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Logarithmic singularities of solutions to nonlinear partial differential equations

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Abstract. We construct a family of singular solutions to some nonlinear partial differential equations which have resonances in the sense of a paper due to T. Kobayashi. The leading term of a solution in our family contains a logarithm, possibly multiplied by a monomial. As an application, we study nonlinear wave equations with quadratic nonlinearities. The proof is done by the reduction to a Fuchsian equation with singular coefficients.

Introduction.

In this paper, we study singular solutions to nonlinear partial differential equations with holomorphic (or real-analytic) coefficients. The solutions to be constructed shall be singular along a noncharacteristic hypersurface. This phenomenon presents a striking contrast to the corresponding results of the linear theory. Probably, the most well-known example in this direction is the KdV equation:

$$u_{ttt} - 6uu_t + u_x = 0 \quad (t, x \in \boldsymbol{C}). \tag{0.1}$$

Although the surface $t = 0$ is noncharacteristic, the equation (0.1) has solutions of the form

$$u = \frac{2}{t^2} + gt^2 + ht^4 - \frac{1}{24} g_x t^5 + \cdots,$$

where $g = g(x)$ and $h = h(x)$ are arbitrary functions. Note that many solutions have been obtained in the form of Laurent series for some integrable PDEs (a

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useful reference is [**1**]).

In [**7**], Kichenassamy and Srinivasan introduced an expansion of a generalized form in order to solve PDEs with polynomial nonlinearities. In their paper, the solutions behave asymptotically

$$u(t,x) \sim u_0(x)t^{\nu} \quad (\text{as } t \to 0),$$

where ν is a rational number. The remainder term may contain logarithms. Besides this general result, specific cases are dealt with in [**5**], [**6**] and [**8**]: in these papers, it is proved that the Liouville equation $\Box u = e^u$ and Einstein's vacuum equations admit solutions led by logarithmic terms.

On the other hand, in [**9**], Kobayashi considered a certain kind of nonlinear PDEs, mainly those with polynomial nonlinearities, and constructed solutions of the form

$$u(t,x) = t^{\sigma_c} \sum_{k=0}^{\infty} u_k(x) t^{k/p},$$

where $\sigma_c \in \boldsymbol{Q}, p \in \boldsymbol{N}^* = \{1,2,\dots\}$. The exponent σ_c is determined by the nonlinear terms of the equation and Kobayashi imposed a kind of generalized non-resonance condition on it. In particular, he assumed $\sigma_c \neq 0,1,2,\dots,m-2$, where m is the order of the equation. One has $\sigma_c = -2$ for the KdV equation (0.1) and it is within the framework of his result. In the present paper, the authors shall deal with the excluded cases and construct solutions with a logarithm in the leading term. If $\sigma_c = l \in \{0,1,2,\dots,m-2\}$, then the asymptotic behavior of the solutions is

$$u(t,x) \sim a(x)t^l \log t \quad (\text{as } t \to 0) \tag{0.2}$$

and the remainder term involves an arbitrary holomorphic function in x. Note that the case where $\sigma_c = 0$ has already been treated in [**14**] in a different formulation. This result about nonlinear wave equations shall be improved in Part 1, Section 3.

Note that Tahara extended Kobayashi's result for first-order equations with entire nonlinearities in [**11**] and [**12**].

All the above mentioned authors employ the method of Fuchsian Reduction: the leading terms can be found by formal calculation and the remainder terms are obtained by solving nonlinear Fuchsian equations. In the present paper, we generalize this method in the sense that we employ Fuchsian equations with singular coefficients.

The organization of the present paper is as follows: In Part 1, we shall study an equation of the form

$$\partial_t^m u = f\big(t, x, (\partial_t^j \partial_x^\alpha u)\big),$$

where f is holomorphic (real-analytic) in its arguments, and construct solutions with the asymptotic behavior $u(t,x) \sim a(x)t^l \log t$, $l \in \{0, 1, \ldots, m-2\}$. We shall explain how this equation is reduced to a Fuchsian equation with singular coefficients:

$$(t\partial_t)^m u = F\big(t, x, ((t\partial_t)^j \partial_x^\alpha u)\big).$$

Here $F(t,x,Z)$ is singular at $t = 0$. The latter equation shall be solved in Part 2 by using the techniques developed in [**13**].

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Part 1. Logarithmic singularities.

1. Main result.

Let $(t,x) = (t, x_1, \ldots, x_n) \in \boldsymbol{C} \times \boldsymbol{C}^n$, fix $m \in \boldsymbol{N}^*$ and set: $I_m = \{(j,\alpha) \in \boldsymbol{N} \times \boldsymbol{N}^n\,;\, j + |\alpha| \le m \text{ and } j < m\}$, $N =$ the cardinal of I_m, and $U = (U_{j,\alpha})_{(j,\alpha)\in I_m} \in \boldsymbol{C}^N$. We set $\partial_t = \partial/\partial t$, $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$.

We study nonlinear PDEs of the form

$$\partial_t^m u = f\big(t, x, (\partial_t^j \partial_x^\alpha u)_{(j,\alpha)\in I_m}\big). \tag{1.1}$$

Here $f(t,x,U)$ is holomorphic in $\{(t,x) \in \boldsymbol{C}_t \times \boldsymbol{C}_x^n; |t| < r_0, |x| < R_0\} \times \boldsymbol{C}_U^N$, where r_0 and R_0 are positive constants. Note that Kobayashi assumed that f was a polynomial in U. We shall deal with several infinite sums closely related to f. Their convergence follows from the fact that $f(t,x,U)$ is entire in U (see Proposition 2.3 for example).

We may write

$$f(t,x,U) = \sum_{\mu\in\mathscr{M}} f_\mu(t,x) U^\mu, \quad \mu = (\mu_{j,\alpha})_{(j,\alpha)\in I_m}, \quad U^\mu = \prod_{(j,\alpha)\in I_m} U_{j,\alpha}^{\mu_{j,\alpha}}$$

for some subset $\mathscr{M}$ of $\boldsymbol{N}^N$, the set of $\boldsymbol{N}$-valued functions on I_m. We assume that $f_\mu(t,x)$ does not vanish identically if $\mu \in \mathscr{M}$ (then $\mathscr{M}$ is unique).

We expand $f_\mu(t,x)$ in t:

$$f_\mu(t,x) = t^{k_\mu} \sum_{k=0}^{\infty} f_{\mu,k}(x) t^k.$$

We assume that $f_{\mu,0}(x)$ does not vanish identically. Summing up, we have

$$f(t,x,U) = \sum_{\mu \in \mathscr{M}} f_\mu(t,x) U^\mu = \sum_{\mu \in \mathscr{M}} \left(t^{k_\mu} \sum_{k=0}^{\infty} f_{\mu,k}(x) t^k \right) U^\mu. \tag{1.2}$$

We set

$$|\mu| = \sum_{(j,\alpha) \in I_m} \mu_{j,\alpha}, \quad \gamma(\mu) = \sum_{(j,\alpha) \in I_m} j \mu_{j,\alpha}.$$

Lemma 1.1. *For any integer l, we have*

$$t^{\gamma(\mu) - l|\mu|} U^\mu = t^{\gamma(\mu) - l|\mu|} \prod_{(j,\alpha) \in I_m} U_{j,\alpha}^{\mu_{j,\alpha}} = \prod_{(j,\alpha) \in I_m} (t^{j-l} U_{j,\alpha})^{\mu_{j,\alpha}}.$$

We assume the following:

(A0) $\displaystyle\sup_{\mu \in \mathscr{M}, |\mu| \geq 2} \frac{\gamma(\mu) - m - k_\mu}{|\mu| - 1} = l \in \{0, 1, 2, \ldots, m-2\}.$

Note that the left-hand side is the "characteristic exponent"*[1] σ_c in the terminology of [**9**]. (It is proved in [**9**] that $\sigma_c < m-1$ holds if the supremum is attained by some μ.)

(A1) $\mathscr{M}_0 \underset{\text{def}}{=} \{\mu \in \mathscr{M}; |\mu| \geq 2, (\gamma(\mu) - m - k_\mu)/(|\mu| - 1) = l\}$ is non-empty: i.e. the supremum in (A0) is attained.

(A2) If $\mu \in \mathscr{M}_0$ and $\mu_{j,\alpha} \neq 0$, then $j \geq l+1$ and $\alpha = 0$.

(A3) For a sufficiently small positive constant $C > 0$, we have

$$m - l + k_\mu - \gamma(\mu) + l|\mu| \geq C \sum_{\substack{(j,\alpha) \in I_m \\ j \leq l}} \mu_{j,\alpha}$$

*[1]It should not be confused with characteristic exponents of a Fuchsian equation. See [**13**].

for any $\mu \in \mathscr{M} \setminus \mathscr{M}_0$. (This is trivial if f is a polynomial by Lemma 2.1 below.)

Example 1.2. The prototype is the ODE

$$u^{(m)} = t^k \{u^{(m-1)}\}^2, \; l = m - k - 2.$$

This equation is satisfied if $u^{(m-1)} = C_1 t^{-k-1}$, where C_1 is a suitable constant. In this case, we have $u \sim C_2 t^l \log t$ as $t \to 0$, where C_2 is another constant.

