0, \infty$, and the height formula gives (cf. the proof of Lemma 5.2)

$$\langle P, P\rangle \geq \mu\big(\mathrm{MW}(F^{(n)}_{gen})\big) - \frac{2}{3}\cdot n \tag{6.20}$$

where the value of μ is given by (5.8). If $P \neq O$ is a torsion section, we have $\langle P, P\rangle = 0$, which implies that $n = 2, 4, 6$ or $n = 3$. But $n = 3$ is impossible, because if $P \neq O$ is a torsion then $P' = 2P$ should satisfy $\langle P', P'\rangle = 0$ too; writing down the height formula for P', we find easily a contradiction.

Thus a non-trivial torsion in case (iii) can occur only for the case (a), and similarly, a non-trivial torsion in case (iv) can occur only for the case (b). Moreover the torsion subgroup, if any, must be as stated in the lemma.

Conversely, we exhibit the torsion points. First, for (a), $n = 2$:

$$F^{(2)}_{0,1} :\ y^2 = x^3 + \left(t^2 + \frac{1}{t^2} - 2\right), \tag{6.21}$$

we have 3-torsion points $\pm(0, t - 1/t)$. Next, for (b), $n = 3$:

$$F^{(3)}_{1,0} :\ y^2 = x^3 - 3x + \left(t^3 + \frac{1}{t^3}\right), \tag{6.22}$$

we have a 2-torsion point $(-(t + 1/t), 0)$ and two more by replacing t by ωt with cube roots of unity ω. This completes the proof of Lemma 6.2. □

Remark. As for the statement about the torsion parts of the Mordell-Weil group in question, an alternative approach is to apply Shimada's classification result [**11**].

7. Explicit generators and structure of lattices.

In this section, we consider the structure and generators of the Mordell-Weil lattices $\mathrm{MW}(F_{gen}^{(n)})$ and $\mathrm{MW}(F_{\alpha,\beta}^{(n)})$, and prove Theorems 2.5 and 2.6.

7.1. Review of the case $n = 1, 2$.

In our previous paper [**16**], we have studied the case $n = 1, 2$ in detail (over any algebraically closed base field k of characteristic different from 2 and 3). Recall [**16**, Theorem 1.1, 1.2, 6.1]:

Theorem 7.1. *Assume $j_1 \neq j_2$. Then*

$$\mathrm{MW}\big(F_{\alpha,\beta}^{(1)}\big) \cong \mathrm{Hom}(C_1, C_2), \tag{7.1}$$

(*where the norm of $\phi \in \mathrm{Hom}(C_1, C_2)$ is $2\deg(\phi)$*), *and*

$$M = \mathrm{MW}\big(F_{\alpha,\beta}^{(2)}\big) \supset M_0 \oplus M_1, \quad [M : M_0 \oplus M_1] = 2^h \tag{7.2}$$

where $M_0 = \mathrm{MW}(F_{\alpha,\beta}^{(1)})[2]$ and $M_1 = \mathrm{MW}(E_{\alpha,\beta}^{(1)})[2]$. Here

$$E_{\alpha,\beta}^{(1)} : y^2 = x^3 - 3\alpha t^2 x + t^2(t^2 - 2\beta t + 1) \tag{7.3}$$

is a rational elliptic surface with $\mathrm{MW}(E_{\alpha,\beta}^{(1)}) \cong (A_2^*)^{\oplus 2}$. In particular, we have

$$\mathrm{MW}\big(F_{gen}^{(1)}\big) = \{0\}, \quad \mathrm{MW}\big(F_{gen}^{(2)}\big) \cong A_2^*[2]^{\oplus 2}. \tag{7.4}$$

The generators (2.6) given in Theorem 2.5 for $n = 2$ are, up to sign, the minimal vectors of norm $4/3$ of two copies of $A_2^*[2]$ which are similar to hexagonal lattices.

Note that both $E^{(1)}$ and $F^{(1)}$ above become isomorphic to $F^{(2)}$ by the quadratic base change $t \to t^2$, i.e. they are the twist of each other with respect to the quadratic extension.

