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Subordinate fibers of Takamura splitting families for stellar singular fibers

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Abstract. Takamura constructed a theory on splitting families of degenerations of Riemann surfaces. We call them Takamura splitting families. In a Takamura splitting family, there appear two kinds of singular fibers, called a main fiber and subordinate fibers. In this paper, when the original singular fiber is stellar and the core is a projective line, we determine the number of subordinate fibers and describe the types of singular points, which are nodes.

1. Introduction.

A degeneration of Riemann surfaces of genus g is a proper surjective holomorphic map $\pi : M \to \Delta$ from a smooth complex surface M to the unit open disk $\Delta = \{s \in \boldsymbol{C} : \ |s| < 1\}$, and the fiber over the origin is singular and any other fiber is a smooth complex curve of genus g ($g \geq 1$). A smooth fiber is called a *general fiber*. The classification problem of degenerations has been studied in the algebraic geometry and many results are obtained. On the other hand, Matsumoto-Montesinos [**6**] proved that topological equivalent classes of degenerations of Riemann surfaces correspond to conjugate classes of topological monodromies.

In this paper, we are rather interested in deformations of degenerations, that is, splitting families. Let $\Delta^\dagger = \{t \in \boldsymbol{C} : \ |t| < \varepsilon\}$ be a sufficient small disk, let $\mathscr{M}$ be a smooth complex 3-manifold, and let $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ be a proper flat surjective holomorphic map. We denote $\Delta \times \{t\}$ by Δ_t, $\Psi^{-1}(\Delta_t)$ by M_t and $\Psi |_{M_t}$: $M_t \to \Delta_t$ by π_t ($t \in \Delta^\dagger$). Then M_t is a complex 2-manifold and π_t is a surjective holomorphic map. We say that $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is a deformation family of the degeneration $\pi : M \to \Delta$ if for $t = 0$, $\pi_0 : M_0 \to \Delta_0$ coincides with the original degeneration $\pi : M \to \Delta$. Moreover if for $t \neq 0$, $\pi_t : M_t \to \Delta_t$ has more than two singular fibers, then we call $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ a *splitting family* of $\pi : M \to \Delta$.

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There are lots of open problems of splitting families on a classification of atomic degenerations and on relations between monodromies and splitting families.

Motivated by the classification problem of atomic degenerations, Takamura [**13**] gave a new method of construction of splitting families, which he named *barking deformations*, and we call them *Takamura splitting families.* In this paper, we confine ourselves to the case when the original singular fiber X is stellar and the core is a projective line (see Definition 2.2 for the definition of stellar and the core). He defined a simple crust Y of X. Y is a subdivisor of X which satisfies some conditions (see Definition 2.5). He showed that if a singular fiber X has a simple crust, then there exists a splitting family $\pi_t : M_t \to \Delta_t$. For $t \neq 0$, the singular fiber $X_0 = \pi_t^{-1}(0)$ is called the *main fiber* of π_t and the others singular fibers of π_t are called *subordinate fibers.*

In order to state our theorem, we need to introduce the notion of "J-generic" and "proportional". We define an arrangement polynomial $J(z)$ in Section 2. The polynomial $J(z)$ plays an important role to prove our main theorem. A splitting family π_t is J-generic if $J(z) = 0$ has no multiple root. The proportionality of subbranches was introduced by Takamura [**13**]. He showed that if a simple crust Y has proportional subbranches, then the number of singular points in the neighborhood of the core gets fewer but there appear some singular points around the edges of the proportional subbranches.

The main theorem of this paper is as follows:

Theorem 1.1. *Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber X whose core is a projective line. Let $\pi_t : M_t \to \Delta_t$ be a Takamura splitting family associated with a simple crust Y. Suppose that $\pi_t : M_t \to \Delta_t$ is J-generic and any subbranch of Y is not proportional. Then*

(1) *The number of singular points of all subordinate fibers is $n_0(N - 2 + c)$.*
(2) *The number of subordinate fibers is at most $(n_0/d)(N - 2 + c)$.*
(3) *Any singular point of a subordinate fiber is a node.*

Here N, m_0, n_0, c and d are the integers associated with π and π_t. (See the notation above the expression (1).*)*

See the proof of Proposition 3.1 for (1) and (2) of Theorem 1.1, and the proof of Proposition 3.6 for (3) of Theorem 1.1. Using these results, the first author developed the software *Splitica* [**3**] to draw computer graphics of splitting phenomena.

Takamura [**13**] determine the types of singular points of subordinate fibers for the general case (the core may have arbitrary genus). He determines the number and the types of a singular points of subordinate fibers by using a *plot function.* Our proof is a little different from his.

In Section 2, we will explain the terminologies in the above statement, and we prepare notation and introduce Takamura's theorems. We prove our theorem from Section 3 to Section 5.

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2. Preparation.

After we review results on degenerations and Takamura splitting families given by Takamura [**12**] and [**13**], we give the local expression of splitting families around the core and the notion of a Milnor fiber.

Lemma 2.1 (Takamura [**12**]). *Suppose that integers $m_0, m_1, \dots, m_\lambda$ satisfy the following conditions*:

(1) $m_0 > m_1 > \cdots > m_\lambda > 0$,
(2) $r_i := (m_{i-1} + m_{i+1})/m_i$ *is an integer greater than one* $(i = 1, 2, \dots, \lambda - 1)$,
(3) $r_\lambda := m_{\lambda-1}/m_\lambda$ *is an integer greater than one.*

Then there exists a degeneration of Riemann surfaces with the singular fiber $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$. Here Δ_0 is an open disk and $\Theta_1, \Theta_2, \dots, \Theta_\lambda$ are projective lines. The components Θ_i and Θ_{i+1} (resp. *Δ_0 and Θ_1*) *intersect transversely at only one point and m_i* (resp. *m_0*) *is the multiplicity of Θ_i* (resp. *Δ_0*).

Definition 2.2. Let $\pi : M \to \Delta$ be a degeneration of Riemann surfaces. The singular fiber $X = \pi^{-1}(0)$ is said to be *stellar* if X satisfies the following three properties:

(1) The singular fiber X is expressed as $X = m_0\Theta_0 + \sum_{j=1}^{N} Br^{(j)}$, where $N \geq 2$, $Br^{(j)} = \sum_{i=1}^{\lambda_j} m_i^{(j)}\Theta_i^{(j)}$ and the integers $m_0, m_1^{(j)}, m_2^{(j)}, \dots, m_{\lambda_j}^{(j)}$ satisfy the conditions in Lemma 2.1 for each j.
(2) The irreducible component Θ_0 is a Riemann surface, and each $\Theta_i^{(j)}$ $(j = 1, 2, \dots, N,\ i = 1, 2 \dots, \lambda_j)$ is a projective line. The components $\Theta_i^{(j)}$ and $\Theta_{i+1}^{(j)}$ (also $\Theta_1^{(j)}$ and Θ_0) intersect transversely at one point $(j = 1, 2, \dots, N$, $i = 1, 2, \dots, \lambda_j - 1)$.
(3) Set $r_0 := 1/m_0 \sum_{j=1}^{N} m_1^{(j)}$. Then r_0 is a positive integer.

See Figure 1.

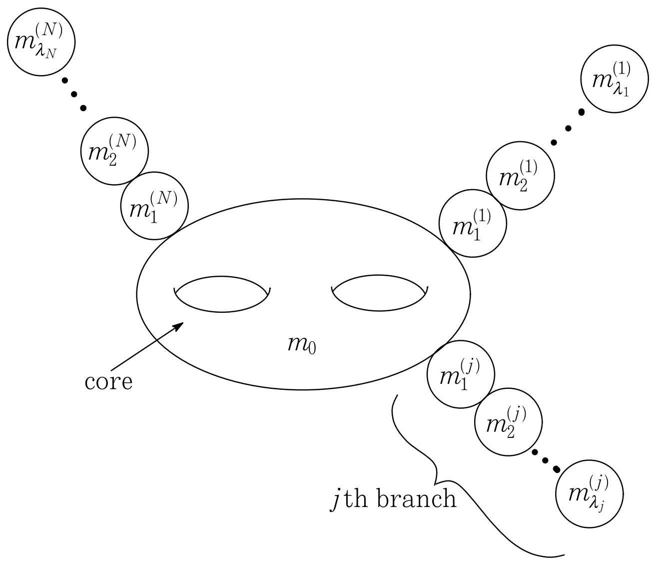

Figure 1. A stellar singular fiber.

