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On the topology of the Newton boundary at infinity

Dedicated to Professor Hà Huy Vui on his sixtieth birthday

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Abstract. We are interested in a global version of Lê-Ramanujam μ-constant theorem from the Newton polyhedron point of view. More precisely, we prove a stability theorem which says that the global monodromy fibration of a polynomial function with Newton non-degenerate is uniquely determined by its Newton boundary at infinity. Furthermore, the continuity of atypical values for a family of complex polynomial functions also is considered.

1. Introduction.

Let $f\colon \boldsymbol{C}^n \to \boldsymbol{C}$ be a complex polynomial function. It is well known that there exists a (minimal) finite set $B(f)$ in $\boldsymbol{C}$, called the bifurcation set of f, such that the restriction:

$$f\colon \boldsymbol{C}^n \setminus f^{-1}(B(f)) \to \boldsymbol{C} \setminus B(f)$$

is a C^∞-locally trivial fibration (see, for example, [**39**], [**40**], [**23**], [**33**], [**14**]). This fibration permits us to introduce the global monodromy fibration of f. Namely, for $r > \max\{|c| \mid c \in B(f)\}$ and $\boldsymbol{S}^1_r := \{c \in \boldsymbol{C} \mid |c| = r\}$, this is the restriction

$$f\colon f^{-1}(\boldsymbol{S}^1_r) \to \boldsymbol{S}^1_r.$$

The problem of studying the global monodromy fibration of complex polynomials was considered by many authors, see for example: [**26**], [**16**], [**13**], [**17**], [**37**], [**35**], [**1**], [**2**], [**10**], [**27**], [**11**], [**36**], [**5**], [**6**], [**7**], etc. However, most of them treat only polynomial functions, which have isolated singularities affine and at infinity (see [**32**] or [**34**] for the last notion). It seems more difficult to obtain similar results in the general case.

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In the present work we will be interested in a global version of Lê-Ramanujam μ-constant theorem from the Newton polyhedron point of view. We will prove a stability theorem which says that the global monodromy fibration of a polynomial function with Newton non-degenerate is uniquely determined by its Newton boundary at infinity.

In order to formulate the main result at hand we first need some definitions about Newton polyhedra, see [**20**], [**25**], [**9**], [**3**]. Let $f\colon \boldsymbol{C}^n \to \boldsymbol{C}$ be a polynomial function. We express f as follows: $f(z) := \sum_{\alpha \in \boldsymbol{N}^n} a_\alpha z^\alpha$. The support $\mathrm{supp}(f)$ is defined to be $\{\alpha \mid a_\alpha \neq 0\}$. We denote $\Gamma_-(f)$ to be the convex hull of the set $\{0\} \cup \mathrm{supp}(f)$. The Newton boundary at infinity $\Gamma_\infty(f)$ is by definition the union of the closed faces of the polyhedron $\Gamma_-(f)$ which do not contain the origin. Here and below, by face we shall understand face of any dimension. For each closed face Δ of $\Gamma_\infty(f)$ we denote by f_Δ the polynomial $\sum_{\alpha \in \Delta} a_\alpha z^\alpha$. The polynomial f is called (Newton) non-degenerate if for each face $\Delta \in \Gamma_\infty(f)$, the system of equations

$$\frac{\partial f_\Delta}{\partial z_1} = \frac{\partial f_\Delta}{\partial z_2} = \cdots = \frac{\partial f_\Delta}{\partial z_n} = 0$$

has no solutions in $(\boldsymbol{C} - \{0\})^n$. The polynomial f is called convenient if the intersection of $\mathrm{supp}(f)$ with each coordinate axis is non-empty.

The main result of this paper is the following:

THEOREM 1.1. *Let f and g be two complex polynomial functions in n variables such that the following conditions hold*

(i) $\Gamma_\infty(f) = \Gamma_\infty(g) \neq \emptyset$; *and*
(ii) f *and* g *are non-degenerate.*

Then the global monodromy fibrations of f and g are isomorphic.

REMARK 1.2. It is worth noting that the polynomials f and g can have non-isolated singularities, affine and at infinity. Moreover, there is no restriction on the dimension n.

REMARK 1.3. Theorem 1.1 can be considered as a global version of the local results considered by Oka M. [**28**], [**29**], [**30**] (see also [**31**]). For convenient and non-degenerate polynomial functions, this result was obtained in [**26**]. See also [**15**], [**16**], [**17**], [**5**], [**6**], [**7**] for related results.

Let us now sketch the basic idea of the proof of Theorem 1.1. We first connect f to g by a family $F_t, t \in [0, 1]$, of complex polynomial functions with $F_0 \equiv f$ and $F_1 \equiv g$ such that $\Gamma_\infty(F_t)$ is constant and such that F_t are non-degenerate for all

$t \in [0,1]$. We next show that (i) the finite set $B(F_t) \subset \boldsymbol{C}$ is contained in some open disc of radius independent of t; and (ii) all the fibers of the whole family F_t over a large circle are transversal to all sufficiently large spheres. Then, we may show as in [**17**] (see also [**5**]) that the global monodromy fibrations of F_t are isomorphic. As an application of this procedure, we also find (see Theorem 4.2) that atypical values of F_t, given in [**25**], depends continuously on t.

The results obtained by Lê D. T. and Ramanujam C. P. and by Oka M. have played the inspiring role in undertaking this research. On the other hand, the proof, based on the results of Némethi A. and Zaharia A., uses only the curve selection lemma as a tool.