7.2. Proof of Theorem 2.5 for $n > 2$.

Assume $2 < n \leq 6$. With the notation in Theorem 2.5 or 2.6, we consider

$$M = \mathrm{MW}\big(F_{\alpha,\beta}^{(n)}\big), \quad L = \mathrm{MW}\big(F_{\alpha,\beta}^{(n)+}\big)[2], \quad M_0 = \mathrm{MW}\big(F_{\alpha,\beta}^{(1)}\big)[n]. \tag{7.5}$$

By introducing the elliptic curve $E/k(w)$:

$$E : y^2 = x^3 - 3\alpha x + w - 2\beta, \tag{7.6}$$

and letting

$$s = t + \frac{1}{t}, \quad T = t^n, \quad w = T + \frac{1}{T}, \tag{7.7}$$

we have (by ignoring the lattice structure)

$$M = E(k(t)), \quad L = E(k(s)), \quad M_0 = E(k(T)). \tag{7.8}$$

We note that $E \cong F^{(1)+}_{\alpha,\beta}$. Note also (cf. [**17**]) that $k(t)/k(w)$ is a Galois extension with the Galois group $G = \langle \sigma, \tau \rangle$ where

$$\sigma : t \to \zeta_n \cdot t, \quad \tau : t \to \frac{1}{t} \tag{7.9}$$

(ζ_n a primitive n-th root of unity). Since $k(T)$ corresponds to $\langle \sigma \rangle$ and $k(s)$ to $\langle \tau \rangle$ by Galois theory, we have also $M^{\langle \sigma \rangle} = M_0$ and $M^{\langle \tau \rangle} = L$.

Lemma 7.2. *$L \cap M_0 = \{0\}$ and $L \oplus M_0$ is an orthogonal direct sum.*

Proof. Since $L \cap M_0 = E(k(s) \cap k(T))$ and $k(s) \cap k(T) = k(w)$ by Galois theory, the first assertion follows from $E(k(w)) = \{0\}$. The latter holds because $E/k(w)$ defines a rational elliptic surface with a singular fibre of type II^* at $w = \infty$.

Next take any $P \in L$ and $Q \in M_0$, and let $\langle P, Q \rangle = a$. Applying τ, we have $\langle P^\tau, Q^\tau \rangle = a$ too, since the height pairing is invariant under an automorphism ([**12**, Proposition 8.13]). It follows that $\langle P, Q + Q^\tau \rangle = 2a$. But, since $Q + Q^\tau$ belongs to $L \cap M_0 = \{0\}$, we have $2a = 0$, implying $a = 0$. This proves the second assertion. □

Lemma 7.3. *Let $L' \subset M$ be the image of L under σ; in other words, let $L' = E(k(s'))$ where $s' = s^\sigma = \zeta_n t + 1/(\zeta_n t)$. Then $L \cap L'$ is equal to $\{0\}$ in case n is odd, and to $\mathrm{MW}(F^{(2)+})$ in case n is even.*

Proof. We have $L \cap L' = E(k(s) \cap k(s'))$, and $k(s) \cap k(s')$ corresponds to $H = \langle \tau, \tau' \rangle \subset G$ where $\tau' = \sigma^{-1}\tau\sigma$. By rewriting $\tau' = \sigma^{-2}\tau$, we see that $H = \langle \tau, \sigma^{-2} \rangle$ is equal to $\langle \tau, \sigma \rangle = G$ if n is odd, but H is of index 2 in G if n is even. It follows that $L \cap L' = E(k(w)) = \{0\}$ in case n is odd and that $L \cap L' \cong \mathrm{MW}(F^{(2)+})$ in case n is even. □

Now we refer to [**12**, Theorem 10.10] for the fact that the rational points of the form (2.8) in Theorem 2.5, Section 2, generate the Mordell-Weil group $E(k(s))$ for any $E/k(s)$ which defines a rational elliptic surface.

LEMMA 7.4. *Assume $h = 0$ and let $n = 3$ or $n = 5$. Then* $\det(L + L')$ *is equal to* $3^4/4^2 (n = 3)$ *or* $5^4 (n = 5)$*, and we have* $M = L + L'$.

PROOF. By Lemma 7.3, $L + L'$ has the same rank as M under the assumption. If we admit the assertion about the determinant, then $L + L' = M \cong M^{(n)}_{gen}$ follows in view of Theorem 2.4. This will prove the lemma and hence Theorem 2.5 for $n = 3$ or $n = 5$. Thus the proof is reduced to computing the determinant of the height matrix, which we omit here but it is similar to that given in [**17**, Section 6, pp. 59–64] or [**19**, III, pp. 184–186]. □