The irreducible component Θ_0 is called the *core*, and a chain of projective lines $Br^{(j)}$ is called the jth *branch*. We call $\overline{Br}^{(j)} = m_0^{(j)}\Delta_0^{(j)} + \sum_{i=1}^{\lambda_j} m_i^{(j)}\Theta_i^{(j)}$ the jth *fringed branch*, where Δ_0 is a small disk around the intersection point of Θ_0 and $Br^{(j)}$. $m_0\Delta_0^{(j)}$ is called its fringe. From this forth, for simplicity, we call a fringed branch a branch. The negative integer $-r_0$ is equal to $\Theta_0 \cdot \Theta_0$, the self-intersection number of Θ_0. We remark that a degeneration $\pi : M \to \Delta$ has a stellar singular fiber precisely when the monodromy map of $\pi : M \to \Delta$ is periodic.

Theorem 2.3 (Takamura [**12**]). *Set $X := m_0\Theta_0 + \sum_{j=1}^{N} Br^{(j)}$, where $Br^{(j)} := \sum_{i=1}^{\lambda_j} m_i^{(j)}\Theta_i^{(j)}$. If integers $m_0, m_1^{(j)}, \dots, m_{\lambda_j}^{(j)}$ $(j = 1, 2, \dots, N)$ satisfy the conditions in Lemma* 2.1 *and $r_0 = 1/m_0 \sum_{j=1}^{N} m_1^{(j)}$ is a positive integer, then there exists a linear degeneration $\pi : M \to \Delta$ such that the singular fiber $\pi^{-1}(0)$ is equal to X.*

He constructed a degeneration $\pi : M \to \Delta$ with the singular fiber X by resoluting cyclic quotient singularities. The construction is as follows: Let C be a complex curve of genus g, and let $h : C \to C$ be a periodic automorphism of order m_0 with the valency data $\left(m_1^{(1)}/m_0, m_1^{(2)}/m_0, \dots, m_1^{(N)}/m_0\right)$. First, he constructed a minimal resolution map $r : M \to (C\times\Delta)/G$, where M is a resolution space of the quotient space $(C \times \Delta)/G$ and G is a cyclic group generated by $g : C \times \Delta \to C \times \Delta$, $g(x, s) = (h^{-1}(x), e^{2\pi \mathrm{i}/m_0} s)$. Secondly, he defines the function $\overline{\phi} : (C \times \Delta)/G \to \Delta$ induced by the G-invariant function $\phi : C \times \Delta \to \Delta$, $\phi(x, s) = s^{m_0}$. Set $\pi := \overline{\phi} \circ r : M \to \Delta$, and π is a degeneration with the singular fiber X. On the other hand, Matsumoto-Montesinos [**6**] constructed a degeneration by open-book construction.

Definition 2.4. Let $\pi : M \to \Delta$ be a degeneration with the singular fiber $\pi^{-1}(0) = X$. The degeneration $\pi : M \to \Delta$ is normally minimal if

(1) any singularity of the reduced part of X is a node, and
(2) an irreducible component of X is a exceptional curve, then it intersects other irreducible components at more than two points.

From now on, we assume that $\pi : M \to \Delta$ is normally minimal.

Set $X := m_0\Theta_0 + \sum_{j=1}^{N} Br^{(j)}$, where $Br^{(j)} := \sum_{i=1}^{\lambda_j} m_i^{(j)}\Theta_i^{(j)}$, and set $Y := n_0\Theta_0 + \sum_{j=1}^{N} br^{(j)}$, where $br^{(j)} := \sum_{i=1}^{e_j} n_i^{(j)}\Theta_i^{(j)}$. We say that Y is *a subdivisor* of X if Y satisfies (1) $\lambda_j \geq e_j \geq 0$ and (2) $m_i^{(j)} \geq n_i^{(j)} > 0$ for each i and j. Suppose that Y is a subdivisor of X. Let $\overline{Br}^{(j)} = m_0\Delta_0^{(j)} + \sum_{i=0}^{\lambda_j} m_i^{(j)}\Theta_i^{(j)}$ be the jth branch of a stellar singular fiber X. We call $\overline{br}^{(j)} = n_0\Delta_0^{(j)} + \sum_{i=0}^{e_j} n_i^{(j)}\Theta_i^{(j)}$ a *subbranch* of the jth branch $\overline{Br}^{(j)}$ if $\overline{br}^{(j)}$ satisfies one of the following conditions: (1) $e_j = 0, 1$ or (2) $e_j \geq 2$ and $n_{i-1}^{(j)} = r_i^{(j)} n_i^{(j)} - n_{i+1}^{(j)}$ $(i = 1, 2, \ldots, e_j - 1)$, where $n_0^{(j)} = n_0$. We note that if $\overline{br}^{(j)}$ is the subbranch of $\overline{Br}^{(j)}$ and $m_i^{(j)} \geq l n_i^{(j)}$ $(i = 1, 2, \ldots, e_j)$ for a positive integer l, then $l\overline{br}^{(j)} = l n_0 \Delta_0^{(j)} + \sum_{i=0}^{e_j} l n_i^{(j)}\Theta_i^{(j)}$ is also a subbranch of $\overline{Br}^{(j)}$. There are three types of subbranches introduced by Takamura [**13**]:

- Type A_l: For a positive integer l, a subbranch $\overline{br}^{(j)}$ of a branch $\overline{Br}^{(j)}$ is of type A_l if the following two conditions are satisfied: (1) $m_i^{(j)} \geq l n_i^{(j)}$ $(i = 0, 1, \ldots, e_j)$ and (2) $n_{e_j - 1}^{(j)}/n_{e_j}^{(j)} \geq r_{e_j}^{(j)}$, where $m_0^{(j)} = m_0$.
- Type B_l: For a positive integer l, a subbranch $\overline{br}^{(j)}$ of a branch $\overline{Br}^{(j)}$ is of type B_l if the following three conditions are satisfied: (1) $m_i^{(j)} \geq l n_i^{(j)}$, (2) $l = m_{e_j}^{(j)}$ and (3) $n_{e_j}^{(j)} = 1$.
- Type C_l: For a positive integer l, a subbranch $\overline{br}^{(j)}$ of a branch $\overline{Br}^{(j)}$ is of type C_l if the following four conditions are satisfied: (1) $m_i^{(j)} \geq l n_i^{(j)}$, (2) $n_{e_j - 1}^{(j)}$ is divided by $n_{e_j}^{(j)}$, (3) $r_{e_j} > n_{e_j - 1}^{(j)}/n_{e_j}^{(j)}$ and (4) l is divided by $u := (m_{e_j - 1}^{(j)} - l n_{e_j - 1}^{(j)}) - (r_{e_j}^{(j)} - 1)(m_{e_j}^{(j)} - l n_{e_j}^{(j)})$.

There exists a subbranch satisfying the conditions of both type A_l and type

B_l. It is said to be of type AB_l. A subbranch is said to be *proportional* if it satisfies $m_0 n_1^{(j)} = n_0 m_1^{(j)}$. In particular, a proportional subbranch satisfies $m_0/n_0 = m_1^{(j)}/n_1^{(j)} = \cdots = m_{e_j}^{(j)}/n_{e_j}^{(j)}$. We note that (i) a subbranch of type C_l is not proportional and (ii) if a subbranch of type B_l is proportional, then it is of type AB_l.

We need a "special" subdivisor of X. It is a criterion of a existence of a splitting family.

Definition 2.5 (Simple crust for stellar singular fiber). For a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^N Br^{(j)}$, let Y be a subdivisor of X, where $Y = n_0\Theta_0 + \sum_{j=1}^N br^{(j)}$ and $br^{(j)} = \sum_{i=1}^{e_j} n_i^{(j)}\Theta_i^{(j)}$. A subdivisor Y of X is a simple crust with the barking multiplicity l if the following two conditions are satisfied:

(1) There exists a positive integer l such that for each $j = 1, 2, \ldots, N$, $n_0\Delta_0^{(j)} + br^{(j)}$ is a subbranch of $m_0\Delta_0^{(j)} + Br^{(j)}$ of the type A_l, B_l or C_l. Here $\Delta_0^{(j)}$ is an open disk around $p_j = \Theta_0 \cap \Theta_1^{(j)}$.
(2) $N^{\otimes n_0} \cong \mathscr{O}_{\Theta_0}(-\sum_{j=1}^N n_1^{(j)}p_1^{(j)} + D)$, where $D = \sum_{i=1}^c a_i q_i$ and $\mathscr{O}_{\Theta_0}$ stands for the sheaf of germs of holomorphic functions on Θ_0.