The paper is organized as follows. Some necessary results on the bifurcation set of complex polynomials are recalled in Section 2. The proof of Theorem 1.1 is given in Section 3. Finally, the continuity of some critical values for a family of complex polynomials is considered in Section 4.

2. Preliminaries.

2.1. Notations.

Let $f\colon \boldsymbol{C}^n \to \boldsymbol{C}$ be a polynomial function. By $\operatorname{grad} f$ we denote the vector $\operatorname{grad} f := \left(\overline{\frac{\partial f}{\partial z_1}}, \overline{\frac{\partial f}{\partial z_2}}, \ldots, \overline{\frac{\partial f}{\partial z_n}}\right)$, so the chain rule may be expressed by the inner product $\partial f/\partial \mathbf{v} = \langle \mathbf{v}, \operatorname{grad} f\rangle$.

For each $r > 0$, we will denote $D_r := \{c \in \boldsymbol{C} \mid |c| < r\}$ for the open disc and let $\boldsymbol{B}_r^{2n} := \{z \in \boldsymbol{C}^n \mid \|z\| < r\}$ be the open ball in $\boldsymbol{C}^n$ centered in the origin and with radius r. We will also write $\boldsymbol{S}_r^{2n-1} := \{z \in \boldsymbol{C}^n \mid \|z\| = r\}$ for the sphere.

For $I \subset \{1, 2, \ldots, n\}$, let $\boldsymbol{R}^I := \{\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \boldsymbol{R}^n \mid \alpha_i = 0$ for $i \notin I\}$. $\boldsymbol{C}^I$ is defined similarly.

2.2. Bifurcation values of a polynomial function.

Let $f\colon \boldsymbol{C}^n \to \boldsymbol{C}$ be a non-constant polynomial function. One can check that the bifurcation set $B(f)$ always contains the set of critical values $\Sigma_0(f)$ of f; in particular, if $n = 1$ then $B(f) = \Sigma_0(f)$. However, besides the critical values of f, the set $B(f)$ may contain some extra values, corresponding to the so-called "critical values at infinity". This may happen since f is not proper for $n \geq 2$ and we cannot apply Ehresmann's Fibration Theorem. Therefore, the problem of describing the bifurcation set $B(f)$ is not easy in general. Until now it is solved only for a few cases. We send to [**12**], [**38**], [**18**] for more details.

We recall now the result of Némethi A. and Zaharia A. [**25**] on how to estimate the bifurcation set. For this purpose, let $\overline{\operatorname{supp}(f)}$ be the convex hull of the set $\operatorname{supp}(f) \setminus \{0\}$. A closed face Δ of $\overline{\operatorname{supp}(f)}$ is called bad if:

(i) the affine subspace of the minimal dimension spanned by Δ contains the origin, and

(ii) there exists a hyperplane $H \subset \boldsymbol{R}^n$ with equation $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the coordinates in $\boldsymbol{R}^n$, such that:

 (ii$_a$) there exist i and j with $a_i \cdot a_j < 0$, and

 (ii$_b$) $H \cap \overline{\mathrm{supp}(f)} = \Delta$.

More geometrically, Condition (ii$_a$) says that the hyperplane H intersects the interior of the positive octant of $\boldsymbol{R}^n$. We denote by $\mathscr{B}$ the set of bad faces of $\overline{\mathrm{supp}(f)}$. For $\Delta \in \mathscr{B}$, we define:

$$\Sigma_0^{'}(f_\Delta) := \big\{ f_\Delta(z^0) \mid z^0 \in (\boldsymbol{C} - \{0\})^n \text{ and } \mathrm{grad} f_\Delta(z^0) = 0 \big\}.$$

Let $\Sigma_\infty(f) := \cup_{\Delta \in \mathscr{B}} \Sigma_0^{'}(f_\Delta)$. It is clear that $\Sigma_0^{'}(f_\Delta) \subset \Sigma_0(f_\Delta)$. This, together with an algebraic version of Sard's theorem (see [**4**]), yields that $\Sigma_\infty(f)$ is a finite set.

The following result give an estimation for the bifurcation set $B(f)$ of f in terms of its Newton boundary at infinity.

PROPOSITION 2.1 ([**20**], [**8**], [**25**] (see also, [**41**], [**19**], [**6**])). *Let $f \colon \boldsymbol{C}^n \to \boldsymbol{C}$ be a non-degenerate polynomial function. Then the following statements hold*

(i) *If f is convenient, then $B(f) = \Sigma_0(f)$.*

(ii) *If f is not convenient, then $B(f) \subset \Sigma_0(f) \cup \Sigma_\infty(f) \cup \{f(0)\}$.*

For the sake of completeness, we also recall the following lemma, that will help us to prove our second result (see Theorem 4.2).