We shall construct solutions to (1.1) which behave like (0.2). In order to give a precise statement, we introduce a function class $\widetilde{\mathscr{O}}_+$.

We use the following notation:

- $\mathscr{R}(\boldsymbol{C} \setminus \{0\})$, the universal covering space of $\boldsymbol{C} \setminus \{0\}$,
- $S_\theta = \{t \in \mathscr{R}(\boldsymbol{C} \setminus \{0\}) \; ; \; |\arg t| < \theta\}$, a sector in $\mathscr{R}(\boldsymbol{C} \setminus \{0\})$,
- $S(\varepsilon(y)) = \{t \in \mathscr{R}(\boldsymbol{C} \setminus \{0\}); 0 < |t| < \varepsilon(\arg t)\}$, where $\varepsilon(y)$ is a positive continuous function on $\boldsymbol{R}_y$,
- $D_r = \{x = (x_1, \ldots, x_n) \in \boldsymbol{C}^n \; ; \; |x_i| < r \text{ for } i = 1, \ldots, n\}$.

Definition 1.3. $\widetilde{\mathscr{O}}_+$ denotes the set of all $v(t, x)$ satisfying the following two conditions:

i) $v(t, x)$ is a holomorphic function on $S(\varepsilon(y)) \times D_r$ for some positive continuous function $\varepsilon(y)$ on $\boldsymbol{R}_y$ and some constant $r > 0$.

ii) There exists a constant $a > 0$ such that for any $\tilde{r} \in]0, r[$ and $\theta > 0$ we have

$$\max_{x \in D_{\tilde{r}}} |v(t, x)| = O(|t|^a) \quad (\text{as } t \longrightarrow 0 \text{ in } S_\theta).$$

Remark 1.4. We can replace $\varepsilon(y)$ by another function with smaller values. Therefore we have only to consider those functions with some decreasing property. See Section 5 for details.

Our main result is the following:

Theorem 1.5. *Assume* (A0)–(A3) *and set* $\beta_{j,l} = (-1)^{j-l-1} l!(j-l-1)!$ *for* $j \geq l+1$. *Let* $A = a(x)$ *be a solution to*

$$\sum_{\mu \in \mathscr{M}_0} f_{\mu,0}(x) \left(\prod_{j=l+1}^{m-1} \beta_{j,l}^{\mu_{j,0}} \right) A^{|\mu|-1} = \beta_{m,l}. \tag{1.3}$$

Then, for any holomorphic function $b(x)$ *in a neighborhood of* $x = 0$, *there exists* *a function* $v(t,x) \in \widetilde{\mathscr{O}}_+$ *such that*

$$\begin{aligned} u(t,x) &= a(x)t^l \log t + t^l b(x) + t^l v(t,x) \\ &= t^l\{a(x)\log t + b(x) + v(t,x)\} \end{aligned}$$

is a solution to (1.1).

The convergence of the sum in the left-hand side of (1.3) shall be proved in Proposition 2.3 and Theorem 1.5 itself shall be proved in Section 2.

Note that the possibilities are $0 \le l \le m-2$ by (A0) and we give examples of the cases where

- $(m,l) = (m, m-k-2)$, quadratic nonlinearity
- $(m,l) = (3,0), (3,1), (4,0), (4,1)$, cubic or higher-order nonlinearity
- $(m,l) = (4,2)$, quadratic or higher-order nonlinearity.

In all of them, the set $\mathscr{M}_0$ in (A1) consists of a single element and we denote it by μ. It attains the supremum in (A0). See Theorem 3.1 for an example of the case $(m,l) = (2,0)$.

Example 1.6. $[l = m-k-2;\ |\mu| = 2, \gamma(\mu) = 2(m-1)]$

$$\partial_t^m u = t^k(\partial_t^{m-1}u)^2 + t^{k'}(\partial_t^j \partial_x^\alpha u)^2,$$

where $j \le m-1$, $|\alpha| \le m-j$, $2(m-1)-k \ge 2j-k'$. This is just a PDE version of the ODE explained in Example 1.2. The second term on the right-hand side is a perturbation.

Example 1.7. $[m=3,\ l=0;\ \gamma(\mu) = 3+k]$

$$\partial_t^3 u = t^k(\partial_t^2 u)^a(\partial_t u)^{3+k-2a}u^b + t^{k'}(\partial_t^2 u)^p(\partial_t u)^q(\partial_x^\alpha u)^r,$$

where $2p+q-k'-3 < 0$, $|\alpha| \le 3$. If the numerator $\gamma(\mu)-m-k_\mu$ vanishes in (A0), the ratio also does and $|\mu|$ in the denominator is irrelevant. It makes it easy to construct examples of the case $l=0$. Because of the irrelevance, we omit the entry of $|\mu|$ between the brackets [ ].

Example 1.8. $[m=3,\ l=1;\ |\mu|=3,\ \gamma(\mu)=6]$

$$\partial_t^3 u = t(\partial_t^2 u)^3 + t^k(\partial_t^j \partial_x^\alpha u)^3,$$

where $3j-k \leq 4$, $|\alpha| \leq 3-j$.

EXAMPLE 1.9. $[m=4,\ l=0;\ \gamma(\mu)=k+4]$

$$\partial_t^4 u = t^k(\partial_t^3 u)^a(\partial_t^2 u)^b(\partial_t u)^{4+k-3a-2b}u^c + t^{k'}(\partial_t^3 u)^p(\partial_t^2 u)^q(\partial_t u)^r(\partial_x^\alpha u)^s,$$

where $3p+2q+r-k'-4<0$, $|\alpha| \leq 4$. Here $|\mu|$ is irrelevant as in Example 1.7 and its entry is omitted.

EXAMPLE 1.10. $[m=4,\ l=1;\ |\mu|=3,\ \gamma(\mu)=6]$

$$\partial_t^4 u = (\partial_t^3 u)^2 u + \partial_t^2 u \cdot \partial_t u \cdot \partial_x^\alpha u,$$

where $|\alpha| \leq 4$.

EXAMPLE 1.11. $[m=4,\ l=2;\ |\mu|=k+2,\ \gamma(\mu)=3(k+2)]$

$$\partial_t^4 u = t^k(\partial_t^3 u)^{k+2} + t^{k'}(\partial_t^j \partial_x^\alpha u)^{k+2},$$

where $k' \geq k$, $0 \leq j \leq 2$, $|\alpha| \leq 4-j$.

REMARK 1.12. We constructed solutions with the growth order $|u| = O(|t^l \log t|)$ under the conditions (A0)–(A3). If u is a solution with the growth order $|u| = O(|t^{l'} \log t|)$, $l' > l$, then we have $|u| = O(|t^{(l+l')/2}|)$, $(l+l')/2 > l$ and a result in [**9**] implies that it can be extended as a holomorphic solution up to some neighborhood of the origin.

2. Reduction to a Fuchsian equation.

In this section, we reduce the equation (1.1) to a Fuchsian equation with singular coefficients. The latter shall be the topic of Part 2.

The assumption (A0) is equivalent to the following:

$$m-l = \sup_{\mu \in \mathscr{M}, |\mu| \geq 2} (\gamma(\mu) - l|\mu| - k_\mu).$$

Then we have

$$m-l = \sup_{\mu \in \mathscr{M}} (\gamma(\mu) - l|\mu| - k_\mu), \tag{2.1}$$

because $\gamma(\mu) - l|\mu| - k_\mu$ is smaller than $m-l$ if $|\mu| = 0, 1$. Therefore, LEMMA 2.1. *We have*

$$m - l + k_\mu = \gamma(\mu) - l|\mu| \quad \textit{if} \quad \mu \in \mathscr{M}_0,$$
$$m - l + k_\mu > \gamma(\mu) - l|\mu| \quad \textit{if} \quad \mu \in \mathscr{M} \setminus \mathscr{M}_0.$$

REMARK 2.2. Following [**9**], we say that the term $t^{k_\mu + k} f_{\mu,k}(x)(\partial_t^j \partial_x^\alpha u)^{\mu_{j,\alpha}}$ has weight $(k_\mu + k) - \gamma(\mu) + l|\mu|$. Lemma 2.1 means that $\mathscr{M}_0$ consists of those μ's that corresponds to the terms of the smallest weight, which is $-m + l$.