If $\Theta_0 = \boldsymbol{P}^1$, then we may replace the condition (2) by $r_0' \geq r_0$, where $r_0' = 1/n_0 \cdot \sum_{j=1}^N n_1^{(j)}$ and $r_0 = 1/m_0 \sum_{j=1}^N m_1^{(j)}$. We state Takamura's theorem as follows:

Theorem 2.6 (Takamura [**13**]). *Let $\pi : M \to \Delta$ be a linear degeneration with the singular fiber $\pi^{-1}(0) = X$. If X has a simple crust Y, then $\pi : M \to \Delta$ admits a splitting family $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$.*

He constructs a splitting family of a degeneration by barking a simple crust Y from a singular fiber X. So it is called a *barking deformation*. We review only the case that a degeneration $\pi : M \to \Delta$ has a stellar singular fiber. But Takamura [**13**] gave criteria of all linear degenerations. For $t \in \Delta^\dagger$, we set $\Delta_t := \Delta \times \{t\}$, $M_t := \Psi^{-1}(\Delta_t)$ and $\pi_t := \Psi\mid_{M_t}: M_t \to \Delta_t$. A barking deformation $\pi_t : M_t \to \Delta_t$ has a singular fiber $\pi_t^{-1}(0)$ $(t \neq 0)$, which we call it the *main fiber*. For $t \neq 0$, $\pi_t : M_t \to \Delta_t$ has some singular fibers other than $\pi_t^{-1}(0)$. We call them *subordinate fibers*. We call $s \in \Delta$ is a *singular value* if $\pi_t^{-1}(s)$ is a singular fiber.

In this paper, we confine ourselves to the case when X is stellar. In the sequel, we assume that X is a stellar singular fiber and the core Θ_0 is $\boldsymbol{P}^1$. We give notation:

- m_0 : the multiplicity of the core of the singular fiber X,
- N : the number of the branches of X,
- $r_0 := 1/m_0 \sum_{j=1}^N m_1^{(j)}$; $-r_0$ is the self-intersection number of the core,
- n_0 : the multiplicity of the core of the simple crust Y,

- Let (z,ζ) and (w,η) be coordinates of a tubular neighborhood of the core Θ_0. Here the base coordinates z and w on Θ_0 satisfy the relation $z=1/w$, and the fiber coordinates ζ and η satisfy the relation $\zeta=w^{r_0}\eta$.
- $c:=\sum_{j=1}^{N} n_1^{(j)}-r_0n_0$,
- $d:=\gcd(m_0,n_0)$,
- $f(z)$: a polynomial of degree c, the roots are denoted by $q_1,q_2,\dots,q_c$. We call $q_1,q_2,\dots,q_c$ *auxiliary points.*
- $p_1,p_2,\dots,p_N$: the attachment points of branches to the core, that is, $p_j=\Theta_0\cap\Theta_1^{(j)}$ $(j=1,2,\dots,N)$. We assume that they are mutually distinct complex numbers in generic positions. Moreover we assume $p_N=\infty$.

We give the local expression of deformation π_t around the core as follows:

$$\pi_t(z,\zeta)=\zeta^{m_0-ln_0}\prod_{j=1}^{N-1}(z-p_j)^{m_1^{(j)}-ln_1^{(j)}}\left\{\zeta^{n_0}\prod_{j=1}^{N-1}(z-p_j)^{n_1^{(j)}}+tf(z)\right\}^l, \tag{1}$$

$$\begin{aligned}\pi_t(w,\eta)&=\eta^{m_0-ln_0}w^{m_N-ln_N}\prod_{j=1}^{N-1}(1-p_jw)^{m_1^{(j)}-ln_1^{(j)}}\\&\quad\times\left\{\eta^{n_0}w^{n_N}\prod_{j=1}^{N-1}(1-p_jw)^{n_1^{(j)}}+tf(w)\right\}^l.\end{aligned}$$

Takamura propagates this along branches to construct a splitting family. We give an important polynomial $J(z)$ on the core:

$$J(z):=\sum_{i=1}^{N-1}\big(m_0n_1^{(i)}-n_0m_1^{(i)}\big)G_i(z)f(z)-m_0\prod_{j=1}^{N-1}(z-p_j)f'(z),$$

where $f'(z)=df(z)/dz$ and

$$G_i(z):=(z-p_1)(z-p_2)\cdots(z-p_{i-1})(z-p_{i+1})\cdots(z-p_{N-1}).$$

We call $J(z)$ an *arrangement polynomial.* The equation $J(z)=0$ determines z-coordinates of singular points of subordinate fibers.

Definition 2.7. A Takamura splitting family $\Psi:\mathscr{M}\to\Delta\times\Delta^\dagger$ is said to be *J-generic* if $J(z)=0$ has no multiple roots.

This condition depends on the points $p_1,p_2,\dots,p_{N-1}$, $q_1,q_2,\dots,q_c$. We also say that $p_1, p_2, \ldots, p_{N-1}$, $q_1, q_2, \ldots, q_c$ are *in J-generic positions* under the same condition. We note that we can take $p_1, p_2, \ldots, p_{N-1}$, $q_1, q_2, \ldots, q_c$ are in J-generic positions (see Section 4).

Now we review the Milnor fiber, which is needed for showing our theorem. Let $f : \boldsymbol{C}^{n+1} \to \boldsymbol{C}$ be a holomorphic function. For sufficiently small positive numbers ε and δ, we set $B_\varepsilon := \{(z_1, z_2, \ldots, z_{n+1}) \in \boldsymbol{C}^{n+1} : |z_1|^2 + |z_2|^2 + \cdots + |z_{n+1}|^2 < \varepsilon\}$ and $D_\delta := \{s \in \boldsymbol{C} : |s| < \delta\}$. For non-zero value $s \in D_\delta$, we set $X_s := \{(z_1, z_2, \ldots, z_{n+1}) \in \boldsymbol{C}^{n+1} : f(z_1, z_2, \ldots, z_{n+1}) = s\}$ and $F_s := X_s \cap B_\varepsilon$. We call F_s a *Milnor fiber*. Let μ be the first Betti number of F_s. We call μ the *Milnor number* of F_s. We introduce two well-known facts of a Milnor fiber and the Milnor number.

THEOREM 2.8 (Milnor [**9**]). *If $(0, 0, \ldots, 0)$ is an isolated singular point, then the Milnor fiber F_s is $(n-1)$-connected. In particular, when $n = 1$, the Milnor fiber F_s is connected.*

THEOREM 2.9 (Milnor [**9**]). *Let $f : \boldsymbol{C}^{n+1} \to \boldsymbol{C}$ be a holomorphic function. For a non-zero value $s \in D_\delta$, let F_s be a Milnor fiber and let μ be the Milnor number of F_s. Then*

(1) $(0, 0, \ldots, 0)$ *is a singular point if and only if the Milnor number μ of F_s is greater than zero.*
(2) $(0, 0, \ldots, 0)$ *is a node (A_1-singularity) if and only if the Milnor number μ of F_s is equal to one.*

From this theorem, if $(0, 0, \ldots, 0)$ is a singular point, then the Milnor fiber is not a disk. (Note: The first Betti number of a disk is zero.) Moreover, if the Milnor fiber F_s is an annulus, then the Milnor number is one, and $(0, 0, \ldots, 0)$ is a node.

3. Proof of main Theorem.

Now we prove our main theorem. First, we show

PROPOSITION 3.1. *Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^N Br^{(j)}$ ($Br^{(j)} = \sum_{i=1}^{\lambda_j} m_i^{(j)}\Theta_i^{(j)}$) such that the core Θ_0 is a projective line. Let $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ be a Takamura splitting family associated with a simple crust $Y = n_0\Theta_0 + \sum_{j=1}^N br^{(j)}$ ($br^{(j)} = \sum_{i=1}^{e_j} n_i^{(j)}\Theta_i^{(j)}$). Suppose that $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is J-generic and any subbranch $br^{(j)}$ is not proportional. Then*

(1) *The number of singular points of all subordinate fibers is $n_0(N - 2 + c)$.*
(2) *The number of subordinate fibers is at most $(n_0/d)(N - 2 + c)$.*

Here $d = \gcd(m_0, n_0)$ *and* $c = \sum_{j=1}^{N} n_1^{(j)} - r_0 n_0$.

REMARK 3.2. In almost all cases, there are $(n_0/d)(N - 2 + c)$ subordinate fibers. But the number of subordinate fibers may decrease since the positions of subordinate fibers depend on $p_1, p_2, \ldots, p_N$, $q_1, q_2, \ldots, q_c$, that is, the subordinate fibers may coincide to each other.

Before we give the proof of Proposition 3.1, we need the following lemma due to Takamura [**13**, Proposition 7.2.1. p. 124, Proposition 10.2.1. p. 180, and Proposition 11.4.2. p. 200].

LEMMA 3.3 (Takamura [**13**]). *Let* $\overline{br}^{(j)}$ *be a subbranch of a branch* $\overline{Br}^{(j)}$. *If* $\overline{br}^{(j)}$ *is not a proportional subbranch, then singular points of subordinate fibers are not on the deformations of tubular neighborhoods of branches.*

We note that if a subbranch is proportional, then the deformation of tubular neighborhoods of branches has singular points.

