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A classification of graded extensions in a skew Laurent polynomial ring

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Abstract. Let V be a total valuation ring of a division ring K with an automorphism σ and let $A = \oplus_{i\in\boldsymbol{Z}} A_i X^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$, the skew Laurent polynomial ring. We classify A by distinguishing four different types based on the properties of A_1 and A_{-1}. A complete description of A_i for all $i \in \boldsymbol{Z}$ is given in the case where A_1 is a finitely generated left $O_l(A_1)$-ideal.

Introduction.

Let K be a division ring with an automorphism σ and let V be a total valuation ring of K, that is, for any non-zero $k \in K$, either $k \in V$ or $k^{-1} \in V$. A graded subring $A = \oplus_{i\in\boldsymbol{Z}} A_i X^i$ of $K[X, X^{-1}; \sigma]$, the skew Laurent polynomial ring, is called a graded total valuation ring of $K[X, X^{-1}; \sigma]$ if for any non-zero homogeneous element aX^i of $K[X, X^{-1}; \sigma]$, either $aX^i \in A$ or $(aX^i)^{-1} \in A$, where $\boldsymbol{Z}$ is the ring of integers. A graded total valuation ring A of $K[X, X^{-1}; \sigma]$ is said to be a graded extension of V in $K[X, X^{-1}; \sigma]$ if $A_0 = V$.

A Gauss extension S of V in $K(X, \sigma)$, the quotient ring of $K[X, X^{-1}; \sigma]$, was defined in [**1**] as a total valuation ring of $K(X, \sigma)$ with $S \cap K = V$ that satisfies the following conditin:

$$\alpha S = a_i X^i S$$

for any $\alpha = \Sigma a_j X^j \in K[X, X^{-1}; \sigma]$ with $a_i X^i S \supseteq a_j X^j S$ for all j. Then the following results were obtained:

Theorem 0.1. *There is a one-to-one correspondence between the set of all Gauss extensions of V in $K(X, \sigma)$ and the set of all graded extensions of V in*

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$K[X, X^{-1}; \sigma]$, which is given by $S \longrightarrow S \cap K[X, X^{-1}; \sigma]$, where S is a Gauss extension of V in $K(X, \sigma)$ ([**1**, (1.8)]).

Theorem 0.2. *Let S be a Gauss extension of V in $K(X, \sigma)$ and let $A = S \cap K[X, X^{-1}; \sigma]$. Then*

(1) *The mapping $\varphi : I \longrightarrow I_g = I \cap K[X, X^{-1}; \sigma]$ is a one-to-one correspondence between the set of all (right) ideals of S and the set of all graded (right) ideals of A.*

(2) *φ induces a one-to-one correspondence between the set of all prime ideals of S and the set of all graded prime ideals of A* ([**1**, (2.1)]).

We note that Gauss extensions in [**1**] were considered in a more general context. Total valuation rings in Ore extensions or in skew polynomial rings have been studied in [**2**], [**3**], [**6**] and [**7**].

Theorems show that it suffices, in some sense, to study graded extensions in order to study the Gauss extensions (in particular, ideal theory of Gauss extensions).

The aim of the paper is to classify the graded extensions of V in $K[X, X^{-1}; \sigma]$ and to study the structure of them.

In Section 1, we will give some basic properties of graded extensions. Let $A = \oplus_{i \in \boldsymbol{Z}} A_i X^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$ and let $W = O_l(A_1)$ be an overring of V. There are two cases: namely, either A_1 is a finitely generated left W-ideal, say, $A_1 = Wa$ for some $a \in A_1$, or A_1 is not a finitely generated left W-ideal.

In this paper, we will concentrate on the case where $A_1 = Wa$ (in the case where A_1 is not a finitely generated left W-ideal, we will study the graded extensions in a forthcoming paper). If $A_1 = Wa$, then it is shown that either $A_{-1} = \sigma^{-1}(a^{-1}J(W))$ or $A_{-1} = W\sigma^{-1}(a^{-1})$, where $J(W)$ is the Jacobson radical of W. From this information, in Section 2, we will classify graded extensions A into four cases and will give complete descriptions of A_i for all $i \in \boldsymbol{Z}$. Except for the case where $A_1 = Wa = a\sigma(W)$, $A_{-1} = \sigma^{-1}(a^{-1}J(W))$ and $J(W) \supset J(W)^2$, A is uniquely determined (see Theorem 2.2 and Theorems 2.4 $\sim$ 2.6). However, in the case $A_1 = Wa = a\sigma(W)$, $A_{-1} = \sigma^{-1}(a^{-1}J(W))$ and $J(W) \supset J(W)^2$, there are infinitely many different graded extensions (the cardinality is at least $\aleph$).

To give a complete description of the graded extensions, we need a map from $\boldsymbol{Z}$ to $\boldsymbol{Z}$ which is called a nice map (see Section 2 for the definition of nice maps).

In Section 3, we will give a complete description of nice maps.

Section 4 contains an example of total valuation rings V and W with $W \supset V$ and $J(W) \supset J(W)^2$ such that the cardinality of the set of all graded extensions $B = \oplus_{i \in \boldsymbol{Z}} B_i X^i$ of V in $K[X, X^{-1}; \sigma]$ with $A_1 = Wa = a\sigma(W) = B_1$ and $A_{-1} = \sigma^{-1}(a^{-1}J(W)) = B_{-1}$ is larger than $\aleph$.

Let I be a right V-submodule of K. Then I is called a right V-ideal if $aI \subseteq V$ for some non-zero $a \in K$. Left V-ideals are defined similarly. It is well known that the set of all right (left) V-ideals is linearly ordered by inclusion, which is used without reference. Let I be an additive subgroup of K. Then the right and left order of I are defined to be

$$O_r(I) = \{k \in K \mid Ik \subseteq I\} \text{ and } O_l(I) = \{k \in K \mid kI \subseteq I\}.$$

Furthermore, for any subsets I and J of K, we use the notation:

$$\begin{aligned}(J : I)_r &= \{k \in K \mid Ik \subseteq J\},\\ (J : I)_l &= \{k \in K \mid kI \subseteq J\} \text{ and}\\ I^- &= \{c^{-1} \mid c \in I, c \neq 0\}.\end{aligned}$$

We refer the readers to [**5**] for some basic properties of non-commutative valuation rings.

1. Some basic properties of graded extensions.

Throughout this paper, V will denote a total valuation ring of a division ring K with an automorphism σ of K. $K[X, X^{-1}; \sigma]$ will be the skew Laurent polynomial ring with its quotient division ring $K(X, \sigma)$. In this section, we will give some basic properties of graded extensions of V in $K[X, X^{-1}; \sigma]$. We start with the following easy lemma.

Lemma 1.1. *Let $A = \oplus_{i \in \boldsymbol{Z}} A_i X^i$ be a subset of $K[X, X^{-1}; \sigma]$ with $A_0 = V$. Then A is a graded extension of V if and only if*

(1) $A_i \sigma^i(A_j) \subseteq A_{i+j}$ for all $i, j \in \boldsymbol{Z}$ and A_i is an additive subgroup of K for all $i \in \boldsymbol{Z}$ and

(2) $A_i \cup \sigma^i(A_{-i}^-) = K$ for all $i \in \boldsymbol{Z}$.

Proof. Suppose that $A = \oplus_{i \in \boldsymbol{Z}} A_i X^i$ is a graded extension of V in $K[X, X^{-1}; \sigma]$. Then (1) easily follows, because $A_i X^i A_j X^j = A_i \sigma^i(A_j) X^{i+j} \subseteq A_{i+j} X^{i+j}$. To prove (2), let $a \in K, a \neq 0$. If $aX^i \in A$, then $a \in A_i$. If $aX^i \notin A$, then $A \ni (aX^i)^{-1} = X^{-i} a^{-1} = \sigma^{-i}(a^{-1}) X^{-i}$ so that $\sigma^{-i}(a^{-1}) \in A_{-i}$. Hence $a \in \sigma^i(A_{-i}^-)$, showing that $A_i \cup \sigma^i(A_{-i}^-) = K$.

Suppose that (1) and (2) hold. Then A is a graded subring of $K[X, X^{-1}; \sigma]$ by (1). To prove that A is a graded extension of V in $K[X, X^{-1}; \sigma]$, let $aX^i \in K[X, X^{-1}; \sigma]$. If $a \in A_i$, then $aX^i \in A$. If $a \notin A_i$, then $a \in \sigma^i(A_{-i}^-)$, i.e., $\sigma^{-i}(a^{-1}) \in A_{-i}$. Hence, $(aX^i)^{-1} = \sigma^{-i}(a^{-1}) X^{-i} \in A$. □

The following lemma is more or less known.

Lemma 1.2. *Let W be a total valuation ring. Then*

(1) *Let I and J be left W-ideals of K such that $J(W)I \subseteq J \subseteq I$. Then either $J = I$ or $J = J(W)I$.*

(2) *Let I and J be right W-ideals of K such that $IJ(W) \subseteq J \subseteq I$. Then either $J = I$ or $J = IJ(W)$.*

Proof.

(1) Suppose that $J(W)I \subset J \subset I$. Then there exist $b \in J \setminus J(W)I$ and $c \in I \setminus J$. Thus $J(W)I \subset Wb \subset Wc$, because the set of all left W-ideals is linearly ordered by inclusion. Hence $bc^{-1} \in J(W)$ and so $b \in J(W)c \subseteq J(W)I$, a contradiction. Therefore, we have either $J = I$ or $J = J(W)I$.

(2) This is just a right version of (1). $\square$

The following results will be used in the investigation of graded extensions.