LEMMA 2.2 ([**41**, Lemma 5.2]). *Let $f \colon \boldsymbol{C}^n \to \boldsymbol{C}$ be a non-degenerate polynomial function. Suppose that the hypersurface $f^{-1}(c) \subset \boldsymbol{C}^n$ has non-isolated singularities. Then either $c \neq f(0)$ and there exists a bad face Δ of $\overline{\mathrm{supp}(f)}$ such that $c \in \Sigma_0^{'}(f_\Delta)$, or $c = f(0)$.*

3. Proof of Theorem 1.1.

In this section, we give a proof of Theorem 1.1. So let $f, g \colon \boldsymbol{C}^n \to \boldsymbol{C}$ be two polynomial functions with the same Newton boundary at infinity such that they are non-degenerate. Since the non-degeneracy condition is an open condition (see [**20**, Théorème 6.1], [**28**, Appendix]), we can take a piecewise analytic family $F(z,t)$ such that

(i) $F(z,0) = f(z), F(z,1) = g(z)$; and

(ii) $F_t(z) := F(z,t)$ as a function of z is a non-degenerate polynomial with $\Gamma_\infty(F_t) = \Gamma_\infty(f)$ for each t.

Hence, we reduce Theorem 1.1 to the case when the family $F(z,t)$ is analytic. Therefore, with no loss of generality, we may well assume that the family $F(z,t)$ satisfies these conditions.

We shall need later on the following well known result.

LEMMA 3.1. *Let r and R_0 be positive numbers such that the following conditions hold*

(i) *the bifurcation set $B(F_t)$ is contained in the open disc D_r; and*
(ii) *for all $c \in \boldsymbol{S}^1_r$ and for all $R \geq R_0$, the fiber $F_t^{-1}(c)$ intersects the sphere $\boldsymbol{S}^{2n-1}_R$ transversally.*

Then the global monodromy fibration of F_t:

$$F_t \colon F_t^{-1}(\boldsymbol{S}^1_r) \to \boldsymbol{S}^1_r$$

is isomorphic to the following fibration

$$F_t \colon F_t^{-1}(\boldsymbol{S}^1_r) \cap \boldsymbol{B}^{2n}_R \to \boldsymbol{S}^1_r$$

for all $R \geq R_0$.

PROOF. The following proof is adapted from [**17**]. We can find a smooth vector field tangent to the fibers of F_t and pointing out the spheres $\boldsymbol{S}^{2n-1}_R$. In fact, since $\operatorname{grad} F_t(z)$ and z are $\boldsymbol{C}$-linearly independent vectors for all $z \in F_t^{-1}(\boldsymbol{S}^1_r) \cap \boldsymbol{B}^{2n}_R$, there exists a smooth vector field $\mathbf{v}(z)$ such that

(i) $\langle \mathbf{v}(z), \operatorname{grad} F_t(z)\rangle = 0$,
(ii) $\langle \mathbf{v}(z), z\rangle > 0$.

(We can construct such a vector field locally, then extend it over $F_t^{-1}(\boldsymbol{S}^1_r)$ by a smooth partition of unity.) Put

$$\mathbf{w}(z) = \frac{\mathbf{v}(z)}{2\langle \mathbf{v}(z), z\rangle}(\|z\|^4 + 1).$$

This field is completely integrable. So let $p_{z^0}(\tau)$ be its integral curve with $p_{z^0}(0) = z^0$. By Condition (i), if $z^0 \in F_t^{-1}(c) \cap \boldsymbol{B}^{2n}_R$, then $p_{z^0}(\tau) \in F_t^{-1}(c)$. Moreover,

$$\begin{aligned}\frac{d\|p_{z^0}(\tau)\|^2}{d\tau} &= \left\langle \frac{dp_{z^0}(\tau)}{d\tau}, p_{z^0}(\tau)\right\rangle + \left\langle p_{z^0}(\tau), \frac{dp_{z^0}(\tau)}{d\tau}\right\rangle = 2\mathrm{Re}\left\langle \frac{dp_{z^0}(\tau)}{d\tau}, p_{z^0}(\tau)\right\rangle \\ &= 2\mathrm{Re}\langle \mathbf{w}(p_{z^0}(\tau)), p_{z^0}(\tau)\rangle = \|p_{z^0}(\tau)\|^4 + 1.\end{aligned}$$

Hence

$$\arctan \|p_{z^0}(\tau)\|^2 - \arctan \|z^0\|^2 = \tau.$$

Or equivalently,

$$\|p_{z^0}(\tau)\|^2 = \tan(\tau + \arctan \|z^0\|^2).$$

Let $\tau^0 := (\pi/2) - \arctan R^2$. Then $p_{z^0}(\tau^0) \to \infty$ as $\|z^0\| \to R$. Thus, it induces a diffeomorphism

$$F_t^{-1}(\boldsymbol{S}_r^1) \cap \boldsymbol{B}_R^{2n} \to F_t^{-1}(\boldsymbol{S}_r^1), \quad z^0 \mapsto p_{z^0}(\tau^0),$$

which completes the proof of the lemma. □

In the next paragraphs we shall show that

(i) there exists a positive constant r such that

$$\Sigma_0(F_t) \cup \Sigma_\infty(F_t) \cup \{F_t(0)\} \subset D_r \quad \text{for all} \quad t \in [0,1];$$

(ii) there exists a positive number R_0 such that for all $R \geq R_0$, for all $t \in [0,1]$, and all $c \in \boldsymbol{S}_r^1$, the fiber $F_t^{-1}(c)$ intersects the sphere $\boldsymbol{S}_R^{2n-1}$ transversally.