PROOF OF PROPOSITION 3.1. We fix $t \in \Delta \setminus \{0\}$, and we determine the number of singular points and singular values. By assumption, the simple crust Y does not have a proportional subbranch. So by Lemma 3.3, singular points are not on the deformations of tubular neighborhoods of branches. Hence in order to know the positions and the numbers of singular points of subordinate fibers, we solve the following system of equations:

$$\pi_t(z,\zeta) = s \neq 0, \quad \frac{\partial \pi_t}{\partial \zeta}(z,\zeta) = 0 \quad \text{and} \quad \frac{\partial \pi_t}{\partial z}(z,\zeta) = 0,$$

where $\pi_t(z,\zeta)$ is expressed by (1). (Note: Any attachment point p_i is not a base coordinate of the singular points of subordinate fibers since for $s \neq 0$, (p_i, ζ) (resp. $(1/p_i, \eta)$) does not satisfy the equation $\pi_t(z,\zeta) - s = 0$ (resp. $\pi_t(w,\eta) - s = 0$). Hence $p_N = \infty$ is not a base coordinate of singular points. So we only consider the expression (1).) The calculations are very complicated, so we give the details in Section 5. Here we give a survey of the proof.

From $\pi_t(z,\zeta) \neq 0$ and $\partial \pi_t(z,\zeta)/\partial \zeta = 0$, we obtain

$$t = -\frac{m_0 \zeta^{n_0} \prod_{j=1}^{N-1} (z - p_j)^{n_1^{(j)}}}{(m_0 - l n_0) f(z)}. \tag{2}$$

(See Section 5.1.) Substitute (2) into $\pi_t(\zeta, z) = s$, which yields

$$s=\left(-\frac{ln_0}{m_0-ln_0}\right)^l\zeta^{m_0}\prod_{j=1}^{N-1}(z-p_j)^{m_1^{(j)}}. \tag{3}$$

From (2) and $\partial\pi_t(z,\zeta)/\partial z=0$, the following equation holds:

$$J(z)=\sum_{i=1}^{N-1}(m_0n_1^{(i)}-n_0m_1^{(i)})G_i(z)f(z)-m_0\prod_{j=1}^{N-1}(z-p_j)f'(z)=0,$$

where $f'(z)=df(z)/dz$ and

$$G_i(z):=(z-p_1)(z-p_2)\cdots(z-p_{i-1})(z-p_{i+1})\cdots(z-p_{N-1}).$$

(See Section 5.2.) The coefficient of the highest degree term of $J(z)$ is

$$\begin{aligned}\sum_{i=1}^{N-1}\bigl(m_0n_1^{(i)}-n_0m_1^{(i)}\bigr)-m_0c&=-n_0\sum_{i=1}^{N-1}m_1^{(i)}+m_0\Bigl(\sum_{i=1}^{N-1}n_1^{(i)}-c\Bigr)\\&=-n_0\bigl(r_0m_0-m_1^{(N)}\bigr)+m_0\bigl(r_0n_0-n_1^{(N)}\bigr)\\&=n_0m_1^{(N)}-m_0n_1^{(N)}.\end{aligned} \tag{4}$$

Since a simple crust Y has no proportional subbranches, $n_0m_1^{(N)}-m_0n_1^{(N)}$ is not equal to zero. So the degree of $J(z)$ is $(N-2+c)$. From the assumption, the splitting family $\Psi:\mathscr{M}\to\Delta\times\Delta^\dagger$ is J-generic. Hence the equation has $(N-2+c)$ roots. We denote the roots of $J(z)=0$ by α_k $(k=1,2,\dots,N-2+c)$. From (2), we have

$$\zeta^{n_0}=\frac{(ln_0-m_0)t}{m_0\prod_{j=1}^{N-1}(\alpha_k-p_j)^{n_1^{(j)}}}, \tag{5}$$

and thus for each α_k, the number of roots of (5) is n_0. Hence there are $n_0(N-2+c)$ singular points.

REMARK 3.4. From above discussion, singular points of subordinate fibers are isolated.

Deleting the term in ζ from (2) and (3), we have

$$\frac{t^{m_0/d}}{s^{n_0/d}} = \frac{m_0^{m_0/d}(m_0 - ln_0)^{(ln_0-m_0)/d}}{(-ln_0)^{ln_0/d}}\left\{\prod_{j=1}^{N-1}(z-p_j)^{(m_0n_1^{(j)}-n_0m_1^{(j)})/d}\right\} \times \frac{1}{f(z)^{m_0/d}}. \tag{6}$$

We set

$$g(z) := \frac{m_0^{m_0/d}(m_0 - ln_0)^{(ln_0-m_0)/d}}{(-ln_0)^{ln_0/d}}\left\{\prod_{j=1}^{N-1}(z-p_j)^{(m_0n_1^{(j)}-n_0m_1^{(j)})/d}\right\} \times \frac{1}{f(z)^{m_0/d}}.$$

Namely, $g(z)$ is the right hand side of the equation (6). For each α_k $(k = 1, 2, \ldots, N-2+c)$ of the root of $J(z) = 0$, we solve the equation in s

$$s^{n_0/d} = \frac{t^{m_0/d}}{g(\alpha_k)}. \tag{7}$$

(Note: $g(\alpha_k)$ is not zero since α_k is not either of $p_1, p_2, \ldots, p_N$, $q_1, q_2, \ldots$, or q_c.) For each α_k, the number of the singular values s is n_0/d. We note that there may exist α_i and α_j $(i \neq j)$ such that $g(\alpha_i) = g(\alpha_j)$, that is, the right hand side of the equation (7) coincides to each other. So there appear at most $(n_0/d)(N-2+c)$ subordinate fibers. This completes the proof of Proposition 3.1. □

Remark 3.5. If $g(\alpha_1), g(\alpha_2), \ldots$, and $g(\alpha_{N-2+c})$ are all different, then there exist $(n_0/d)(N-2+c)$ subordinate fibers and each subordinate fiber has d singular points.

Rewriting the equation (6), then we obtain

$$\prod_{j=1}^{N-1}(z-p_j)^{(m_0n_1^{(j)}-n_0m_1^{(j)})/d} - \frac{(-ln_0)^{ln_0/d}t^{m_0/d}}{m_0^{m_0/d}(m_0 - ln_0)^{(ln_0-m_0)/d}s^{n_0/d}}f(z)^{m_0/d} = 0. \tag{8}$$

We denote by $D_s(z)$ the left hand side of this equation. Namely,

$$D_s(z) := \prod_{j=1}^{N-1}(z-p_j)^{(m_0n_1^{(j)}-n_0m_1^{(j)})/d} - \frac{(-ln_0)^{ln_0/d}t^{m_0/d}}{m_0^{m_0/d}(m_0 - ln_0)^{(ln_0-m_0)/d}s^{n_0/d}}f(z)^{m_0/d}.$$

For each s, we define a branched covering map

$$h_s : \pi_t^{-1}(s) \to \boldsymbol{C}\backslash\{p_1, p_2, \ldots, p_{N-1}\}$$

by

$$h_s(z,\zeta) := z.$$

This is the map from a fiber $\pi_t^{-1}(s)$ to the core of the singular fiber $\pi^{-1}(0)$. For $z_0 \in \boldsymbol{C} \setminus \{p_1, p_2, \dots, p_{N-1}\}$, the inverse image $h_s^{-1}(z_0) = \{(z_0,\zeta) : \pi_t(z_0,\zeta) - s = 0\}$. It follows that $(z_0,\zeta_0) \in h_s^{-1}(z_0)$ is a ramification point if and only if (z_0,ζ_0) satisfies $\pi_t(z_0,\zeta_0) - s = 0$ and $\partial\pi_t(z_0,\zeta_0)/\partial\zeta = 0$. So the branch points of h_s is given by the roots of this equation $D_s(z) = 0$ since the equation (8) is obtained only from the system of equations

$$\pi_t(z,\zeta) - s = 0 \quad \text{and} \quad \frac{\partial \pi_t}{\partial \zeta}(z,\zeta) = 0.$$

Next we show

PROPOSITION 3.6. *Let X be a stellar singular fiber such that the core is a projective line. Suppose that $\Psi : \mathscr{M} \to \Delta \times \Delta^\dagger$ is J-generic and the simple crust Y has no proportional subbranch. Then any subordinate fiber has only nodes.*

(Note: This statement is nothing other than (3) of Theorem 1.1.) To prove Proposition 3.6, we need to show some results.