To cover the case $n = 4, 6$, we can modify the above argument as follows. First, by Theorem 2.4, we may assume that $\alpha = \beta = 0$, since this corresponds to the case where $j_1 = 0, j_2 = 1$, i.e. $C_1 : y^2 = x^3 - 1$ and $C_2 : y^2 = x^3 - x$, which are non-isogenous elliptic curves ($h = 0$). In this case, the elliptic surface

$$F^{(n)}_{0,0} : y^2 = x^3 + t^n + \frac{1}{t^n} \tag{7.10}$$

for any n has an automorphism

$$(x, y, t) \to \big(- x, \sqrt{-1}y, \zeta_{2n} \cdot t\big). \tag{7.11}$$

Let $\tilde{\sigma}$ be the automorphism of $M = \mathrm{MW}(F^{(n)}_{0,0}) \cong \mathrm{MW}(F^{(n)}_{gen})$ and redefine L' as the image of L under $\tilde{\sigma}$.

LEMMA 7.5. *Assume $h = 0$ and let $n = 4$ or $n = 6$. Then the intersection $L \cap L' = \{0\}$, and* $\det(L + L')$ *is equal to* $4^4/3^2 (n = 4)$ *or* $6^4 (n = 6)$.

As before, Lemma 7.5 will complete the proof of Theorem 2.5 for all $n \leq 6$ in view of Theorem 2.4. The lemma can be proven by computing the height determinant. We refer to Usui's paper in preparation [**19**, IV] for more detail, who uses such results for the determination of Mordell-Weil lattice of $y^2 = x^3 + t^m + 1$ for all m.

7.3. Proof of Theorem 2.6 for $n > 2$.

Now we assume only the condition $j_1 \neq j_2$ (which is weaker than $h = 0$) in the situation of Section 7.2. For n fixed, the trivial lattice of $F^{(n)}_{\alpha,\beta}$ is the same as that for general parameter α, β, and hence the argument in Section 6.2 applies.

By the inclusion (6.12), we have

$$\mathrm{MW}\big(F^{(n)}_{gen}\big) \cong M_1 = L + L' \subset \mathrm{MW}\big(F^{(n)}_{\alpha,\beta}\big). \tag{7.12}$$

On the other hand, $M_0 = \mathrm{MW}(F^{(1)}_{\alpha,\beta})[n]$ is orthogonal to M_1 in $M = \mathrm{MW}(F^{(n)}_{\alpha,\beta})$ by Lemma 7.2. Hence, comparing the rank, we see that $M_0 \oplus M_1$ is of finite index in M. This proves Theorem 2.6. □

N.B. Actually we have shown in the above that $L + L'$ has the right rank, only in case n is odd (Lemma 7.3), since we have omitted the proof of lemma 7.5. In order to make the proof of Theorem 2.6 self-contained, we note an alternative proof in case $n = 4, 6$ similar to the case $n = 2$ mentioned in Section 7.1.

Namely, if $n = 2m$, then we have

$$M = \mathrm{MW}\big(F^{(n)}_{\alpha,\beta}\big) \supset \mathrm{MW}\big(F^{(m)}_{\alpha,\beta}\big)[2] \oplus \mathrm{MW}\big(E^{(m)}_{\alpha,\beta}\big)[2] \tag{7.13}$$

as a sublattice of finite index, where $E^{(m)}$ is the quadratic twist of $F^{(m)}$ for $m = 2$ or 3 given by

$$E^{(2)}_{\alpha,\beta} : y^2 = x^3 - 3\alpha t^2 x + t(t^4 - 2\beta t^2 + 1), \tag{7.14}$$

$$E^{(3)}_{\alpha,\beta} : y^2 = x^3 - 3\alpha t^2 x + (t^6 - 2\beta t^3 + 1). \tag{7.15}$$

Since Theorem 2.6 is proven for $m = 2, 3$, it suffices to note that $E^{(m)}$ is a rational elliptic surface for $m = 2, 3$ (e.g. with MWL of type E_8 for general values of α, β; cf. [**12**, Section 10]). This proves Theorem 2.6. □

8. Application to singular K3 surfaces.

Let X be a singular K3 surface, i.e. a complex K3 surface with maximal Picard number $\rho(X) = 20$, and let $T(X)$ denote the lattice of transcendental cycles on X, given with the natural orientation. Let Q_X denote the Gram matrix of $T(X)$ with respect to an oriented basis:

$$Q_X = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}. \tag{8.1}$$

Let us recall the following:

(1) The correspondence $X \to Q_X$ defines a bijection from the set of singular K3

surfaces (up to isomorphisms) to the set of positive-definite even integral matrices up to $SL_2(\boldsymbol{Z})$-equivalence (see [**4**]).
(2) Every singular K3 surface X is isomorphic to an elliptic surface $F^{(1)}_{\alpha,\beta}$ for some α, β (not necessarily unique) ([**4**], [**15**]).

