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# On nonseparable Erdős spaces

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**Abstract.** In 2005, Dijkstra studied subspaces $\mathscr{E}$ of the Banach spaces $\ell^p$ that are constructed as 'products' of countably many zero-dimensional subsets of $\boldsymbol{R}$, as a generalization of Erdős space and complete Erdős space. He presented a criterion for deciding whether a space of the type $\mathscr{E}$ has the same peculiar features as Erdős space, which is one-dimensional yet totally disconnected and has a one-dimensional square. In this paper, we extend the construction to a nonseparable setting and consider spaces $\mathscr{E}_\mu$ corresponding to products of $\mu$ zero-dimensional subsets of $\boldsymbol{R}$ in nonseparable Banach spaces. We are able to generalize both Dijkstra's criterion and his classification of closed variants of $\mathscr{E}$. We can further generalize the latter to complete spaces and we find that a one-dimensional complete space $\mathscr{E}_\mu$ is homeomorphic to a product of complete Erdős space with a countable product of discrete spaces. Among the applications, we find coincidence of the small and large inductive dimension for $\mathscr{E}_\mu$.

## 1. Introduction.

Let $A$ be an arbitrary set and let $p \geq 1$. We consider a generalization of the Banach space $\ell^p$, given by

$$\ell^p_A = \left\{ x = (x_\alpha)_{\alpha \in A} \in \boldsymbol{R}^A : \sum_{\alpha \in A} |x_\alpha|^p < \infty \right\}.$$

The topology on $\ell^p_A$ is generated by the norm $\|x\|_p = \left(\sum_{\alpha \in A} |x_\alpha|^p\right)^{1/p}$. Let $(E_\alpha)_{\alpha \in A}$ be a fixed collection of subsets of $\boldsymbol{R}$. We define

$$\mathscr{E}_A = \left( \prod_{\alpha \in A} E_\alpha \right) \cap \ell^p_A = \{x \in \ell^p_A : x_\alpha \in E_\alpha \text{ for } \alpha \in A\}.$$

We extend the domain of $\|\cdot\|_p$ to $\boldsymbol{R}^A$ by putting $\|x\|_p = \infty$ if $x \in \boldsymbol{R}^A \setminus \ell^p_A$. Sets of cardinality equal to that of $A$ generate a Banach space that is isomorphic to $\ell^p_A$,

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so in general we think of $A$ as being an infinite cardinal number $\mu$ and we consider the spaces $\ell^p_\mu$ and $\mathscr{E}_\mu$.

For $\mu = \omega$, $p = 2$ and $E_n = \boldsymbol{Q}$ for every $n \in \omega$, we find *Erdős space* $\mathfrak{E}$ and $E_n = \{0\} \cup \{1/m : m \in \boldsymbol{N}\}$ for every $n \in \omega$ yields *complete Erdős space* $\mathfrak{E}_\mathrm{c}$. Both spaces were introduced by Erdős [**8**] who proved that these spaces have the peculiar property that they are one-dimensional, yet totally disconnected and homeomorphic to their own squares. The spaces $\mathfrak{E}$, $\mathfrak{E}_\mathrm{c}$, and also $\mathfrak{E}_\mathrm{c}^\omega$ were characterized by Dijkstra and van Mill [**4**], [**6**], [**5**] and Dijkstra [**3**]. General separable metrizable spaces $\mathscr{E}_\omega$ are studied in Dijkstra [**2**]. Our aim is to generalize this last paper to arbitrary cardinal numbers $\mu$ and hence to a nonseparable setting. In particular, in Section 3 we shall prove the following generalization of [**2**, Theorem 1].

Theorem 1. *Assume that $\mathscr{E}_\mu$ is not empty and that* $\operatorname{ind} E_\alpha = 0$ *for every $\alpha \in \mu$. For each $m \in \boldsymbol{N}$ we let $\eta(m) \in \boldsymbol{R}^\mu$ be given by its coordinates*

$$\eta(m)_\alpha = \sup\{|a| : a \in E_\alpha \cap [-1/m, 1/m]\},$$

*for $\alpha \in \mu$, where* $\sup \emptyset = 0$. *The following statements are equivalent*:

1. (1) $\|\eta(m)\|_p = \infty$ *for each* $m \in \boldsymbol{N}$;
2. (2) *there exists an* $x \in \prod_{\alpha\in\mu} E_\alpha$ *with* $\|x\|_p = \infty$ *and* $\lim_{\alpha\in\mu} x_\alpha = 0$;
3. (3) *every nonempty clopen subset of $\mathscr{E}_\mu$ is unbounded*; *and*
4. (4) $\operatorname{ind} \mathscr{E}_\mu > 0$.

The expression $\lim_{\alpha\in\mu} x_\alpha = 0$ means that $\{\alpha \in \mu : |x_\alpha| \geq \varepsilon\}$ is finite for every $\varepsilon > 0$. One obtains [**2**, Theorem 1] by substituting $\mu = \omega$ in the theorem above.

Section 4, Section 5, and part of Section 6 are devoted to the proof of the following main result, a classification theorem for complete spaces $\mathscr{E}_\mu$.

Theorem 2. *If $\mathscr{E}_\mu$ is topologically complete,* $\operatorname{ind} \mathscr{E}_\mu > 0$, *and each $E_\alpha$ is zero-dimensional, then there exist discrete spaces $X$ and $Y$ such that $\mathscr{E}_\mu$ is homeomorphic to $\mathfrak{E}_\mathrm{c} \times X^\omega \times Y$.*

This theorem extends Theorem 23(b) in Dijkstra and van Mill [**5**], which states that $\mathscr{E}_\omega$ is homeomorphic to $\mathfrak{E}_\mathrm{c}$, whenever $\mathscr{E}_\omega$ is topologically complete, $\operatorname{ind} \mathscr{E}_\omega > 0$, and each $E_n$ is zero-dimensional.

We also establish the universality of the spaces $\mathfrak{E}_\mathrm{c} \times X^\omega \times Y$ of Theorem 2 which shows that the interesting aspect of the $\mathscr{E}_\mu$, namely one-dimensionality, is essentially separable metric in nature; see Corollary 35 and Theorem 41.

In Section 6 we also discuss applications of our generalized theorems, the most important of which is the coincidence of the three most common dimension functions for spaces of the type $\mathscr{E}_\mu$.

THEOREM 3. $\operatorname{ind} \mathscr{E}_\mu = \operatorname{Ind} \mathscr{E}_\mu = \dim \mathscr{E}_\mu$.

The existence of the nonseparable, completely metrizable *Roy space* [**12**], in which small and large inductive dimension differ, shows that this is not a triviality.

## 2. Preliminaries.

We denote the zero vector in $\ell^p_A$ and $\boldsymbol{R}^A$ by $\mathbf{0}_A$ or simply by $\mathbf{0}$. For $x \in \ell^p_A$ and $\varepsilon > 0$ we put $B_A(x,\varepsilon) = \{y \in \ell^p_A : \|x-y\|_p < \varepsilon\}$. If $I \subset A$, then we write $(x_\alpha)_{\alpha\in A}\restriction I = (x_\alpha)_{\alpha \in I}$ so if $x \in \mathscr{E}_A$ then $x\restriction I \in \mathscr{E}_I$. Note that $\ell^p_A = \ell^p_I \times \ell^p_{A\setminus I}$ and $\mathscr{E}_A = \mathscr{E}_I \times \mathscr{E}_{A\setminus I}$. Sometimes it is useful to identify $\ell^p_I$ with $\ell^p_I \times \{\mathbf{0}_{A\setminus I}\} \subset \ell^p_A$ so we can also write $\ell^p_A = \ell^p_I + \ell^p_{A\setminus I}$ and $\mathscr{E}_A = \mathscr{E}_I + \mathscr{E}_{A\setminus I}$.

REMARK 4. The $\ell^p$-norm is a Kadec norm. That is, the norm topology on $\ell^p$ is the weakest topology that makes all coordinate projections $z \mapsto z_i$ and the norm function continuous. A straightforward generalization of the proof gives that $\ell^p_\mu$ has a Kadec norm as well. Thus, the graph of the norm function when seen as a function from $\ell^p_\mu$ with the product topology (or any other topology that lies between the product topology and the norm topology) to $\boldsymbol{R}$ is homeomorphic to $\ell^p_\mu$ by the obvious map. Thus the space $\mathscr{E}_\mu$ is homeomorphic to a subspace of $\big(\prod_{\alpha\in\mu} E_\alpha\big) \times \boldsymbol{R}$ which immediately leads to:

PROPOSITION 5. *If every $E_\alpha$ is zero-dimensional then* $\operatorname{ind} \mathscr{E}_\mu \le 1$.

DEFINITION 6. A Hausdorff space (or a topology) is called *zero-dimensional* if the small inductive dimension is at most 0, that is, if there is a basis consisting of clopen sets. A Hausdorff space $X$ is called *almost zero-dimensional* if there exists a second Hausdorff topology $\mathscr{W}$ on $X$ that *witnesses the almost zero-dimensionality of $X$*, which means that $\mathscr{W}$ is zero-dimensional and weaker than the given topology and that every point of $X$ has a neighbourhood basis in $X$ consisting of sets that are closed in $(X,\mathscr{W})$. The topology $\mathscr{W}$ is also called a witness to the almost zero-dimensionality of $X$, or a witness topology for short.

Let $\hat{\boldsymbol{R}}$ denote the compactification $[-\infty,\infty]$ of $\boldsymbol{R}$. Recall that a function $\varphi : X \to \hat{\boldsymbol{R}}$ is called *upper semicontinuous* (*USC*) if $\{x \in X : \varphi(x) < t\}$ is open in $X$ for every $t \in \boldsymbol{R}$. If $-\varphi$ is USC, then $\varphi$ is called *lower semicontinuous* (*LSC*). It is easily seen that the $p$-norm is an LSC function on the product space $\boldsymbol{R}^\mu$. We extend the norm over the hypercube $\hat{\boldsymbol{R}}^\mu$ by putting $\|x\|_p = \infty$ for every $x \in \hat{\boldsymbol{R}}^\mu \setminus \ell^p_\mu$. The extended norm function is also LSC on $\hat{\boldsymbol{R}}^\mu$. If every $E_\alpha$ is zero-dimensional, then the topology $\mathscr{E}_\mu$ inherits from the (zero-dimensional) product topology on $\prod_{\alpha\in\mu} E_\alpha$ witnesses the almost zero-dimensionality of $\mathscr{E}_\mu$ and we will call this topology the *standard witness topology.* This follows immediately from the fact that the norm function is LSC with respect to the product topology. It now follows that $\mathscr{E}_\mu$ is almost zero-dimensional and hence totally disconnected.

Definition 7. If $\varphi, \psi : X \to [0, \infty]$, define the following subspaces of $X \times [0, \infty]$:

$$G_0^\varphi = \{(x, \varphi(x)) : x \in X,\ \varphi(x) > 0\},$$

$$L_0^\varphi = \{(x, t) : x \in X,\ 0 \le t \le \varphi(x)\},$$

$$G_\psi^\infty = \{(x, \psi(x)) : x \in X,\ \psi(x) < \infty\},$$

$$L_\psi^\infty = \{(x, t) : x \in X,\ \psi(x) \le t \le \infty\}.$$

If $X$ is nonempty, zero-dimensional, separable, and metrizable, then a USC function $\varphi$ is said to be a *U-Lelek function* if $G_0^\varphi$ is dense in $L_0^\varphi$. If $\varphi$ is a U-Lelek function on a compact domain and one identifies $X \times \{0\} \subset L_0^\varphi$ to a single point, then the obtained quotient space $L_0^\varphi/0$ is called a *Lelek fan*, with *endpoint set* $G_0^\varphi$; see Lelek [**11**]. Similarly, an LSC function $\psi$ is an *L-Lelek function* if $G_\psi^\infty$ is dense in $L_\psi^\infty$.

If we put $1/\infty = 0$ and $1/0 = \infty$ then it is clear that a function $\varphi$ is U-Lelek if and only if $1/\varphi$ is L-Lelek. It was shown by Kawamura, Oversteegen, and Tymchatyn [**10**] that $G_0^\varphi$ is homeomorphic to complete Erdős space whenever $\varphi$ is a U-Lelek function with a compact domain; see also Dijkstra [**2**].