PROPOSITION 2.3. *The sum in the left-hand side of* (1.3) *is an entire function in $A \in \boldsymbol{C}$ and hence has at most one exceptional value in the sense of Picard's theorem in the value distribution theory of complex analysis.*

PROOF. By Lemma 2.1, we have

$$\sum_{\mu \in \mathscr{M}_0} f_{\mu,0}(x) \prod_{j=l+1}^{m-1} W_{j,0}^{\mu_{j,0}} = r^{m-l} \sum_{\mu \in \mathscr{M}_0} f_{\mu,0}(x) r^{k_\mu} \prod_{j=l+1}^{m-1} (W_{j,0}/r^{j-l})^{\mu_{j,0}} \tag{2.2}$$

for any $W = (W_{j,\alpha})_{(j,\alpha)}$ and $r > 0$. Except for the factor r^{m-l}, the sum in the right-hand side is nothing but a partial sum of (1.2), evaluated at $t = r, U = W/r^{j-l}$ by (A2). Hence it is convergent if $r > 0$ is sufficiently small and so is the left-hand side. Set $W_{j,0} = \beta_{j,l} A$, then we have

$$\sum_{\mu \in \mathscr{M}_0} f_{\mu,0}(x) \prod_{j=l+1}^{m-1} W_{j,0}^{\mu_{j,0}} = \sum_{\mu \in \mathscr{M}_0} f_{\mu,0}(x) \biggl(\prod_{j=l+1}^{m-1} \beta_{j,l}^{\mu_{j,0}} \biggr) A^{|\mu|}.$$

This sum is convergent for any A. □

We set

$$[\rho; j] = \Gamma(\rho + 1)/\Gamma(\rho - j + 1) = \rho(\rho - 1)(\rho - 2) \cdots (\rho - j + 1).$$

Here Γ denotes the Gamma function. Note that $[\rho, 0] = 1$ and that $[\rho; j] = 0$ if $j - \rho$ is a positive integer. We define the sequence $\{b_{j,l}\}_j$ by

$$b_{0,l} = 0, \ b_{j+1,l} = [l; j] + (l - j) b_{j,l}.$$

Then we have

LEMMA 2.4.

$$\partial_t^j(t^l\log t)=\begin{cases} t^{l-j}\{[l;j]\log t+b_{j,l}\} & (j\le l),\\ \beta_{j,l}t^{-(j-l)} & (j\ge l+1).\end{cases}$$

We introduce new unknown functions $a(x)$ and $v(t,x)$ by setting

$$u(t,x)=a(x)t^l\log t+t^lb(x)+t^lv(t,x).$$

Here the function $b(x)$ is arbitrary.

By using $t^{j-l}\partial_t^jt^l=[t\partial_t+l;j]$ and $t^l\in\operatorname{Ker}\partial_t^{l+1}$, we obtain

LEMMA 2.5.

$$t^{j-l}\partial_t^j\partial_x^\alpha u=\begin{cases} \partial_x^\alpha a(x)\{[l;j]\log t+b_{j,l}\}+[t\partial_t+l;j]\partial_x^\alpha(b+v) & (j\le l),\\ \beta_{j,l}\partial_x^\alpha a(x)+[t\partial_t+l;j]\partial_x^\alpha v & (j\ge l+1).\end{cases}$$

Let us calculate the right-hand side of (1.1). We have

$$f\big(t,x,(\partial_t^j\partial_x^\alpha u)_{(j,\alpha)\in I_m}\big)=\sum_{\mu\in\mathscr{M}}\left(\sum_{k=0}^{\infty}f_{\mu,k}(x)t^{k_\mu+k}\right)\prod_{(j,\alpha)\in I_m}(\partial_t^j\partial_x^\alpha u)^{\mu_{j,\alpha}}.$$

We extract the terms of the smallest weight and set

$$S=\sum_{\mu\in\mathscr{M}_0}f_{\mu,0}(x)t^{k_\mu}\prod_{(j,\alpha)\in I_m}(\partial_t^j\partial_x^\alpha u)^{\mu_{j,\alpha}}.$$

Then by (A2), it can be written in a simpler form

$$S=\sum_{\mu\in\mathscr{M}_0}f_{\mu,0}(x)t^{k_\mu}\prod_{j=l+1}^{m-1}(\partial_t^j u)^{\mu_{j,0}}$$

and it is free of logarithms. The sum $\sum_{\mu\in\mathscr{M}_0}f_{\mu,0}(x)t^{k_\mu}\prod_{j=l+1}^{m-1}U_{j,0}^{\mu_{j,0}}$ is absolutely convergent because it is a partial sum of $f(t,x,U)$. Therefore S is also absolutely convergent. Note that

$$m-l+k_\mu=\gamma(\mu)-l|\mu|=\sum_{j=l+1}^{m-1}(j-l)\mu_{j,0}$$

for $\mu \in \mathscr{M}_0$. Hence by Lemma 1.1 we have

$$
\begin{aligned}
t^{m-l} S &= \sum_{\mu \in \mathscr{M}_0} f_{\mu,0}(x) \prod_{j=l+1}^{m-1} (t^{j-l} \partial_t^j u)^{\mu_{j,0}} \\
&= \sum_{\mu \in \mathscr{M}_0} f_{\mu,0}(x) \prod_{j=l+1}^{m-1} \big\{ \beta_{j,l} a(x) + [t\partial_t + l; j] v \big\}^{\mu_{j,0}}.
\end{aligned}
$$

This quantity consists of terms of weight 0. All the remaining parts of $t^{m-l} f$ consists of terms of positive weight (the weight of $\log t$ is 0).

By binomial expansion, we obtain

$$
t^{m-l} S = T_0 + T_1 + T_2,
$$

where

$$
\begin{aligned}
T_0 &= \sum_{\mu \in \mathscr{M}_0} f_{\mu,0}(x) \prod_{j=l+1}^{m-1} \{\beta_{j,l} a(x)\}^{\mu_{j,0}} = \sum_{\mu \in \mathscr{M}_0} f_{\mu,0}(x) \Bigg(\prod_{j=l+1}^{m-1} \beta_{j,l}^{\mu_{j,0}} \Bigg) a(x)^{|\mu|}, \\
T_1 &= \sum_{\mu \in \mathscr{M}_0} f_{\mu,0}(x) \sum_{j=l+1}^{m-1} \Bigg(\prod_{i \neq j} \{\beta_{i,l} a(x)\}^{\mu_{i,0}} \Bigg) \mu_{j,0} \{\beta_{j,l} a(x)\}^{\mu_{j,0}-1} [t\partial_t + l; j] v \\
&= \sum_{\mu \in \mathscr{M}_0} f_{\mu,0}(x) \sum_{j=l+1}^{m-1} \Bigg(\prod_{i \neq j} \beta_{i,l}^{\mu_{i,0}} \Bigg) \mu_{j,0} \beta_{j,l}^{\mu_{j,0}-1} a(x)^{|\mu|-1} [t\partial_t + l; j] v,
\end{aligned}
$$

$T_2 =$ a polynomial in $(t\partial_t)^k v$ $(k = 0, 1, \ldots, m-1)$ free of terms of degree ≤ 1.

Note that T_0 is free of v and its derivatives.

On the other hand, we have

$$
t^{m-l} \partial_t^m u = \beta_{m,l} a(x) + [t\partial_t + l; m] v.
$$

We multiply both sides of (1.1) by t^{m-l}. If $a(x)$ is determined by (1.3), the function $u = a t^l \log t + t^l b + t^l v$ is a solution to (1.1) if and only if v is a solution to the equation below:

$$
[t\partial_t + l; m] v - T_1 = t^{m-l} f - T_0 - T_1. \tag{2.3}
$$

Set

$$\delta(\mu) = m - l + k_\mu - \gamma(\mu) + l|\mu|, \qquad |\mu|_l = \sum_{\substack{(j,\alpha)\in I_m \\ j\le l}} \mu_{j,\alpha}.$$

Since $\delta(\mu)$ is an integer, Lemma 2.1 implies that $\delta(\mu) = 0$ for $\mu \in \mathscr{M}_0$ and that $\delta(\mu) \ge 1$ for $\mu \in \mathscr{M} \setminus \mathscr{M}_0$. Moreover, (A3) can be written in a simple form:

$$\delta(\mu) \ge C|\mu|_l, \qquad \mu \in \mathscr{M} \setminus \mathscr{M}_0.$$

These two estimates imply that

$$\delta(\mu) = \frac{\delta(\mu)}{3} + \frac{\delta(\mu)}{3} + \frac{\delta(\mu)}{3} \ge \frac{1}{3} + \frac{\delta(\mu)}{3} + \frac{C|\mu|_l}{3} \tag{2.4}$$

holds for any $\mu \in \mathscr{M} \setminus \mathscr{M}_0$.

By the way, we have

$$\begin{aligned} t^{m-l} f(t,x,U) &= t^{m-l} \sum_{\mu\in\mathscr{M}} \biggl(t^{k_\mu} \sum_{k=0}^{\infty} f_{\mu,k}(x) t^k \biggr) U^\mu \\ &= \sum_{\mu\in\mathscr{M}} \biggl(\sum_{k=0}^{\infty} f_{\mu,k}(x) t^{\delta(\mu)+k} \biggr) \prod_{(j,\alpha)\in I_m} (t^{j-l} U_{j,\alpha})^{\mu_{j,\alpha}}. \end{aligned} \tag{2.5}$$

Lemma 2.6. *Let ρ be an indeterminate. Then $\{[\rho + l; j]\}_{j=0,\dots,m-1}$ is equivalent to $\{\rho^j\}_{j=0,\dots,m-1}$ in the sense that there is a bijective correspondence between these quantities.*

Proof. We introduce two column vectors $\vec{v}$ and $\vec{w}$ by

$$\begin{aligned} \vec{v} &= {}^t([\rho + l; j])_{j=0,\dots,m-1} \\ &= {}^t(1, \rho + l, (\rho + l)(\rho + l - 1), \dots, (\rho + l)(\rho + l - 1) \cdots (\rho + l - m + 1)), \\ \vec{w} &= {}^t(\rho^j)_{j=0,\dots,m-1} = (1, \rho, \rho^2, \dots, \rho^{m-1}). \end{aligned}$$

There exists a lower triangular matrix M such that $\vec{v} = M\vec{w}$. All the diagonal elements of M are 1. Hence M is invertible. □

Proposition 2.7. *The equation* (2.3) *satisfies the conditions* C$_1$) *and* C$_2$) *in Part* 2. *Here our unknown function $v(t,x)$ plays the role of u in Part* 2.