LEMMA 3.7. *If α is a kth root of $D_s(z) = 0$, then α is a $(k-1)$st root of $J(z) = 0$.*

PROOF. For simplicity, we set

$$M_j := \frac{m_0 n_1^{(j)} - n_0 m_1^{(j)}}{d} \quad \text{and} \quad a(s) := \frac{(-l n_0)^{l n_0/d} t^{m_0/d}}{m_0^{m_0/d}(m_0 - l n_0)^{(l n_0 - m_0)/d} s^{n_0/d}}.$$

Then we have

$$D_s(z) = \prod_{j=1}^{N-1} (z - p_j)^{M_j} - a(s) f(z)^{m_0/d}.$$

We compute $\partial D_s(z)/\partial z$:

$$\begin{aligned}\frac{\partial D_s}{\partial z}(z) &= \sum_{j=1}^{N-1} M_j G_j(z) \prod_{i=1}^{N-1} (z-p_i)^{M_i-1} - a(s)\frac{m_0}{d} f(z)^{m_0/d-1} f'(z) \\ &= \frac{\sum_{j=1}^{N-1} M_j G_j(z) \prod_{i=1}^{N-1} (z-p_i)^{M_i-1} f(z) - a(s) m_0 f(z)^{m_0/d} f'(z)}{d \times f(z)}.\end{aligned}$$

For a root α of $D_s(z) = 0$, the equation $a(s) f(\alpha)^{m_0/d} = \prod_{i=1}^{N-1} (\alpha - p_i)^{M_i}$ holds. We substitute this equation into the last expression. Then we have

$$\begin{aligned}\frac{\partial D_s}{\partial z}(\alpha) &= \frac{\sum_{j=1}^{N-1} M_j G_j(\alpha) \prod_{i=1}^{N-1} (\alpha-p_i)^{M_i-1} f(\alpha) - m_0 \prod_{i=1}^{N-1} (\alpha-p_i)^{M_i} f'(\alpha)}{d \times f(\alpha)} \\ &= \frac{\prod_{i=1}^{N-1} (\alpha-p_i)^{M_i-1} \big[\sum_{j=1}^{N-1} M_j G_j(\alpha) f(\alpha) - m_0 \prod_{i=1}^{N-1} (\alpha-p_i) f'(\alpha) \big]}{d \times f(\alpha)} \\ &= \frac{\prod_{i=1}^{N-1} (\alpha-p_i)^{M_i-1} J(\alpha)}{d \times f(\alpha)}. \qquad (9)\end{aligned}$$

On the other hand, if α is a root of $D_s(z) = 0$ of multiplicity k, then we have

$$D_s(z) = (z-\alpha)^k g(z), \quad g(\alpha) \neq 0.$$

The derivative is given by

$$\begin{aligned}\frac{\partial D_s(z)}{\partial z} &= k(z-\alpha)^{k-1} g(z) + (z-\alpha)^k g'(z) \\ &= (z-\alpha)^{k-1} [k g(z) + (z-\alpha) g'(z)]. \qquad (10)\end{aligned}$$

From (9) and (10), we obtain

$$J(z) = (z-\alpha)^{k-1} [k g(z) + (z-\alpha) g'(z)] \frac{d \times f(z)}{\prod_{i=1}^{N-1} (z-p_i)^{M_i-1}}.$$

Since $(\alpha - p_i) \neq 0$, $f(\alpha) \neq 0$ and $[k g(\alpha) + (\alpha - \alpha) g'(\alpha)] \neq 0$, α is a $(k-1)$st multiple root of the equation $J(z) = 0$. It follows that if α is a root of $D_s(z) = 0$ of multiplicity k, then α is a root of $J(z) = 0$ of multiplicity $k-1$. This completes the proof of Lemma 3.7. □

If $k = 1$, then α is not a root of $J(z) = 0$, that is, α is not a z-coordinate of a singular point. For $p_1, p_2, \ldots, p_{N-1}$, $q_1, q_2, \ldots, q_c$ which are in J-generic positions, if α is a root of $D_s(z) = 0$ of multiplicity greater than two, then α is a multiple root of $J(z) = 0$ by Lemma 3.7. This contradicts the assumption of J-generic. So $D_s(z) = 0$ has at most double roots. Hence we get

Corollary 3.8. *If $p_1, p_2, \ldots, p_{N-1}$, $q_1, q_2, \ldots, q_c$ are in J-generic positions, then the equation $D_s(z) = 0$ has at most double roots.*

Lemma 3.9. *The following two conditions are equivalent*:

(1) *The equation $D_s(z) = 0$ has multiple roots.*
(2) *The value s is singular, that is, $\pi_t^{-1}(s)$ is a subordinate fiber.*

Proof. (1)$\Rightarrow$(2): If $D_s(z) = 0$ has a multiple root α, then α is a root of $J(z) = 0$. Since a root of $J(z)$ is a z-coordinate of a singular point, s is a singular value.

(2)$\Rightarrow$(1): For a singular value s, let α be a z-coordinate of a singular point of $\pi_t^{-1}(s)$. Then $J(\alpha) = 0$, and s and α satisfy the equation (7):

$$s^{n_0/d} = \frac{t^{m_0/d}}{g(\alpha)}.$$

The equation $D_s(z) = 0$ is the deformation of (7). This means that s and α satisfy $D_s(\alpha) = 0$, that is, α is a root of $D_s(z) = 0$. Moreover, by Lemma 3.7, α is a multiple root of $D_s(z) = 0$. □

Next we show

Lemma 3.10. *For a root α of the equation $D_s(z) = 0$, the equation $\pi_t(\alpha, \zeta) - s = 0$ in ζ has double roots. Moreover, the equation $\pi_t(\alpha, \zeta) - s = 0$ in ζ does not have a multiple root whose multiplicity is greater than two.*

Proof. The equation $D_s(z) = 0$ is a discriminant of the equation $\pi_t(z, \zeta) - s = 0$ in ζ. Since $D_s(z) = 0$ is obtained by the system of equations;

$$\pi_t(z, \zeta) - s = 0 \quad \text{and} \quad \frac{\partial \pi_t}{\partial \zeta}(z, \zeta) = 0,$$

if α is a root of $D_s(z) = 0$, then the equation $\pi_t(\alpha, \zeta) - s = 0$ in ζ has multiple roots. On the other hand, the root of $\partial \pi_t(\alpha, \zeta)/\partial \zeta = 0$ is obtained by solving the following equation:

$$\zeta^{m_0-ln_0-1}\prod_{j=1}^{N-1}(\alpha-p_j)^{m_1^{(j)}-ln_1^{(j)}}\left\{\zeta^{n_0}\prod_{j=1}^{N-1}(\alpha-p_j)^{n_1^{(j)}}+tf(\alpha)\right\}^{l-1}$$
$$\times\left[m_0\zeta^{n_0}\prod_{j=1}^{N-1}(\alpha-p_j)^{n_1^{(j)}}-(m_0-ln_0)tf(\alpha)\right]=0.$$

Since $\pi_t(\alpha,\zeta)\neq 0$, we have $\zeta^{m_0-ln_0-1}\neq 0$, $\zeta^{n_0}\prod_{j=1}^{N-1}(\alpha-p_j)^{n_1^{(j)}}+tf(\alpha)\neq 0$ and

$$m_0\zeta^{n_0}\prod_{j=1}^{N-1}(\alpha-p_j)^{n_1^{(j)}}-(m_0-ln_0)tf(\alpha)=0,$$

which is rewritten

$$\zeta^{n_0}=\frac{(ln_0-m_0)t}{m_0\prod_{j=1}^{N-1}(\alpha-p_j)^{n_1^{(j)}}}.$$

This equation in ζ has simple roots only. Hence the equation $\pi_t(\alpha,\zeta)-s=0$ in ζ has double roots. Moreover, the equation $\pi_t(\alpha,\zeta)-s=0$ in ζ does not have a multiple root whose multiplicity is greater than two. This completes the proof of Lemma 3.10. □

We use the following Lemma.

Lemma 3.11. *Let $h: M\to\Delta$ be an m-fold branched covering map with two branch points b_1 and b_2, where M is a complex surface and Δ is a disk. If all ramification indices are two, then the inverse image $h^{-1}(\Delta)$ consists of annuli and disks.*

Proof. Let D_1 be a small disk centered at b_1, such that D_1 does not contain b_2. Around a ramification point $\widetilde{b}_1$ of b_1 whose ramification index is two, the inverse image $h^{-1}(D_1)$ is a disk given by pasting two disks together. (See Figure 2.) So $h^{-1}(\Delta)$ is constructed by the following way: Prepare m disks. According to the position of ramification points, the way of pasting is either of four cases in Figure 3. Hence it follows that $h^{-1}(\Delta)$ consists of annuli and disks. This completes the proof of Lemma 3.11. □

Proof of Proposition 3.6. Now we complete the proof of Proposition 3.6. We fix $t\in\Delta^\dagger\setminus\{0\}$. For a singular value $s_0\in\Delta\setminus\{0\}$, we take a non-singular value s_1 near s_0. The fiber $\pi_t^{-1}(s_0)$ is a subordinate fiber and the fiber $\pi_t^{-1}(s_1)$ is a

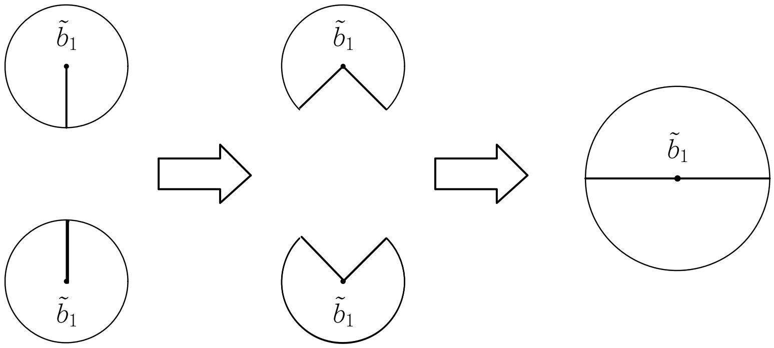

Figure 2. Cut lines of two disks, and paste together along the lines.