## 3. The small inductive dimension of $\mathscr{E}_\mu$.

In this section, we shall prove Theorem 1 which gives a criterion for deciding whether a space of the type $\mathscr{E}_\mu$ is zero-dimensional or not. Recall that for each $m \in \boldsymbol{N}$ we let $\eta(m) \in \boldsymbol{R}^\mu$ be given by

$$\eta(m)_\alpha = \sup\{|a| : a \in E_\alpha \cap [-1/m, 1/m]\}, \quad \alpha \in \mu,$$

where $\sup \emptyset = 0$. Theorem 1 is a generalization of the following theorem of Dijkstra [**2**], from $\omega$ to an arbitrary cardinal number $\mu$.

Theorem 8. *Assume that $\mathscr{E}_\omega$ is not empty and that $E_n$ is zero-dimensional for each $n \in \omega$. The following statements are equivalent*:

(1) $\|\eta(m)\|_p = \infty$ *for each* $m \in \boldsymbol{N}$;

(2) *there exists an* $x \in \prod_{n=0}^\infty E_n$ *with* $\|x\|_p = \infty$ *and* $\lim_{n\to\infty} x_n = 0$;

(3) *every nonempty clopen subset of* $\mathscr{E}_\omega$ *is unbounded*; *and*
(4) $\operatorname{ind} \mathscr{E}_\omega > 0$.

We shall use this theorem for the proof of Theorem 1. Afterwards, we shall consider some consequences of the criterion, among which the relation between the minimal weight of nonempty open subsets of $\mathscr{E}_\mu$ and the small inductive dimension.

REMARK 9. We write $\lim_{\alpha\in\mu} x_\alpha = 0$ if for each $\varepsilon > 0$ the set $\{\alpha \in \mu : |x_\alpha| \geq \varepsilon\}$ is finite. Note that $\lim_{\alpha\in\mu} x_\alpha = 0$ if and only if there exists a countable subset $I = \{\alpha_i : i \in \omega\}$ of $\mu$ such that $\lim_{i\to\infty} x_{\alpha_i} = 0$ and $x_\beta = 0$ whenever $\beta \in \mu \setminus I$. If $\|x\|_p < \infty$ then $\lim_{\alpha\in\mu} x_\alpha = 0$ and $x$ has countable support. Thus, if $\mathscr{E}_\mu \neq \emptyset$ then $0 \in E_\alpha$ for all but countably many $\alpha \in \mu$.

PROOF OF THEOREM 1. (1) $\Rightarrow$ (2). For every $m \in \boldsymbol{N}$, the equality $\|\eta(m)\|_p = \infty$ implies the existence of a countable (infinite) subset $I_m \subset \mu$ such that $\|\eta(m)\restriction I_m\|_p = \infty$. Let $z \in \mathscr{E}_\mu$ and let $I$ be a countable subset of $\mu$ such that $\bigcup_{m\in\boldsymbol{N}} I_m \subset I$ and $z_\alpha = 0$ for each $\alpha \in \mu \setminus I$. Then $\mathscr{E}_I$ satisfies condition (1) of Theorem 8 and hence there exists a point $x \in \prod_{\alpha\in I} E_\alpha$ with $\|x\|_p = \infty$ and $\lim_{\alpha\in I} x_\alpha = 0$. Then $(x, \mathbf{0}_{\mu\setminus I}) \in \prod_{\alpha\in\mu} E_\alpha$ is as required by condition (2).

(2) $\Rightarrow$ (3). Assume that $x \in \prod_{\alpha\in\mu} E_\alpha$ is such that $\|x\|_p = \infty$ and $\lim_{\alpha\in\mu} x_\alpha = 0$. Let $C$ be a nonempty clopen subset of $\mathscr{E}_\mu$ and choose $a \in C$. Put $I = \{\alpha \in \mu : x_\alpha \neq 0 \text{ or } a_\alpha \neq 0\}$ and note that $|I| = \omega$. Note that $a \in C' = C \cap \mathscr{E}_I$ where we identify $\mathscr{E}_I$ with $\mathscr{E}_I \times \{\mathbf{0}_{\mu\setminus I}\}$ as a subset of $\mathscr{E}_\mu$. Apply (2) $\Rightarrow$ (3) of Theorem 8 to $\mathscr{E}_I$ and $x\restriction I$ to find that $C'$ and hence $C$ are unbounded.

The implication (3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1). We prove this implication by contraposition. Let $m \in \boldsymbol{N}$ be such that $\|\eta(m)\|_p < \infty$ and let $a \in \mathscr{E}_\mu$. Select a countable infinite $I \subset \mu$ such that $\eta(m)_\alpha = a_\alpha = 0$ for each $\alpha \in \mu \setminus I$. Applying Theorem 8 to $\mathscr{E}_I$ we find a clopen neighbourhood basis $\{C_n : n \in \boldsymbol{N}\}$ for $a\restriction I$ in $\mathscr{E}_I$. Since $\eta(m)_\alpha = 0$ for $\alpha \in \mu \setminus I$ we have that $\mathbf{0}$ is an isolated point in $\mathscr{E}_{\mu\setminus I}$. Thus $\{C_n \times \{\mathbf{0}_{\mu\setminus I}\} : n \in \boldsymbol{N}\}$ is a clopen neighbourhood basis for $a$ in $\mathscr{E}_\mu$. The conclusion is that $\operatorname{ind} \mathscr{E}_\mu = 0$. As with the proof of Theorem 8 the zero-dimensionality of the $E_\alpha$'s is used only for this implication. □

Interestingly, if $\mathscr{E}_\mu$ is not zero-dimensional, then the space contains a (closed) copy of $\mathfrak{E}_\mathrm{c}$.

COROLLARY 10. *If every* $E_\alpha$ *is zero-dimensional, then* $\operatorname{ind} \mathscr{E}_\mu > 0$ *if and only if there is a countable set* $I \subset \mu$ *such that* $\mathscr{E}_I \times \{\mathbf{0}_{\mu\setminus I}\} \subset \mathscr{E}_\mu$ *and* $\mathscr{E}_I$ *contains a closed copy of* $\mathfrak{E}_\mathrm{c}$.

PROOF. Since $\operatorname{ind} \mathfrak{E}_\mathrm{c} = 1$ the "if" part is trivial. If $\operatorname{ind} \mathscr{E}_\mu > 0$ then by

Theorem 1 there exists an $x \in \prod_{\alpha\in\mu} E_\alpha$ satisfying condition (2). Let $z \in \mathscr{E}_\mu$ and consider the sets given by $E'_\alpha = \{x_\alpha, z_\alpha\} \subset E_\alpha$ and the space $\mathscr{E}'_\mu = \big(\prod_{\alpha\in\mu} E'_\alpha\big) \cap \ell^p_\mu$ they generate. Let $I$ be the countable infinite set $\{\alpha \in \mu : E'_\alpha \neq \{0\}\}$. Then by results of Dijkstra [**2**] (quoted as Theorems 8 and 16 in this paper) we have $\mathscr{E}'_\mu \approx \mathscr{E}'_I \approx \mathfrak{E}_\mathrm{c}$ and $\mathscr{E}'_\mu$ is clearly imbedded as a closed subset in $\mathscr{E}_I \times \{\mathbf{0}_{\mu\setminus I}\} \subset \mathscr{E}_\mu$. □

The following three propositions link the 'richness' of the sets $E_\alpha$ near 0 to the topological properties local weight and dimension.

We define $\limsup_{\alpha\in\mu} E_\alpha = \bigcap_{F\in\mathscr{F}} \overline{\bigcup_{\alpha\in\mu\setminus F} E_\alpha} \subset \boldsymbol{R}$, where $\mathscr{F}$ denotes the collection of finite subsets of $\mu$.

Proposition 11. *If* 0 *is a cluster point of* $\limsup_{\alpha\in\mu} E_\alpha$ *then* $\operatorname{ind} \mathscr{E}_\mu \neq 0$.

Proof. Assume that $\operatorname{ind} \mathscr{E}_\mu = 0$ so $\mathscr{E}_\mu \neq \emptyset$ and by Theorem 1 there is an $m \in \boldsymbol{N}$ such that $\|\eta(m)\|_p < \infty$. Select a $t \in (-1/m, 1/m) \cap \limsup_{\alpha\in\mu} E_\alpha$ with $t \neq 0$. Note that $\lim_{\alpha\in\mu} \eta(m)_\alpha = 0$ so $F = \{\alpha \in \mu : \eta(m)_\alpha > |t|/2\}$ is finite. If $\alpha \in \mu\setminus F$ then $E_\alpha$ is disjoint from the neighbourhood $(-1/m, -|t|/2) \cup (|t|/2, 1/m)$ of $t$ thus $t \notin \limsup_{\alpha\in\mu} E_\alpha$, contradicting the choice of $t$. □

Definition 12. The *weight* of a space $X$ is given by

$$w(X) = \min\{|\mathscr{B}| : \mathscr{B} \text{ a basis for the topology of } X\} + \omega$$

and the *local weight* is given by

$$lw(X) = \min\{w(U) : U \text{ an open nonempty subset of } X\}.$$

Proposition 13. *If* $\mathscr{E}_\mu \neq \emptyset$, *then*

$$\begin{aligned} w(\mathscr{E}_\mu) &= |\{\alpha : E_\alpha \neq \{0\}\}| + \omega \text{ and} \\ lw(\mathscr{E}_\mu) &= \min_{n\in\boldsymbol{N}} |\{\alpha : \exists t \in E_\alpha,\ 0 < |t| < 1/n\}| + \omega \\ &= \min\{w(U) : x \in U \subset \mathscr{E}_\mu,\ U \text{ open}\},\ \forall x \in \mathscr{E}_\mu. \end{aligned}$$

*In particular,* $\mathscr{E}_\mu$ *is weight homogeneous.*

Proof. Since $\mathscr{E}_\mu \neq \emptyset$, at most countably many of the $E_\alpha$ do not contain zero. Since separable metric factors do not change the cardinals involved we may assume without loss of generality that $0 \in E_\alpha$ for every $\alpha \in \mu$. We put

$$\kappa = |\{\alpha : E_\alpha \neq \{0\}\}| + \omega;$$
$$\lambda = \min_{n \in \boldsymbol{N}} |\{\alpha : \exists t \in E_\alpha,\ 0 < |t| < 1/n\}| + \omega;$$
$$\lambda_x = \min\{w(U) : x \in U \subset \mathscr{E}_\mu,\ U \text{ open}\},\ \forall x \in \mathscr{E}_\mu.$$

We can clearly imbed $\mathscr{E}_\mu$ in the space $\ell^p_\kappa$, which implies that $w(\mathscr{E}_\mu) \leq \kappa$. If $\kappa = \omega$ then trivially $\kappa = w(\mathscr{E}_\mu)$. If $\kappa > \omega$ then $K = \{\alpha : E_\alpha \setminus \{0\} \neq \emptyset\}$ has cardinality $\kappa$. Choose $t_\alpha \in E_\alpha \setminus \{0\}$ for every $\alpha \in K$ and define the point $a^\alpha \in \mathscr{E}_\mu$ by $a^\alpha_\beta = t_\alpha$ if $\beta = \alpha$ and $a^\alpha_\beta = 0$ if $\beta \neq \alpha$. Then for distinct $\alpha, \beta \in K$, we find $\|a^\alpha - a^\beta\|_p > |t_\alpha|$. Thus $\{a^\alpha : \alpha \in K\}$ is a discrete subspace of $\mathscr{E}_\mu$ of cardinality $\kappa$ and we have $w(\mathscr{E}_\mu) = \kappa$. By definition we have $lw(\mathscr{E}_\mu) = \min\{\lambda_x : x \in \mathscr{E}_\mu\}$ thus it suffices to prove that $\lambda = \lambda_x$ for each $x \in \mathscr{E}_\mu$. Let $x \in \mathscr{E}_\mu$ be arbitrary and let $n$ be such that $|\{\alpha : \exists t \in E_\alpha,\ 0 < |t| < 1/n\}| + \omega = \lambda$. Let $I$ be the countable set $\{\alpha : x_\alpha \neq 0\}$. Define $E'_\alpha = E_\alpha$ if $\alpha \in I$ and $E'_\alpha = E_\alpha \cap (-1/n, 1/n)$ if $\alpha \in \mu \setminus I$ and let $\mathscr{E}'_\mu$ be the corresponding subspace of $\ell^p_\mu$. Note that $B_\mu(x, 1/n) \cap \mathscr{E}_\mu \subset \mathscr{E}'_\mu$ and that $w(\mathscr{E}'_\mu) = \lambda$ by the first part of this proposition. Thus $\lambda_x \leq \lambda$. If $\lambda = \omega$ then clearly $\lambda_x = \lambda$ so assume $\lambda > \omega$. If $k \in \boldsymbol{N}$ then $K = \{\alpha \in \mu \setminus I : \exists t \in E_\alpha,\ 0 < |t| < 1/k\}$ has cardinality at least $\lambda$. Choose $t_\alpha \in E_\alpha$ with $0 < |t_\alpha| < 1/k$ for every $\alpha \in K$ and define the point $a^\alpha \in \mathscr{E}_\mu$ by $a^\alpha_\beta = t_\alpha$ if $\beta = \alpha$ and $a^\alpha_\beta = x_\alpha$ if $\beta \neq \alpha$. Then $\{a^\alpha : \alpha \in K\}$ is a discrete subspace of $B_\mu(x, 1/k) \cap \mathscr{E}_\mu$ of cardinality $\geq \lambda$ thus we have $\lambda \leq \lambda_x$. □

Remark 14. It follows easily from Proposition 13 that the character of the standard witness topology that $\mathscr{E}_\mu$ inherits from the product space $\prod_{\alpha \in \mu} E_\alpha$ equals $w(\mathscr{E}_\mu)$ at every point.