Now suppose $X \cong F^{(1)}_{\alpha,\beta}$. Then X has a double cover $F^{(2)}_{\alpha,\beta}$ which is isomorphic to a Kummer surface $\mathrm{Km}(C_1 \times C_2)$ such that the absolute invariants j_1, j_2 of C_1, C_2 are given by (1.4). We have $\rho(X) = 20$ iff C_1, C_2 are isogenous elliptic curves with complex multiplications. As a special case of Theorem 2.1, we have the following result:

THEOREM 8.1. *If a singular K3 surface $X \cong F^{(1)}_{\alpha,\beta}$ corresponds to the matrix Q_X above, then for each $n \leq 6$, $X^{(n)} = F^{(n)}_{\alpha,\beta}$ is the singular* K3 *surface corresponding to*

$$Q_{X^{(n)}} = nQ_X = n \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}. \tag{8.2}$$

EXAMPLE 8.1. Let $j_1 = j_2 = 0$, i.e. $\alpha = 0, \beta = \pm 1$; we choose $\beta = 1$. Then $X = F^{(1)}_{0,1}$ with the defining equation: $y^2 = x^3 + t + 1/t - 2$ corresponds to $Q_X = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. By Theorem 8.1, $F^{(n)}_{0,1} : y^2 = x^3 + t^n + 1/t^n - 2$ corresponds to nQ_X for any $n \leq 6$. In particular, we have

$$\det \mathrm{T}\big(F^{(n)}_{0,1}\big) = \det \mathrm{NS}\big(F^{(n)}_{0,1}\big) = 3n^2 \ (n \leq 6).$$

EXAMPLE 8.2. Let $j_1 = j_2 = 1$, so that we can take $\alpha = 1, \beta = 0$. Then $X = F^{(1)}_{1,0}$ corresponds to $Q_X = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. By Theorem 8.1, $F^{(n)}_{0,1} : y^2 = x^3 - 3x + t^n + 1/t^n$ corresponds to nQ_X, and we have

$$\det \mathrm{T}\big(F^{(n)}_{1,0}\big) = \det \mathrm{NS}\big(F^{(n)}_{1,0}\big) = 4n^2 \ (n \leq 6).$$

N.B. In the above example, the elliptic curve C_1 with $j_1 = 1$ has a defining equation $y^2 = x^4 - 1$. Then $F^{(2)}_{1,0}$ is isomorphic to the Kummer surface $\mathrm{Km}(C_1 \times C_1)$ with the defining equation $(x_1^4 - 1)t^2 = x_2^4 - 1$, where t is the "elliptic parameter" giving the elliptic fibration $F^{(2)}$ (cf. [**7**], [**16**, Section 5]). Since $Y = F^{(4)}_{1,0}$ is obtained from $F^{(2)}_{1,0}$ by the base change $t = u^2$, it is birational to $(x_1^4 - 1)u^4 = x_2^4 - 1$. Hence Y is isomorphic to the Fermat quartic surface and we see that $Q_Y = 4Q_X$ with det $= 64$ (as is well-known). This argument is applicable to more general situation (cf. [**2**]).

Example 8.3 (cf. [**17**]). Next take

$$C_1 : y^2 = x^3 - 1 \ (j_1 = 0), \quad C_2 : y^2 = x^3 - 15x + 22 \ (j_2 = 5^3/4). \tag{8.3}$$

(There is a 2-isogeny $\varphi : C_1 \to C_2$, and C_2 is a unique elliptic curve with this property up to isomorphism.) Then we have $\alpha = 0$, $\beta = \pm 11/2\sqrt{-1}$, and a simple coordinate change makes the defining equation of $F^{(n)}_{\alpha,\beta}$ to be:

$$y^2 = x^3 + t^n - \frac{1}{t^n} - 11. \tag{8.4}$$

The singular K3 surface $X = F^{(n)}_{\alpha,\beta}$ corresponds to $Q_X = 2n\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. The case $n = 5$ has been studied in detail in [**17**]: the Mordell-Weil lattice $\mathrm{MW}(F^{(5)}_{\alpha,\beta})$ of rank 18 and $\det = 3 \cdot 10^2$.

In closing, we remark that the method in this paper has some application to supersingular K3 surfaces, which will be discussed elsewhere.

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