These facts, together with Lemma 3.1, imply that the global monodromy fibration of the polynomial function F_t is isomorphic to the fibration $F_t\colon F_t^{-1}(\boldsymbol{S}_r^1) \cap \boldsymbol{B}_R^{2n} \to \boldsymbol{S}_r^1$. Hence, the original method of proof of Lê D. T. [**21**] is applicable; and we can show that the fibrations $F_t\colon F_t^{-1}(\boldsymbol{S}_r^1) \cap \boldsymbol{B}_R^{2n} \to \boldsymbol{S}_r^1, t \in [0,1]$, are isomorphic. Consequently, the global monodromy fibrations of F_t are isomorphic.

3.1. Boundedness of affine singularities.

The following result says that the set $\Sigma_0(F_t)$ of critical values of F_t is contained in some open disc of radius independent of t.

Lemma 3.2. *There exists a positive number r such that*

$$\Sigma_0(F_t) \subset D_r \quad \textit{for all} \quad t \in [0,1].$$

Proof. Suppose that this is not the case. Then, by the Curve Selection Lemma (see [**24**], [**26**]), there exist an analytic curve $p(s) := (p_1(s), p_2(s), \ldots, p_n(s))$ and a real analytic function $t(s), s \in (0, \epsilon)$, such that:

(i) $\lim_{s\to 0}\|p(s)\| = \infty$;
(ii) $\lim_{s\to 0} t(s) = t^0 \in [0,1]$;
(iii) $\lim_{s\to 0} F(p(s), t(s)) = \infty$; and
(iv) $\operatorname{grad} F(p(s), t(s)) \equiv 0$.

Let $I := \{i \mid p_i \not\equiv 0\}$. By Condition (i), $I \neq \emptyset$. For $i \in I$, expand the coordinate p_i in terms of the parameter: say

$$p_i(s) = z_i^0 s^{a_i} + \text{higher order terms in } s,$$

where $z_i^0 \neq 0$ and $\min_{i\in I} a_i < 0$. Moreover, it follows from Condition (iii) that $\Gamma_\infty(F_t) \cap \boldsymbol{R}^I \neq \emptyset$. Let d be the minimal value of the linear function $\sum_{i\in I} a_i\alpha_i$ on $\Gamma_-(F_t) \cap \boldsymbol{R}^I$, and let Δ be the (unique) maximal face of $\Gamma_-(F_t) \cap \boldsymbol{R}^I$ where the linear function takes this value. By a direct calculation, then

$$F(p(s), t(s)) = F_\Delta(z^0, t^0)s^d + \text{higher order terms in } s,$$

here and below, we put $z^0 := (z_1^0, z_2^0, \dots, z_n^0)$ with $z_i^0 = 1$ for $i \notin I$. By Condition (iii), $d < 0$. Consequently, Δ is a face of $\Gamma_\infty(F_t)$.

On the other hand, we have

$$\frac{\partial F}{\partial z_i}(p(s), t(s)) = \frac{\partial F_\Delta}{\partial z_i}(z^0, t^0)s^{d-a_i} + \text{higher order terms in } s.$$

Hence, it follows from Condition (iv) that

$$\frac{\partial F_\Delta}{\partial z_i}(z^0, t^0) = 0, \quad i = 1, 2, \dots, n,$$

which is a contradiction to the non-degeneracy assumption of the polynomial function $F_{t^0}(z)$. $\square$

3.2. Boundedness of singularities at infinity.

We next observe the

LEMMA 3.3. *There exists a positive number r such that*

$$\Sigma_\infty(F_t) \subset D_r \ \textit{ for all } \ t \in [0,1].$$

PROOF. Assuming the contrary and using the Curve Selection Lemma [**24**], [**26**] we can find a bad face Δ of $\overline{\operatorname{supp}(F_t)}$ and a real analytic curve $(p(s), t(s)) \in$

$(\boldsymbol{C}-\{0\})^n \times [0,1], s \in (0,\epsilon)$, such that

(i) $\lim_{s\to 0} \|p(s)\| = \infty$;
(ii) $\lim_{s\to 0} t(s) = t^0 \in [0,1]$;
(iii) $\lim_{s\to 0} F_\Delta(p(s),t(s)) = \infty$; and
(iv) $\mathrm{grad} F_\Delta(p(s),t(s)) \equiv 0$.

Since $p(s) \in (\boldsymbol{C}-\{0\})^n$, we may write

$$p_i(s) = z_i^0 s^{a_i} + \text{higher order terms in } s,$$

where $z_i^0 \neq 0$ and $\min_{i=1,2,\ldots,n} a_i < 0$. Let Δ' be the maximal face of Δ where the linear function $\sum_{i=1}^n a_i\alpha_i$ defined on Δ takes its minimal value, say d. Then we may write

$$F_\Delta(p(s),t(s)) = F_{\Delta'}(z^0,t^0)s^d + \text{higher order terms in } s.$$

By Condition (iii), $d < 0$. Consequently, Δ' is a face of $\Gamma_\infty(F_t)$.

On the other hand, we also have the following Taylor expansions

$$\frac{\partial F_\Delta}{\partial z_i}(p(s),t(s)) = \frac{\partial F_{\Delta'}}{\partial z_i}(z^0,t^0)s^{d-a_i} + \text{higher order terms in } s, \quad i = 1,2,\ldots,n.$$

This fact, combined with Condition (iv), yields that

$$\frac{\partial F_{\Delta'}}{\partial z_i}(z^0,t^0) = 0, \quad i = 1,2,\ldots,n,$$

which is a contradiction to the non-degeneracy assumption of the polynomial function $F_{t^0}(z)$. □

LEMMA 3.4. *There exists a positive number r such that*

$$\{F_t(0)\} \subset D_r \quad \textit{for all} \quad t \in [0,1].$$

PROOF. The claim easily follows from the continuity of the family $F(z,t)$. □

Other properties of the sets $\Sigma_0(F_t), \Sigma_\infty(F_t)$ and $\{F_t(0)\}$ will be given in Theorem 4.2.