Proposition 15. *If* $lw(\mathscr{E}_\mu) > \omega$ *then* $\operatorname{ind} \mathscr{E}_\mu > 0$.

Proof. Note that $lw(\mathscr{E}_\mu) > \omega$ implies that $\mathscr{E}_\mu \neq \emptyset$. We have, according to Proposition 13,

$$\min_{n \in \boldsymbol{N}} |\{\alpha : \exists t \in E_\alpha,\ 0 < |t| < 1/n\}| > \omega.$$

Consequently, $\eta(n)_\alpha > 0$ for uncountably many $\alpha$ and hence $\|\eta(n)\|_p = \infty$ for each $n \in \boldsymbol{N}$. By Theorem 1 we have $\operatorname{ind} \mathscr{E}_\mu > 0$. □

## 4. Closed nonseparable Erdős spaces.

The purpose of this section is to extend the following theorem of Dijkstra [**2**, Theorem 3] to the nonseparable setting.

Theorem 16. *If $E_n$ is closed in $\boldsymbol{R}$ for every $n \in \omega$, then $\mathscr{E}_\omega$ is homeomorphic to complete Erdős space $\mathfrak{E}_{\mathrm{c}}$ if and only if* $\operatorname{ind} \mathscr{E}_\omega > 0$ *and every $E_n$ is zero-dimensional.*

We first generalize the concept of complete Erdős space.

Definition 17. For $\omega \le \lambda \le \kappa$, let

$$E_\alpha = \begin{cases} \{0\} \cup \{1/n : n \in \boldsymbol{N}\}, & \text{if } \alpha \in \lambda; \\ \{0,1\}, & \text{if } \alpha \in \kappa \setminus \lambda. \end{cases}$$

We define $\mathfrak{E}_{\mathrm{c}}^p(\lambda,\kappa) = (\prod_{\alpha\in\kappa} E_\alpha) \cap \ell_\kappa^p$. It is clear that $\mathfrak{E}_{\mathrm{c}}^p(\lambda,\kappa) \subset \mathfrak{E}_{\mathrm{c}}^p(\kappa,\kappa)$ and that $\mathfrak{E}_{\mathrm{c}}^p(\lambda,\kappa)$ is complete as a closed subset of $\ell_\kappa^p$.

Remark 18. Note that $\mathfrak{E}_{\mathrm{c}} = \mathfrak{E}_{\mathrm{c}}^2(\omega,\omega)$ and that by Theorems 8 and 16 every $\mathfrak{E}_{\mathrm{c}}^p(\omega,\omega)$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}}$. Note that $\mathfrak{E}_{\mathrm{c}}^p(\lambda,\kappa)$ contains $\mathfrak{E}_{\mathrm{c}}^p(\omega,\omega)$ as a factor thus we have by [**5**, Theorem 23(a)] that $\mathscr{E}_\omega \times \mathfrak{E}_{\mathrm{c}}^p(\lambda,\kappa) \approx \mathfrak{E}_{\mathrm{c}}^p(\lambda,\kappa)$ for every nonempty topologically complete $\mathscr{E}_\omega$ with zero-dimensional $E_n$'s. We find it convenient to work mainly in $\ell_\mu^1$ and to use the $\mathfrak{E}_{\mathrm{c}}^1(\lambda,\kappa)$ as standard models for the generalized complete Erdős spaces.

Remark 19. According to Proposition 13 and the definitions, we have that $lw(\mathfrak{E}_{\mathrm{c}}^p(\lambda,\kappa)) = \lambda$ and $w(\mathfrak{E}_{\mathrm{c}}^p(\lambda,\kappa)) = \kappa$. This implies that $\mathfrak{E}_{\mathrm{c}}^p(\lambda,\kappa) \not\approx \mathfrak{E}_{\mathrm{c}}^p(\mu,\nu)$, whenever $(\lambda,\kappa) \ne (\mu,\nu)$.

Remark 20. We consider $\mathfrak{E}_{\mathrm{c}}^p(\lambda,\kappa)$ when $\lambda < \kappa$. Note that $\mathfrak{E}_{\mathrm{c}}^p(\lambda,\kappa)_{\kappa\setminus\lambda}$ consists of the elements of $\{0,1\}^{\kappa\setminus\lambda}$ with only finitely many nonzero coordinates thus the space has cardinality $\kappa$. Furthermore, the metric on $\mathfrak{E}_{\mathrm{c}}^p(\lambda,\kappa)_{\kappa\setminus\lambda}$ that is generated by the $p$-norm assumes only the values $n^{1/p}$ for $n \in \omega$ and hence the space is homeomorphic to $\kappa_{\mathrm{D}}$, which stands for $\kappa$ with the discrete topology. Since $\mathfrak{E}_{\mathrm{c}}^p(\lambda,\kappa)_\lambda = \mathfrak{E}_{\mathrm{c}}^p(\lambda,\lambda)$ we have that $\mathfrak{E}_{\mathrm{c}}^p(\lambda,\kappa)$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}}^p(\lambda,\lambda) \times \kappa_{\mathrm{D}}$.

Consider now the following controlled version of the uniqueness of the Lelek fan taken from Dijkstra and van Mill [**6**, Theorem 6.2]. For a function $f\colon X \to \hat{\boldsymbol{R}}$, the number $M(f)$ stands for $\sup\{|f(x)| : x \in X\} \in [0,\infty]$. We use the convention $\sup\emptyset = 0$, so in case $X = \emptyset$ we have $M(f) = 0$.

Theorem 21. *If $\varphi : C \to [0,\infty)$ and $\psi : D \to [0,\infty)$ are U-Lelek functions with compact domain and if $t > |\log(M(\varphi)/M(\psi))|$, then there are a homeomorphism $h : C \to D$ and a continuous $f : C \to (0,\infty)$ such that $\psi \circ h = f \cdot \varphi$ and $M(\log f) < t$.*

To generalize Theorem 16, we need a variation on this theorem:

LEMMA 22. *Let $\varepsilon > 0$ be given. If $\varphi : C \to [0,\infty)$ and $\psi : D \to [0,\infty]$ are L-Lelek functions with compact domain and if $\varphi^{-1}(0)$ and $\psi^{-1}(0)$ are singletons, then there are a homeomorphism $h : C \to D$ and a continuous $f : C \to [0,\infty)$ such that $\psi \circ h = f \cdot \varphi$ and $M(\log f) < \varepsilon$.*

PROOF. Let $\{c\} = \varphi^{-1}(0)$ and $\{d\} = \psi^{-1}(0)$ and choose metrics for $C$ and $D$ that are bounded by 1. We construct by recursion sequences of clopen sets $U_0 \subsetneq U_1 \subsetneq \cdots \subset C$ and $V_0 \subsetneq V_1 \subsetneq \cdots \subset D$ such that for each $i \in \omega$,

(1) $c \notin U_i$ and $d \notin V_i$;
(2) $\operatorname{diam}(C \setminus U_i) \leq 2^{-i}$ and $\operatorname{diam}(D \setminus V_i) \leq 2^{-i}$; and
(3) if $i \geq 1$ then

$$\left|\log \frac{M\big(\frac{1}{\varphi}\restriction(U_i \setminus U_{i-1})\big)}{M\big(\frac{1}{\psi}\restriction(V_i \setminus V_{i-1})\big)}\right| < \varepsilon 2^{-i}.$$

For the base step put $U_0 = V_0 = \emptyset$.

Assume now that $U_i$ and $V_i$ have been found. Let $A \subsetneq C \setminus U_i$ and $B \subsetneq D \setminus V_i$ be clopen such that $c \in A$, $\operatorname{diam} A \leq 2^{-i-1}$, $d \in B$, and $\operatorname{diam} B \leq 2^{-i-1}$. Note that $\frac{1}{\varphi}\restriction(C \setminus (A \cup U_i))$ is a USC function into $[0,\infty)$ with compact domain thus $t = M\big(\frac{1}{\varphi}\restriction(C \setminus (A \cup U_i))\big) < \infty$. Similarly, $s = M\big(\frac{1}{\psi}\restriction(D \setminus (B \cup V_i))\big) < \infty$. By symmetry we may assume that for instance $t \geq s$. Since $\frac{1}{\psi}\restriction B$ is clearly a U-Lelek function with $\frac{1}{\psi}(d) = \infty$ we can find a $b \in B$ such that $\big|\log\big(t/\frac{1}{\psi}(b)\big)\big| < \varepsilon 2^{-i-2}$. By upper semicontinuity of $\frac{1}{\psi}\restriction B$ we can find a clopen neighbourhood $O$ of $b$ in $B$ which misses the point $d$, such that

$$\left|\log \frac{\frac{1}{\psi}(b)}{M\big(\frac{1}{\psi}\restriction O\big)}\right| < \varepsilon 2^{-i-2}$$

and hence $\big|\log\big(t/M\big(\frac{1}{\psi}\restriction O\big)\big)\big| < \varepsilon 2^{-i-1}$. Putting $U_{i+1} = C \setminus A$ and $V_{i+1} = (D \setminus B) \cup O$ we easily see that the induction hypotheses are satisfied.

The induction being complete we consider for $i \in \boldsymbol{N}$ the U-Lelek functions $\frac{1}{\varphi}\restriction(U_i \setminus U_{i-1})$ and $\frac{1}{\psi}\restriction(V_i \setminus V_{i-1})$ and we note that they do not assume the value $\infty$. Noting that the domains of these functions are compact and that they satisfy hypothesis (3) we apply Theorem 21 and find a homeomorphism

$$h_i \colon U_i \setminus U_{i-1} \to V_i \setminus V_{i-1}$$

and a continuous function

$$f_i\colon U_i \setminus U_{i-1} \to (0,\infty)$$

such that $M(\log f_i) < \varepsilon 2^{-i}$ and

$$\frac{1}{\psi} \circ h_i = f_i \cdot \left(\frac{1}{\varphi} \restriction (U_i \setminus U_{i-1})\right).$$

We define $h\colon C \to D$ and $g\colon C \to (0,\infty)$ by

$$\begin{cases} h(x) = d \text{ and } g(x) = 1, & \text{if } x = c,\\ h(x) = h_i(x) \text{ and } g(x) = \frac{1}{f_i}(x), & \text{if } x \in U_i \setminus U_{i-1} \text{ for } i \in \boldsymbol{N}. \end{cases}$$

By hypotheses (1) and (2) we obtain a well-defined homeomorphism $h$. Also, $g$ is a continuous function because $M(\log f_i) < \varepsilon 2^{-i}$. Furthermore, $\psi \circ h = g \cdot \varphi$ holds everywhere and $M(\log g) < \varepsilon$. □

Lemma 23. *Suppose that every $E_\alpha \subset \boldsymbol{R}$ is closed and zero-dimensional and that $p = 1$. If we have $\lambda = lw(\mathscr{E}_\mu) = w(\mathscr{E}_\mu) > \omega$, then $\mathscr{E}_\mu \approx \mathfrak{E}^1_{\mathrm{c}}(\lambda,\lambda)$.*

Proof. Since $\lambda > \omega$ we have $\mathscr{E}_\mu \neq \emptyset$ and hence $I = \{\alpha \in \mu : 0 \notin E_\alpha\}$ is countable. Since by Remark 18 we have $\mathscr{E}_I \times \mathfrak{E}^1_{\mathrm{c}}(\lambda,\lambda) \approx \mathfrak{E}^1_{\mathrm{c}}(\lambda,\lambda)$ we may assume that $0 \in E_\alpha$ for each $\alpha \in \mu$.