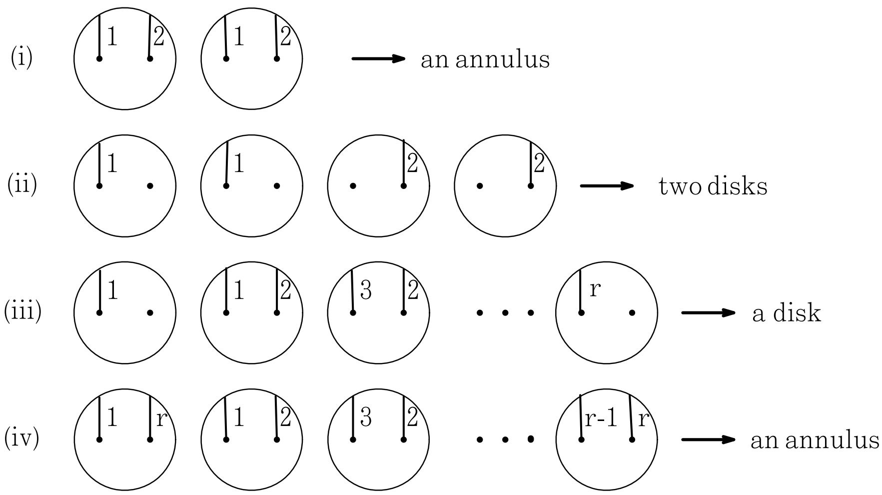

Figure 3. Four cases: Cut off the lines and paste the lines of the same numbers.

general fiber. Let γ be a path from s_1 to s_0 in Δ, which does not pass the others singular values. Namely, $\gamma : [0,1] \to \Delta \backslash S$ is a path such that $\gamma(0) = s_1$ and $\gamma(1) = s_0$, where S is the set of the other singular values: $S = \{s \in \Delta \setminus \{s_0\} : \pi_t^{-1}(s)$ is a singular fiber$\}$. (See Figure 4.) By Corollary 3.8 and Lemma 3.9, for the singular value s_0, the equation $D_{s_0}(z) = 0$ has double roots. We denote by $\alpha_1, \alpha_2, \ldots, \alpha_r$ the double roots of $D_{s_0}(z) = 0$ (r is the number of double roots of $D_{s_0}(z) = 0$), and denote by $\alpha_{r+1}, \alpha_{r+2}, \ldots, \alpha_{r+u}$ the simple roots of $D_{s_0}(z) = 0$ (u is the number of simple roots of $D_{s_0}(z) = 0$). We note that, by Lemma 3.7, $\alpha_1, \alpha_2, \ldots, \alpha_r$ are the z-coordinates of singular points of the subordinate fiber $\pi_t^{-1}(s_0)$. For a non-singular value s_1, let $\alpha_1^{(1)}, \alpha_1^{(2)}, \alpha_2^{(1)}, \alpha_2^{(2)}, \ldots, \alpha_r^{(1)}, \alpha_r^{(2)}, \alpha'_{r+1}, \ldots, \alpha'_{r+u}$ be the

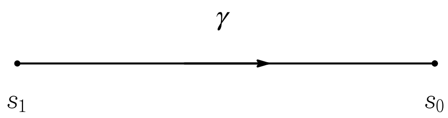

Figure 4.

roots of $D_{s_1}(z)=0$ ($2r+u$ is the number of the roots of $D_{s_1}(z)=0$). We note that, by Lemma 3.9, since s_1 is a non-singular value, $D_{s_1}(z)=0$ has simple roots only. We assume that when we vary the parameter s from s_1 to s_0 along the path γ, two simple roots $\alpha_i^{(1)}$ and $\alpha_i^{(2)}$ converge to α_i ($i=1,2,\dots,r$). We set $L:=\{z\in \boldsymbol{C}: D_s(z)=0, s\in\gamma\}$. Then L is not connected. So we denote the connected components of L by $\ell_1,\ell_2,\dots,\ell_r,\ell_{r+1},\dots,\ell_{r+u}$. Here for $i=1,2,\dots,r$, $\partial\ell_i=\{\alpha_i^{(1)},\alpha_i^{(2)}\}$ and ℓ_i contains α_i. (See Figure 5.) For $i=r+1,r+2,\dots,r+u$, $\partial\ell_i=\{\alpha_i',\alpha_i\}$.

We defined a branched covering map $h_s:\pi_t^{-1}(s)\to \boldsymbol{C}\setminus\{p_1,p_2,\dots,p_{N-1}\}$ by $h_s(z,\zeta)=z$. The inverse image $h_s^{-1}(z_0)$ of $z_0\in\boldsymbol{C}\setminus\{p_1,p_2,\dots,p_{N-1}\}$ is a set $\{(z_0,\zeta):\pi_t(z_0,\zeta)-s=0\}$. For a branch point z_0, a point $(z_0,\zeta_0)\in h_s^{-1}(z_0)$ is a ramification point whose ramification index is n if and only if ζ_0 is an nth root of the equation $\pi_t(z_0,\zeta)-s=0$ in ζ. By Lemma 3.10, for the roots $\alpha_i^{(1)}$ and $\alpha_i^{(2)}$ of $D_{s_1}(z)=0$, $\pi_t(\alpha_i^{(j)},\zeta)-s=0$ ($j=1,2$) has double roots, that is, the ramification index of the ramification point over the branch point $\alpha_i^{(j)}$ is two. We denote by Δ_i a disk which rounds ℓ_i and does not contain the others trace ℓ_k ($k\neq i$). By Lemma 3.11, the inverse image $h_{s_1}^{-1}(\Delta_i)$ consists of annuli and disks.

Let $x_i:=(\alpha_i,\beta_i)$ be a singular point of the subordinate fiber $\pi_t^{-1}(s_0)$, and let F_{s_1} be a Milnor fiber, and let μ be the Milnor number of F_{s_1}. (Note that F_{s_1} is the intersection of $\pi_t^{-1}(s_1)$ and a small ball B_ε whose center is x_i.) Since the Milnor fiber F_{s_1} is a subset of $h_{s_1}^{-1}(\Delta_i)$, from the above discussion on the branched covering h_{s_1}, F_{s_1} consists of annuli and disks. Since the singular point x_i is isolated (see Remark 3.4), the Milnor fiber F_{s_1} is connected by Theorem 2.8. So F_{s_1} is either an annulus or a disk. By Theorem 2.9 (1), if the Milnor fiber F_{s_1} is a disk, then the Milnor number μ is zero and x_i is not a singular point. This contradicts that x_i is a singular point. Hence the Milnor fiber F_{s_1} is an annulus and the Milnor number μ is one. (Note: The first Betti number of an annulus is equal to one.) By Theorem 2.9 (2), it follows that the singular point x_i is a node. This completes the proof of Proposition 3.6. $\square$

Moreover we can describe vanishing cycles. Let c_i be a simple closed curve on the annulus such that $h_{s_1}(c_i)=\ell_i$ ($i=1,2,\dots,r$). (See Figure 5 and Figure 6.) When the parameter s moves from s_1 to s_0 along the path γ, the simple closed curve c_i on the annulus contracts to a singular point x_i (Figure 6). So the

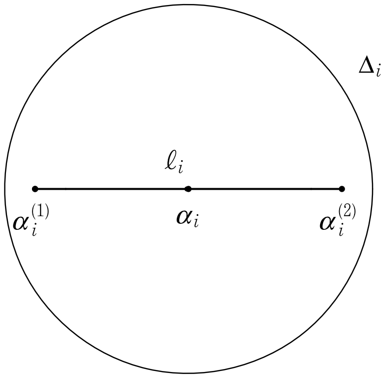

Figure 5.