Proof. Set

$$a^{(\alpha)} = \partial_x^\alpha a(x), \quad b^{(\alpha)} = \partial_x^\alpha b(x),$$
$$W_{j,\alpha} = t^{j-l} U_{j,\alpha} - a^{(\alpha)} t^{j-l} \partial_t^j (t^l \log t) - [l; j] b^{(\alpha)},$$

then $W_{j,\alpha}$ corresponds to $t^{j-l}\partial_t^j \partial_x^\alpha (t^l v) = [t\partial_t + l; j]\partial_x^\alpha v(t,x)$. By Lemma 2.6, $\{[t\partial_t + l; j]v\}_{j=0,\dots,m-1}$ is equivalent to $\{(t\partial_t)^j v\}_{j=0,\dots,m-1}$, the latter being used in Part 2.

For brevity, we set

$$\begin{aligned}\widetilde{W}_{j,\alpha} &= W_{j,\alpha} + a^{(\alpha)} t^{j-l} \partial_t^j (t^l \log t) + [l; j] b^{(\alpha)} (= t^{j-l} U_{j,\alpha}) \\ &= \begin{cases} W_{j,\alpha} + a^{(\alpha)}\{[l; j]\log t + b_{j,l}\} + [l; j] b^{(\alpha)} & (j \le l), \\ W_{j,\alpha} + \beta_{j,l} a^{(\alpha)} & (j \ge l+1). \end{cases}\end{aligned}$$

Here we have used Lemma 2.4 and the fact that $[l; j] = 0$ if $j \ge l+1$. By (2.5) we have

$$\{t^{m-l} f(t,x,U) - T_0 - T_1\} - T_2 = t^{m-l} f(t,x,U) - t^{m-l} S = I_1 + I_2,$$

where

$$I_1 = \sum_{\mu \in \mathscr{M}_0} \sum_{k \ge 1} f_{\mu,k}(x) t^k \prod_{j=l+1}^{m-1} \widetilde{W}_{j,0}^{\mu_{j,0}},$$
$$I_2 = \sum_{\mu \notin \mathscr{M}_0} \sum_{k=0}^{\infty} f_{\mu,k}(x) t^{\delta(\mu)+k} \prod_{\substack{(j,\alpha) \in I_m \\ j \le l}} \widetilde{W}_{j,\alpha}^{\mu_{j,\alpha}} \prod_{\substack{(j,\alpha) \in I_m \\ j \ge l+1}} \widetilde{W}_{j,\alpha}^{\mu_{j,\alpha}}.$$

The convergence of I_1 can be proved by the method of Proposition 2.3. If $|t| \le r$, then the following estimates holds:

$$|I_1| \le |t| r^{m-l-1} \sum_{\mu \in \mathscr{M}_0} \sum_{k \ge 1} |f_{\mu,k}(x)| r^{k_\mu + k} \prod_{j=l+1}^{m-1} \big(|\widetilde{W}_{j,0}|/r^{j-l}\big)^{\mu_{j,0}}.$$

We see that I_1 and its derivatives in W are of order $O(|t|)$.

Next let us consider I_2. The trivial fact $\delta(\mu) > 0, \mu \notin \mathscr{M}_0$ helps, to be sure, but it is not good enough. If $j \le l$, the quantity $\widetilde{W}_{j,\alpha}$ contains a logarithm, whose unboundedness is the greatest obstacle. We overcome it by assuming (A3). The trick is the following fact: if $C > 0$, then $t^C \widetilde{W}_{j,\alpha}$ is bounded as $t \to 0$.

If $|t| < r < 1$, the inequality (2.4) implies that

$$|t^{\delta(\mu)}| \leq |t|^{1/3} \times (r^{1/3})^{\delta(\mu)} \times (|t|^{C/3})^{|\mu|_l}$$
$$= |t|^{1/3} r^{(m-l+k_\mu)/3} \times \prod_{\substack{(j,\alpha)\in I_m \\ j \leq l}} \left(\frac{|t|^{C/3}}{r^{(j-l)/3}} \right)^{\mu_{j,\alpha}} \times \prod_{\substack{(j,\alpha)\in I_m \\ j \geq l+1}} \left(\frac{1}{r^{(j-l)/3}} \right)^{\mu_{j,\alpha}}.$$

Therefore

$$|I_2| \leq |t|^{1/3} \sum_{\mu \notin \mathscr{M}_0} \sum_{k=0}^{\infty} |f_{\mu,k}(x)| (r^{1/3})^{m-l+k_\mu} |t|^k$$
$$\times \prod_{\substack{(j,\alpha)\in I_m \\ j \leq l}} \left(r^{(l-j)/3} |t|^{C/3} |\widetilde{W}_{j,\alpha}| \right)^{\mu_{j,\alpha}} \times \prod_{\substack{(j,\alpha)\in I_m \\ j \geq l+1}} \left(r^{(l-j)/3} |\widetilde{W}_{j,\alpha}| \right)^{\mu_{j,\alpha}}.$$

If $r > 0$ is sufficiently small, then I_2 is convergent in $|t| < r < r^{1/3}$. We see that I_2 and its derivatives in W are of order $O(|t|^{1/3})$. □

End of Proof of Theorem 1.5. Theorem 1.5 follows from Proposition 2.7 and Theorem 4.1 of Part 2.

Remark 2.8. An arbitrary function $b(x)$ is contained in the family of solutions in Theorem 1.5. In some cases, there may be more: as is stated in Remark 4.2 of Part 2, the equation (2.3) may admit a family of solutions involving one or more arbitrary functions in x.

3. Nonlinear wave equation.

We can relax the condition imposed in [**14**]. Although we formulate our result in the complex domain, it is trivial that an analogous result holds in the real-analytic category.

We consider

$$\Box u(s,y) = g(s,y;u,\partial_s u,\nabla_y u) \tag{3.1}$$

in an open set of $\boldsymbol{C}^{n+1} = \boldsymbol{C}_s \times \boldsymbol{C}_y^n$. Here $\Box = \partial^2/\partial s^2 - \sum_{i=1}^n \partial^2/\partial y_i^2$, $\nabla_y u = (\partial u/\partial y_1, \ldots, \partial u/\partial y_n)$. We assume that $g(s,y;z,\sigma,\eta)$ is a holomorphic function in all its arguments and is entire in (z,σ,η). Moreover we assume that it is a polynomial of degree 2 in (σ, η). Its homogeneous part of degree 2 is denoted by g_2.

Let $\psi(y)$ be a holomorphic function with

$$1 - \{\nabla_y \psi(y)\}^2 \neq 0, \tag{3.2}$$

where $\nabla_y \psi(y) = \big(\psi_1(y), \dots, \psi_n(y)\big)$, $\psi_i(y) = \partial\psi(y)/\partial y_i$ $(i = 1, 2, \dots, n)$, $\{\nabla_y \psi(y)\}^2 = \sum_{i=1}^n \psi_i(y)^2$. Moreover we assume that

$$g_2\big(\psi(y), y; 0, 1, -\nabla_y \psi(y)\big) \neq 0. \tag{3.3}$$

Note that this assumption corresponds to $k_\mu = 0$, where k_μ is as in Section 1.

THEOREM 3.1. *Assume* (3.2) *and* (3.3). *Then, in a neighborhood of the hypersurface* $\Sigma = \{s = \psi(y)\}$, *there exists a family of solutions* $u(s, y)$ *to* (3.1) *with the asymptotic behavior*

$$u(s, y) \sim -\frac{1 - \{\nabla_y \psi(y)\}^2}{g_2\big(\psi(y), y; 0, 1, -\nabla_y \psi(y)\big)} \log\big(s - \psi(y)\big) \quad \text{as} \quad s \to \psi(y).$$

This asymptotic expansion is uniform when y *remains on a compact set and* $\arg(s - \psi(y))$ *is bounded. The remainder term involves an arbitrary holomorphic function on* Σ.