Let us consider $\mathscr{E}_\mu$ and a second space given by $\mathscr{E}'_\mu = \left(\prod_{\alpha\in\mu} E'_\alpha\right) \cap \ell^1_\mu$, with weight and local weight equal to $\lambda$ as well. We shall prove that $\mathscr{E}_\mu \approx \mathscr{E}'_\mu$ from which the theorem follows by Remark 19. By Proposition 13 we have $\lambda = |\{\alpha : E_\alpha \neq \{0\}\}| = |\{\alpha : E'_\alpha \neq \{0\}\}|$ and hence we may assume that $\mu = \lambda$.

Using transfinite recursion we construct a collection $\{A_\alpha : \alpha \in \lambda\}$ of pairwise disjoint countable subsets of $\mu$ such that for each $\alpha \in \lambda$,

(1) $\operatorname{ind} \mathscr{E}_{A_\alpha} > 0$ and
(2) $\alpha \in \bigcup_{\beta \le \alpha} A_\beta$.

Assume that $\alpha \in \lambda$ is such that $A_\beta$ has been found for $\beta < \alpha$. We have by Proposition 13 that

$$\lambda = \min_{n\in\boldsymbol{N}} |\{\alpha : \exists t \in E_\alpha,\, 0 < |t| < 1/n\}|.$$

Let $B = \bigcup_{\beta<\alpha} A_\beta$ and note that $|B| < \lambda$. Thus we have that $lw(\mathscr{E}_{\mu\setminus B}) = \lambda$ and hence there is by Proposition 15 and Corollary 10 a countable set $A_\alpha \subset \mu \setminus B$

such that $\operatorname{ind}\mathscr{E}_{A_\alpha} > 0$. If $\alpha \notin B$ then we add $\alpha$ to $A_\alpha$. Thus we have a partition $\{A_\alpha : \alpha \in \lambda\}$ of $\mu$ into countable sets such that $\operatorname{ind}\mathscr{E}_{A_\alpha} > 0$ for every $\alpha$. Of course there is a similar partition $\{A'_\alpha : \alpha \in \lambda\}$ for $\mathscr{E}'_\mu$.

Consider the closure $\overline{E_\alpha}$ in $\hat{\boldsymbol{R}}$ of $E_\alpha$. Since $E_\alpha$ is already closed in $\boldsymbol{R}$ we have $\overline{E_\alpha} \setminus E_\alpha \subset \{\pm\infty\}$ and hence $\mathscr{E}_{A_\beta} = \big(\prod_{\alpha\in A_\beta} \overline{E_\alpha}\big) \cap \ell^1_{A_\beta}$. For $\beta \in \lambda$ we let $\varphi_\beta$ stand for the (extended) norm on $\prod_{\alpha\in A_\beta} \overline{E_\alpha}$. Similarly $\psi_\beta$ stands for the norm on $\prod_{\alpha\in A'_\beta} \overline{E'_\alpha}$. The proof in Dijkstra [**2**] of Theorem 16 shows that both $\varphi_\beta$ and $\psi_\beta$ are L-Lelek functions on a compact domain. Since the zero vector belongs to the domain of both functions there exist by Lemma 22 a homeomorphism $h_\beta\colon \prod_{\alpha\in A_\beta} \overline{E_\alpha} \to \prod_{\alpha\in A'_\beta} \overline{E'_\alpha}$ and a continuous function $f_\beta : \prod_{\alpha\in A_\beta} \overline{E_\alpha} \to (1/2, 2)$, with $\psi_\beta \circ h_\beta = f_\beta \cdot \varphi_\beta$. By the Kadec property (see Remark 4) we have that $\mathscr{E}_{A_\beta} \approx G^\infty_{\varphi_\beta}$ and $\mathscr{E}'_{A'_\beta} \approx G^\infty_{\psi_\beta}$ via the natural maps. It is then clear that by Lemma 22 there is a homeomorphism

$$H : \prod_{\beta\in\lambda} \mathscr{E}_{A_\beta} \to \prod_{\beta\in\lambda} \mathscr{E}'_{A'_\beta} \text{ given by } x = (x{\restriction}A_\beta)_{\beta\in\lambda} \mapsto (h_\beta(x{\restriction}A_\beta))_{\beta\in\lambda}.$$

Thus,

$$\|H(x)\|_1 = \sum_{\beta\in\lambda} \psi_\beta(h_\beta(x{\restriction}A_\beta)) = \sum_{\beta\in\lambda} f_\beta(x{\restriction}A_\beta)\varphi_\beta(x{\restriction}A_\beta)$$

and of course $\|x\|_1 = \sum_{\beta\in\lambda} \varphi_\beta(x{\restriction}A_\beta)$. In particular, we find

$$\frac{1}{2}\|x\|_1 \le \|H(x)\|_1 \le 2\|x\|_1$$

and hence $H(\mathscr{E}_\mu) = \mathscr{E}'_\mu$. Note that $H{\restriction}\mathscr{E}_\mu\colon \mathscr{E}_\mu \to \mathscr{E}'_\mu$ is a homeomorphism on the level of the topologies that the spaces inherit from $\prod_{\beta\in\lambda} \mathscr{E}_{A_\beta}$ and $\prod_{\beta\in\lambda} \mathscr{E}'_{A'_\beta}$, respectively. To prove that the spaces are also homeomorphic with respect to the norm topologies, we only have to show that $x \mapsto \|H(x)\|_1$ is continuous on $(\mathscr{E}_\mu, \|\cdot\|_1)$; see Remark 4. A symmetric argument then automatically gives continuity of the inverse and completes the proof.

Let $x \in \mathscr{E}_\mu$ and $\varepsilon > 0$ be given. Then there exists a finite set $I \subset \lambda$ such that, if we denote $A = \bigcup_{\beta\in I} A_\beta$, then $\|x{\restriction}(\mu\setminus A)\|_1 < \varepsilon/7$. Let us also put $A' = \bigcup_{\beta\in I} A'_\beta$. Since $f_\beta$ is continuous on $\mathscr{E}_{A_\beta}$ (which follows from its continuity with respect to the even weaker standard witness topology); the norm $\varphi_\beta$ is continuous on $\mathscr{E}_{A_\beta}$; and $I$ is finite, we know that the following map is continuous on $\prod_{\beta\in I} \mathscr{E}_{A_\beta}$: for $x \in \prod_{\beta\in\lambda} \mathscr{E}_{A_\beta}$,

$$x\restriction A \mapsto \sum_{\beta\in I} f_\beta(x\restriction A_\beta)\varphi_\beta(x\restriction A_\beta) = \sum_{\beta\in I}\psi_\beta(h_\beta(x\restriction A_\beta)) = \|(H(x))\restriction A'\|_1.$$

Hence, there exists an open neighbourhood $U \subset \prod_{\beta\in I}\mathscr{E}_{A_\beta}$ of $x\restriction A$ such that for every $y\restriction A \in U$, we have $\big|\|(H(x))\restriction A'\|_1 - \|(H(y))\restriction A'\|_1\big| < \varepsilon/7$. Now consider the open set $V \subset \mathscr{E}_\mu$ defined by

$$V = \Bigl(U \times \prod_{\beta\in\lambda\setminus I}\mathscr{E}_{A_\beta}\Bigr) \cap \{y : \|x-y\|_1 < \varepsilon/7\}.$$

Then for every $y \in V$, we have

$$\begin{aligned}
&\big|\|H(x)\|_1 - \|H(y)\|_1\big| \\
&\quad\le \big|\|(H(x))\restriction A'\|_1 - \|(H(y))\restriction A'\|_1\big| + \|(H(x))\restriction(\mu\setminus A')\|_1 + \|(H(y))\restriction(\mu\setminus A')\|_1 \\
&\quad< \frac{\varepsilon}{7} + 2\|x\restriction(\mu\setminus A)\|_1 + 2\|y\restriction(\mu\setminus A)\|_1 \\
&\quad< \frac{3\varepsilon}{7} + 2\bigl(\|y-x\|_1 + \|x\restriction(\mu\setminus A)\|_1\bigr) \\
&\quad< \varepsilon.
\end{aligned}$$

We thus find that $\mathscr{E}_\mu \approx \mathscr{E}'_\mu$ also holds with respect to the norm topologies. □

We are now ready to generalize Theorem 16.

Theorem 24. *If $\mathscr{E}_\mu$ is such that every $E_\alpha$ is closed in $\boldsymbol{R}$, then $\mathscr{E}_\mu \approx \mathfrak{E}^1_{\mathrm{c}}(lw(\mathscr{E}_\mu), w(\mathscr{E}_\mu))$ if and only if every $E_\alpha$ is zero-dimensional and* $\operatorname{ind}\mathscr{E}_\mu > 0$.

In view of Proposition 15 we also have the following result.

Corollary 25. *If $\mathscr{E}_\mu$ is such that every $E_\alpha$ is closed in $\boldsymbol{R}$ and $lw(\mathscr{E}_\mu) > \omega$, then $\mathscr{E}_\mu \approx \mathfrak{E}^1_{\mathrm{c}}(lw(\mathscr{E}_\mu), w(\mathscr{E}_\mu))$ if and only if every $E_\alpha$ is zero-dimensional.*

Proof of Theorem 24. Put $\kappa = w(\mathscr{E}_\mu)$ and $\lambda = lw(\mathscr{E}_\mu)$. Suppose that $\mathscr{E}_\mu \approx \mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)$. Then every $E_\alpha$ is imbeddable in $\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)$. Thus, since $\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)$ is totally disconnected, every $E_\alpha$ is zero-dimensional as a subset of $\boldsymbol{R}$. Of course, $\operatorname{ind}\mathscr{E}_\mu = \operatorname{ind}\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa) \ge \operatorname{ind}\mathfrak{E}_{\mathrm{c}} > 0$.

For the converse, assume that every $E_\alpha$ is zero-dimensional and that $\operatorname{ind}\mathscr{E}_\mu > 0$. If $\kappa = \omega$, then we use Proposition 13 and Theorem 16 to find $\mathscr{E}\mu \approx \mathfrak{E}_{\mathrm{c}} \approx \mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)$. So we may assume that $\kappa > \omega$. If $p \ne 1$ then we let the homeomorphism $\vartheta\colon \boldsymbol{R} \to \boldsymbol{R}$ be given by $\vartheta(t) = \operatorname{sgn}(t)|t|^p$. If we replace every $E_\alpha$ by $\vartheta(E_\alpha)$ then we obtain a space $\left(\prod_{\alpha\in\mu}\vartheta(E_\alpha)\right)\cap\ell^1_\mu$ that is homeomorphic to $\mathscr{E}_\mu$. Thus we may assume that $p=1$. By Proposition 13 we may also assume that $\mu=\kappa$ and $E_\alpha\neq\{0\}$ for each $\alpha$. If $\lambda=\kappa$ then we use Lemma 23. We may assume that $\kappa>\lambda$.