Lemma 1.3. *Let W be a total valuation ring of K, $\alpha \in K$ with $\alpha \neq 0$, $i \in \boldsymbol{Z}$ with $i \neq 0$ and let I and J be subsets of K. Then*

(1) *If $I = W\alpha$ and $J \supseteq \sigma^{-i}(\alpha^{-1}J(W))$, then $I \cup \sigma^i(J^-) = K$.*

(2) *If $I = \alpha\sigma^i(W)$ and $J \supseteq J(W)\sigma^{-i}(\alpha^{-1})$, then $I \cup \sigma^i(J^-) = K$.*

(3) *If $I = \alpha\sigma^i(J(W))$ and $J = W\sigma^{-i}(\alpha^{-1})$, then $I \cup \sigma^i(J^-) = K$.*

(4) *If $I = J(W)\alpha$ and $J = \sigma^{-i}(\alpha^{-1})\sigma^{-i}(W)$, then $I \cup \sigma^i(J^-) = K$.*

Proof.

(1) Let $b \in K \setminus I$. Then $Wb \supset I$ and so $\alpha = wb$ for some $w \in J(W)$. Thus $\sigma^{-i}(b^{-1}) = \sigma^{-i}(\alpha^{-1}w) \in \sigma^{-i}(\alpha^{-1}J(W)) \subseteq J$, i.e., $b \in \sigma^i(J^-)$. Hence $I \cup \sigma^i(J^-) = K$ follows.

(2) This is proved as in (1).

(3) Let $b \in K \setminus I$. Then it follows that $\alpha^{-1}b \notin \sigma^i(J(W))$ so that $\sigma^i(W)\alpha^{-1}b \supset \sigma^i(J(W))$. Thus $\sigma^i(W)\alpha^{-1}b \supseteq \sigma^i(W)$ and so $\sigma^i(W)\alpha^{-1} \supseteq \sigma^i(W)b^{-1}$. Let $b^{-1} = \sigma^i(w)\alpha^{-1}$ for some $w \in W$. Then $\sigma^{-i}(b^{-1}) = w\sigma^{-i}(\alpha^{-1}) \in J$ and so $b^{-1} \in \sigma^i(J)$, i.e., $b \in \sigma^i(J^-)$. Hence $I \cup \sigma^i(J^-) = K$ follows.

(4) This is proved as in (3). $\square$

Let $A = \oplus_{i\in\boldsymbol{Z}} A_iX^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$, $O_l(A_i) = W$ and $O_r(A_i) = \sigma^i(U)$. Then note that W and U are both overrings of V, because A_i is a left V and right $\sigma^i(V)$-ideal. The following lemma is crucial for the classification of graded extensions.

Lemma 1.4. *Let $A = \oplus_{i\in\boldsymbol{Z}} A_iX^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$ with $O_l(A_i) = W$ and $O_r(A_i) = \sigma^i(U)$ for a fixed $i \in \boldsymbol{N}$, where $\boldsymbol{N}$ is the set of all* *natural numbers.*

(1) *Suppose that* $A_i = W\alpha$ *for some non-zero* $\alpha \in K$. *Then*

(i) *If* $W = V$ *and* $V\alpha = \alpha\sigma^i(V)$, *then either* $A_{-i} = V\sigma^{-i}(\alpha^{-1})$ *or* $A_{-i} = J(V)\sigma^{-i}(\alpha^{-1})$.

(ii) *If either* $W \supset V$ *or* $V\alpha \supset \alpha\sigma^i(V)$ *(when* $W = V$*), then* $A_{-i} = \sigma^{-i}(\alpha^{-1}J(W))$.

(2) *Suppose that* $A_i = \alpha\sigma^i(U)$ *for some non-zero* $\alpha \in K$ *and* $\alpha\sigma^i(U) \supset U\alpha$. *Then* $A_{-i} = J(U)\sigma^{-i}(\alpha^{-1})$.

Proof.

(1) First we will prove that $\sigma^i(A_{-i}) \supseteq \alpha^{-1}J(W)$. Since $\sigma^i(A_{-i})$ is a right V-ideal, we have either $\sigma^i(A_{-i}) \supseteq \alpha^{-1}J(W)$ or $\sigma^i(A_{-i}) \subset \alpha^{-1}J(W)$. If $\sigma^i(A_{-i}) \subset \alpha^{-1}J(W)$, then take any element $b = \alpha^{-1}w$ for some $w \in J(W)$ with $b \notin \sigma^i(A_{-i})$. It is clear that $b^{-1} \notin \sigma^i(A_{-i}^-)$ and $b^{-1} = w^{-1}\alpha \notin W\alpha = A_i$. Thus $b^{-1} \notin A_i \cup \sigma^i(A_{-i}^-) = K$, a contradiction by Lemma 1.1. Hence $\sigma^i(A_{-i}) \supseteq \alpha^{-1}J(W)$ follows. Furthermore, from $A_i\sigma^i(A_{-i}) \subseteq V$, we derive $\sigma^i(A_{-i}) \subseteq (V : A_i)_r$. First suppose that $W \supset V$, then $(V : A_i)_r = \alpha^{-1}J(W)$ by [**4**, the right version of Lemma 1.1]. Hence $\sigma^i(A_{-i}) = \alpha^{-1}J(W)$, i.e., $A_{-i} = \sigma^{-i}(\alpha^{-1}J(W))$. Next suppose that $W = V$, then $\alpha^{-1}J(V) \subseteq \sigma^i(A_{-i}) \subseteq (V : A_i)_r = \alpha^{-1}V$. Thus we have either $\sigma^i(A_{-i}) = \alpha^{-1}V$ or $\sigma^i(A_{-i}) = \alpha^{-1}J(V)$ by Lemma 1.2, i.e., either $A_{-i} = \sigma^{-i}(\alpha^{-1}V)$ or $A_{-i} = \sigma^{-i}(\alpha^{-1}J(V))$. Hence, in the case when $V\alpha = \alpha\sigma^i(V)$, either $A_{-i} = V\sigma^{-i}(\alpha^{-1})$ or $A_{-i} = J(V)\sigma^{-i}(\alpha^{-1})$. Finally in the case when $W = V$ and $V\alpha \supset \alpha\sigma^i(V)$, if $\sigma^i(A_{-i}) = \alpha^{-1}V$, then $V \supseteq A_{-i}\sigma^{-i}(A_i) = \sigma^{-i}(\alpha^{-1}V\alpha)$, and so $\sigma^i(V) \supseteq \alpha^{-1}V\alpha$, a contradiction. Hence $A_{-i} = \sigma^{-i}(\alpha^{-1}J(V))$.

(2) This is proved in the same way as in (1), noticing $\sigma^i(A_{-i}) \supseteq \sigma^i(J(U)\alpha^{-1})$ first. □

Corollary 1.5. *Under the same notation and assumption as in Lemma 1.4, we have*

(1) *Suppose that* $A_i = W\alpha$ *for some non-zero* $\alpha \in K$. *Then* $A_i\sigma^i(A_{-i}) \supseteq J(W)$.

(2) *Suppose that* $A_i = \alpha\sigma^i(U)$ *for some non-zero* $\alpha \in K$. *Then* $\sigma^i(A_{-i})A_i \supseteq \sigma^i(J(U))$.

Proof.

(1) This easily follows, because $\sigma^i(A_{-i}) \supseteq \alpha^{-1}J(W)$ by the proof of Lemma 1.4.

(2) This is proved in the same way as in (1). □

Lemma 1.6. *Let* $A = \oplus_{i\in \boldsymbol{Z}} A_iX^i$ *be a graded extension of* V *in* $K[X, X^{-1}; \sigma]$ *with* $O_l(A_1) = W$ *and* $O_r(A_1) = \sigma(U)$. *Then*

(1) *If* $A_1 = Wa$ *for some non-zero* $a \in K$*, then* $J(W)A_{i+1} \subseteq A_1\sigma(A_i) \subseteq A_{i+1}$ *for all* $i \in \boldsymbol{N}$*.*

(2) *If* $A_1 = a\sigma(U)$ *for some non-zero* $a \in K$*, then* $A_{i+1}\sigma^{i+1}(J(U)) \subseteq A_i\sigma^i(A_1) \subseteq A_{i+1}$ *for all* $i \in \boldsymbol{N}$*.*

Proof.

(1) It is clear that $A_1\sigma(A_i) \subseteq A_{i+1}$ and $A_{-1}\sigma^{-1}(A_{i+1}) \subseteq A_i$. So $\sigma(A_{-1})A_{i+1} \subseteq \sigma(A_i)$. Thus $A_1\sigma(A_{-1})A_{i+1} \subseteq A_1\sigma(A_i)$ follows. Since $A_1\sigma(A_{-1}) \supseteq J(W)$ by Corollary 1.5, we have $J(W)A_{i+1} \subseteq A_1\sigma(A_{-1})A_{i+1} \subseteq A_1\sigma(A_i) \subseteq A_{i+1}$.

(2) This is proved in the same way as in (1). $\square$

2. A classification of graded extensions of V in $K[X, X^{-1}; \sigma]$ with $A_1 = Wa$.

Let $A = \oplus_{i\in\boldsymbol{Z}} A_iX^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$ with $O_l(A_1) = W$, an overring of V. Suppose that A_1 is a finitely generated left W-ideal. Then it is principal, say, $A_1 = Wa$. Since A_1 and $a\sigma(W)$ are both right $\sigma(V)$-ideals, by Lemma 1.4, we can distinguish the following four cases for A:

(a) $W = V, A_1 = Va = a\sigma(V)$ and $A_{-1} = V\sigma^{-1}(a^{-1})$.

(b) $A_1 = Wa \supset a\sigma(W)$.

(c) $A_1 = Wa \subset a\sigma(W)$ (in this case, $W \supset V$).

(d) $A_1 = Wa = a\sigma(W)$ and $A_{-1} = \sigma^{-1}(a^{-1}J(W))$ (in this case, we must consider two cases, $J(W) = J(W)^2$ and $J(W) \supset J(W)^2$).