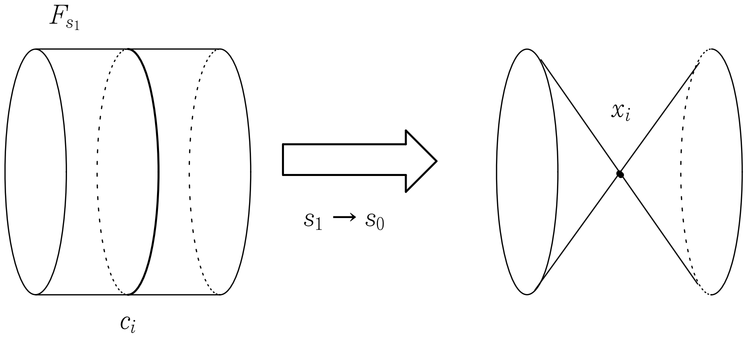

Figure 6.

simple closed curve c_i, one of the lifts of the trace ℓ_i, is a vanishing cycle of the subordinate fiber $\pi_t^{-1}(s_0)$.

Takamura [**13**], showed a stronger result than ours: The singular points of subordinate fibers are A_r-singularities. Here r is the integer associated with the ordinal singular fiber X and the simple crust Y. In this paper, it is the order of zeros of $J(z)$. His proof is as follows: We outline his proof when the order of zeros of the arrangement polynomial $J(z)$ is one. Let α be a double root of $D_{s_0}(z)=0$, and let β be a double root of the equation $\pi_t(\alpha,\zeta)-s_0=0$ in ζ. (Note: By Lemma 3.10, if α is a double root of $D_{s_0}(z)=0$, then $\pi_t(\alpha,\zeta)-s_0=0$ also has double roots.) Then the fiber $\pi_t^{-1}(s_0)$ near the point $(z,\zeta)=(\alpha,\beta)$ is locally defined by the equation

$$(\zeta-\beta)^2+(\zeta-\beta)Q(z-\alpha)+R(z-\alpha)=0.$$

We denote the left hand side by $T(z,\zeta)$. Then $\partial T(z,\zeta)/\partial\zeta = 0$ is equivalent to

$$\frac{\partial T}{\partial \zeta}(z,\zeta) = 2(\zeta-\beta) + Q(z-\alpha) = 0.$$

We substitute this for $T(z,\zeta) = (\zeta-\beta)^2 + (\zeta-\beta)Q(z-\alpha) + R(z-\alpha) = 0$, and we have

$$T(z,\zeta) = (\zeta-\beta)^2 - 2(\zeta-\beta)^2 + R(z-\alpha) = -(\zeta-\beta)^2 + R(z-\alpha) = 0.$$

We rewrite this:

$$(\zeta-\beta)^2 = R(z-\alpha). \tag{11}$$

On the other hand, since the equation $T(z,\zeta) = 0$ in ζ is a double root at $(z,\zeta) = (\alpha,\beta)$ by Lemma 3.10, we have

$$Q(z-\alpha)^2 - 4R(z-\alpha) = 0.$$

We substitute this for (11), we obtain the equation

$$(\zeta-\beta)^2 = \frac{Q(z-\alpha)^2}{4}.$$

We set $\zeta' := \zeta - \beta$ and $z' := Q(z-\alpha)/2$. Then we have

$$(\zeta')^2 = (z')^2.$$

This means that $(z,\zeta) = (\alpha,\beta)$ is a node, that is, an A_1-singularity.

4. J-generic position.

In this section, we prove the following lemma.

LEMMA 4.1. *There exist points $p_1, p_2, \ldots, p_N$, $q_1, q_2, \ldots, q_c$ in J-generic positions.*

PROOF. Recall that

$$J(z) := \sum_{i=1}^{N-1} \bigl(m_0 n_1^{(i)} - n_0 m_1^{(i)}\bigr) G_i(z) f(z) - m_0 \prod_{j=1}^{N-1} (z - p_j) f'(z),$$

where $f'(z) = df(z)/dz$. We assume that $p_1, p_2, \ldots, p_{N-1},\ q_1, q_2 \ldots, q_c$ are also parameters, and denote $J(z)$ by $J(x)$, where $x = (z, p, q) \in \boldsymbol{C}^{N+c}$, $p = (p^{(1)}, p^{(2)}, \ldots, p^{(N-1)})$ and $q = (q^{(1)}, q^{(2)}, \ldots, q^{(c)})$. Set $V := \{x = (z, p, q) \in \boldsymbol{C}^{N+c} : J(x) = 0\}$ and $\Delta := \{(p, q) \in \boldsymbol{C}^{N-1+c} : J(x) = \partial J(x)/\partial z = 0\}$. First of all, we show that V is a smooth complex manifold. Assume that V has a singular point, and we deduce a contradiction. Suppose that $x_0 = (z_0, p_0, q_0)$ is a singular point of V, where $p_0 = (p_0^{(1)}, p_0^{(2)}, \ldots, p_0^{(N-1)})$ and $q_0 = (q_0^{(1)}, q_0^{(2)}, \ldots, q_0^{(c)})$. Then

$$\frac{\partial J}{\partial z}(x_0) = \frac{\partial J}{\partial p_i}(x_0) = \frac{\partial J}{\partial q_j}(x_0) = 0 \quad (i = 1, 2, \ldots, N-1,\ j = 1, 2, \ldots, c).$$

For simplicity, we set $G_{1,i}(x) := (z-p_2)(z-p_3)\cdots(z-p_{i-1})(z-p_{i+1})\cdots(z-p_{N-1})$. Then we obtain the following equation:

$$\frac{\partial J}{\partial p_1}(x_0) = -d \times f(x_0) \sum_{i=2}^{N-1} M_i G_{1,i}(x_0) + m_0 G_1(x_0) f'(x_0) = 0,$$

that is,

$$m_0 G_1(x_0) f'(x_0) = d \times f(x_0) \sum_{i=2}^{N-1} M_i G_{1,i}(x_0). \tag{12}$$

We substitute (12) for $J(x_0)$, and we have

$$\begin{aligned}
J(x_0) &= d \sum_{i=1}^{N-1} M_i G_i(x_0) f(x_0) - m_0 \prod_{j=1}^{N-1} \bigl(z_0 - p_j^{(0)}\bigr) f'(x_0) \\
&= d M_1 G_1(x_0) f(x_0) + d \sum_{i=2}^{N-1} M_i G_i(x_0) f(x_0) - m_0 \prod_{j=1}^{N-1} \bigl(z_0 - p_j^{(0)}\bigr) f'(x_0) \\
&= d M_1 G_1(x_0) f(x_0) + (z_0 - p_1^{(0)}) d \sum_{i=2}^{N-1} M_i G_{1,i}(x_0) f(x_0) \\
&\quad - m_0 \prod_{j=1}^{N-1} \bigl(z_0 - p_j^{(0)}\bigr) f'(x_0)
\end{aligned}$$

$$
\begin{aligned}
&= dM_1G_1(x_0)f(x_0) + \bigl(z_0 - p_1^{(0)}\bigr)m_0G_1(x_0)f'(x_0) \\
&\quad - m_0 \prod_{j=1}^{N-1} \bigl(z_0 - p_j^{(0)}\bigr) f'(x_0) \\
&= dM_1G_1(x_0)f(x_0) - m_0\bigl(z_0 - p_1^{(0)}\bigr)f'(x_0)\Biggl[- G_1(x_0) + \prod_{j=2}^{N-1} \bigl(z_0 - p_j^{(0)}\bigr)\Biggr] \\
&= dM_1G_1(x_0)f(x_0).
\end{aligned}
$$

However z_0 is not either of $p_1^{(0)}, p_2^{(0)}, \dots, p_{N-1}^{(0)}$, $q_1^{(0)}, q_2^{(0)}, \dots, q_{c-1}^{(0)}$, or $q_c^{(0)}$. So $G_1(x_0) \neq 0$ and $f(x_0) \neq 0$, and $dM_1G_1(x_0)f(x_0) \neq 0$. It yields a contradiction to $J(x_0) = 0$. Hence V has no singular point, and V is a smooth manifold.

Let $pr : V \to \boldsymbol{C}^{N-1+c}$, $pr(z,p,q) := (p,q)$ be a projection. Since V is a smooth manifold, we may apply Sard's Theorem: The set of critical values of pr is measure zero. The set of critical values is equal to $\Delta = \{x = (p,q) : J(x) = \partial J(x)/\partial z = 0\}$. Hence Δ is a measure zero set. If we take a point $(p_1, p_2, \dots, p_N, q_1, q_2, \dots, q_c)$ in $\boldsymbol{C}^{N-2+c} \setminus \Delta$, then $p_1, p_2, \dots, p_N$, $q_1, q_2, \dots, q_c$ are in J-generic positions. This completes the proof of Lemma 4.1. $\square$

5. Completion of the proof of Proposition 3.1.

In this section, we compute $\partial\pi_t(z,\zeta)/\partial\zeta$ and $\partial\pi_t(z,\zeta)/\partial z$, and then we solve the system of equations

$$
\pi_t(z,\zeta) - s = 0, \quad \frac{\partial \pi_t}{\partial \zeta}(z,\zeta) = 0 \quad \text{and} \quad \frac{\partial \pi_t}{\partial z}(z,\zeta) = 0.
$$

Finally, we derive the relation (2) and give the explicit form of the arrangement polynomial $J(z)$.