We have by Corollary 10 that there is a countable set $I\subset\mu$ such that $\operatorname{ind}\mathscr{E}_I>0$ and $0\in E_\alpha$ for each $\alpha\in\mu\setminus I$. Let $n$ be such that

$$A=I\cup\{\alpha:\exists t\in E_\alpha,\,0<|t|<1/n\}$$

has cardinality $\lambda$. If $\lambda=\omega$ then Theorem 16 ensures that $\mathscr{E}_A\approx\mathfrak{E}_{\mathrm{c}}\approx\mathfrak{E}^1_{\mathrm{c}}(\lambda,\lambda)$. If $\lambda>\omega$ then $\mathscr{E}_A\approx\mathfrak{E}^1_{\mathrm{c}}(\lambda,\lambda)$ follows from Lemma 23. Note that for each $\alpha\in\mu\setminus A$ we have $E_\alpha\cap(-1/n,1/n)=\{0\}\neq E_\alpha$. It is easily seen that then

$$\mathscr{E}_{\mu\setminus A}\approx\bigoplus_{F\in\mathscr{F}}\prod_{\alpha\in F}(E_\alpha\setminus\{0\}),$$

where $\mathscr{F}$ denotes the collection of all finite subsets of $\mu\setminus A$. Note that $|\mathscr{F}|=\kappa$ because $|\mu\setminus A|=\kappa>|A|$. Therefore,

$$\begin{aligned}\mathscr{E}_\mu=\mathscr{E}_A\times\mathscr{E}_{\mu\setminus A}&\approx\mathfrak{E}^1_{\mathrm{c}}(\lambda,\lambda)\times\bigoplus_{F\in\mathscr{F}}\prod_{\alpha\in F}(E_\alpha\setminus\{0\})\\&\approx\bigoplus_{F\in\mathscr{F}}\left(\mathfrak{E}^1_{\mathrm{c}}(\lambda,\lambda)\times\prod_{\alpha\in F}(E_\alpha\setminus\{0\})\right)\\&\approx\bigoplus_{F\in\mathscr{F}}\mathfrak{E}^1_{\mathrm{c}}(\lambda,\lambda)\approx\mathfrak{E}^1_{\mathrm{c}}(\lambda,\lambda)\times\kappa_{\mathrm{D}}\\&\approx\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa),\end{aligned}$$

where we used Remarks 18 and 20. □

We have the following generalization of [**2**, Corollary 4].

COROLLARY 26. *Let $\kappa\geq\lambda\geq\omega$ and let every $E_\alpha$ be closed in $\boldsymbol{R}$. We have $\mathscr{E}_\mu\times\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)\approx\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)$ if and only if every $E_\alpha$ is zero-dimensional, $\mathscr{E}_\mu\neq\emptyset$, $\lambda\geq lw(\mathscr{E}_\mu)$, and $\kappa\geq w(\mathscr{E}_\mu)$.*

## 5. Complete nonseparable Erdős spaces.

In Dijkstra and van Mill [**5**, Theorem 23(b)] Theorem 16 was generalized to topologically complete spaces $\mathscr{E}_\omega\subset\ell^p$ as follows.

Theorem 27. *The space $\mathscr{E}_\omega$ is homeomorphic to $\mathfrak{E}_c$ if and only if* $\operatorname{ind}\mathscr{E}_\omega > 0$ *and $E_n$ is a zero-dimensional $G_\delta$-set in $\boldsymbol{R}$ for every $n \in \omega$.*

We intend to generalize Theorem 24 in the nonseparable setting to complete spaces as well.

Proposition 28. *A nonempty $\mathscr{E}_\mu$ is topologically complete if and only if every $E_\alpha$ is a $G_\delta$-set in $\boldsymbol{R}$.*

Proof. If $\mathscr{E}_\mu \neq \emptyset$ is topologically complete, then every set $E_\alpha$ can be considered as a closed imbedded subspace of $\mathscr{E}_\mu$. Thus, it is a topologically complete subset of $\boldsymbol{R}$ and hence a $G_\delta$-set.

Conversely, assume that every $E_\alpha$ is a $G_\delta$-set in $\boldsymbol{R}$. Let $I$ be the countable set $\{\alpha \in \mu : 0 \notin E_\alpha\}$ and note that $\mathscr{E}_I$ is a $G_\delta$-set in $\ell^p_I$, since the set $\prod_{\alpha\in I} E_\alpha$ is already a $G_\delta$-set in $\boldsymbol{R}^I$. It thus suffices to consider spaces $\mathscr{E}_\mu$ for which 0 is contained in every $E_\alpha$. For each $\alpha \in \mu$ we can write $\boldsymbol{R}\setminus E_\alpha = \bigcup_{n\in\omega} A^\alpha_n$ for certain closed sets $A^\alpha_n \subset \boldsymbol{R}$. We can easily arrange that $A^\alpha_n \cap (-2^{-n+1}, 2^{-n+1}) = \emptyset$ for each $n$ and $\alpha$.

Consider the sets $F_n = \{x \in \ell^p_\mu : \exists\alpha \in \mu,\ x_\alpha \in A^\alpha_n\}$. If $x \notin F_n$, then $x_\alpha \notin A^\alpha_n$ for every $\alpha \in \mu$. Due to summability, there exists a finite set $I \subset \mu$ with $\alpha \in I$ if and only if $|x_\alpha| \geq 2^{-n}$. Let $\varepsilon = \min(\{2^{-n}\} \cup \{d(x_\alpha, A^\alpha_n) : \alpha \in I\})$. If $\|x-y\|_p < \varepsilon$ and $\alpha \in I$, then $|x_\alpha - y_\alpha| < \varepsilon$ for every $\alpha \in I$ and hence $y_\alpha \notin A^\alpha_n$. Furthermore, if $\alpha \notin I$ then $|x_\alpha| < 2^{-n}$ and $|x_\alpha - y_\alpha| < 2^{-n}$ imply that $|y_\alpha| < 2^{-n+1}$ and again $y_\alpha \notin A^\alpha_n$. Thus, if $\|x-y\|_p < \varepsilon$, then also $y \notin F_n$, which proves that $F_n$ is closed. It is then easy to see that $\bigcup_{n\in\omega} F_n = \ell^p_\mu \setminus \mathscr{E}_\mu$, whence $\mathscr{E}_\mu$ is $G_\delta$ in $\ell^p_\mu$ and the space is topologically complete. □

The following theorem is essentially a controlled version of the Negligibility Theorem for Lelek fans from Kawamura, Oversteegen and Tymchatyn [**10**, Theorem 6].

Theorem 29. *Let $\varphi : C \to [0,\infty)$ be a U-Lelek function with compact domain and let $A = \bigcup_{i\in\omega} A_i$ be given such that for every $i \in \omega$ the set $A_i$ is closed in $C$ and $G_0^{\varphi\restriction A_i}$ is nowhere dense in $G_0^\varphi$. If $\varepsilon > 0$ then there exist a compact space $D$; a continuous surjection $g : C \to D$ with the additional property that $g\restriction(C\setminus A) : C\setminus A \to D$ is a bijection; and a continuous function $\alpha : C \to [0,\infty)$ with $M(\log\alpha) < \varepsilon$, such that the function $\psi : D \to [0,\infty)$ defined by*

$$\psi(g(x)) = \alpha(x)\varphi(x)$$

*for every $x \in C \setminus A$, is a U-Lelek function and $G_0^{\varphi\restriction(C\setminus A)} \approx G_0^\psi$, via the homeomorphism $(x,\varphi(x)) \mapsto (g(x),\psi(g(x)))$.*

Proof. We choose a compatible metric $d$ on $C$ and for $Y \subset C$ and $\varepsilon > 0$ we put $U_\varepsilon(Y) = \{x \in C : \exists y \in Y,\, d(x,y) < \varepsilon\}$. Since $C$ is zero-dimensional (it is a Cantor set), we may assume that the $A_i$ are pairwise disjoint.

The set $G_0^\varphi$ is complete (it is even shown to be homeomorphic to $\mathfrak{E}_\mathrm{c}$ by Kawamura, Oversteegen and Tymchatyn in [**10**]), so by the assumptions we can apply the Baire Category Theorem, to find that

$$G_0^{\varphi\restriction(C\setminus A)} = G_0^\varphi \setminus \bigcup_{i\in\omega} G_0^{\varphi\restriction A_i}$$

is dense in $G_0^\varphi$ and hence in $L_0^\varphi$ as well.

Inductively, we shall construct for every $i \in \omega$ a (finite) clopen partition $\mathscr{U}_i$ of $C$; points $b_B \in C \setminus A$; sets $B^+ = B \cup \{b_B\}$ for every

$$B \in \mathscr{A}_i = \{U \cap A_i : U \in \mathscr{U}_i\} \setminus \{\emptyset\};$$

the set

$$A_i^+ = \bigcup \{B^+ : B \in \mathscr{A}_i\};$$

and a continuous function $\alpha_i \colon C \to (0,\infty)$, with the following properties:

(i) if $i > 0$ then $\mathscr{U}_i$ refines $\mathscr{U}_{i-1}$;
(ii) if $i > 0$ and $U \in \mathscr{U}_i$ satisfies $U \cap \bigcup_{n<i} A_n^+ = \emptyset$, then $\operatorname{diam} U < 2^{-i}$;
(iii) for every $B \in \bigcup_{n\le i} \mathscr{A}_n$, there exists a $U \in \mathscr{U}_i$ such that $U \cap \bigcup_{n\le i} A_n^+ = B^+$ and $U \subset U_{2^{-i}}(B^+)$;
(iv) if $i > 0$ then $\alpha_i \restriction \bigcup_{n<i} A_n^+ = \alpha_{i-1} \restriction \bigcup_{n<i} A_n^+$;
(v) if $i > 0$ then $M(\log(\alpha_i/\alpha_{i-1})) < \varepsilon 2^{-i}$; and
(vi) if $B \in \mathscr{A}_i$ then $\alpha_i(b_B)\varphi(b_B) > M((\alpha_i \cdot \varphi)\restriction B)$.

For the base step we can arrange that $A_0 = \emptyset$. Put $\mathscr{U}_0 = \{C\}$ and $\alpha_0 = 1$ and note that the induction hypotheses are void.

Suppose now that the induction hypotheses are all satisfied up to and including $i-1$. We can find a finite discrete open refinement $\mathscr{V}$ of $\mathscr{U}_{i-1}$ such that the mesh is smaller than $\min\{2^{-i}, d(A_i, \bigcup_{n<i} A_n^+)\}$. If $B \in \bigcup_{n<i} \mathscr{A}_n$ we put $V_B = \mathrm{St}(B^+, \mathscr{V})$. Defining

$$\mathscr{U}_i = \left\{V_B : B \in \bigcup_{n<i} \mathscr{A}_n\right\} \cup \left\{V \in \mathscr{V} : V \cap \bigcup_{n<i} A_n^+ = \emptyset\right\}$$

we note that $\mathscr{U}_i$ is a clopen partition of $C$ and that hypotheses (i) and (ii) are satisfied. Let $U \in \mathscr{U}_i$ be such that $B = A_i \cap U \neq \emptyset$ thus $B \in \mathscr{A}_i$. Density of $G_0^{\varphi\restriction(C\setminus A)}$ in $L_0^\varphi$ implies density of $G_0^{(\alpha_{i-1}\cdot\varphi)\restriction(C\setminus A)}$ in $L_0^{\alpha_{i-1}\cdot\varphi}$. If $M((\alpha_{i-1}\cdot\varphi)\restriction B) = 0$ then we choose a point $b_B \in U \setminus A$ with $\varphi(b_B) > 0$ and we put $t_B = 1$. Otherwise there exists a point $b_B \in U \setminus A$ satisfying

$$0 < \log \frac{M((\alpha_{i-1}\cdot\varphi)\restriction B)}{(\alpha_{i-1}\cdot\varphi)(b_B)} < \varepsilon 2^{-i}$$

and a number $t_B$ for which

$$1 < \frac{M((\alpha_{i-1}\cdot\varphi)\restriction B)}{(\alpha_{i-1}\cdot\varphi)(b_B)} < t_B < e^{\varepsilon 2^{-i}}.$$

Let $C_B$ be a clopen set with $b_B \in C_B \subset U \setminus B$. Define

$$\alpha_i(x) = \begin{cases} t_B \alpha_{i-1}(x), & \text{if } x \in C_B \text{ for some } B \in \mathscr{A}_i; \\ \alpha_{i-1}(x), & \text{otherwise.} \end{cases}$$

Then, by the choice of $t_B$ we find

$$(\alpha_i\cdot\varphi)(b_B) = t_B\alpha_{i-1}(b_B)\varphi(b_B) > M((\alpha_{i-1}\cdot\varphi)\restriction B) = M((\alpha_i\cdot\varphi)\restriction B).$$

Furthermore,

$$M\left(\log\frac{\alpha_i}{\alpha_{i-1}}\right) = \max_{B\in\mathscr{A}_i} \log t_B < \varepsilon 2^{-i}.$$

Together, this yields (iii)–(vi) and the induction process is finished.