The aim of this section is to describe the structure of A_i and A_{-i} based on the properties of A_1 and A_{-1} according to the classification above.

In the remainder of this section, we assume that $A = \oplus_{i\in\boldsymbol{Z}} A_iX^i$ is a subset of V in $K[X, X^{-1}; \sigma]$ with $A_0 = V$ and $A_1 = Wa$ for some $a \in K$, where $W \subset K$ is an overring of V.

For a fixed non-zero $a \in K$, we set

$$\alpha_i = a\sigma(a)\cdots\sigma^{i-1}(a), \alpha_{-i} = \sigma^{-i}(\alpha_i^{-1}) \text{ for all } i \in \boldsymbol{N} \text{ and } \alpha_0 = 1.$$

Then we have

$$\alpha_{-i} = \sigma^{-1}(a^{-1})\sigma^{-2}(a^{-1})\cdots\sigma^{-i}(a^{-1}) \text{ for all } i \in \boldsymbol{N}, \alpha_i = \sigma^i(\alpha_{-i}^{-1})$$

and

$$\alpha_i\sigma^i(\alpha_j) = \alpha_{i+j} \text{ for all } i, j \in \boldsymbol{Z},$$

which are freely used in this section.

In Lemma 2.1, we will use the following general property of total valuation rings: If $W \supset V$, then $J(V) \supset J(W)$ and $J(V)J(W) = J(W)$.

Lemma 2.1. *Let W and U be overrings of V and let $0 \neq a \in K$ as above. Then*

(1) *Suppose that $Wa = a\sigma(W)$. Then $W\alpha_i = \alpha_i\sigma^i(W)$, $J(W)\alpha_i = \alpha_i\sigma^i(J(W))$ for all $i \in \boldsymbol{Z}$. Furthermore, if $J(W)$ is principal, say, $J(W) = Wb^{-1} = b^{-1}W$ for some $b^{-1} \in J(W)$, then $J(W)^j\alpha_i = \alpha_i\sigma^i(J(W)^j)$ for all $i, j \in \boldsymbol{Z}$, where $J(W)^j = Wb^{-j}$.*

(2) *Suppose that $Wa \supset a\sigma(W)$. Then $W\alpha_i \supset \alpha_i\sigma^i(W)$, $J(W)\alpha_i \subset \alpha_i\sigma^i(J(W))$ and $J(W)\alpha_{-i}\sigma^{-i}(J(W)) = \alpha_{-i}\sigma^{-i}(J(W))$ for all $i \in \boldsymbol{N}$. In particular, $W\alpha_i$ is a right $\sigma^i(W)$-ideal and $\alpha_{-i}\sigma^{-i}(J(W))$ is a left W-ideal.*

(3) *Suppose that $a\sigma(U) \supset Ua$. Then $\alpha_i\sigma^i(U) \supset U\alpha_i$, $\alpha_i\sigma^i(J(U)) \subset J(U)\alpha_i$, and $J(U)\alpha_{-i}\sigma^{-i}(J(U)) = J(U)\alpha_{-i}$ for all $i \in \boldsymbol{N}$. In particular, $\alpha_i\sigma^i(U)$ is a left U-ideal and $J(U)\alpha_{-i}$ is a right $\sigma^{-i}(J(U))$-ideal.*

Proof.

(1) For any $i \in \boldsymbol{Z}$, the formulas $W\alpha_i = \alpha_i\sigma^i(W)$, $J(W)\alpha_i = \alpha_i\sigma^i(J(W))$ are easily proved by induction on i. In the case when $J(W)$ is principal, $J(W)\alpha_i = \alpha_i\sigma^i(J(W))$ implies $J(W)^{-1}\alpha_i = \alpha_i\sigma^i(J(W)^{-1})$ and so $J(W)^j\alpha_i = \alpha_i\sigma^i(J(W)^j)$ is also proved by induction on j for any $j \in \boldsymbol{Z}$.

(2) We inductively have: $\alpha_i^{-1}W\alpha_i \supset \sigma(\alpha_{i-1}^{-1}W\alpha_{i-1}) \supset \cdots \supset \sigma^i(W)$ and so $W\alpha_i \supset \alpha_i\sigma^i(W)$ follows. From $\alpha_i^{-1}W\alpha_i \supset \sigma^i(W)$, we derive $\alpha_i^{-1}J(W)\alpha_i \subset \sigma^i(J(W))$ and so $J(W)\alpha_i \subset \alpha_i\sigma^i(J(W))$ follows. Furthermore, $\sigma^i(J(W)) \supset \alpha_i^{-1}J(W)\alpha_i$ implies $J(W) \supset \sigma^{-i}(\alpha_i^{-1})\sigma^{-i}(J(W))\sigma^{-i}(\alpha_i)$. So it follows that $J(W)\sigma^{-i}(\alpha_i^{-1})\sigma^{-i}(J(W))\sigma^{-i}(\alpha_i) = \sigma^{-i}(\alpha_i^{-1})\sigma^{-i}(J(W))\sigma^{-i}(\alpha_i)$. Thus $J(W)\alpha_{-i}\sigma^{-i}(J(W)) = \alpha_{-i}\sigma^{-i}(J(W))$, because $\alpha_{-i} = \sigma^{-i}(\alpha_i^{-1})$. The last statement is now clear.

(3) This is proved in a similar way as in (2). □

We start with the case (a) which is the simplest one.

Theorem 2.2. *Let $A = \oplus_{i\in\boldsymbol{Z}} A_iX^i$ be a subset of $K[X, X^{-1}; \sigma]$ with $A_0 = V$, $A_1 = Va = a\sigma(V)$ and $A_{-1} = V\sigma^{-1}(a^{-1})$. Then $A = \oplus_{i\in\boldsymbol{Z}} A_iX^i$ is a graded extension of V in $K[X, X^{-1}; \sigma]$ if and only if $A_i = V\alpha_i$ for all $i \in \boldsymbol{Z}$.*

Proof. Suppose that $A = \oplus_{i\in\boldsymbol{Z}} A_iX^i$ is a graded extension of V in $K[X, X^{-1}; \sigma]$. We will prove that $A_i = V\alpha_i$ for all $i \in \boldsymbol{N}$ by induction on i. Assume that $A_i = V\alpha_i$ for some $i \in \boldsymbol{N}$. $A_{-1}\sigma^{-1}(A_{i+1}) \subseteq A_i$ implies that $\sigma(A_{-1})A_{i+1} \subseteq \sigma(A_i)$. So $A_{i+1} \subseteq a\sigma(A_i) \subseteq A_{i+1}$. Hence $A_{i+1} = A_1\sigma(A_i) = Va\sigma(V\alpha_i) = V\alpha_{i+1}$. Similarly we have $A_{-i-1} = A_{-1}\sigma^{-1}(A_{-i}) =$

$V\sigma^{-1}(a^{-1})\sigma^{-1}(V\alpha_{-i}) = V\alpha_{-i-1}$.

Conversely, suppose that $A_i = V\alpha_i$ for all $i \in \boldsymbol{Z}$. Then $A = \oplus_{i\in\boldsymbol{Z}} A_i X^i$ is an additive subgroup of $K[X, X^{-1}; \sigma]$. Since $V\alpha_i = \alpha_i\sigma^i(V)$ by Lemma 2.1 and $\alpha_i\sigma^i(\alpha_j) = \alpha_{i+j}$ for all $i, j \in \boldsymbol{Z}$, we have $A_iX^iA_jX^j = A_{i+j}X^{i+j}$. For any $i \in \boldsymbol{Z}$ with $i \neq 0$, we have $A_i = V\alpha_i = \alpha_i\sigma^i(V) = \sigma^i(\alpha_{-i}^{-1})\sigma^i(V) \supseteq \sigma^i(\alpha_{-i}^{-1}J(V))$ and $A_{-i} = V\alpha_{-i} = \alpha_{-i}\sigma^{-i}(V) = \sigma^{-i}(\alpha_i^{-1}V) \supseteq \sigma^{-i}(\alpha_{-i}^{-1}J(V))$. Thus $A_i \cup \sigma(A_{-i}^{-}) = K$ by Lemma 1.3 (1). Hence $A = \oplus_{i\in\boldsymbol{Z}} A_iX^i$ is a graded extension of V in $K[X, X^{-1}; \sigma]$ by Lemma 1.1. □

The following are typical examples of graded extensions of V in $K[X, X^{-1}, \sigma]$.