PROOF. Set

$$t = s - \psi(y), \quad x = y, \quad \Psi = 1 - \{\nabla_y \psi(y)\}^2 (\neq 0).$$

Then, as is proved in [**5**], we have

$$\partial_s = \partial_t, \quad \partial_{y_i} = -\psi_i \partial_t + \partial_{x_i},$$
$$\Box = \Box_{s,y} = \Psi \partial_t^2 + 2\sum_{i=1}^n \psi_i \partial_{x_i} \partial_t + (\Delta\psi)\partial_t - \Delta_x.$$

In a neighborhood of $t = s - \psi(y) = 0$, the original equation (3.1) becomes

$$\partial_t^2 u = (\text{linear part}) + \Psi^{-1} g\big(t + \psi(x), x; u, \partial_t u, \big((-\psi_i \partial_t + \partial_{x_i})u\big)_{i=1,\dots,n}\big).$$

When we expand the right-hand side in a power series in t, we find the term

$$\Psi^{-1} g_2\big(\psi(x), x; 0, 1, -\nabla_y \psi(y)\big)(\partial_t u)^2.$$

It corresponds to μ with $\mu_{1,0} = 2, \mu_{j,\alpha} = 0$ (otherwise). For this μ, we have

$$k_\mu = 0,\ \gamma(\mu) = 2,\ f_{\mu,0}(x) = \Psi^{-1} g_2\big(\psi(x), x; 0, 1, -\nabla_y \psi(y)\big).$$

It is the only element of $\mathscr{M}_0$ and we can apply Theorem 1.5 with $l = 0$. Since $\beta_{2,0} = -1, \beta_{1,0} = 1$, the equation (1.3) reduces to $f_{\mu,0}(x)A = -1$.

Theorem 1.5 enables us to construct a solution in a neighborhood of each point on the hypersurface. In spite of Remark 2.8, these solutions overlap, if $b(x)$ is fixed, because they are constructed in the same way, i.e. by Proposition 5.2, (6.5) and (6.6). □

Part 2. Nonlinear Fuchsian equations with singular coefficients.

We shall generalize the result of [**13**] to the case where the equations have singular coefficients at $t = 0$. We employ the same notation as in Part 1. Two sets have to be introduced:

- $S_\theta(\delta) = \{t \in S_\theta\ ;\ 0 < |t| < \delta\}$, a sectorial domain in $\mathscr{R}(\boldsymbol{C} \setminus \{0\})$,
- $S_\theta(\varepsilon(y)) = S_\theta \cap S(\varepsilon(y))$.

4. An existence theorem.

We consider

$$(t\partial_t)^m u = F\big(t, x, \big((t\partial_t)^j \partial_x^\alpha u\big)_{(j,\alpha)\in I_m}\big) \tag{4.1}$$

with the unknown function $u = u(t, x)$. Here the function F is allowed to be singular at $t = 0$. Typically, it may involve powers of $\log t$. More precisely, we assume:

C$_1$) $F(t, x, Z)$ is a holomorphic function in (t, x, Z), $Z = (Z_{j,\alpha})_{(j,\alpha)\in I_m} \in \boldsymbol{C}^N$ on $S(\varepsilon(y)) \times D_{R_0} \times \{|Z| < L\}$ for a positive continuous function $\varepsilon(y)$ and constants $R_0 > 0$, $L > 0$.

C$_2$) There exist a constant $s > 0$ and holomorphic functions $c_j(x)$ $(0 \le j \le m-1)$ on D_{R_0} such that for any $\theta > 0$, $(j, \alpha) \in I_m$ and $(i, \beta) \in I_m$ we have

i) $\sup_{x\in D_{R_0}} |F(t, x, 0)| = O(|t|^s)$ (as $S_\theta \ni t \longrightarrow 0$),

$$\text{ii)} \sup_{x\in D_{R_0}} \left|\frac{\partial F}{\partial Z_{j,0}}(t,x,0) - c_j(x)\right| = O(|t|^s) \text{ (as } S_\theta \ni t \longrightarrow 0),$$

$$\text{iii)} \sup_{x\in D_{R_0}} \left|\frac{\partial F}{\partial Z_{j,\alpha}}(t,x,0)\right| = O(|t|^s) \text{ (as } S_\theta \ni t \longrightarrow 0) \text{ if } |\alpha|>0,$$

$$\text{iv)} \sup_{x\in D_{R_0},\, |Z|<L} \left|\frac{\partial^2 F}{\partial Z_{j,\alpha}\partial Z_{i,\beta}}(t,x,Z)\right| = O(1) \text{ (as } S_\theta \ni t \longrightarrow 0).$$

Then we have:

Theorem 4.1. *Assume the conditions* C$_1$) *and* C$_2$). *Then, the equation* (4.1) *has a solution* $u(t,x)$ *in the class* $\widetilde{\mathscr{O}}_+$.

Remark 4.2. The solution of (4.1) in $\widetilde{\mathscr{O}}_+$ is not necessarily unique. There may be a family of solutions involving one or more arbitrary functions in x. See [**13**].

Note that this theorem is essential in the proof of Theorem 1.5.

5. Some preparatory discussions.

Before the proof of Theorem 4.1, let us present some preparatory discussions. For a function $\phi(x)$ on D_r, we define the norm $\|\phi\|_r$ by

$$\|\phi\|_r = \sup_{x\in D_r} |\phi(x)|.$$

Let $\varepsilon(y)$ be a positive continuous function on $\boldsymbol{R}_y$. We say that $\varepsilon(y)$ is decreasing in $|y|$ if the following condition holds: $|y_1| \le |y_2|$ implies $\varepsilon(y_1) \ge \varepsilon(y_2)$. In Definition 1.3 we may assume that $\varepsilon(y)$ is decreasing in $|y|$ without loss of generality.

Definition 5.1. (1) For $d \ge 0$, $\theta > 0$ and $R > 0$, we denote by $\widetilde{\mathscr{O}}_d(S_\theta(\varepsilon(y))\times D_R)$ the set of all the holomorphic functions $u(t,x)$ on $S_\theta(\varepsilon(y)) \times D_R$ that satisfy the following estimate: for any $0 < r < R$ there is a constant $C > 0$ such that

$$|u(t,x)| \le C|t|^d \quad \text{on} \quad S_\theta(\varepsilon(y)) \times D_r.$$

(2) We set $\widetilde{\mathscr{O}}_d(S(\varepsilon(y))\times D_R) = \bigcap_{\theta>0} \widetilde{\mathscr{O}}_d(S_\theta(\varepsilon(y))\times D_R)$.

Let $m \in \boldsymbol{N}^*$ and $c_j(x)$ $(j = 0,1,\dots,m-1)$ be as in Section 4. Set

$$C(\lambda,x) = \lambda^m - c_{m-1}(x)\lambda^{m-1} - \cdots - c_1(x)\lambda - c_0(x), \tag{5.1}$$

and denote by $\lambda_1(x), \ldots, \lambda_m(x)$ the roots of $C(\lambda, x) = 0$ in λ, and let us consider the following equation:

$$C(t\partial_t, x)v = g(t, x). \tag{5.2}$$

PROPOSITION 5.2. *Let $a > 0$. Suppose that*

$$\{a, 2a, 3a, \ldots\} \cap \{\mathrm{Re}\,\lambda_1(0), \ldots, \mathrm{Re}\,\lambda_m(0)\} = \emptyset \tag{5.3}$$

and that $\varepsilon(y)$ is decreasing in $|y|$. Then we can take a sufficiently small $R_1 > 0$ satisfying the following properties $(*)_k$ and $(\sharp)_k$ for $k = 1, 2, \ldots$:

$(*)_k$: *For any $g(t,x) \in \widetilde{\mathscr{O}}_{ak}(S(\varepsilon(y)) \times D_{R_1})$, the equation (5.2) has a solution $v(t,x) \in \widetilde{\mathscr{O}}_{ak}(S(\varepsilon(y)) \times D_{R_1})$.*

$(\sharp)_k$: *Moreover, if $g(t,x)$ satisfies*

$$\|g(t,x)\|_r \le C|t|^{ak} \quad \text{on} \quad S_\theta(\varepsilon(y)) \tag{5.4}$$

for some $0 < r < R_1$, $C > 0$ and $\theta > 0$, we have the estimate

$$\|(t\partial_t)^j v(t)\|_r \le \frac{M_\theta}{k^{m-j}}\, C|t|^{ak} \quad (j = 0, 1, \ldots, m-1) \text{ on } S_\theta(\varepsilon(y)), \tag{5.5}$$

where the constant $M_\theta > 0$ is independent of k, $g(t,x)$, r and j.

This proposition can be proved by the same argument as in the proof of Lemma 6 in [**13**]. For the readers' convenience, we will give its proof in Section 7.

LEMMA 5.3 (Nagumo's Lemma). *If $\phi(x)$ is a holomorphic function on D_R and if*

$$\|\phi\|_r \le \frac{C}{(R-r)^b} \quad \text{for any } 0 < r < R$$

holds for some $C \ge 0$ and $b \ge 0$, then we have

$$\left\| \frac{\partial \phi}{\partial x_i} \right\|_r \le \frac{e(b+1)C}{(R-r)^{b+1}} \quad \text{for any } 0 < r < R \quad \text{and } i = 1, \ldots, n.$$

PROOF. See Nagumo [**10**] or Lemma 5.1.3 of Hörmander [**4**]. □

6. Proof of Theorem 4.1.

Assume the conditions C_1) and C_2). Then, by expanding $F(t,x,Z)$ in Z, our equation (4.1) is written in the form

$$\begin{aligned} C(t\partial_t,x)u = a(t,x) &+ \sum_{\substack{(j,\alpha)\in I_m \\ |\alpha|>0}} b_{j,\alpha}(t,x)(t\partial_t)^j\partial_x^\alpha u \\ &+ \sum_{|\nu|\geq 2} g_\nu(t,x) \prod_{(j,\alpha)\in I_m} \bigl[(t\partial_t)^j\partial_x^\alpha u\bigr]^{\nu_{j,\alpha}}, \end{aligned} \tag{6.1}$$

where $\nu = (\nu_{j,\alpha})_{(j,\alpha)\in I_m} \in \boldsymbol{N}^N$ and $|\nu| = \sum_{(j,\alpha)\in I_m} \nu_{j,\alpha} \geq 2$. The coefficients $a(t,x)$, $b_{j,\alpha}(t,x)$ and $g_\nu(t,x)$ are all holomorphic functions on $S(\varepsilon(y)) \times D_{R_0}$ with suitable growth order to be specified below. By replacing $\varepsilon(y)$ if necessary, we may suppose that $0 < \varepsilon(y) \leq 1$ and that $\varepsilon(y)$ is decreasing in $|y|$.