3.3. Transversality in the neighbourhood of infinity.

In order to finish the proof of Theorem 1.1, we need the following result.

LEMMA 3.5 (Compare [**26**, Lemma 19]). *Let r be a positive number such that the conclusions of Lemmas* 3.2, 3.3 *and* 3.4 *are fulfilled. Then there exists R_0 sufficiently large such that for all $R \geq R_0$ and for all $c \in \boldsymbol{S}_r^1$ we have that the fiber $F_t^{-1}(c)$ meets transversally the sphere $\boldsymbol{S}_R^{2n-1}$ for each $t \in [0,1]$.*

PROOF. If the assertion is not true, then by the Curve Selection Lemma [**24**], [**26**] there exist an analytic curve $p(s) := (p_1(s), p_2(s), \ldots, p_n(s))$ and a real analytic function $t(s), s \in (0, \epsilon)$, such that:

(i) $\lim_{s\to 0} \|p(s)\| = \infty$;
(ii) $\lim_{s\to 0} t(s) = t^0 \in [0,1]$;
(iii) $\lim_{s\to 0} F(p(s), t(s)) = c$; and
(iv) $\mathrm{grad} F(p(s), t(s)) = \lambda(s) p(s)$, where $\lambda(s) \in \boldsymbol{C}$.

By Lemma 3.2, $\lambda(s) \not\equiv 0$. Thus we may suppose that

$$\lambda(s) = \lambda^0 s^\delta + \text{higher order terms in } s,$$

here $\lambda^0 \neq 0$ and $\delta \in \boldsymbol{Q}$.

Let $I := \{i \mid p_i \not\equiv 0\}$. By Condition (i), $I \neq \emptyset$. For $i \in I$, let us write

$$p_i(s) = z_i^0 s^{a_i} + \text{higher order terms in } s,$$

where $z_i^0 \neq 0$ and $\min_{i\in I} a_i < 0$. Since $c \in \boldsymbol{S}_r^1$, it follows from Lemma 3.4 that the restriction of F_t on $\boldsymbol{C}^I$ is non-trivial i.e., $\Gamma_\infty(F_t) \cap \boldsymbol{R}^I \neq \emptyset$. Let d be the minimal value of the linear function $\sum_{i\in I} a_i \alpha_i$ on $\Gamma_-(F_t) \cap \boldsymbol{R}^I$, and let Δ be the (unique) maximal face of $\Gamma_-(F_t) \cap \boldsymbol{R}^I$ where the linear function takes this value. Then we may write

$$\frac{\partial F}{\partial z_i}(p(s), t(s)) = \frac{\partial F_\Delta}{\partial z_i}(z^0, t^0) s^{d-a_i} + \cdots = \overline{\lambda^0}\ \overline{z_i^0} s^{\delta + a_i} + \cdots .$$

Let $I' := \{i \in I \mid d - a_i = \delta + a_i\}$. Then $i \notin I'$ if and only if

$$\frac{\partial F_\Delta}{\partial z_i}(z^0, t^0) = 0$$

and in this case $d - a_i < \delta + a_i$. There are two cases to be considered.

CASE 1. *The set I' is empty.* Then it is clear that

$$\frac{\partial F_\Delta}{\partial z_i}(z^0, t^0) = 0, \quad i = 1, 2, \ldots, n.$$

Hence, from the non-degeneracy condition of the polynomial function F_{t^0}, it follows that $d = 0$ and Δ is a bad face of $\overline{\text{supp}(f)}$. Consequently, $c = F_\Delta(z^0, t^0) \in \Sigma_\infty(F_{t^0})$. However, this is a contradiction to Lemma 3.3 because we know that $c \in \boldsymbol{S}^1_r$ and the set $\Sigma_\infty(F_{t^0})$ is contained in the open disc D_r.

CASE 2. *The set I' is non-empty.* The function F restricted on the curve $(p(s), t(s))$ has the form

$$F(p(s), t(s)) = F_\Delta(z^0, t^0)s^d + \text{higher order terms in } s.$$

If $F_\Delta(z^0, t^0) \neq 0$ then $d = 0$ because of Condition (iii). Consequently, $dF_\Delta(z^0, t^0) = 0$. This fact, together with the Euler relation

$$dF_\Delta(z, t) = \sum_{i=1}^{n} a_i z_i \frac{\partial F_\Delta}{\partial z_i}(z, t),$$

yields that

$$0 = dF_\Delta(z^0, t^0) = \sum_{i \in I'} a_i z_i^0 \frac{\partial F_\Delta}{\partial z_i}(z^0, t^0) = \sum_{i \in I'} \frac{d - \delta}{2} z_i^0 \frac{\partial F_\Delta}{\partial z_i}(z^0, t^0).$$

But $d \neq \delta$ because $\min_{i \in I} a_i < 0$ and $d - a_i \leq \delta + a_i$ for all $i \in I$. Thus we obtain the absurd equality

$$0 = \sum_{i \in I'} z_i^0 \frac{\partial F_\Delta}{\partial z_i}(z^0, t^0) = \sum_{i \in I'} \overline{\lambda^0} |z_i^0|^2.$$

The lemma is proved. $\square$

Now we have finished the preliminaries and can complete the proof of the main result.