Proposition 2.3. *Let W and U be overrings of V.*

(1) *Suppose that either $Wa \supset a\sigma(W)$ or $Wa = a\sigma(W)$. Set $A_i = W\alpha_i$, $A_{-i} = \alpha_{-i}\sigma^{-i}(J(W))$ and $A_0 = V$ for all $i \in \boldsymbol{N}$. Then $A = \oplus_{i\in\boldsymbol{Z}} A_iX^i$ is a graded extension of V in $K[X, X^{-1}; \sigma]$.*

(2) *Suppose that $a\sigma(U) \supset Ua$. Set $A_i = \alpha_i\sigma^i(U)$, $A_{-i} = J(U)\alpha_{-i}$ and $A_0 = V$ for all $i \in \boldsymbol{N}$. Then $A = \oplus_{i\in\boldsymbol{Z}} A_iX^i$ is a graded extension of V in $K[X, X^{-1}; \sigma]$.*

Proof. We will only prove this in the case where $A_1 = Wa \supset a\sigma(W)$. It is clear that A is an additive subgroup of $K[X, X^{-1}; \sigma]$ and that $A_i \cup \sigma(A_{-i}^{-}) = K$ for all $i \in \boldsymbol{Z}$ by Lemma 1.3 (1) and (3). Thus it suffices to prove that $A_i\sigma^i(A_j) \subseteq A_{i+j}$ for all $i, j \in \boldsymbol{Z}$ by Lemma 1.1, which will be proved in the following way:

For any $i, j \in \boldsymbol{N}$, by using Lemma 2.1 (2), we have

$$A_i\sigma^i(A_j) = W\alpha_i\sigma^i(W\alpha_j) = W\alpha_i\sigma^i(W)\sigma^i(\alpha_j) = W\alpha_{i+j} = A_{i+j},$$

$$\begin{aligned} A_{-i}\sigma^{-i}(A_{-j}) &= \alpha_{-i}\sigma^{-i}(J(W)A_{-j}) = \alpha_{-i}\sigma^{-i}(A_{-j}) = \alpha_{-i}\sigma^{-i}(\alpha_{-j}\sigma^{-j}(J(W))) \\ &= \alpha_{-i-j}\sigma^{-i-j}(J(W)) = A_{-i-j}, \end{aligned}$$

$$\begin{aligned} A_i\sigma^i(A_{-j}) &= W\alpha_i\sigma^i(\alpha_{-j}\sigma^{-j}(J(W))) = W\alpha_i\sigma^i(\alpha_{-j})\sigma^{i-j}(J(W)) \\ &= W\alpha_{i-j}\sigma^{i-j}(J(W)) \subseteq A_{i-j} \text{ and} \end{aligned}$$

$$A_{-i}\sigma^{-i}(A_j) = \alpha_{-i}\sigma^{-i}(J(W))\sigma^{-i}(W\alpha_j) = \sigma^{-i}(\alpha_i^{-1}J(W)\alpha_j).$$

So if $j \geq i$, then $A_{-i}\sigma^{-i}(A_j) \subseteq \sigma^{-i}(\sigma^i(J(W))\alpha_i^{-1}\alpha_j) = J(W)\,\sigma^{-i}(\alpha_i^{-1})\sigma^{-i}(\alpha_j) = J(W)\alpha_{-i+j} \subseteq A_{-i+j}$. If $i < j$, then $A_{-i}\sigma^{-i}(A_j) = \alpha_{-i}\sigma^{-i}(J(W)\alpha_j) \subseteq \alpha_{-i}\sigma^{-i}(\alpha_j\sigma^j(J(W))) = \alpha_{-i+j}\sigma^{-i+j}(J(W)) = A_{-i+j}$. If either $i = 0$ or $j = 0$, then it is clear that $A_i\sigma^i(A_j) = A_{i+j}$. Hence A is a graded extension of V in $K[X, X^{-1}; \sigma]$.

In the case where $A_1 = Wa = a\sigma(W)$, it is proved in a similar way by using Lemmas 1.1, 2.1 (1), and 1.3 (1) and (3).

(2) This is also proved in a similar way as in (1) by using Lemmas 1.1, 2.1 (3), and 1.3 (2) and (4). $\square$

Second, we will consider the case (b), i.e., $A_1 = Wa \supset a\sigma(W)$.

THEOREM 2.4. *Let W be an overring of V and let $A = \oplus_{i\in \boldsymbol{Z}} A_i X^i$ be a subset of $K[X, X^{-1}; \sigma]$ with $A_0 = V$ and $A_1 = Wa \supset a\sigma(W)$. Then $A = \oplus_{i\in \boldsymbol{Z}} A_i X^i$ is a graded extension of V in $K[X, X^{-1}; \sigma]$ if and only if $A_i = W\alpha_i$ and $A_{-i} = \alpha_{-i}\sigma^{-i}(J(W))$ for all $i \in \boldsymbol{N}$.*

PROOF. Suppose that A is a graded extension of V in $K[X, X^{-1}; \sigma]$. We will prove that $A_i = W\alpha_i$ for all $i \in \boldsymbol{N}$ by induction on i. Assume that $A_i = W\alpha_i$ for some $i \in \boldsymbol{N}$. Then $A_1\sigma(A_i) = Wa\sigma(W)\sigma(\alpha_i) = Wa\sigma(\alpha_i) = W\alpha_{i+1}$, because $Wa\sigma(W) = Wa$. Because of $J(W)A_{i+1} \subseteq A_1\sigma(A_i) \subseteq A_{i+1} \subseteq WA_{i+1}$ by Lemma 1.6, it follows from Lemma 1.2 that either $J(W)A_{i+1} = W\alpha_{i+1}$ or $A_{i+1} = W\alpha_{i+1}$. Assume that $J(W)A_{i+1} = W\alpha_{i+1}$. By Lemma 1.4, $\sigma(A_{-1}) = a^{-1}J(W)$ and so $a^{-1}J(W)A_{i+1} = \sigma(A_{-1})A_{i+1} \subseteq \sigma(A_i) = \sigma(W\alpha_i)$. Thus $W\alpha_{i+1} = J(W)A_{i+1} \subseteq a\sigma(W\alpha_i)$ and so $Wa \subseteq a\sigma(W)$ follows, which is a contradiction. Hence $A_{i+1} = W\alpha_{i+1}$, as desired. It follows from Lemma 1.4 that $A_{-i} = \sigma^{-i}(\alpha_i^{-1}J(W)) = \alpha_{-i}\sigma^{-i}(J(W))$.

Conversely, suppose that $A_i = W\alpha_i$ and $A_{-i} = \alpha_{-i}\sigma^{-i}(J(W))$ for all $i \in \boldsymbol{N}$. Then A is a graded extension of V in $K[X, X^{-1}; \sigma]$ by Proposition 2.3. $\square$

Third, we will consider the case (c), i.e., $A_1 = Wa \subset a\sigma(W)$. In this case, we note that $W \supset V$ and $\sigma(W) \supset a^{-1}Wa = O_r(A_1) = \sigma(U)$. So it follows that $W \supset U \supseteq V$ and $A_1 = a\sigma(U) \supset Ua$. Furthermore, $\sigma(A_{-1}) = a^{-1}J(W) = a^{-1}J(W)aa^{-1} = \sigma(J(U))a^{-1}$. Note that $a\sigma(U) \supset Ua$ implies $a\sigma(W) \supset Wa$. Hence the proof of following theorem will be similar to the proof of Theorem 2.4.

THEOREM 2.5. *Let W be an overring of V and let $A = \oplus_{i\in \boldsymbol{Z}} A_i X^i$ be a subset of $K[X, X^{-1}; \sigma]$ with $A_0 = V$ and $A_1 = Wa = Wa\sigma(V) \subset a\sigma(W)$. Set $a^{-1}Wa = \sigma(U)$ and assume $U \supseteq V$. Then A is a graded extension of V in $K[X, X^{-1}; \sigma]$ if and only if $A_i = \alpha_i\sigma^i(U)$ and $A_{-i} = J(U)\alpha_{-i}$ for all $i \in \boldsymbol{N}$.*

Finally, we will study the case (d), i.e., $A_1 = Wa = a\sigma(W)$ and $A_{-1} = \sigma^{-1}(a^{-1}J(W))$. In this case, we note that $A_{-1} = J(W)\alpha_{-1}$ by Lemma 2.1. We first consider the case where $J(W) = J(W)^2$.

THEOREM 2.6. *Let W be an overring of V and let $A = \oplus_{i\in \boldsymbol{Z}} A_i X^i$ be a subset of $K[X, X^{-1}; \sigma]$ with $A_0 = V$, $A_1 = Wa = a\sigma(W)$ and $A_{-1} = J(W)\alpha_{-1}$.*

Suppose that $J(W)^2 = J(W)$. *Then* A *is a graded extension of* V *in* $K[X, X^{-1}; \sigma]$ *if and only if* $A_i = W\alpha_i$ *and* $A_{-i} = J(W)\alpha_{-i}$ *for all* $i \in \boldsymbol{N}$.

Proof. Suppose that A is a graded extension of V in $K[X, X^{-1}; \sigma]$. We will prove that $A_i = W\alpha_i$ for all $i \in \boldsymbol{N}$ by induction on i. Assume that $A_i = W\alpha_i$ for some $i \in \boldsymbol{N}$. Then $A_1\sigma(A_i) = W\alpha_{i+1}$ since $Wa = a\sigma(W)$. Since $J(W)A_{i+1} \subseteq A_1\sigma(A_i)$ by Lemma 1.6, it follows that $A_{i+1} \subseteq (W\alpha_{i+1} : J(W))_r = W\alpha_{i+1}$, because $J(W)^2 = J(W)$ and $(W : J(W))_r = W$. Hence $A_{i+1} = W\alpha_{i+1}$ follows. By Lemma 1.4, either $A_{-i} = \sigma^{-i}(\alpha_i^{-1}J(W))$ or $A_{-i} = W\sigma^{-i}(\alpha_i^{-1}) = \sigma^{-i}(\alpha_i^{-1}W)$. Assume that $A_{-l} = \sigma^{-l}(\alpha_l^{-1}W)$ for some $l \in \boldsymbol{N}$ (we may assume that l is the smallest natural number for this possibility). Then $l > 1$ and so we have, by Lemma 2.1,

$$\begin{aligned} A_{-1} \supseteq A_{-l}\sigma^{-l}(A_{l-1}) &= \sigma^{-l}(\alpha_l^{-1}W \cdot W\alpha_{l-1}) = \alpha^{-l}(\alpha_l^{-1}\alpha_{l-1}\sigma^{l-1}(W)) \\ &= \sigma^{-l}(\sigma^{l-1}(a^{-1}W)) = \sigma^{-1}(a^{-1}W) \supset \sigma^{-1}(a^{-1}J(W)) = A_{-1}, \end{aligned}$$

which is a contradiction. Hence $A_{-i} = \sigma^{-i}(\alpha_i^{-1}J(W)) = \alpha_{-i}\sigma^{-i}(J(W)) = J(W)\alpha_{-i}$ for all $i \in \boldsymbol{N}$ by Lemma 2.1.