Let us construct a formal solution. By taking $a > 0$ suitably we may suppose that $0 < a \leq s$ and

$$\{a, 2a, 3a, \ldots\} \cap \bigl\{\mathrm{Re}\,\lambda_1(0), \ldots, \mathrm{Re}\,\lambda_m(0)\bigr\} = \emptyset \tag{6.2}$$

hold. By Proposition 5.2, we have such an $R_1 > 0$ that the properties $(*)_k$ and $(\sharp)_k$ are valid for $k = 1, 2, \ldots$. Since $R_1 > 0$ can be very small, we may assume that $a(t,x)$, $b_{j,\alpha}(t,x)$ and $g_\nu(t,x)$ have the following properties:

i) $a(t,x) \in \widetilde{\mathscr{O}}_a(S(\varepsilon(y)) \times D_{R_1})$,
ii) $b_{j,\alpha}(t,x) \in \widetilde{\mathscr{O}}_a(S(\varepsilon(y)) \times D_{R_1})$ for $(j,\alpha) \in I_m$, $|\alpha| > 0$,
iii) $g_\nu(t,x) \in \widetilde{\mathscr{O}}_0(S(\varepsilon(y)) \times D_{R_1})$ for $|\nu| \geq 2$.

We shall construct a formal solution of (6.1) in the form

$$u(t,x) = \sum_{k\geq 1} u_k(t,x), \quad u_k(t,x) \in \widetilde{\mathscr{O}}_{ak}\bigl(S(\varepsilon(y)) \times D_{R_1}\bigr). \tag{6.3}$$

Let us decompose our equation (6.1). We have formally

$$\begin{aligned} \sum_{k\geq 1} C(t\partial_t,x)u_k = a(t,x) &+ \sum_{k\geq 1} b_{j,\alpha}(t,x)(t\partial_t)^j\partial_x^\alpha u_k \\ &+ \sum_{|\nu|\geq 2} g_\nu(t,x) \prod_{(j,\alpha)\in I_m} \Bigl[\sum_{k\geq 1}(t\partial_t)^j\partial_x^\alpha u_k\Bigr]^{\nu_{j,\alpha}}. \end{aligned} \tag{6.4}$$

Therefore, the equation (6.1) is satisfied if $\{u_k(t,x)\,;\,k=1,2,\ldots\}$ is determined by the following recurrent family of equations (6.5) and (6.6):

$$C(t\partial_t,x)u_1 = a(t,x) \tag{6.5}$$

and for $k \ge 2$,

$$\begin{aligned} C(t\partial_t,x)u_k = &\sum_{(j,\alpha)\in I_m} b_{j,\alpha}(t,x)(t\partial_t)^j\partial_x^\alpha u_{k-1} \\ &+ \sum_{2\le|\nu|\le k} g_\nu(t,x)\sum_{|k(\nu)|=k}\prod_{(j,\alpha)\in I_m}\prod_{l=1}^{\nu_{j,\alpha}}(t\partial_t)^j\partial_x^\alpha u_{k_{j,\alpha}(l)}, \end{aligned} \tag{6.6}$$

where

$$\begin{aligned} k_{j,\alpha}(l) &\in \boldsymbol{N}^*, \\ k(\nu) &= \{(k_{j,\alpha}(l))\,;\,(j,\alpha)\in I_m, 1\le l\le \nu_{j,\alpha}\}, \\ |k(\nu)| &= \sum_{(j,\alpha)\in I_m}(k_{j,\alpha}(1)+\cdots+k_{j,\alpha}(\nu_{j,\alpha})). \end{aligned}$$

It should be remarked that in the right-hand side of (6.6) only the terms $u_1,\ldots,u_{k-1}$ and their derivatives appear. Thus, by applying Proposition 5.2 to (6.5) and (6.6) ($k\ge 2$) inductively on k we can obtain a solution $\{u_k(t,x)\,;\,k=1,2,\ldots\}$ of the recurrent family (6.5) and (6.6) ($k\ge 2$) such that

$$(t\partial_t)^j\partial_x^\alpha u_k \in \widetilde{\mathscr{O}}_{ak}(S(\varepsilon(y))\times D_{R_1}) \quad \text{for } (j,\alpha)\in I_m,\ k=1,2,\ldots. \tag{6.7}$$

This proves

Proposition 6.1. *In the above situation, we can construct*

$$u(t,x) = \sum_{k\ge 1} u_k(t,x) \tag{6.8}$$

with the condition (6.7) *which solves the equation* (6.1) *formally in the sense that* $\{u_k(t,x)\,;\,k=1,2,\ldots\}$ *satisfies* (6.5) *and* (6.6) ($k\ge 2$).

Set $f_1(t)=a(t,x)$ and

$$f_k(t) = \sum_{(j,\alpha)\in I_m} b_{j,\alpha}(t,x)(t\partial_t)^j \partial_x^\alpha u_{k-1}$$

$$+ \sum_{2\le|\nu|\le k} g_\nu(t,x) \sum_{|k(\nu)|=k} \prod_{(j,\alpha)\in I_m} \prod_{l=1}^{\nu_{j,\alpha}} (t\partial_t)^j \partial_x^\alpha u_{k_{j,\alpha}(l)} \quad (k \ge 2). \tag{6.9}$$

Then, by $(\sharp)_k$ of Proposition 5.2 we see also

Proposition 6.2. *In Proposition* 6.1*, we have the following additional property: if f_k satisfies*

$$\|f_k(t)\|_r \le C|t|^{ak} \quad \text{on } S_\theta(\varepsilon(y)) \tag{6.10}$$

for some $0 < r < R_1$, $C > 0$ and $\theta > 0$, we have the estimate

$$\|(t\partial_t)^j u_k(t)\|_r \le \frac{M_\theta}{k^{m-j}} C|t|^{ak} \ (j = 0, 1, \dots, m-1) \quad \text{on } S_\theta(\varepsilon(y)). \tag{6.11}$$

Next, let us prove the convergence of the formal solution (6.8). Our aim is to show that (6.8) gives an $\widetilde{\mathscr{O}}_+$-solution of (6.1). Take any $\theta > 0$ and $0 < R < R_1$ (with $0 < R \le 1$) and fix them. It is sufficient to prove that the formal solution (6.8) is convergent in $\widetilde{\mathscr{O}}_a(S_\theta(\delta) \times D_{R/2})$ for some $\delta > 0$.

By the assumption there exist constants $B_{j,\alpha} \ge 0$ ($(j,\alpha) \in I_m$) and $G_\nu \ge 0$ ($|\nu| \ge 2$) satisfying the following properties:

i) $|b_{j,\alpha}(t,x)| \le B_{j,\alpha}|t|^a$ on $S_\theta(\varepsilon(y)) \times D_R$,
ii) $|g_\nu(t,x)| \le G_\nu$ on $S_\theta(\varepsilon(y)) \times D_R$,
iii) $\sum_{|\nu|\ge 2} G_\nu Z^\nu$ is convergent in a neighborhood of $Z = 0 \in \boldsymbol{C}^N$.

By (6.7) we have $(t\partial_t)^j \partial_x^\alpha u_1 \in \widetilde{\mathscr{O}}_a(S(\varepsilon(y)) \times D_{R_1})$ for any $(j,\alpha) \in I_m$; therefore we can take a constant $A_1 \ge 0$ such that

$$\left\|(t\partial_t)^j \partial_x^\alpha u_1(t)\right\|_R \le A_1|t|^a \text{ on } S_\theta(\varepsilon(y)), \quad (j,\alpha) \in I_m. \tag{6.12}$$

Set $\beta = (em)^m$. Using these A_1, β, $B_{j,\alpha}$ and G_ν, we consider the following holomorphic functional equation with respect to $Y = Y(z), z \in \boldsymbol{C}$:

$$Y = A_1 z + M_\theta \left[\sum_{(j,\alpha)\in I_m} \frac{B_{j,\alpha} z(\beta Y)}{(R-r)^m} + \sum_{|\nu|\ge 2} \frac{G_\nu}{(R-r)^{m(|\nu|-1)}} (\beta Y)^{|\nu|} \right], \tag{6.13}$$

where r is a parameter with $0 < r < R$.