PROOF OF THEOREM 1.1. We fix a large number $r \in (0, +\infty)$ such that:

(i) for the open disc D_r the conclusions of Lemmas 3.2, 3.3 and 3.4 are fulfilled i.e., we have

$$\Sigma_0(F_t) \cup \Sigma_\infty(F_t) \cup \{F_t(0)\} \subset D_r \quad \text{for all } t \in [0,1].$$

(ii) for R sufficiently large the conclusion of Lemma 3.5 is satisfied.

Then, by Proposition 2.1, we get

$$B(F_t) \subset D_r \quad \text{for all } t \in [0,1].$$

Hence, it follows from Lemma 3.1 that the global monodromy fibration of the polynomial function F_t:

$$F_t \colon F_t^{-1}(\boldsymbol{S}_r^1) \to \boldsymbol{S}_r^1.$$

is isomorphic to the following fibration

$$F_t \colon F_t^{-1}(\boldsymbol{S}_r^1) \cap \boldsymbol{B}_R^{2n} \to \boldsymbol{S}_r^1.$$

Now, with arguments similar to the ones used in the proof of the classical Lê D. T. and Ramanujam C. P. theorem (see [**17**, Lemma 2.1] or [**5**, Lemma 12]), we have that the fibrations $F_t \colon F_t^{-1}(\boldsymbol{S}_r^1) \cap \boldsymbol{B}_R^{2n} \to \boldsymbol{S}_r^1, t \in [0,1]$, are isomorphic. As a conclusion, the global monodromy fibrations of the polynomials F_t are isomorphic. This completes the proof of Theorem 1.1. □

Remark 3.6. As in the remark after the proof of Theorem 3 in [**5**], we can improve the proof of Lemma 3.1 (and then of Theorem 1.1) in order to get a trivialization of the whole family of global monodromy fibrations of F_t, that is to say the family of fibrations $F_t^{-1}(\boldsymbol{S}_r^1) \to \boldsymbol{S}_r^1, t \in [0,1]$, is topologically a product family. We shall leave to the reader to verify these facts.

We will illustrate Theorem 1.1 with two examples.

Example 3.7. We study a family of polynomials $F_t(x,y) := 2x^4 - 3tx^2y^2 + x^2y^3$. An easy computation show that $\Gamma_\infty(F_t)$ is constant and the polynomial function F_t is non-degenerate for all $t \in [0,1]$. By Theorem 1.1, the global monodromy fibrations of F_0 and F_1 are isomorphic. We notice that F_t has non-isolated critical points, $\Sigma_0(F_0) = \{0\}$ and for $t \neq 0$, $\Sigma_0(F_t) = \{0, -2t^6\}$. Moreover, it is not hard to see that $B(F_t) = \Sigma_0(F_t)$.

Example 3.8. Let us consider $F_t(x,y,z) := x + tx^2yz + x^3y^2z^2$. By a direct calculation, $\Gamma_\infty(F_t)$ is constant and the polynomial function F_t is non-degenerate for all $t \in [0,1]$. By again Theorem 1.1, the global monodromy fibrations of F_0

and F_1 are isomorphic. In this example, the polynomial function $F_t\colon \boldsymbol{C}^3 \to \boldsymbol{C}$ has no critical points. Moreover, it is easy to check that $\Sigma_\infty(F_t) = \emptyset$ and the fiber $F_t^{-1}(0)$ is not topologically equivalent to any other fiber. Consequently, $B(F_t) = \{0\}$. Let us notice that the polynomial function F_t has "non-isolated singularities at infinity" (see [**32**] or [**34**] for this notion).

4. Continuity of the critical values.

Let $F_t, t \in [0,1]$, be a family of complex polynomial functions in n variables. For the remainder of the paper, motivated by the works of Bodin A. [**5**], [**6**], [**7**] we shall prove that the multi-valued functions $t \mapsto \Sigma_\infty(F_t) \cup \{F_t(0)\}$ and $\mapsto \Sigma_0(F_t) \cup \Sigma_\infty(F_t) \cup \{F_t(0)\}$ are closed continuous functions under some assumptions.

We say that a multi-valued function $t \mapsto \mathfrak{F}(t)$ is continuous if at each point t^0 and at each value $c^0 \in \mathfrak{F}(t^0)$ there is a neighborhood U of t^0 such that for all $t \in U$, there exists $c(t) \in \mathfrak{F}(t)$ near c^0. $\mathfrak{F}$ is closed, if, for all points t^0, for all sequences $c(t) \in \mathfrak{F}(t), t \neq t^0$, such that $c(t) \to c^0 \in \boldsymbol{C}$ as $t \to t^0$, then $c^0 \in \mathfrak{F}(t^0)$.