Conversely, suppose that $A_i = W\alpha_i$ and $A_{-i} = J(W)\alpha_{-i}$ for all $i \in \boldsymbol{N}$. Then A is a graded extension of V in $K[X, X^{-1}; \sigma]$ by Lemma 2.1 and Proposition 2.3. □

As it has been seen in Theorems 2.2 and 2.4 $\sim$ 2.6, the graded extension $A = \oplus_{i\in\boldsymbol{Z}} A_i X^i$ is uniquely determined by A_1 and A_{-1} in the cases (a), (b), (c) and (d) with $J(W) = J(W)^2$. However, in the case (d) with $J(W) \supset J(W)^2$, A is not uniquely determined by A_1 and A_{-1}. In fact, we will show in Section 3 that the cardinality of the set of all graded extensions is at least $\aleph$.

In the remainder of this section, we assume that $J(W) \supset J(W)^2$, i.e., $J(W)$ is principal, say, $J(W) = b^{-1}W = Wb^{-1}$ for some $b^{-1} \in J(W)$ as well as $A_1 = Wa = a\sigma(W)$ and $A_{-1} = J(W)\alpha_{-1}$.

Lemma 2.7. *Let* $A = \oplus_{i\in\boldsymbol{Z}} A_i X^i$ *be a graded extension of* V *in* $K[X, X^{-1}; \sigma]$ *with* $A_1 = Wa = a\sigma(W)$ *and* $A_{-1} = J(W)\alpha_{-1}$. *Suppose that* $J(W) = b^{-1}W = Wb^{-1}$. *Then for any* $i \in \boldsymbol{Z}$, *there is an element* $k \in \boldsymbol{Z}$ *such that* $Wb^{k-1}\alpha_i \subset A_i \subseteq Wb^k\alpha_i$ *and* $WA_i = Wb^k\alpha_i$. *In particular,* WA_i *is a right* $\sigma^i(W)$*-ideal.*

Proof. First note that $J(W)^k\alpha_i = \alpha_i\sigma^i(J(W)^k)$ for all $i, k \in \boldsymbol{Z}$ by Lemma 2.1.

We will first prove that $Wb^{k-1}\alpha_i \subset A_i \subseteq Wb^k\alpha_i$ for any $i \in \boldsymbol{N}$ by induction on i. If $i = 1$, then $k = 0$ and so we may assume that $Wb^{k-1}\alpha_i \subset A_i \subseteq Wb^k\alpha_i$

for some $k \geq 0$. Then $WA_i = Wb^k\alpha_i$ by Lemma 1.2 and so $A_1\sigma(A_i) = Wb^k\alpha_{i+1}$. Thus, from $J(W)A_{i+1} \subseteq A_1\sigma(A_i) \subseteq A_{i+1}$, we have either $WA_{i+1} = Wb^k\alpha_{i+1}$ or $b^{-1}WA_{i+1} = Wb^k\alpha_{i+1}$, i.e., $WA_{i+1} = Wb^{k+1}\alpha_{i+1}$. So $Wb^{k-1}\alpha_{i+1} \subset A_{i+1} \subseteq Wb^k\alpha_{i+1}$ in the former case and $Wb^k\alpha_{i+1} \subset A_{i+1} \subseteq Wb^{k+1}\alpha_{i+1}$ in the latter case. Since $A_{-1} = J(W)\alpha_{-1}$, we can prove that for any $i \in \boldsymbol{N}$, $Wb^{k-1}\alpha_{-i} \subset A_{-i} \subseteq Wb^k\alpha_{-i}$ and $WA_{-i} = Wb^k\alpha_{-i}$ for some $k < 0$ in the same way. It is clear that WA_i is a right $\sigma^i(W)$-ideal for all $i \in \boldsymbol{Z}$. □

In Lemma 2.7, for any $i \in \boldsymbol{Z}$, $WA_i = Wb^k\alpha_i$ for some $k \in \boldsymbol{Z}$. More generally, we have

Lemma 2.8. *Let W be an overring of V and let $\gamma_i \in K$ be nonzero elements such that $W\gamma_i = \gamma_i\sigma^i(W)$, $\gamma_i\sigma^i(\gamma_j) = \gamma_{i+j}$ and $\gamma_0 = 1$ for all $i, j \in \boldsymbol{Z}$. Suppose that $J(W) = b^{-1}W = Wb^{-1}$ and that, for any $i \in \boldsymbol{Z}$, there is an $f(i) \in \boldsymbol{Z}$ with $f(0) = 0$. Set $B_i = Wb^{f(i)}\gamma_i$ for all $i \in \boldsymbol{Z}$ with $i \neq 0$ and $B_0 = V$. Then*

(1) *For any $i, j \in \boldsymbol{Z}$ with $j \neq -i$, $B_i\sigma^i(B_j) \subseteq B_{i+j}$ if and only if $f(i)+f(j) \leqq f(i+j)$.*

(2) *For any $i \in \boldsymbol{Z}$, $B_i \cup \sigma^i(B_{-i}^-) = K$ if and only if $f(i) + f(-i) \geq -1$.*

Proof.

(1) Because of $\gamma_i\sigma^i(J(W)^k) = J(W)^k\gamma_i$ as in Lemma 2.7, for any $i, k \in \boldsymbol{Z}$, we have $B_i\sigma^i(B_j) = Wb^{f(i)}\gamma_i\sigma^i(Wb^{f(j)})\sigma^i(\gamma_j) = Wb^{f(i)+f(j)}\gamma_i\sigma^i(\gamma_j) = Wb^{f(i)+f(j)}\gamma_{i+j}$. Hence $B_i\sigma^i(B_j) \subseteq B_{i+j}$ if and only if $f(i) + f(j) \leqq f(i + j)$ for all $i, j \in \boldsymbol{Z}$ with $j \neq -i$.

(2) Suppose that $B_i \cup \sigma^i(B_{-i}^-) = K$ for all $i \in \boldsymbol{Z}$. If $f(i) + f(-i) \leqq -2$ for some $i \in \boldsymbol{Z}$, then $i \neq 0$ and $c = b^{f(i)+1}\gamma_i \notin B_i$. Then we have

$$
\begin{aligned}
c^{-1}W &= \gamma_i^{-1}b^{-f(i)-1}W = \sigma^i(Wb^{-f(i)-1})\gamma_i^{-1} \supset \sigma^i(Wb^{-f(i)-2}\gamma_{-i}) \\
&\supseteq \sigma^i(Wb^{f(-i)}\gamma_{-i}) = \sigma^i(B_{-i}).
\end{aligned}
$$

Thus $c \notin B_i \cup \sigma^i(B_{-i}^-) = K$, a contradiction. Hence $f(i) + f(-i) \geq -1$ for all $i \in \boldsymbol{Z}$.

Conversely, suppose that $f(i) + f(-i) \geq -1$ for all $i \in \boldsymbol{Z}$. If $c \in K$ with $c \notin B_i$, then $b^{f(i)}\gamma_i c^{-1} \in J(W) = b^{-1}W$ and so $b^{f(i)+1}\gamma_i c^{-1} \in W$. Thus we have

$$
\begin{aligned}
c^{-1} &\in \gamma_i^{-1}b^{-f(i)-1}W \subseteq \gamma_i^{-1}b^{f(-i)}W = \sigma^i(Wb^{f(-i)})\gamma_i^{-1} \\
&= \sigma^i(Wb^{f(-i)})\sigma^i(\gamma_{-i}) = \sigma^i(B_{-i}).
\end{aligned}
$$

So $c \in \sigma^i(B_{-i}^-)$ and hence $B_i \cup (\sigma^i(B_{-i}^-) = K$ follows. □

From Lemma 2.8 we have the following definition:

A map $f : \boldsymbol{Z} \longrightarrow \boldsymbol{Z}$ is called a graded map if $f(0) = 0$, $f(i) + f(j) \leqq f(i+j)$ and $f(i) + f(-i) \geq -1$ for all $i, j \in \boldsymbol{Z}$.

A graded map f is called a nice map if $f(1) = 0, f(-1) = -1$.

If f is a graded map, then we note that either $f(i) + f(-i) = -1$ or $f(i) + f(-i) = 0$ for any $i \in \boldsymbol{Z}$, because $-1 \leqq f(i) + f(-i) \leqq f(i + (-i)) \leqq f(0) = 0$. Furthermore, $f(i) + f(j) = f(i + j)$ or $f(i) + f(j) = f(i + j) - 1$ for any $i, j \in \boldsymbol{Z}$, because $f(i) \geq f(i + j) + f(-j) \geq f(i + j) - f(j) - 1$.

Assume that $W \neq V$ is an overring of V. Then, under the notation and assumption in Lemma 2.8, we have $B = \oplus_{i \in \boldsymbol{Z}} B_i X^i$ is a graded extension of V in $K[X, X^{-1}; \sigma]$ if and only if f is a graded map with $f(i) + f(-i) = -1$ for any $i \neq 0$. Furthermore, B is a graded extension of V in $K[X, X^{-1}; \sigma]$ with $B_1 = W\gamma_1$ and $B_{-1} = J(W)\gamma_{-1}$ if and only if f is a nice map with $f(i) + f(-i) = -1$ for any $i \neq 0$.

Now under the notation and assumption in Lemma 2.7, for any $i \in \boldsymbol{Z}$, $WA_i = Wb^k\alpha_i$ for some $k \in \boldsymbol{Z}$. We define $f(i) = k$. Then we have

Lemma 2.9.

(1) *The map f defined above is a nice map.*

(2) *$Wb^{f(i)-1}\alpha_i \subset A_i \subset Wb^{f(i)}\alpha_i$ for some $i \in \boldsymbol{Z}$ with $|i| \geq 2$ if and only if $W \neq V$ and $f(i) + f(-i) = 0$.*

Proof.