By the implicit function theorem we see that the equation (6.13) has a unique holomorphic solution $Y(z)$ with $Y(0) = 0$ in a neighborhood of $z = 0$. We expand $Y(z)$ into powers of z:

$$Y(z) = \sum_{k \geq 1} Y_k z^k.$$

We see that the coefficients Y_k $(k = 1, 2, \ldots)$ are uniquely determined by the following recurrence formulas:

$$Y_1 = A_1 \tag{6.14}$$

and for $k \geq 2$,

$$Y_k = M_\theta \Bigg[\sum_{(j,\alpha) \in I_m} \frac{B_{j,\alpha} \beta Y_{k-1}}{(R-r)^m} + \sum_{2 \leq |\nu| \leq k} \frac{G_\nu}{(R-r)^{m(|\nu|-1)}} \sum_{|k(\nu)|=k} \prod_{(j,\alpha) \in I_m} \prod_{l=1}^{\nu_{j,\alpha}} \beta Y_{k_{j,\alpha}(l)} \Bigg]. \tag{6.15}$$

Moreover, by induction on k we see that each Y_k has the form

$$Y_k = \frac{C_k}{(R-r)^{m(k-1)}} \quad (k = 1, 2, \ldots), \tag{6.16}$$

where $C_1 = A_1$ and $C_k \geq 0$ $(k \geq 2)$ are constants independent of the parameter r.

The following lemma guarantees that $Y(z)$ can be used as a majorant series of the formal solution (6.8).

Proposition 6.3. *For any* $0 < r < R$ *and* $(j, \alpha) \in I_m$*, we have*

$$\big\| (t\partial_t)^j \partial_x^\alpha u_k(t) \big\|_r \leq \beta Y_k |t|^{ak} \ (k = 1, 2, \ldots) \quad \text{on } S_\theta(\varepsilon(y)). \tag{6.17}$$

Proof. By (6.12) we have

$$\big\| (t\partial_t)^j \partial_x^\alpha u_1(t) \big\|_r \leq A_1 |t|^a = Y_1 |t|^a \leq \beta Y_1 |t|^a \quad \text{on } S_\theta(\varepsilon(y)).$$

This proves (6.17) for $k = 1$.

Let us show the general case by induction on k. Suppose that $k \geq 2$ and that (6.17) has already been proved for $u_1, \dots, u_{k-1}$. Then, by (6.9) we have

$$\begin{aligned} \|f_k(t)\|_r \leq & \sum_{(j,\alpha)\in I_m} B_{j,\alpha}|t|^a \times \beta Y_{k-1}|t|^{a(k-1)} \\ & + \sum_{2\leq|\nu|\leq k} G_\nu \sum_{|k(\nu)|=k} \prod_{(j,\alpha)\in I_m} \prod_{l=1}^{\nu_{j,\alpha}} \beta Y_{k_{j,\alpha}(l)}|t|^{ak_{j,\alpha}(l)} \\ = & |t|^{ak}\Bigg[\sum_{(j,\alpha)\in I_m} B_{j,\alpha}\beta Y_{k-1} + \sum_{2\leq|\nu|\leq k} G_\nu \sum_{|k(\nu)|=k} \prod_{(j,\alpha)\in I_m} \prod_{l=1}^{\nu_{j,\alpha}} \beta Y_{k_{j,\alpha}(l)}\Bigg]. \end{aligned}$$

Therefore, by comparing this with (6.15) and by using $1/(R-r) > 1$ we have

$$\|f_k(t)\|_r \leq \frac{(R-r)^m}{M_\theta} Y_k|t|^{ak} = \frac{C_k}{M_\theta(R-r)^{m(k-2)}}|t|^{ak} \quad \text{on } S_\theta(\varepsilon(y))$$

for any $0 < r < R$. Hence, by Proposition 6.2 we have

$$\left\|(t\partial_t)^j u_k(t)\right\|_r \leq \frac{1}{k^{m-j}} \frac{C_k}{(R-r)^{m(k-2)}}|t|^{ak} \text{ on } S_\theta(\varepsilon(y)) \tag{6.18}$$

for any $0 < r < R$ and $j = 0, 1, \dots, m-1$. By Lemma 5.3 we have

$$\begin{aligned} \left\|(t\partial_t)^j \partial_x^\alpha u_k\right\|_r & \leq \frac{1}{k^{m-j}} \frac{\{m(k-2)+1\}\cdots\{m(k-2)+|\alpha|\}e^{|\alpha|}C_k}{(R-r)^{m(k-2)+|\alpha|}}|t|^{ak} \\ & \leq \frac{1}{k^{m-j-|\alpha|}} \frac{m^{|\alpha|}e^{|\alpha|}C_k}{(R-r)^{m(k-2)+|\alpha|}}|t|^{ak} \\ & \leq \frac{\beta C_k}{(R-r)^{m(k-2)+|\alpha|}}|t|^{ak} \leq \frac{\beta C_k}{(R-r)^{m(k-1)}}|t|^{ak} = \beta Y_k|t|^{ak} \end{aligned}$$

on $S_\theta(\varepsilon(y))$ for any $0 < r < R$ and $(j,\alpha) \in I_m$; this proves (6.17).

Thus, we have proved Proposition 6.3. □

Let us complete the proof of Theorem 4.1. Set $r = R/2$ and fix it. Since $Y(z) = \sum_{k\geq1} Y_k z^k$ is convergent, we can take a small constant $\delta > 0$ so that $C = \sum_{k\geq1} \beta Y_k \delta^{ak} < \infty$ and $S_\theta(\delta) \subset S_\theta(\varepsilon(y))$ hold. Then, by Proposition 6.3, for any $(t,x) \in S_\theta(\delta) \times D_{R/2}$ we have

$$\sum_{k\geq 1}|u_k(t,x)| \leq \sum_{k\geq 1}\|u_k(t)\|_{R/2} \leq \sum_{k\geq 1}\beta Y_k|t|^{ak} \leq \sum_{k\geq 1}\beta Y_k\delta^{ak}\frac{|t|^a}{\delta^a} \leq \frac{C|t|^a}{\delta^a}.$$

This proves that the formal solution (6.8) is convergent in $\widetilde{\mathscr{O}}_a(S_\theta(\delta)\times D_{R/2})$.

Thus, we have proved Theorem 4.1.

7. Proof of Proposition 5.2.

The proof of Proposition 5.2 is almost the same as that of Lemma 6 of [**13**]. For the readers' convenience we will give its proof.

Let $p \in \boldsymbol{N}^*$. For bounded holomorphic functions $a_i(x)$ $(i = 1,\dots,p)$ on D_R, set

$$P(\xi,x) = \xi^p + a_1(x)\xi^{p-1} + \cdots + a_{p-1}(x)\xi + a_p(x), \tag{7.1}$$

and denote by $\xi_1(x),\dots,\xi_p(x)$ the roots of $P(\xi,x)=0$. Let $\varepsilon(y)$ be a positive continuous function on $\boldsymbol{R}_y$ which is decreasing in $|y|$.

For $d>0$, let us consider

$$P(t\partial_t,x)u = f(t,x) \quad \text{in } \widetilde{\mathscr{O}}_d(S(\varepsilon(y))\times D_R). \tag{7.2}$$

Set

$$E(\tau_1,\dots,\tau_p,x) = \frac{1}{p!}\sum_{\pi\in S_p}(\tau_1)^{-\xi_{\pi(1)}(x)}\cdots(\tau_p)^{-\xi_{\pi(p)}(x)},$$

where S_p is the permutation group on p letters. Then, by Lemma 2 of [**13**] we have

LEMMA 7.1. *Assume that*

$$d - \operatorname{Re}\xi_i(x) \geq L \quad (i=1,\dots,p) \tag{7.3}$$

holds on D_R for some $L>0$. Then, we have:

(1) *For any $f(t,x)\in\widetilde{\mathscr{O}}_d(S(\varepsilon(y))\times D_R)$, the equation* (7.2) *has a unique solution $u(t,x)\in\widetilde{\mathscr{O}}_d(S(\varepsilon(y))\times D_R)$ and it is represented by the following integral formula*:

$$u(t,x) = \int_0^t\frac{d\tau_p}{\tau_p}\int_0^{\tau_p}\frac{d\tau_{p-1}}{\tau_{p-1}}\cdots\int_0^{\tau_2}\frac{d\tau_1}{\tau_1}\left[E\left(\frac{\tau_1}{\tau_2},\dots,\frac{\tau_{p-1}}{\tau_p},\frac{\tau_p}{t},x\right)f(\tau_1,x)\right].$$

(2) *If $f(t,x)$ satisfies the estimate*

$$|f(t,x)| \le C|t|^d \quad on \quad S_\theta(\varepsilon(y)) \times D_R \tag{7.4}$$

for some $C > 0$ and $\theta > 0$, the solution $u(t,x)$ satisfies

$$|(t\partial_t)^j u(t,x)| \le \frac{A^j}{L^{p-j}} C|t|^d \quad (j = 0, 1, \dots, p) \quad on \quad S_\theta(\varepsilon(y)) \times D_R \tag{7.5}$$

for any constant $A > 0$ with

$$A \ge \max_{1 \le i \le p} \left[1 + \frac{1}{L} \sup_{x \in D_R} |\xi_i(x)| \right].$$

Next, set $\delta = \varepsilon(0)$, and let us consider the Cauchy problem with initial data on $\{t = \delta\}$:

$$\begin{cases} P(t\partial_t, x)u = f(t,x) \quad \text{in } \widetilde{\mathscr{O}}_d(S(\varepsilon(y)) \times D_R), \\ (t\partial_t)^j u\big|_{t=\delta} = \lim_{t \to \delta} (t\partial_t)^j u = 0 \ (j = 0, \dots, p-1). \end{cases} \tag{7.6}$$

Then, by Lemma 3 of [**13**] we obtain Lemma 7.2 below. This is where we employ the assumption that $\varepsilon(y)$ is decreasing in $|y|$. It enables us to choose a suitable path of integration.