It is well known (see, for example, [**6**]) that the multi-valued function $t \mapsto \Sigma_0(F_t)$ is not closed. Moreover, it is not necessarily continuous even if $\Gamma_\infty(F_t)$ is constant and F_t is non-degenerate for all $t \in [0,1]$. For instance:

EXAMPLE 4.1. Let the family $F_t(x,y) := x^3 - x^2y + tx$. Then it is easy to see that $\Gamma_\infty(F_t)$ is constant and F_t is non-degenerate for all $t \in [0,1]$. We have $\Sigma_0(F_0) = \{0\}$ and for $t \neq 0$, $\Sigma_0(F_t) = \emptyset$. We notice that even if $t \mapsto \Sigma_0(F_t)$ is not continuous and closed, the maps $t \mapsto \Sigma_\infty(F_t) \cup \{F_t(0)\}$ and $t \mapsto \Sigma_0(F_t) \cup \Sigma_\infty(F_t) \cup \{F_t(0)\}$ are continuous and closed. This is expressed in the following result.

THEOREM 4.2 (Compare [**6**, Theorem 1]). *Let $F_t, t \in [0,1]$, be an analytic family of complex polynomials such that $\emptyset \neq \Gamma_\infty(F_t)$ is constant and such that F_t is non-degenerate for all $t \in [0,1]$. Then the multi-valued functions*

$$t \mapsto \Sigma_\infty(F_t) \cup \{F_t(0)\}$$

and

$$t \mapsto \Sigma_0(F_t) \cup \Sigma_\infty(F_t) \cup \{F_t(0)\}$$

are continuous and closed.

PROOF OF THE CONTINUITY. We first suppose that $c^0 \in \Sigma_0(F_{t^0})$ for some $t^0 \in [0,1]$. If the hypersurface $\{F_{t^0}(z) = c^0\}$ has only isolated singularities, then it follows easily from [**8**, Proposition 2.1] that for all t near t^0 there exists a critical value $c(t) \in \Sigma_0(F_t)$ near c^0, and we get the continuity. Otherwise, if the hypersurface $\{F_{t^0}(z) = c^0\}$ has non-isolated singularities, then from Lemma 2.2, we obtain $c^0 \in \Sigma_\infty(F_{t^0}) \cup \{F_{t^0}(0)\}$. Therefore, it suffices to show that the multi-valued function $t \mapsto \Sigma_\infty(F_t) \cup \{F_t(0)\}$ is continuous.

Clearly, the function $t \mapsto \{F_t(0)\}$ is continuous. Moreover, for simplicity of notations, we assume $F_t(0) = 0$, that is to say the constant term of F_t is zero.

We suppose that $c^0 \in \Sigma_\infty(F_{t^0})$ and that $c^0 \neq F_{t^0}(0)$. By Proposition 2.1, there exists a face Δ of $\Gamma_-(F_{t^0})$ that contains the origin such that $c^0 \in \Sigma_0^{'}((F_{t^0})_\Delta))$. We shall use induction on $\dim(\Delta)$ to get the continuity.

The case $\dim(\Delta) = 1$: we have to adapt the beginning of the proof of [**6**, Lemma 11]. As $\Gamma_\infty(F_t)$ is constant, Δ is a face of $\Gamma_-(F_t)$ for all t. There exists a family of polynomials $P_t \in \boldsymbol{C}[u]$ and a monomial $z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$ ($\alpha_i > 0$, $\gcd(\alpha_1, \alpha_2, \dots, \alpha_n) = 1$) such that $(F_t)_\Delta(z) = P_t(z^\alpha)$. The family P_t is continuous (in t) and is of constant degree (because $\Gamma_-(F_t)$ is constant). The set $\Sigma_0^{'}((F_{t^0})_\Delta)$ and more generally the set $\Sigma_0^{'}((F_t)_\Delta)$ can be computed by

$$\Sigma_0^{'}((F_t)_\Delta) = \{P_t(u) \mid u \in \boldsymbol{C} - \{0\} \quad \text{and} \quad P_t'(u) = 0\}.$$

Since $c^0 \in \Sigma_0^{'}((F_{t^0})_\Delta)$, there exists a value $u^0 \in \boldsymbol{C} - \{0\}$ with $P_{t^0}'(u^0) = 0$, and for t near t^0 there is a value $u(t) \in \boldsymbol{C} - \{0\}$ near u^0 with $P_t'(u(t)) = 0$ (because $P_t'(u)$ is a continuous function in t and is of constant degree in u). Then $c(t) := P_t(u(t)) \in \Sigma_\infty(F_t)$ near c^0 and we get the continuity.

We now assume that $\dim(\Delta) > 1$. If the hypersurface $\{(F_{t^0})_\Delta(z) = c^0\}$ has only isolated singularities, then, based again on Proposition 2.1 of [**8**], we easily get the continuity. Otherwise, it follows from Lemma 2.2 and $c^0 \neq F_{t^0}(0)$ that there exists a bad face $\Delta' \subsetneq \Delta$ of $\overline{\mathrm{supp}(F_{t^0})}$ with $\dim(\Delta') < \dim(\Delta)$ such that $c^0 \in \Sigma_0^{'}((F_{t^0})_{\Delta'})$. By induction, we get the continuity.

Proof of the closeness. Based on the Curve Selection Lemma (see [**24**], [**26**]), it suffices to verify the claim on analytic curves. Note that, the multi-valued function $t \mapsto \{F_t(0)\}$ is closed because the family $F(z,t)$ is continuous. Hence, the claim follows immediately from Lemmas 4.3 and 4.4 below. Consequently, the theorem is proved. □

Lemma 4.3. *Let $(p(s), t(s)), s \in (0, \epsilon)$, be a real analytic curve such that*:

(i) $\lim_{s\to 0} t(s) = t^0 \in [0,1]$;
(ii) $\lim_{s\to 0} F(p(s), t(s)) = c^0$; *and*
(iii) $\mathrm{grad} F(p(s), t(s)) \equiv 0$.