(1) It is clear that $f(0) = 0 = f(1)$ and $f(-1) = -1$, since $A_1 = Wa, A_0 = V$ and $A_{-1} = J(W)\alpha_{-1} = Wb^{-1}\alpha_{-1}$. Now let $B_i = WA_i = Wb^{f(i)}\alpha_i$ for any $i \in \boldsymbol{Z}$ with $i \neq 0$ and $B_0 = V$. Then $B_i\sigma^i(B_j) = WA_i\sigma^i(W)\sigma^i(A_j) = WA_i\sigma^i(A_j) \subseteq WA_{i+j} = B_{i+j}$ if $j \neq -i$, since WA_i is a right $\sigma^i(W)$-ideal. Furthermore, it is clear that $B_i \supseteq A_i$ and $B_{-i} \supseteq A_{-i}$ for all $i \in \boldsymbol{Z}$. Hence f is a nice map by Lemmas 1.1 and 2.8.

(2) Suppose that $Wb^{f(i)-1}\alpha_i \subset A_i \subset Wb^{f(i)}\alpha_i$ for some $i \in \boldsymbol{Z}$. Then $|i| \geq 2$ since $A_1 = Wa$ and $A_{-1} = Wb^{-1}\alpha_{-1}$. By Lemma 1.2, $W \neq V$. Assume that $f(i) + f(-i) = -1$. Let $\beta = wb^{f(i)}\alpha_i \in Wb^{f(i)}\alpha_i \setminus A_i$, then w is a unit of W and so

$$\begin{aligned}\beta^{-1}W &= \alpha_i^{-1}b^{-f(i)}W = \sigma^i(b^{f(-i)}W)\alpha_i^{-1} = \sigma^i(Wb^{f(-i)+1})\alpha_i^{-1}\\ &= \sigma^i(Wb^{f(-i)+1}\alpha_{-i}) \supset \sigma^i(A_{-i}),\end{aligned}$$

which shows $\beta \notin \sigma^i(A_{-i}^-) \cup A_i = K$, a contradiction. Hence $f(i) + f(-i) = 0$.

Conversely, suppose that $W \neq V$ and $f(i) + f(-i) = 0$. Then it is clear that $|i| \geq 2$, because $f(1) = 0$ and $f(-1) = -1$. Assume that $A_i = Wb^{f(i)}\alpha_i$. Then, by Lemma 1.4, we have $A_{-i} = \sigma^{-i}(\alpha_i^{-1} b^{-f(i)} J(W)) = \alpha_{-i}\sigma^{-i}(Wb^{f(-i)-1}) = Wb^{f(-i)-1}\alpha_{-i}$, a contradiction. Hence $Wb^{f(i)-1}\alpha_i \subset A_i \subset Wb^{f(i)}\alpha_i$ follows. □

LEMMA 2.10. *Let f be a graded map with $f(l) + f(-l) = 0$ for some $l \in \boldsymbol{N}$. Then*

(1) *$f(i+l) = f(i) + f(l)$ and $f(i-l) = f(i) + f(-l)$ for all $i \in \boldsymbol{Z}$.*

(2) *Suppose that l is the smallest natural number with $f(l)+f(-l) = 0$. Then $f(j) + f(-j) = 0$ if and only if $j \in l\boldsymbol{Z}$.*

PROOF.

(1) For any $i \in \boldsymbol{Z}$, $f(i) = f(i+l-l) \geq f(i+l) + f(-l)$. So $f(i+l) \geq f(i)+f(l) \geq f(i+l)+f(-l)+f(l) = f(i+l)$, which shows $f(i+l) = f(i)+f(l)$. Similarly, we have $f(i-l) = f(i) + f(-l)$.

(2) If $j = lq$ for some $q \in \boldsymbol{Z}$, then $f(j) + f(-j) = qf(l) + qf(-l) = 0$ by (1). Conversely, suppose that $f(j) + f(-j) = 0$ and let $j = lp + i$ for some $p, i \in \boldsymbol{Z}$ with $0 \leqq i < l$. Then $0 = f(j) + f(-j) = f(i) + f(-i)$ by (1), which shows $i = 0$, i.e., $j \in l\boldsymbol{Z}$. □

Now we are ready to describe the case (d) with $J(W) \supset J(W)^2$.

THEOREM 2.11. *Let W be an overring of V and let $A = \oplus_{i\in\boldsymbol{Z}} A_i X^i$ be a subset of $K[X, X^{-1}; \sigma]$ with $A_0 = V$, $A_1 = Wa = a\sigma(W)$ and $A_{-1} = J(W)\alpha_{-1}$. Suppose that $J(W) = b^{-1}W = Wb^{-1}$ for some $b^{-1} \in J(W)$. Then A is a graded extension of V in $K[X, X^{-1}; \sigma]$ if and only if the following properties hold*:

(1) *There is a nice map f such that $WA_i = Wb^{f(i)}\alpha_i$ for all $i \in \boldsymbol{Z}$.*

(2) (a) *If either $W = V$ or $f(i) + f(-i) = -1$ for all $i \in \boldsymbol{Z}$ with $i \neq 0$, then $A_i = Wb^{f(i)}\alpha_i$ for all $i \in \boldsymbol{Z}$ with $i \neq 0$.*

(b) *If $W \neq V$ and there is an $l \in \boldsymbol{N}$ $(l \geq 2)$ with $f(l) + f(-l) = 0$ (assume l is the smallest natural number for this property), then $A_i = Wb^{f(i)}\alpha_i$ for all $i \notin l\boldsymbol{Z}$ and $B = \oplus_{j\in\boldsymbol{Z}} A_{jl}X^{jl}$ is a graded extension of V in $K[X^l, X^{-l}; \sigma^l]$ with $Wb^{f(jl)-1}\alpha_{jl} \subset A_{jl} \subset Wb^{f(jl)}\alpha_{jl}$ for all $j \in \boldsymbol{Z}$.*

PROOF. Suppose that $A = \oplus_{i\in\boldsymbol{Z}} A_i X^i$ is a graded extension of V in $K[X, X^{-1}; \sigma]$. Then there is a nice map f such that $WA_i = Wb^{f(i)}\alpha_i$ for all $i \in \boldsymbol{Z}$ by Lemmas 2.7 and 2.9. In the case (2) (a), it follows from Lemmas 2.7 and 2.9 that $A_i = Wb^{f(i)}\alpha_i$ for all $i \in \boldsymbol{Z}$ with $i \neq 0$. In the case (2) (b), the statement follows from Lemmas 2.9 and 2.10.

Conversely, suppose that (1) and either (2) (a) or (2) (b) hold. Then A is an additive subgroup of $K[X, X^{-1}; \sigma]$ with $A_0 = V$. In order to prove that $A_i \cup \sigma^i(A_{-i}^-) = K$ for all $i \in \boldsymbol{Z}$, we may assume that $f(i)+f(-i) = -1$, $A_i = Wb^{f(i)}\alpha_i$ and $A_{-i} = Wb^{f(-i)}\alpha_{-i}$ by the assumption. Then $A_{-i} = \alpha_{-i}\sigma^{-i}(Wb^{f(-i)}) =$

$\alpha_{-i}\sigma^{-i}(Wb^{-f(i)-1}) = \sigma^{-i}(\alpha_i^{-1}b^{-f(i)}J(W))$. Hence $A_i \cup \sigma^i(A_{-i}^-) = K$ by Lemma 1.3 (1).

Finally we will prove that A is a ring. Note that $A_i\sigma^i(A_j) \subseteq Wb^{f(i)}\alpha_i\sigma^i(Wb^{f(j)}\alpha_j) = Wb^{f(i)+f(j)}\alpha_i\sigma^i(\alpha_j) = Wb^{f(i)+f(j)}\alpha_{i+j}$ by Lemma 2.1. So if $i+j \notin l\boldsymbol{Z}$, then $A_i\sigma^i(A_j) \subseteq A_{i+j}$ follows. In the case when $i+j \in l\boldsymbol{Z}$, there are two cases, i.e., either $i,j \in l\boldsymbol{Z}$ or $i,j \notin l\boldsymbol{Z}$. If $i,j \in l\boldsymbol{Z}$, then $A_i\sigma^i(A_j) \subseteq A_{i+j}$, since B is a graded extension of V in $K[X, X^{-1};\sigma]$.

If $i,j \notin l\boldsymbol{Z}$, then $j = kl - i$ for some $k \in \boldsymbol{Z}$. So we have $f(i) + f(j) = f(i) + f(-i+kl) = f(i) + f(-i) + f(kl) = -1 + f(kl) = -1 + f(i+j)$ by Lemma 2.10. Thus $A_i\sigma^i(A_j) \subseteq Wb^{f(i)+f(j)}\alpha_{i+j} = Wb^{f(i+j)-1}\alpha_{i+j} \subset A_{i+j}$ by Lemma 2.7. Hence A is a graded extension of V in $K[X, X^{-1};\sigma]$ by Lemma 1.1. □

3. Description of nice maps.

As it has been seen in Section 2, nice maps are useful in the study of graded extensions of V in $K[X, X^{-1};\sigma]$. In this section we will give a full description of nice maps.

Lemma 3.1. *Let f be a nice map. Then*
(1) $f(i) + 1 \geq f(i+1) \geq f(i)$ *for all* $i \in \boldsymbol{Z}$.
(2) $0 \leqq f(i) < i$ *for all* $i \in \boldsymbol{N}$.

Proof.
(1) Since $f(i+1) \geq f(i)+f(1) = f(i)$ and $f(i) \geq f(i+1)+f(-1) = f(i+1)-1$, we have $f(i) + 1 \geq f(i+1) \geq f(i)$.
(2) This easily follows from (1) by induction on i. □

Let f be a nice map. Then $0 \leqq f(i)/i < 1$ for all $i \in \boldsymbol{N}$ by Lemma 3.1. Let $\gamma = \sup\{f(i)/i \mid i \in \boldsymbol{N}\}$. We will use this γ to describe all nice maps.