Lemma 7.2. *Assume that*

$$d - \operatorname{Re} \xi_i(x) \le -L \quad (i = 1, \dots, p) \tag{7.7}$$

holds on D_R for some $L > 0$. Then, we have:

(1) *For any $f(t,x) \in \widetilde{\mathscr{O}}_d(S(\varepsilon(y)) \times D_R)$, the equation* (7.6) *has a unique solution $u(t,x) \in \widetilde{\mathscr{O}}_d(S(\varepsilon(y)) \times D_R)$ and it is represented by the following integral formula*:

$$u(t,x) = (-1)^p \int_t^\delta \frac{d\tau_p}{\tau_p} \int_{\tau_p}^\delta \frac{d\tau_{p-1}}{\tau_{p-1}} \cdots \int_{\tau_2}^\delta \frac{d\tau_1}{\tau_1} \left[E\left(\frac{\tau_1}{\tau_2}, \dots, \frac{\tau_{p-1}}{\tau_p}, \frac{\tau_p}{t}, x \right) f(\tau_1, x) \right].$$

(2) *If $f(t,x)$ satisfies the estimate* (7.4) *for some $C > 0$ and $\theta > 0$, and if $|\operatorname{Im} \xi_i(x)| \le M$ $(i = 1, \dots, p)$ holds on D_R, the solution $u(t,x)$ satisfies*

$$|(t\partial_t)^j u(t,x)| \le A^j \left(e^{M\theta}\left(\theta + \frac{1}{L}\right)\right)^{p-j} C|t|^d \quad (j=0,1,\dots,p) \tag{7.8}$$

on $S_\theta(\varepsilon(y)) \times D_R$ *for any constant* $A > 0$ *with*

$$A \ge \max_{1\le i\le p}\left[1 + \left(e^{M\theta}\left(\theta + \frac{1}{L}\right)\right) \sup_{x\in D_R} |\xi_i(x)|\right].$$

Now, by using these lemmas let us give a proof of Proposition 5.2.

Proof of Proposition 5.2. By the assumption (5.3), we can take sufficiently small $R_1 > 0$ and $L > 0$ so that $|ak - \operatorname{Re}\lambda_i(x)| \ge L$ holds for any $k = 1,2,\dots$, $i = 1,2,\dots,m$ and $x \in D_{R_1}$. Then, by taking a sufficiently small constant $c > 0$ we may suppose that $|ak - \operatorname{Re}\lambda_i(x)| \ge ck$ holds for any $k = 1,2,\dots$, $i = 1,2,\dots,m$ and $x \in D_{R_1}$.

Take any $k \in \{1,2,\dots\}$ and fix it: let us show the properties $(*)_k$ and $(\sharp)_k$. To do so, we set $I_1 = \{i\,;\, \operatorname{Re}\lambda_i(0) < ak\}$ and $I_2 = \{i\,;\, \operatorname{Re}\lambda_i(0) > ak\}$, and set

$$C_j(\lambda,x) = \prod_{i\in I_j}(\lambda - \lambda_i(x)), \quad j = 1,2.$$

Then, $C_j(\lambda,x)$ $(j=1,2)$ are polynomials in λ whose coefficients are bounded holomorphic functions on D_{R_1} and we can apply Lemmas 7.1 and 7.2 to $C_1(t\partial_t,x)u = f$ and $C_2(t\partial_t,x)u = f$, respectively. Let $p = \#I_2$ be the cardinal of the set I_2. Then the degrees of $C_1(\lambda,x)$ and $C_2(\lambda,x)$ are $m-p$ and p respectively.

Let us show $(*)_k$. For $g(t,x) \in \widetilde{\mathscr{O}}_{ak}(S(\varepsilon(y)) \times D_{R_1})$, let us consider the equation

$$C(t\partial_t,x)v = g(t,x), \quad v \in \widetilde{\mathscr{O}}_{ak}(S(\varepsilon(y)) \times D_{R_1}). \tag{7.9}$$

Since $C(t\partial_t,x) = C_2(t\partial_t,x)C_1(t\partial_t,x)$ holds, we can obtain a solution $v(t,x)$ of (7.9) by solving the following two equations:

$$\begin{cases} C_2(t\partial_t,x)u = g(t,x), \quad u \in \widetilde{\mathscr{O}}_{ak}(S(\varepsilon(y)) \times D_R), \\ (t\partial_t)^j u\big|_{t=\delta} = 0 \quad (j=0,\dots,p-1), \end{cases} \tag{7.10}$$

and

$$C_1(t\partial_t,x)v = u(t,x), \quad v \in \widetilde{\mathscr{O}}_{ak}(S(\varepsilon(y)) \times D_{R_1}). \tag{7.11}$$

Thus, by Lemmas 7.1 and 7.2 we have a solution $v(t,x) \in \widetilde{\mathscr{O}}_{ak}(S(\varepsilon(y)) \times D_{R_1})$ of (7.9).

Next, let us show $(\sharp)_k$. Suppose that

$$\|g(t)\|_r \le C|t|^{ak} \quad \text{on} \quad S_\theta(\varepsilon(y)) \tag{7.12}$$

for some $0 < r < R_1$, $C > 0$ and $\theta > 0$. Then, by applying Lemma 7.2 to (7.10) we have the estimates

$$\|(t\partial_t)^j u(t)\|_r \le A_2^j \Bigl(e^{M\theta}\Bigl(\theta + \frac{1}{ck}\Bigr)\Bigr)^{p-j} C|t|^{ak} \quad (j = 0, 1, \ldots, p) \quad \text{on } S_\theta(\varepsilon(y)) \tag{7.13}$$

for any constants $M > 0$ and $A_2 > 0$ such that

$$M \ge \max_{1 \le i \le m} \sup_{x \in D_{R_1}} \bigl|\mathrm{Im}\,\lambda_i(x)\bigr|,$$

$$A_2 \ge \max_{1 \le i \le m} \Bigl[1 + \Bigl(e^{M\theta}\Bigl(\theta + \frac{1}{ck}\Bigr)\Bigr) \sup_{x \in D_{R_1}} |\lambda_i(x)|\Bigr].$$

We consider a variant of (7.11) in view of (7.13):

$$C_1(t\partial_t, x)w = (t\partial_t)^j u(t,x) \in \widetilde{\mathscr{O}}_{ak}(S(\varepsilon(y)) \times D_{R_1}). \tag{7.14}$$

We can find a unique solution $w \in \widetilde{\mathscr{O}}_{ak}(S(\varepsilon(y)) \times D_{R_1})$ by Lemma 7.1. Moreover we have $w = (t\partial_t)^j v$ because both are solutions to (7.14) and are of order $O(|t|^{ak})$ as $t \to 0$. Uniqueness of such a solution follows from the fact that $\mathrm{Re}\,\lambda_i(0) < ak$ $(i \in I_1)$.

Let $A_1 > 0$ be an arbitrary constant with

$$A_1 \ge \max_{1 \le i \le m} \Bigl[1 + \frac{1}{ck} \sup_{x \in D_R} |\xi_i(x)|\Bigr].$$

Then, by applying Lemma 7.1 to (7.14) we have the estimate

$$\|(t\partial_t)^{i+j} v(t)\|_r = \|(t\partial_t)^i w(t)\|_r \le \frac{{A_1}^i}{(ck)^{m-p-i}} {A_2}^j \Bigl(e^{M\theta}\Bigl(\theta + \frac{1}{ck}\Bigr)\Bigr)^{p-j} C|t|^{ak} \tag{7.15}$$

on $S_\theta(\varepsilon(y))$ for $i = 0, 1, \ldots, m-p$ and $j = 0, 1, \ldots, p$.

If k satisfies

$$k > \mu_0 = \frac{1}{a} \max_{1 \le i \le m} \operatorname{Re} \lambda_i(0),$$

we have $p = \# I_2 = 0$ and so (7.15) is reduced to

$$\|(t\partial_t)^i v(t)\|_r \le \frac{{A_1}^i}{(ck)^{m-i}} C|t|^{ak} \quad (i = 0, 1, \dots, m) \quad \text{on } S_\theta(\varepsilon(y)). \tag{7.16}$$

Since $\{k\, ;\, k < \mu_0\}$ is a finite set, we have the estimate (5.5) for any $k \ge 1$. □

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