For every $b_B$ found in the construction, we define $g\restriction B \equiv b_B$, and $g(x) = x$ otherwise. So $g\colon C \to C \setminus A$ is a surjection and we let $D$ be the set $C \setminus A$ equipped with the quotient topology. Note that $g\restriction(C \setminus A)$ is the identity and that $D$ is compact. We shall show that $D$ is totally disconnected. Then it is Hausdorff and hence compact metric as a continuous image of a compact metric space. We may conclude that $D$ is zero-dimensional as a compact and totally disconnected space.

Note that it follows from hypotheses (i) and (iii) that $g^{-1}(g(U)) = U$ and hence $g(U)$ is clopen in $D$ for every $U \in \bigcup_{i\in\omega}\mathscr{U}_i$. Thus to prove that $D$ is totally disconnected, it suffices to show that points can be separated by images of elements of $\mathscr{U}_k$'s. Let $x \neq y$ in $D$ and assume first that $x = b_B$ for some $B$. Then $g^{-1}(x) = B^+$ and we can find an $n \in \boldsymbol{N}$ such that $2^{-n} < d(y, B^+)$. Note that by (iii) there is a $U \in \mathscr{U}_n$ such that $B^+ \subset U$ and $y \notin U$. Since $g^{-1}(g(U)) = U$ we have that $g(U)$ separates $x$ from $y$. Now we consider the remaining case that neither $x$ nor $y$ is equal to some $b_B$ for $B \in \bigcup_{i\in\omega} \mathscr{A}_i$. Let $n \in \boldsymbol{N}$ be such that $2^{-n} < d(x,y)$ and let $x \in U \in \mathscr{U}_n$. If $U \cap \bigcup_{i<n} A_i^+ = \emptyset$ then $y \notin U$ by (ii) and we are done. So let $B \in \bigcup_{i<n} \mathscr{A}_i$ be such that $B^+ \subset U$. By (iii) there is a highest index $m \geq n$ such that $\{x\} \cup B^+ \subset V$ for some $V \in \mathscr{U}_m$. Let $W \in \mathscr{U}_{m+1}$ be such that $x \in W$. Then $W \subset V \setminus B^+$ and since $V \cap \bigcup_{i\leq m} A_i^+ = B^+$ we have $W \cap \bigcup_{i\leq m} A_i^+ = \emptyset$. This means by (ii) that $\operatorname{diam} W < 2^{-m-1} < 2^{-n}$ thus $y \notin W$.

By (v), $(\log \alpha_i)_{i\in\omega}$ is a uniform Cauchy sequence of continuous functions and hence $\alpha = \lim_{i\to\infty} \alpha_i$ is well-defined and continuous. Moreover, since $\alpha_0 = 1$ we have

$$M(\log \alpha) \leq \sum_{i=1}^{\infty} M\left(\log \frac{\alpha_i}{\alpha_{i-1}}\right) < \sum_{i=1}^{\infty} \varepsilon 2^{-i} = \varepsilon.$$

If $B \in \mathscr{A}_i$ then hypotheses (iv) and (vi) imply

$$(\alpha \cdot \varphi)(b_B) = (\alpha_i \cdot \varphi)(b_B) > M((\alpha_i \cdot \varphi){\restriction}B) = M((\alpha \cdot \varphi){\restriction}B). \tag{1}$$

The function $\psi\colon D \to [0,\infty)$ is simply defined by $\psi = (\alpha \cdot \varphi){\restriction}(C \setminus A)$. Note that the function $\alpha \cdot \varphi$ is USC on $C$. We show that $\psi$ is USC on $D$. Let $x \in D$ and let $t > \psi(x)$. Then $O = (\alpha \cdot \varphi)^{-1}([0,t))$ is an open set in $C$ that contains the fibre $g^{-1}(x)$, where we used formula (1) for the case that $x$ equals some $b_B$. By compactness we have that $D \setminus g(C \setminus O)$ is an open neighbourhood of $x$ that is mapped in $[0,t)$ by $\psi$.

Note that $\varphi{\restriction}(C\setminus A)$ is a U-Lelek function thus $(\alpha\cdot\varphi){\restriction}(C\setminus A)$ is also a U-Lelek function. Since the topology on $D$ is weaker than the topology on $C \setminus A$ it follows immediately that $\psi$ is a U-Lelek function.

Finally, we show $(x,\varphi(x)) \mapsto (x,\psi(x))$ defines a homeomorphism from $G_0^{\varphi{\restriction}(C\setminus A)}$ to $G_0^{\psi}$. By continuity of $\alpha$, we find that $(x,\varphi(x)) \mapsto (x,\alpha(x)\varphi(x))$ defines a homeomorphism from $G_0^{\varphi}$ to $G_0^{\alpha\cdot\varphi}$. Thus we are left with showing that the identity map $h$ from $G_0^{(\alpha\cdot\varphi){\restriction}(C\setminus A)}$ to $G_0^{\psi}$ is a homeomorphism. Note that the topology of the first space is inherited from $(C \setminus A) \times \boldsymbol{R}$ instead of $D \times \boldsymbol{R}$. Hence, bijectivity and continuity are no problem for $h$. To show that $h^{-1}$ is continuous let $\lim_{n\to\infty} x_n = x$ in $D$ such that $\lim_{n\to\infty} \psi(x_n) = \psi(x)$. If $\lim_{n\to\infty} x_n = x$ in $C$ then we are done. So assume that $(x_n)_{n\in\omega}$ does not converge to $x$ in $C$. By compactness there is a subsequence $(x_{n_k})_{k\in\omega}$ that converges to some $x' \in C \setminus \{x\}$. Since $g$ is continuous we have $g(x') = x$ and hence $x = b_B$ for some $B$. Then $x' \in B = g^{-1}(x) \setminus \{x\}$. Thus, by inequality (1) and upper semicontinuity,

$$\limsup_{k\to\infty} \psi(x_{n_k}) \le \alpha(x')\varphi(x') \le M((\alpha\cdot\varphi)\restriction B) < \psi(b_B).$$

This result contradicts the assumption that $\lim_{n\to\infty}\psi(x_n) = \psi(x)$. The proof is complete. □

We shall extend Lemma 23 from closed to complete spaces. The following definition is needed.

Definition 30. Let $\varphi : X \to [0,\infty]$ be a function and let $X$ be a subspace of a space $Y$. We define $\mathrm{ext}_Y\ \varphi : Y \to [0,\infty]$ by

$$(\mathrm{ext}_Y\ \varphi)(y) = \inf\big\{M(\varphi\restriction(X\cap U)) : U \text{ open in } Y,\ y\in U\big\} \quad \text{for } y \in Y.$$

Lemma 31. *Suppose that every $E_\alpha \subset \boldsymbol{R}$ is a zero-dimensional $G_\delta$-set in $\boldsymbol{R}$ and that $p = 1$. If we have $\lambda = lw(\mathscr{E}_\mu) = w(\mathscr{E}_\mu) > \omega$, then $\mathscr{E}_\mu \approx \mathfrak{E}^1_{\mathrm{c}}(\lambda,\lambda)$.*

Proof. As in the proof of Lemma 23 we may assume that $0 \in E_\alpha$ for each $\alpha$ and that we have a partition $\{A_\beta : \beta\in\lambda\}$ of $\mu = \lambda$ into countable sets such that $\mathrm{ind}\,\mathscr{E}_{A_\beta} > 0$ for each $\beta$. Let $\varphi_\beta$ be the norm on $\prod_{\alpha\in A_\beta} E_\alpha$. By the same argument as employed in [**2**, Theorem 3] we have that $\varphi_\beta$ is L-Lelek. Select a Cantor set $C$ that compactifies $\prod_{\alpha\in A_\beta} E_\alpha$ and put $\chi = \mathrm{ext}_C \frac{1}{\varphi_\beta}$. According to [**6**, Lemma 4.8] $\chi$ is a USC function that extends $1/\varphi_\beta$ and $\chi$ is a U-Lelek function because $G_0^{1/\varphi_\beta}$ is dense in $L_0^\chi$. Select a clopen neighbourhood basis $C = B_0 \supsetneq B_1 \supsetneq \cdots$ of $\mathbf{0}$ and put $C_i = B_i \setminus B_{i+1}$ for every $i\in\omega$. Note that $\chi\restriction C_i$ is a U-Lelek function on a compact domain. Let $i\in\omega$ and consider the neighbourhood $C\setminus C_i$ of $\mathbf{0}$. By the product topology there are a finite subset $F$ of $A_\beta$ and an $\varepsilon_i > 0$ such that $\big(\prod_{\alpha\in F}(E_\alpha\cap(-\varepsilon_i,\varepsilon_i))\big)\times\prod_{\alpha\in A_\beta\setminus F} E_\alpha \subset C\setminus C_i$. Thus for each $x\in Z_i = C_i\cap\prod_{\alpha\in A_\beta} E_\alpha$ we have $\varphi_\beta(x)\ge\varepsilon_i$. Since $C_i$ is clopen we have $\chi\restriction C_i = \mathrm{ext}_{C_i}\big(\frac{1}{\varphi_\beta}\restriction Z_i\big)$ and hence $\chi\restriction C_i$ is bounded by $1/\varepsilon_i$.

Since $\prod_{\alpha\in A_\beta} E_\alpha$ is topologically complete, the set $P_i = C_i\setminus\prod_{\alpha\in A_\beta} E_\alpha$ is $\sigma$-compact. Also, since $G_0^{1/\varphi_\beta}$ is dense in $L_0^\chi$ we have that $G_0^{\chi\restriction P_i}$ has a dense complement in $G_0^{\chi\restriction C_i}$. Thus we can apply Theorem 29 to $\chi\restriction C_i$ and $P_i$, yielding a Cantor set $D^i$, a continuous bijection $g^i : C_i\setminus P_i\to D^i$, and a continuous function $f^i : C_i\setminus P_i\to(0,\infty)$ such that $\tau^i : D^i\to[0,\infty)$ given by $\tau^i\circ g^i = f^i\cdot(\chi\restriction(C_i\setminus P_i))$ is a U-Lelek function, $(x,\chi(x))\mapsto(g^i(x), f^i(x)\chi(x))$ defines a homeomorphism from $G_0^{\chi\restriction(C_i\setminus P_i)}$ to $G_0^{\tau^i}$, and $M(\log f^i) < 2^{-i}$.

Let $D_\beta$ be the Cantor set that is obtained by adding a compactifying point $\Omega$ to the topological sum $\bigoplus_{i\in\omega} D^i$. Define the continuous bijection $g_\beta : \prod_{\alpha\in A_\beta} E_\alpha \to D_\beta$ by $g_\beta(\mathbf{0}) = \Omega$ and $g_\beta\restriction(C_i\setminus P_i) = g^i$ for $i\in\omega$ and the function $f_\beta : \prod_{\alpha\in A_\beta} E_\alpha \to$

$(0,\infty)$ by $f_\beta(\mathbf{0}) = 1$ and $f_\beta \restriction (C_i \setminus P_i) = f^i$ for $i \in \omega$. Since $M(\log f_i) < 2^{-i}$ it is clear that $f_\beta$ is continuous and that $M(\log f_\beta) < 1$. We define the function $\zeta_\beta \colon D_\beta \to [0,\infty]$ by $\zeta_\beta \circ g_\beta = \varphi_\beta / f_\beta$. Note that $\zeta_\beta \restriction D^i = 1/\tau^i$ for $i \in \omega$ and $\zeta_\beta(\Omega) = 0$ thus $\zeta_\beta$ is an L-Lelek function with compact domain and $g_\beta$ determines a homeomorphism from $G^\infty_{\varphi_\beta}$ to $G^\infty_{\zeta_\beta}$. Observe that the natural map from $\mathscr{E}_{A_\beta}$ to $G^\infty_{\varphi_\beta}$ is a homeomorphism and that the multiplication factor $1/f_\beta$ that links $\varphi_\beta$ to $\zeta_\beta$ is limited to the interval $(e^{-1}, e)$. For any space $\mathscr{E}'_\mu$ we obtain the corresponding data $A'_\beta$, $\varphi'_\beta$ and $\zeta'_\beta$ $(\beta \in \lambda)$. Lemma 22 applies to $\zeta_\beta$ and $\zeta'_\beta$ and we can finish the proof of this lemma in the same way as the proof of Lemma 23. □

Theorem 32. *The space $\mathscr{E}_\mu$ is homeomorphic to $\mathfrak{E}^1_{\rm c}(lw(\mathscr{E}_\mu), w(\mathscr{E}_\mu))$ if and only if every $E_\alpha$ is a zero-dimensional $G_\delta$-subset of $\boldsymbol{R}$ and* $\operatorname{ind} \mathscr{E}_\mu > 0$.