Then $c^0 \in \Sigma_0(F_{t^0}) \cup \Sigma_\infty(F_{t^0}) \cup \{F_{t^0}(0)\}$.

PROOF. Indeed, if $\lim_{s\to 0} p(s) = z^0 \in \boldsymbol{C}^n$, then $c^0 = F(z^0, t^0) \in \Sigma_0(F_{t^0})$, and there is nothing to prove. So we suppose that $\lim_{s\to 0} \|p(s)\| = \infty$. Let $I := \{i \mid p_i \not\equiv 0\}$. We have that $I \neq \emptyset$. If $\Gamma_\infty(F_t) \cap \boldsymbol{R}^I = \emptyset$, then it is easy to see that $c^0 = F_{t^0}(0)$, and there is nothing to prove. Thus, with no loss of generality, we may well assume that $\Gamma_\infty(F_t) \cap \boldsymbol{R}^I \neq \emptyset$. For $i \in I$, expand the coordinate p_i in terms of the parameter: say

$$p_i(s) = z_i^0 s^{a_i} + \text{higher order terms in } s,$$

where $z_i^0 \neq 0$ and $\min_{i\in I} a_i < 0$. Let d be the minimal value of the linear function $\sum_{i\in I} a_i\alpha_i$ on $\Gamma_-(F_t)\cap\boldsymbol{R}^I$, and let Δ be the (unique) maximal face of $\Gamma_-(F_t)\cap\boldsymbol{R}^I$ where the linear function takes this value. Then we may write

$$\frac{\partial F}{\partial z_i}(p(s), t(s)) = \frac{\partial F_\Delta}{\partial z_i}(z^0, t^0)s^{d-a_i} + \text{higher order terms in } s.$$

This fact, together with Condition (iii), yields that

$$\frac{\partial F_\Delta}{\partial z_i}(z^0, t^0) = 0, \quad i = 1, 2, \ldots, n.$$

By the non-degeneracy assumption for the polynomial function $F_{t^0}(z)$, we find that $d = 0$ and Δ is a bad face of $\overline{\operatorname{supp}(F_t)}$. Moreover, it is not difficult to see that $c^0 = F_\Delta(z^0, t^0)$. Consequently, $c^0 \in \Sigma_\infty(F_{t^0})$. This proves the lemma. $\square$

LEMMA 4.4. *Let Δ be a bad face of $\overline{\operatorname{supp}(F_t)}$ and let $(p(s), t(s)) \in (\boldsymbol{C} - \{0\})^n \times [0,1], s \in (0, \epsilon)$, be a real analytic curve such that*

(i) $\lim_{s\to 0} t(s) = t^0 \in [0,1]$;
(ii) $\lim_{s\to 0} F_\Delta(p(s), t(s)) = c^0$; *and*
(iii) $\operatorname{grad} F_\Delta(p(s), t(s)) \equiv 0$.

Then $c^0 \in \Sigma_\infty(F_{t^0}) \cup \{F_{t^0}(0)\}$.

PROOF. In fact, we may write

$$p_i(s) = z_i^0 s^{a_i} + \text{higher order terms in } s,$$

where $z_i^0 \neq 0$ and $a_i \in \boldsymbol{Q}$, $i = 1, 2, \ldots, n$. If $a_1 = a_2 = \cdots = a_n = 0$, then it is clear that $c^0 \in \Sigma_0'(F_{t^0}) \subset \Sigma_\infty(F_{t^0})$, and there is nothing to prove. In the converse case, let Δ' be the maximal face of Δ where the linear function $\sum_{i=1}^n a_i\alpha_i$ defined on Δ

takes its minimal value, say d. Let I be the smallest subset of the set $\{1, 2, \ldots, n\}$ such that $\Delta' \subset \boldsymbol{R}^I$. If $\Gamma_\infty((F_t)_\Delta) \cap \boldsymbol{R}^I = \emptyset$, then $c^0 = (F_{t^0})_\Delta(0) = F_{t^0}(0)$, and there is nothing to prove. Thus, with no loss of generality, we may well assume that $\Gamma_\infty((F_t)_\Delta) \cap \boldsymbol{R}^I \neq \emptyset$. We have the Taylor expansions

$$\frac{\partial F_\Delta}{\partial z_i}(p(s), t(s)) = \frac{\partial F_{\Delta'}}{\partial z_i}(z^0, t^0) s^{d - a_i} + \text{higher order terms in } s, \quad i = 1, 2, \ldots, n.$$

This fact, combined with Condition (iii), yields that

$$\frac{\partial F_{\Delta'}}{\partial z_i}(z^0, t^0) = 0, \quad i = 1, 2, \ldots, n.$$

By the non-degeneracy assumption for the polynomial function $F_{t^0}(z)$, hence $d = 0$ and $\Delta' \subset \Delta$ is a bad face of $\overline{\operatorname{supp}(F_t)}$. Moreover, it is easy to see that $c^0 = F_{\Delta'}(z^0, t^0)$. Consequently, $c^0 \in \Sigma_\infty(F_{t^0})$. This ends the proof. $\square$

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