5.1. Computation to obtain the relation (2).

We recall that the deformation π_t around the core is expressed by

$$
\pi_t(z,\zeta) = \zeta^{m_0 - ln_0} \prod_{j=1}^{N-1} (z - p_j)^{m_1^{(j)} - ln_1^{(j)}} \left\{ \zeta^{n_0} \prod_{j=1}^{N-1} (z - p_j)^{n_1^{(j)}} + tf(z) \right\}^l .
$$

The partial derivative π_t of ζ is the following:

$$\begin{aligned}
\frac{\partial\pi_t}{\partial\zeta}(\zeta,z) &= \prod_{j=1}^{N-1}(z-p_j)^{m_1^{(j)}-ln_1^{(j)}} \\
&\quad\times\left[(m_0-ln_0)\zeta^{m_0-ln_0-1}\left\{\zeta^{n_0}\prod_{j=1}^{N-1}(z-p_j)^{n_1^{(j)}}+tf(z)\right\}^l\right. \\
&\quad\left.+l\zeta^{m_0-ln_0}\left\{\zeta^{n_0}\prod_{j=1}^{N-1}(z-p_j)^{n_1^{(j)}}+tf(z)\right\}^{l-1}\times n_0\zeta^{n_0-1}\prod_{j=1}^{N-1}(z-p_j)^{n_1^{(j)}}\right] \\
&= \zeta^{m_0-ln_0-1}\prod_{j=1}^{N-1}(z-p_j)^{m_1^{(j)}-ln_1^{(j)}}\left\{\zeta^{n_0}\prod_{j=1}^{N-1}(z-p_j)^{n_1^{(j)}}+tf(z)\right\}^{l-1} \\
&\quad\times\left[(m_0-ln_0)\left\{\zeta^{n_0}\prod_{j=1}^{N-1}(z-p_j)^{n_1^{(j)}}+tf(z)\right\}+ln_0\zeta^{n_0}\prod_{j=1}^{N-1}(z-p_j)^{n_1^{(j)}}\right] \\
&= \zeta^{m_0-ln_0-1}\prod_{j=1}^{N-1}(z-p_j)^{m_1^{(j)}-ln_1^{(j)}}\left\{\zeta^{n_0}\prod_{j=1}^{N-1}(z-p_j)^{n_1^{(j)}}+tf(z)\right\}^{l-1} \\
&\quad\times\left[m_0\zeta^{n_0}\left(\prod_{j=1}^{N-1}(z-p_j)^{n_1^{(j)}}\right)+(m_0-ln_0)tf(z)\right].
\end{aligned}$$

Since $\partial\pi_t(z,\zeta)/\partial\zeta = 0$ and $\pi_t(\zeta,z)\neq 0$,

$$-m_0\zeta^{n_0}\prod_{j=1}^{N-1}(z-p_j)^{n_1^{(j)}} = (m_0-ln_0)tf(z). \tag{13}$$

We note that $q_1, q_2, \ldots, q_c$ are not z-coordinates of singular points of subordinate fibers. If q_i $(i = 1, 2, \ldots, c)$ is a z-coordinate of a singular point, then $z = q_i$ satisfies the equation (13). But $f(q_i) = 0$ and the left hand side of (13) is not zero. So $z = q_i$ does not satisfy the equation (13), and we may assume $f(z)\neq 0$. Hence we derive the equation (2):

$$t = \frac{-m_0\zeta^{n_0}\prod_{j=1}^{N-1}(z-p_j)^{n_1^{(j)}}}{(m_0-ln_0)f(z)}.$$

5.2. Deduction of $J(z)$.

The partial derivative π_t of z is the following:

$$\frac{\partial \pi_t}{\partial z} = \zeta^{m_0 - ln_0} \prod_{j=1}^{n-1} (z-p_j)^{m_1^{(j)} - ln_1^{(j)} - 1} \left\{ \sum_{i=1}^{N-1} (m_1^{(i)} - ln_1^{(i)}) G_i(z) \right\} \\ \times \left\{ \zeta^{n_0} \prod_{j=1}^{N-1} (z-p_j)^{n_1^{(j)}} + tf(z) \right\}^l \\ + l\zeta^{m_0 - ln_0} \prod_{j=1}^{N-1} (z-p_j)^{m_1^{(j)} - ln_1^{(j)}} \left\{ \zeta^{n_0} \prod_{j=1}^{N-1} (z-p_j)^{n_1^{(j)}} + tf(z) \right\}^{l-1} \times \Omega_1,$$

where

$$\Omega_1 = \zeta^{n_0} \prod_{j=1}^{N-1} (z-p_j)^{n_1^{(j)} - 1} \left(\sum_{i=1}^{N-1} n_1^{(i)} G_i(z) \right) + tf'(z)$$

and $f'(z) = df(z)/dz$. Put this in order,

$$\frac{\partial \pi_t}{\partial z} = \zeta^{m_0 - ln_0} \prod_{j=1}^{N-1} (z-p_j)^{m_1^{(j)} - ln_1^{(j)} - 1} \left\{ \zeta^{n_0} \prod_{j=1}^{N-1} (z-p_j)^{n_1^{(j)}} + tf(z) \right\}^{l-1} \times \Omega_2,$$

where

$$\Omega_2 = \left\{ \sum_{i=1}^{N-1} (m_1^{(i)} - ln_1^{(i)}) G_i(z) \right\} \left\{ \zeta^{n_0} \prod_{j=1}^{N-1} (z-p_j)^{n_1^{(j)}} + tf(z) \right\} + l \prod_{j=1}^{N-1} (z-p_j) \times \Omega_1.$$

Substitute (2) into Ω_1, and we have

$$\Omega_1 = \zeta^{n_0} \prod_{j=1}^{N-1} (z-p_j)^{n_1^{(j)} - 1} \left\{ \sum_{i=1}^{N-1} n_1^{(i)} G_i(z) - \frac{m_0}{m_0 - ln_0} \prod_{j=1}^{N-1} (z-p_j) \frac{f'(z)}{f(z)} \right\}.$$

From (2), we also obtain the following equation:

$$\zeta^{n_0} \prod_{j=1}^{N-1} (z-p_j)^{n_1^{(j)}} + tf(z) = \frac{-ln_0}{m_0 - ln_0} \zeta^{n_0} \prod_{j=1}^{N-1} (z-p_j)^{n_1^{(j)}}.$$

So

$$\begin{aligned}\Omega_2 &= \left\{ \sum_{i=1}^{N-1} (m_1^{(i)} - l n_1^{(i)}) G_i(z) \right\} \left\{ \frac{-l n_0}{m_0 - l n_0} \zeta^{n_0} \prod_{j=1}^{N-1} (z-p_j)^{n_1^{(j)}} \right\} \\ &\quad + l \zeta^{n_0} \prod_{j=1}^{N-1} (z-p_j)^{n_1^{(j)}} \left\{ \sum_{i=1}^{N-1} n_1^{(i)} G_i(z) - \frac{m_0}{m_0 - l n_0} \prod_{j=1}^{N-1} (z-p_j) \frac{f'(z)}{f(z)} \right\} \\ &= l \zeta^{n_0} \prod_{j=1}^{N-1} (z-p_j)^{n_1^{(j)}} \times \Omega_3,\end{aligned}$$

where

$$\begin{aligned}\Omega_3 &= \frac{-n_0}{m_0 - l n_0} \left\{ \sum_{i=1}^{N-1} (m_1^{(i)} - l n_1^{(i)}) G_i(z) \right\} + \sum_{i=1}^{N-1} n_1^{(i)} G_i(z) \\ &\quad - \frac{m_0}{m_0 - l n_0} \prod_{j=1}^{N-1} (z-p_j) \frac{f'(z)}{f(z)} \\ &= \frac{1}{m_0 - l n_0} \left[\sum_{i=1}^{N-1} (m_0 n_1^{(i)} - n_0 m_1^{(i)}) G_i(z) - m_0 \prod_{j=1}^{N-1} (z-p_j) \frac{f'(z)}{f(z)} \right].\end{aligned}$$

Hence the following equivalences hold:

$$\begin{aligned}\pi_t(z,\zeta) \neq 0 \quad \text{and} \quad \frac{\partial \pi_t}{\partial z}(z,\zeta) = 0 &\iff \Omega_2 = 0 \\ &\iff \Omega_3 = 0 \\ &\iff (m_0 - l n_0) f(z) \Omega_3 = 0.\end{aligned}$$

So we obtain

$$J(z) = \sum_{i=1}^{N-1} \left(m_0 n_1^{(i)} - n_0 m_1^{(i)}\right) G_i(z) f(z) - m_0 \prod_{j=1}^{N-1} (z-p_j) f'(z).$$

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