Proof. For the "only if" part use Proposition 28. The "if" part is virtually identical to the proof of Theorem 24: just replace references to Theorem 16 by Theorem 27 and Lemma 23 by Lemma 31. □

Corollary 33. *If $lw(\mathscr{E}_\mu) > \omega$ then the space $\mathscr{E}_\mu$ is homeomorphic to $\mathfrak{E}^1_{\rm c}(lw(\mathscr{E}_\mu), w(\mathscr{E}_\mu))$ if and only if every $E_\alpha$ is a zero-dimensional $G_\delta$-subset of $\boldsymbol{R}$.*

The following immediate consequence of Theorem 32 generalizes [**5**, Theorem 23(a)].

Corollary 34. *Let $\kappa \geq \lambda \geq \omega$. We have $\mathscr{E}_\mu \times \mathfrak{E}^1_{\rm c}(\lambda,\kappa) \approx \mathfrak{E}^1_{\rm c}(\lambda,\kappa)$ if and only if every $E_\alpha$ is a zero-dimensional $G_\delta$-subset of $\boldsymbol{R}$, $\mathscr{E}_\mu \neq \emptyset$, $\lambda \geq lw(\mathscr{E}_\mu)$, and $\kappa \geq w(\mathscr{E}_\mu)$.*

The following result establishes $\mathfrak{E}^1_{\rm c}(\lambda,\kappa)$ as a universal space.

Corollary 35. *If every $E_\alpha$ is zero-dimensional then $\mathscr{E}_\mu$ is imbeddable in $\mathfrak{E}^1_{\rm c}(lw(\mathscr{E}_\mu), w(\mathscr{E}_\mu))$.*

Proof. We may assume that $\mathscr{E}_\mu \neq \emptyset$. Put $\lambda = lw(\mathscr{E}_\mu)$ and $\kappa = w(\mathscr{E}_\mu)$. By Proposition 13 let $L \subset K \subset \mu$ and let $m \in \boldsymbol{N}$ be such that $|L| = \lambda$, $|K| = \kappa$, $E_\alpha = \{0\}$ for each $\alpha \in \mu \setminus K$, and $E_\alpha \cap (-1/m, 1/m) = \{0\}$ for $\alpha \in K \setminus L$. For every $\alpha \in \mu$, we may choose a countable and dense set $D_\alpha \subset \boldsymbol{R} \setminus E_\alpha$ because $E_\alpha$ is zero-dimensional. Define

$$E'_\alpha = \begin{cases} \boldsymbol{R} \setminus D_\alpha, & \text{if } \alpha \in L; \\ \boldsymbol{R} \setminus (D_\alpha \cup (-1/m, 0) \cup (0, 1/m)), & \text{if } \alpha \in K \setminus L; \\ \{0\}, & \text{if } \alpha \in \mu \setminus K. \end{cases}$$

Put $\mathscr{E}'_\mu = \big(\prod_{\alpha\in\mu} E'_\alpha\big) \cap \ell^p_\mu$ and note that this space is nonempty because it contains $\mathscr{E}_\mu$. For the (infinitely many) $\alpha \in L$ we have $\overline{E'_\alpha} = \boldsymbol{R}$ and thus $\operatorname{ind} \mathscr{E}'_\mu > 0$ by Proposition 11. Note that every $E'_\alpha$ is a zero-dimensional $G_\delta$-set and hence $\mathscr{E}'_\mu \approx \mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)$ by Theorem 32 and Proposition 13. Since $\mathscr{E}_\mu \subset \mathscr{E}'_\mu$ the proof is complete. □

## 6. Applications.

In Section 2 and Section 3 we determined the small inductive dimension of $\mathscr{E}_\mu$. In the separable metric case, this dimension function coincides with the large inductive and the covering dimension. For general metric spaces $X$ we have $\operatorname{ind} X \le \operatorname{Ind} X = \dim X$ by the Katětov-Morita Theorem; see [**7**, Theorem 4.1.3]. For nonseparable metric spaces Ind may be greater than ind, as for example in the completely metrizable Roy space [**12**]. A space $X$ with $\operatorname{Ind} X \le 0$ is called *strongly zero-dimensional*. In this section, we shall show that the spaces considered in this paper do have all three dimensions equal. We shall also consider a nonseparable Erdős type space $M$ that is obtained from the separable $\ell^1$ by strengthening the topology. We show that this particular space is homeomorphic to $\mathfrak{E}^1_{\mathrm{c}}(\mathfrak{c},\mathfrak{c})$, where $\mathfrak{c} = |\boldsymbol{R}|$. The method by which the latter result is obtained has an interesting consequence in that it allows us to prove that $\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega \times \kappa_{\mathrm{D}}$. (Recall that we let $X_{\mathrm{D}}$ denote the set $X$ with the discrete topology.) A fourth application involves the fixed point property for one-point connectifications of $\mathscr{E}_\mu$.

The following representation of $\mathfrak{E}^1_{\mathrm{c}}(\lambda,\lambda)$ plays a key role in the first three applications in this section:

$$\mathfrak{E}'_c(\lambda) = \Bigg(\prod_{(n,\alpha)\in\omega\times\lambda} E_{(n,\alpha)}\Bigg) \cap \ell^1_{\omega\times\lambda}, \text{ where } E_{(n,\alpha)} = \{0, 2^{-n}\}.$$

Remark 36. According to Propositions 11 and 13 in combination with Theorem 24, $\mathfrak{E}'_c(\lambda)$ is indeed homeomorphic to $\mathfrak{E}^p_{\mathrm{c}}(\lambda,\lambda)$.

Proposition 37. $\operatorname{Ind} \mathfrak{E}'_{\mathrm{c}}(\lambda) = \operatorname{Ind} \mathfrak{E}^p_{\mathrm{c}}(\lambda,\lambda) = 1$ *for all* $\lambda$ *and* $p$.

Proof. It suffices to prove that $\operatorname{Ind} \mathfrak{E}'_{\mathrm{c}}(\lambda) \le 1$. Note that a rearrangement yields a continuous injection

$$\mathfrak{E}'_{\mathrm{c}}(\lambda) = \Bigg(\prod_{(n,\alpha)\in\omega\times\lambda} E_{(n,\alpha)}\Bigg) \cap \ell^1_{\omega\times\lambda} \subset \prod_{n\in\omega} \mathfrak{E}'_{\mathrm{c}}(\lambda)_{\{n\}\times\lambda},$$

in which all factors

$$Z_n = \mathfrak{E}'_{\mathrm{c}}(\lambda)_{\{n\}\times\lambda} = \left(\prod_{\alpha\in\lambda} E_{(n,\alpha)}\right) \cap \ell^1_{\{n\}\times\lambda},$$

are discrete spaces, since only norms that are multiples of $2^{-n}$ can occur. Thus by [**7**, Theorem 4.1.25] we have that $Z = \prod_{n\in\omega} Z_n$ is a strongly zero-dimensional metric space. The fact that $\ell^1_{\omega\times\lambda}$ has a Kadec norm means that $\mathfrak{E}'_{\mathrm{c}}(\lambda)$ is homeomorphic to the graph of the norm function as a subspace of $\left(\prod_{(n,\alpha)\in\omega\times\lambda} E_{(n,\alpha)}\right) \times \boldsymbol{R}$. Since the topology that $\mathfrak{E}'_{\mathrm{c}}(\lambda)$ inherits from $Z$ lies between the standard witness topology and the norm topology we have that $\mathfrak{E}'_{\mathrm{c}}(\lambda)$ can also be imbedded in $Z \times \boldsymbol{R}$; see Remark 4. Thus by the Subspace Theorem [**7**, Theorem 4.1.7] and the Cartesian Product Theorem [**7**, Theorem 4.1.21] we have $\operatorname{Ind} \mathfrak{E}'_{\mathrm{c}}(\lambda) \le \operatorname{Ind}(Z \times \boldsymbol{R}) = 1$. □

According to Remark 14 the standard witness topology on $\mathscr{E}_\mu$ is nonmetrizable if $\mathscr{E}_\mu$ is nonseparable. Proposition 37 is based on the observation that $\mathfrak{E}'_{\mathrm{c}}(\lambda)$ has a metrizable witness topology that is strongly zero-dimensional. With Corollary 35 we have

Corollary 38. *If every $E_\alpha$ is zero-dimensional then $\mathscr{E}_\mu$ has a metrizable witness topology that is strongly zero-dimensional.*

We are now ready to prove the coincidence of the dimension functions for the spaces $\mathscr{E}_\mu$ as expressed by Theorem 3.

Proof of Theorem 3. We may assume that $\mathscr{E}_\mu \neq \emptyset$ and that $E_\alpha \neq \{0\}$ $(\alpha \in \mu)$. We first prove the theorem for the case that every $E_\alpha$ is zero-dimensional.

If $\operatorname{ind} \mathscr{E}_\mu = 0$ then by Theorem 1 there exist an $m \in \boldsymbol{N}$ such that $\|\eta(m)\|_p < \infty$. Thus, there exists a countable set $I \subset \mu$ such that $E_\alpha \cap [-1/m, 1/m] = \{0\}$, whenever $\alpha \in \mu \setminus I$. Therefore, as in the proof of Theorem 24, we find

$$\mathscr{E}_\mu = \mathscr{E}_I \times \mathscr{E}_{\mu\setminus I} \approx \mathscr{E}_I \times \bigoplus_{F\in\mathscr{F}} \prod_{\alpha\in F} (E_\alpha \setminus \{0\}) \approx \bigoplus_{F\in\mathscr{F}} \left(\mathscr{E}_I \times \prod_{\alpha\in F} (E_\alpha \setminus \{0\})\right),$$

where $\mathscr{F}$ denotes the collection of finite subsets of $\mu \setminus I$. This is a topological sum of separable metrizable zero-dimensional spaces. Hence, applying [**7**, Proposition 2.2.8] yields that $\operatorname{Ind} \mathscr{E}_\mu = 0$.

Now suppose that $\operatorname{ind} \mathscr{E}_\mu > 0$. Since the spaces are metric, we obtain by the Subspace Theorem, Corollary 35, and Proposition 37 that $\operatorname{Ind} \mathscr{E}_\mu = 1$.

Consider now the case that not every $E_\alpha$ is zero-dimensional. Let $I = \{\alpha \in \mu : \operatorname{ind} E_\alpha > 0\}$. If $I$ is infinite then $\mathscr{E}_\mu$ contains an $n$-cube for every $n \in \boldsymbol{N}$ thus

$\operatorname{ind}\mathscr{E}_\mu = \infty$ and there is nothing to prove. Let $n = |I| < \infty$. Then $\mathscr{E}_I$ contains an $n$-cube $K$ and $\mathscr{E}_I \subset \boldsymbol{R}^n$. If $\operatorname{ind}\mathscr{E}_{\mu\setminus I} = 0$ then we have

$$n \leq \operatorname{ind}\mathscr{E}_\mu \leq \operatorname{Ind}\mathscr{E}_\mu \leq \operatorname{Ind}\mathscr{E}_I + \operatorname{Ind}\mathscr{E}_{\mu\setminus I} \leq n + 0$$

by the Product Theorem and this theorem for zero-dimensional $E_\alpha$'s. If $\operatorname{ind}\mathscr{E}_{\mu\setminus I} > 0$ then by Corollary 10 the space contains a copy of $\mathfrak{E}_\mathrm{c}$. According to Hurewicz [**9**] we have $\operatorname{ind}(K \times \mathfrak{E}_\mathrm{c}) = n + 1$ thus

$$n + 1 \leq \operatorname{ind}\mathscr{E}_\mu \leq \operatorname{Ind}\mathscr{E}_\mu \leq \operatorname{Ind}\mathscr{E}_I + \operatorname{Ind}\mathscr{E}_{\mu\setminus I} \leq n + 1.$$

The proof is complete. □

Now Proposition 5 improves to:

Corollary 39. *If every $E_\alpha$ is zero-dimensional then* $\operatorname{Ind}\mathscr{E}_\mu \leq 1$.