Lemma 3.2. *Let f be a nice map with $f(l) + f(-l) = 0$ for some $l \in \boldsymbol{N}$, then $f(i)/i \leqq f(l)/l$ for all $i \in \boldsymbol{N}$.*

Proof. Let $\gamma = f(l)/l$. Suppose, on the contrary, that $f(k)/k > \gamma$ for some $k \in \boldsymbol{N}$. By Lemma 2.10, $f(kl) = kf(l)$ and so $f(kl)/kl = f(l)/l = \gamma$. On the other hand, $f(kl) \geq f(k) + f((l-1)k) \geq \cdots \geq lf(k)$, which implies $f(kl)/kl \geq f(k)/k > \gamma$, a contradiction. Hence $f(i)/i \leqq f(l)/l$ for all $i \in \boldsymbol{N}$. □

Lemma 3.3. *Let f be a nice map and $\gamma = \sup\{f(i)/i \mid i \in \boldsymbol{N}\}$. If $f(i)/i < \gamma$ for all $i \in \boldsymbol{N}$, then $i\gamma > f(i) \geq i\gamma - 1$ for all $i \in \boldsymbol{N}$.*

Proof. We suppose, on the contrary, that $f(k) < k\gamma - 1$ for some $k \in \boldsymbol{N}$.

Then there is a $t_1 \in \boldsymbol{N}$ big enough with $f(k) < k\gamma - 1 - (k/t_1)$. Similarly we take a $t_2 \in \boldsymbol{N}$ with $f(i)/i < \gamma - (1/t_2)$ for all $i \in \boldsymbol{N}$ $(1 \leqq i \leqq k)$. Set $t = \max\{t_1, t_2\}$. Since $\gamma = \sup\{f(i)/i \mid i \in \boldsymbol{N}\}$, there is an $l \in \boldsymbol{N}$ with $f(l)/l > \gamma - (1/t)$ (assume that l is smallest for this property). Note that $l > k$ by the choice of t. $f(k) < k\gamma - 1 - k/t$ implies $f(k) \leqq [k\gamma - k/t] - 1$, where $[\beta]$ is the Gauss' symbol of a real number β. Since $f(-k) \geq -f(k) - 1$, we have $f(-k) \geq -[k\gamma - k/t]$. Furthermore, $f(l)/l > \gamma - (1/t)$ implies $f(l) > l\gamma - l/t$. Thus $f(l-k) \geq f(l) + f(-k) > l(\gamma - (1/t)) - k[\gamma - (1/t)] \geq (l-k)(\gamma - (1/t))$, i.e., $f(l-k)/(l-k) > \gamma - (1/t)$ with $l > l - k > 0$, which is a contradiction to the choice of l. Hence $i\gamma > f(i) \geq i\gamma - 1$ for all $i \in \boldsymbol{N}$. $\square$

The following Lemma is crucial for the description of all nice maps.

LEMMA 3.4. *Let γ be a real number with $0 \leqq \gamma \leqq 1$. Then*

(1) *If $0 < \gamma < 1$ and f_γ is a map from $\boldsymbol{Z}$ to $\boldsymbol{Z}$ defined by $f_\gamma(i) = [i\gamma]$ for all $i \in \boldsymbol{Z}$, then f_γ is a nice map.*

(2) *If $0 < \gamma \leqq 1$ and $f_\gamma^{(1)}$ is a map from $\boldsymbol{Z}$ to $\boldsymbol{Z}$ defined by $f_\gamma^{(1)}(0) = 0$, $i\gamma - 1 \leqq f_\gamma^{(1)}(i) < i\gamma$ and $f_\gamma^{(1)}(-i) = -f_\gamma^{(1)}(i) - 1$ for all $i \in \boldsymbol{N}$, then $f_\gamma^{(1)}$ is a nice map.*

(3) *If $0 \leqq \gamma < 1$ and $f_\gamma^{(-1)}$ is a map from $\boldsymbol{Z}$ to $\boldsymbol{Z}$ defined by $f_\gamma^{(-1)}(0) = 0$, $f_\gamma^{(-1)}(i) = [i\gamma]$ and $f_\gamma^{(-1)}(-i) = -f_\gamma^{(-1)}(i) - 1$ for all $i \in \boldsymbol{N}$, then $f_\gamma^{(-1)}$ is a nice map.*

PROOF.

(1) It is clear that $f_\gamma(0) = 0 = f_\gamma(1)$ and $f_\gamma(-1) = -1$. For any $i, j \in \boldsymbol{Z}$, we have $[i\gamma] + [-i\gamma] \geqq -1$ and $[i\gamma] + [j\gamma] \leqq i\gamma + j\gamma = (i+j)\gamma$ for all $i, j \in \boldsymbol{Z}$. Hence f_γ is a nice map.

(2) If $\gamma = 1$, then it is clear that $f_1^{(1)}(i) = i - 1$ and $f_1^{(1)}(-i) = -i$ for all $i \in \boldsymbol{N}$. Hence $f_1^{(1)}$ is a nice map. For any γ with $0 < \gamma < 1$, if γ is not a rational number, then it is clear that $f_\gamma^{(1)} = f_\gamma$. If γ is a rational number and let l be the smallest natural number with $l\gamma \in \boldsymbol{Z}$. Then, for any $i \in \boldsymbol{N}$, we have $f_\gamma^{(1)}(i) = i\gamma - 1$ if $i \in l\boldsymbol{Z}$ and $f_\gamma^{(1)}(i) = [i\gamma]$ if $i \notin l\boldsymbol{Z}$. So it is easy to see, by tedious calculation case by case that $f_\gamma^{(1)}$ is a nice map.

(3) If $\gamma = 0$, then $f_0^{(-1)}(i) = 0$ and $f_0^{(-1)}(-i) = -1$ for all $i \in \boldsymbol{N}$. So $f_0^{(-1)}$ is a nice map. For any γ with $0 < \gamma < 1$, if γ is not a rational number, then it is clear that $f_\gamma^{(-1)} = f_\gamma$. If γ is a rational number and let l be the smallest natural number with $l\gamma \in \boldsymbol{Z}$. Then, for any $i \in \boldsymbol{N}$, $f_\gamma^{(-1)}(i) = i\gamma$ if and only if $i \in l\boldsymbol{Z}$ and $f_\gamma^{(-1)}(i) = [i\gamma]$ with $[i\gamma] < i\gamma$ if and only if $i \notin l\boldsymbol{Z}$. Hence it is easy to see, by tedious calculation case by case that $f_\gamma^{(-1)}$ is a nice map. $\square$

Now we are in a position to describe all nice maps.

Theorem 3.5. *$\{f_\gamma, f_\gamma^{(1)}, f_\gamma^{(-1)} \mid 0 < \gamma < 1$ and γ is a real number$\} \cup \{f_0^{(-1)}, f_1^{(1)}\}$ is the set of all nice maps.*

Proof. By Lemma 3.4, it suffices to prove that any nice map f is one in the theorem. Let $\gamma = \sup\{f(i)/i \mid i \in \boldsymbol{N}\}$. Then $0 \leqq \gamma \leqq 1$ by Lemma 3.1. If $\gamma = 0$, then it is easy to see that $f = f_0^{(-1)}$. So we may assume that $0 < \gamma \leqq 1$. If $\gamma = 1$, then $f(i)/i < \gamma = 1$ for all $i \in \boldsymbol{N}$ by Lemma 3.1.

Case 1. Suppose that $f(i)/i < \gamma$ for all $i \in \boldsymbol{N}$. Then $i\gamma > f(i) \geq i\gamma - 1$ by Lemma 3.3 and for all $i > 0$, $f(-i) = -f(i) - 1$ by Lemma 3.2. Hence $f = f_\gamma^{(1)}$.

Case 2. There is an $l \in \boldsymbol{N}$ with $\gamma = f(l)/l$. We choose l as the smallest one for this property and may assume that $0 < \gamma < 1$ by the discussion above. We claim that l is the smallest natural number with $l\gamma \in \boldsymbol{Z}$. Let k be the smallest natural number with $k\gamma \in \boldsymbol{Z}$. Then $l = pk$ for some natural number $p \in \boldsymbol{N}$. If $p > 1$, then $f(k)/k < \gamma$, and so $f(k) \leqq k\gamma - 1$. It follows that $f(-k) \geq -k\gamma$, i.e., $-f(-k) \leqq k\gamma$. Furthermore, since $f(k) \geq f(2k) + f(-k)$, $f(2k) \leqq f(k) - f(-k) \leqq 2k\gamma - 1 < 2k\gamma$. Inductively, we have $f(pk) < pk\gamma$, i.e., $f(l) < l\gamma$, a contradiction. Hence, $l = k$, as claimed. We will prove that

$$f(i) = [i\gamma] \text{ for all } i \in \boldsymbol{N}.$$

For any $i \in \boldsymbol{N}$, since $f(il) \geq if(l) = il\gamma$ and $f(il)/il \leqq \gamma$, we have $f(il) = il\gamma = [il\gamma]$. We suppose, on the contrary, that $f(j) \neq [j\gamma]$ for some $j \in \boldsymbol{N}$. Then $f(j) \leqq [j\gamma] - 1$ and $j \notin l\boldsymbol{Z}$. So $[j\gamma] < j\gamma$ and $f(-j) \geq -[j\gamma]$ follows. Let $q \in \boldsymbol{N}$ with $ql > j$. Then $f(ql - j) \geq f(ql) + f(-j) \geq ql\gamma + (-[j\gamma]) > ql\gamma - j\gamma$, which implies $f(ql - j)/(ql - j) > \gamma$, a contradiction. Hence $f(i) = [i\gamma]$ for all $i \in \boldsymbol{N}$. Next we will prove that

$$f(-i) = -f(i) - 1 \text{ and } f(-i) = [-i\gamma] \text{ for all } i \notin l\boldsymbol{Z}.$$