If a space is not written in the form $\left(\prod_{\alpha\in\mu} E_\alpha\right) \cap \ell^p_\mu$, for zero-dimensional subspaces $E_\alpha \subset \boldsymbol{R}$, then our criteria (for instance, Theorem 1) cannot be verified. An example of such a space is a metric space $(M, \rho)$ defined by

$$M = \{x \in \boldsymbol{R}^\omega : \|x\|_1 < \infty\}$$

and

$$\rho(x, y) = \|x - y\|_1 + \sum_{n\in\omega} 2^{-n} D(x_n, y_n).$$

Here $D$ denotes the discrete metric on $\boldsymbol{R}$ that assumes only the values 0 and 1. Let $\boldsymbol{R}_\mathrm{D}$ stand for $\boldsymbol{R}$ equipped with the discrete topology. Note that the topology on the product space $(\boldsymbol{R}_\mathrm{D})^\omega$ is generated by the metric $\sum_{n\in\omega} 2^{-n} D(x_n, y_n)$. The topology on $M$ that is inherited from $(\boldsymbol{R}_\mathrm{D})^\omega$ witnesses the almost zero-dimensionality of $(M, \rho)$. It is easy to see that $(M, \rho)$ is nonseparable and complete. It is also an easy exercise to check that this space is not zero-dimensional, by a variation on the Erdős argument. But it is not obvious that this space can be represented in the form $\mathscr{E}_\mu$.

Proposition 40. $(M, \rho) \approx \mathfrak{E}^1_\mathrm{c}(\mathfrak{c}, \mathfrak{c})$.

Proof. We shall show that $(M, \rho)$ is homeomorphic to $\mathfrak{E}'_\mathrm{c}(\mathfrak{c})$. Note that the norm on $\mathfrak{E}'_\mathrm{c}(\mathfrak{c})_{\{n\}\times\mathfrak{c}}$ assumes only the values $k2^{-n}$ for $k \in \omega$. For every $k \in \omega$, both the set

$$\big[-(k+1)2^{-n}, -k2^{-n}\big) \cup \big(k2^{-n}, (k+1)2^{-n}\big] \text{ and}$$
$$\big\{x \in \mathfrak{E}'_{\mathrm{c}}(\mathfrak{c})_{\{n\}\times\mathfrak{c}} : \|x\|_1 = (k+1)2^{-n}\big\}$$

have cardinality $\mathfrak{c}$. Let for $n \in \omega$, $h_n : \boldsymbol{R}_{\mathrm{D}} \to \mathfrak{E}'_{\mathrm{c}}(\mathfrak{c})_{\{n\}\times\mathfrak{c}}$ be a bijection that satisfies the properties $h_n(0) = \mathbf{0}$ and for every $k \in \omega$,

$$\begin{aligned} &h_n\big([-(k+1)2^{-n}, -k2^{-n}) \cup (k2^{-n}, (k+1)2^{-n}]\big) \\ &\quad = \big\{x \in \mathfrak{E}'_{\mathrm{c}}(\mathfrak{c})_{\{n\}\times\mathfrak{c}} : \|x\|_1 = (k+1)2^{-n}\big\}. \end{aligned}$$

Then $h_n$ is a homeomorphism because both domain and co-domain are discrete. An important property of $h_n$ is that for every $x \in \boldsymbol{R}_{\mathrm{D}}$ we have

$$|x| \le \|h_n(x)\|_1 < |x| + 2^{-n}. \tag{2}$$

Next, consider the homeomorphism

$$H : (\boldsymbol{R}_{\mathrm{D}})^\omega \to \prod_{n\in\omega} \mathfrak{E}'_{\mathrm{c}}(\mathfrak{c})_{\{n\}\times\mathfrak{c}},$$

given by $H(x)_n = h_n(x_n)$ for $x \in (\boldsymbol{R}_{\mathrm{D}})^\omega$ and $n \in \omega$. Furthermore, formula (2) implies

$$\|x\|_1 \le \|H(x)\|_1 < \|x\|_1 + \sum_{n=0}^{\infty} 2^{-n} = \|x\|_1 + 2,$$

thus we find that $H(M) = \mathfrak{E}'_{\mathrm{c}}(\mathfrak{c})$. If $x$ and $y$ in $M$ agree on the first $m$ coordinates (that is, if they are elements of a standard basic open set of the witness topology inherited from $(\boldsymbol{R}_{\mathrm{D}})^\omega$), then we find the following estimate

$$\begin{aligned} &\big|\big(\|H(x)\|_1 - \|H(y)\|_1\big) - \big(\|x\|_1 - \|y\|_1\big)\big| \\ &\quad = \left| \sum_{n=m}^{\infty} \big((\|h_n(x_n)\|_1 - \|h_n(y_n)\|_1) - (|x_n| - |y_n|)\big) \right| \\ &\quad \le \sum_{n=m}^{\infty} \big(\|h_n(x_n)\|_1 - |x_n| + \|h_n(y_n)\|_1 - |y_n|\big) \\ &\quad \le \sum_{n=m}^{\infty} 2^{-n+1} = 2^{-m+2}. \end{aligned} \tag{3}$$

Let $\varepsilon > 0$ and $x \in M$. Now, if $m$ is such that $2^{-m+2} < \varepsilon$, then

$$H\big(\{y \in M : y_n = x_n,\ n < m \text{ and } |\|x\|_1 - \|y\|_1| < \varepsilon - 2^{-m+2}\}\big) \subset \big\{H(y) :\ |\|H(x)\|_1 - \|H(y)\|_1| < \varepsilon\big\}.$$

This proves that $\|H(x)\|_1$ is continuous in $x \in M$. By the Kadec property (see Remark 4) we now have that $H\restriction(M,\rho)\colon (M,\rho) \to \mathfrak{E}'_{\mathrm{c}}(\mathfrak{c})$ is continuous. By a similar argument formula (3) also guarantees that the inverse of $H\restriction M$ is continuous. We conclude that $(M,\rho) \approx \mathfrak{E}'_{\mathrm{c}}(\mathfrak{c}) \approx \mathfrak{E}^1_{\mathrm{c}}(\mathfrak{c},\mathfrak{c})$. □

Using an argument that is similar to the proof of Proposition 40 we are now able to link every $\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)$ directly to the separable complete Erdős space.

Theorem 41. *Let $\omega \le \lambda \le \kappa$. Then $\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega \times \kappa_{\mathrm{D}}$.*

Proof. By Remarks 20 and 36 it suffices to prove that $\mathfrak{E}'_{\mathrm{c}}(\lambda)$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega$. Let $\mathscr{E}_\omega \subset \ell^1$ be determined by $E_n = \{k2^{-n} : k \in \omega\}$ for $n \in \omega$. According to Theorems 8 and 16 we have $\mathscr{E}_\omega \approx \mathfrak{E}_{\mathrm{c}}$.

In analogy to the preceding proof, let for $n \in \omega$, $h_n : E_n \times \lambda_{\mathrm{D}} \to \mathfrak{E}'_{\mathrm{c}}(\lambda)_{\{n\}\times\lambda}$ be a bijection that satisfies the properties $h_n(0,0) = \mathbf{0}$ and for every $k \in \omega$,

$$h_n\big((\{k2^{-n}\} \times (\lambda \setminus \{0\})) \cup \{((k+1)2^{-n}, 0)\}\big) = \big\{x \in \mathfrak{E}'_{\mathrm{c}}(\lambda)_{\{n\}\times\lambda} : \|x\|_1 = (k+1)2^{-n}\big\}.$$

Again, $h_n$ is a homeomorphism because both domain and co-domain are discrete and we clearly have that for every $(t,\alpha) \in E_n \times \lambda_{\mathrm{D}}$,

$$t \le \|h_n(t,\alpha)\|_1 \le t + 2^{-n}.$$

Next, consider the homeomorphism

$$H : \left(\prod_{n\in\omega} E_n\right) \times (\lambda_{\mathrm{D}})^\omega \to \prod_{n\in\omega} \mathfrak{E}'_{\mathrm{c}}(\lambda)_{\{n\}\times\lambda},$$

given by $H(x,y)_n = h_n(x_n,y_n)$ for $x \in \prod_{n\in\omega} E_n$, $y \in (\lambda_{\mathrm{D}})^\omega$, and $n \in \omega$. The same argument as employed in the proof of Proposition 40 now shows that $H\restriction(\mathscr{E}_\omega \times (\lambda_{\mathrm{D}})^\omega)$ is a homeomorphism between $\mathscr{E}_\omega \times (\lambda_{\mathrm{D}})^\omega$ and $\mathfrak{E}'_{\mathrm{c}}(\lambda)$. □

Now we can formulate Theorem 32 as follows.

THEOREM 42. *Let $lw(\mathscr{E}_\mu) = \lambda$ and $w(\mathscr{E}_\mu) = \kappa$. We have $\mathscr{E}_\mu \approx \mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega \times \kappa_{\mathrm{D}}$ if and only if every $E_\alpha$ is a zero-dimensional $G_\delta$-subset of $\boldsymbol{R}$ and* $\operatorname{ind}\mathscr{E}_\mu > 0$.

Note that Theorem 42 contains Theorem 2.

Let $p$ be a point in a space $X$. We say that *$p$ is a fixed point of $X$* if for every nonconstant continuous function $f\colon X \to X$ we have $f(p) = p$. It is clear that if a space contains a fixed point, then it has the *fixed point property*, that is, for each continuous $f\colon X \to X$ there is an $x \in X$ with $f(x) = x$.

Let the space $\mathscr{E}_\mu^+ = \mathscr{E}_\mu \cup \{\infty\}$ be a Hausdorff extension of $\mathscr{E}$ such that for every neighbourhood $U$ of $\infty$ in $\mathscr{E}_\mu^+$ we have that $\mathscr{E}_\mu \setminus U$ is bounded with respect to the $p$-norm. The following result was proved for the case $\mu = \omega$ by Abry, Dijkstra, and van Mill [**1**].

THEOREM 43. *If every $E_\alpha$ is zero-dimensional, then the following statements about $\mathscr{E}_\mu^+$ are equivalent*:

(1) *$\infty$ is a fixed point of $\mathscr{E}_\mu^+$*;
(2) *$\mathscr{E}_\mu^+$ has the fixed point property*;
(3) *$\mathscr{E}_\mu^+$ is connected*; *and*
(4) $\operatorname{ind}\mathscr{E}_\mu \neq 0$.

PROOF. The implications (1) ⇒ (2) ⇒ (3) are trivial.

(3) ⇒ (4). Assume that $\operatorname{ind}\mathscr{E}_\mu = 0$ and select an $a \in \mathscr{E}_\mu$. Let $U$ and $V$ be disjoint and open in $\mathscr{E}_\mu^+$ such that $a \in U$ and $\infty \in V$. Choose a clopen neighbourhood $C$ of $a$ in $\mathscr{E}_\mu$ such that $C \subset U$ and note that $C$ is also clopen in $\mathscr{E}_\mu^+$. Thus $\mathscr{E}_\mu^+$ is disconnected.

(4) ⇒ (1). Assume that $\operatorname{ind}\mathscr{E}_\mu \neq 0$. Since every $E_\alpha$ is zero-dimensional we have that $\mathscr{E}_\mu$ is totally disconnected. Let $U$ be an arbitrary open neighbourhood of $\infty$ in $\mathscr{E}_\mu^+$ such that $A = \mathscr{E}_\mu^+ \setminus U \neq \emptyset$. Let $C$ be the component of $\infty$ in $U$. According to [**1**, Lemma 14] it suffices to show that $C$ is not closed in the space. By Theorem 1 there is an $x \in \prod_{\alpha\in\mu} E_\alpha$ with $\|x\|_p = \infty$ and $\lim_{\alpha\in\mu} x_\alpha = 0$. Select an $a \in A$ and let $I$ be the countable set $\{\alpha \in \mu : x_\alpha \neq 0 \text{ or } a_\alpha \neq 0\}$. Put

$$U' = \big(U \cap (\mathscr{E}_I \times \{\mathbf{0}_{\mu\setminus I}\})\big) \cup \{\infty\}$$

and let $C'$ be the component of $\infty$ in $U'$. According to the proof of [**1**, Theorem 16] the closure of $C'$ intersects $A$. Since $C'$ is a subset of $C$ we have that $C$ is not closed in $\mathscr{E}_\mu^+$ and the proof is complete. □

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