Suppose that there is an $i \in \boldsymbol{N}$ with $i \notin l\boldsymbol{Z}$ such that $f(-i) + f(i) = 0$. Then $f(i)/i = \gamma$ by Lemma 3.2, so that $i \in l\boldsymbol{Z}$, a contradiction. Hence $f(-i) = -f(i) - 1$ for all $i \in \boldsymbol{N}$ with $i \notin l\boldsymbol{Z}$. In particular, $f(-i) = -f(i) - 1 = -[i\gamma] - 1 = [-i\gamma]$ for any $i \in \boldsymbol{N}$ with $i \notin l\boldsymbol{Z}$. Now, $f(l) = l\gamma$ implies either $f(-l) = -l\gamma$ or $f(-l) = -l\gamma - 1$. If $f(-l) = -l\gamma$, then we have $f(-il) = -il\gamma = [-il\gamma]$ for all $i \in \boldsymbol{N}$ by induction on i. Hence $f = f_\gamma$ follows. If $f(-l) = -l\gamma - 1$, then we have $f(-il) = -il\gamma - 1 = -f(il) - 1$ for any $i \in \boldsymbol{N}$ by induction on i, which shows $f = f_\gamma^{(-1)}$. This completes the proof. $\square$

4. The cardinality of the set of the graded extensions.

Let $A = \oplus_{i\in \boldsymbol{Z}} A_i X^i$ be a graded extension of V in $K[X, X^{-1}; \sigma]$ with $A_1 = Wa = a\sigma(W)$, $A_{-1} = J(W)\alpha_{-1}$ and $J(W) = b^{-1}W = Wb^{-1}$ for some $b^{-1} \in J(W)$. Set $\mathscr{S} = \{B = \oplus_{i\in \boldsymbol{Z}} B_i X^i \mid B$ is a graded extension of V in $K[X, X^{-1}; \sigma]$ with $B_1 = Wa$ and $B_{-1} = J(W)\alpha_{-1}\}$. Then it follows from Theorems 2.11 and 3.5 that $|\mathscr{S}| \geq \aleph$. In this section, we will give an example of a total valuation ring V such that $|\mathscr{S}| > \aleph$.

LEMMA 4.1. *Let f be a nice map with $f(l) + f(-l) = 0$ and $l \geq 2$ (assume that l is the smallest natural number for this property) and let $W \supset U \supset V$ be overrings of V with $Wa = a\sigma(W)$ and $J(W) = b^{-1}W = Wb^{-1}$ for some $b^{-1} \in J(W)$. Suppose that $C = \oplus_{j\in \boldsymbol{Z}} C_{jl} X^{jl}$ is a graded extension of V in $K[X^l, X^{-l}; \sigma^l]$ with $C_l = Ub^{f(l)}\alpha_l$. Then $Wb^{f(jl)-1}\alpha_{jl} \subset C_{jl} \subset Wb^{f(jl)}\alpha_{jl}$ for all $j \in \boldsymbol{Z}$.*

PROOF. $C_l = Ub^{f(l)}\alpha_l$ implies $C_{-l} = \sigma^{-l}(\alpha_l^{-1} b^{f(-l)} J(U)) = \alpha_{-l}\sigma^{-l}(b^{f(-l)} J(U))$ by Lemma 1.4. So $WC_l = Wb^{f(l)}\alpha_l$ and, by Lemma 2.1 (1), $WC_{-l} = Wb^{f(-l)}\alpha_{-l}$. In the case where $j \in \boldsymbol{N}$, we will first prove this assertion by induction on j. It is clear that $Wb^{f(l)-1}\alpha_l \subset C_l \subset Wb^{f(l)}\alpha_l$ and so we may assume that $Wb^{f(jl)-1}\alpha_{jl} \subset C_{jl} \subset Wb^{f(jl)}\alpha_{jl}$ for some $j \in \boldsymbol{N}$. Then since $f(l) + f(jl) = f(jl + l)$ by Lemma 2.10, we have

$$\begin{aligned} Wb^{f(jl+l)-1}\alpha_{jl+l} &= Ub^{f(l)} Wb^{f(jl)-1}\alpha_l \sigma^l(\alpha_{jl}) \\ &= Ub^{f(l)}\alpha_l \sigma^l(Wb^{f(jl)-1}\alpha_{jl}) \\ &\subseteq C_l \sigma^l(C_{jl}) \subseteq C_{jl+l}. \end{aligned}$$

To prove $Wb^{f(jl+l)}\alpha_{jl+l} \supseteq C_{jl+l}$, consider the formulas:

$$\begin{aligned} \sigma^l(\alpha_{-l}) b^{f(-l)} W C_{jl+l} &= \sigma^l(\alpha_{-l}\sigma^{-l}(b^{f(-l)}W) C_{jl+l} = \sigma^l(Wb^{f(-l)}\alpha_{-l}) C_{jl+l} \\ &= \sigma^l(WC_{-l}\sigma^{-l}(C_{jl+l})) \subseteq \sigma^l(WC_{jl}). \end{aligned}$$

Hence

$$\begin{aligned} C_{jl+l} &\subseteq b^{f(l)}\sigma^l(\alpha_{-l}^{-1})\sigma^l(WC_{jl}) = b^{f(l)}\alpha_l\sigma^l(Wb^{f(jl)}\alpha_{jl}) \\ &= b^{f(l)} Wb^{f(jl)}\alpha_l \sigma^l(\alpha_{jl}) = Wb^{f(jl+l)}\alpha_{jl+l}. \end{aligned}$$

To prove $Wb^{f(jl)-1}\alpha_{jl} \subset C_{jl} \subset Wb^{f(jl)}\alpha_{jl}$, it suffices to prove that C_{jl} is not a left W-ideal by Lemma 1.2. On the contrary, assume that C_{jl} is a left W-ideal.

Then we have $C_{jl+l} = Wb^{f(jl+l)}\alpha_{jl+l}$, because

$$\begin{aligned} C_{jl+l} &\supseteq WC_l\sigma^l(C_{jl}) = Wb^{f(l)}\alpha_l\sigma^l(C_{jl}) = \alpha_l\sigma^l(Wb^{f(l)}C_{jl}) \\ &= \alpha_l\sigma^l(Wb^{f(jl+l)}\alpha_{jl}) = \alpha_l\sigma^l(\alpha_{jl}\sigma^{jl}(Wb^{f(jl+l)})) \\ &= \alpha_{jl+l}\sigma^{jl+l}(Wb^{f(jl+l)}) = Wb^{f(jl+l)}\alpha_{jl+l}. \end{aligned}$$

Hence

$$\begin{aligned} C_{jl} &\supseteq C_{-l}\sigma^{-l}(C_{jl+l}) = \alpha_{-l}\sigma^{-l}(b^{f(-l)}J(U)Wb^{f(jl+l)}\alpha_{jl+l}) \\ &= \alpha_{-l}\sigma^{-l}(Wb^{f(jl)})\sigma^{-l}(\alpha_{jl+l}) = Wb^{f(jl)}\alpha_{jl} \supset C_{jl}, \end{aligned}$$

which is a contradiction. Since $J(U) \supset J(W) = Wb^{-1}$, we have

$$C_{-l} = \alpha_{-l}\sigma^{-l}(b^{f(-l)}J(U)) \supset \alpha_{-l}\sigma^{-l}(b^{f(-l)}J(W)) = Wb^{f(-l)-1}\alpha_{-l}$$

and

$$C_{-l} \subset \alpha_{-l}\sigma^{-l}(b^{f(-l)}W) = Wb^{f(-l)}\alpha_{-l}.$$

So, by the similar argument above, we have $Wb^{f(-jl)-1}\alpha_{-jl} \subset C_{-jl} \subset Wb^{f(-jl)}\alpha_{-jl}$ for all $j \in \boldsymbol{N}$, completing the proof. $\square$

Lemma 4.2. *Let f be a nice map with $f(l) + f(-l) = 0$ and $l \geq 2$ (assume that l is the smallest natural number for this property) and let $W \supset U \supset V$ be overrings of V with $Wa = a\sigma(W)$ and $J(W) = b^{-1}W = Wb^{-1}$ for some $b^{-1} \in J(W)$. Set $C_l = Ub^{f(l)}\alpha_l$ and suppose that C_l is a right $\sigma^l(V)$-ideal. Then there is a graded extension $C = \oplus_{j\in\boldsymbol{Z}}C_{jl}X^{jl}$ of V in $K[X^l, X^{-l}; \sigma^l]$ with $Wb^{f(jl)-1}\alpha_{jl} \subset C_{jl} \subset Wb^{f(jl)}\alpha_{jl}$ for all $j \in \boldsymbol{Z}$.*

Proof. Since C_l is a right $\sigma^l(V)$-ideal, we have the following three cases; $C_l = Uc_l = c_l\sigma^l(U)$ or $C_l = Uc_l \supset c_l\sigma^l(U)$ or $C_l = Uc_l \subset c_l\sigma^l(U)$, where $c_l = b^{f(l)}\alpha_l$, which are in the same situation as in Section 2. Hence, in all cases, we have a graded extension $C = \oplus_{j\in\boldsymbol{Z}}C_{jl}X^{jl}$ of V in $K[X^l, X^{-l}; \sigma^l]$ by Proposition 2.3 or Theorem 2.5, and so $Wb^{f(jl)-1}\alpha_{jl} \subset C_{jl} \subset Wb^{f(jl)}\alpha_{jl}$ for all $j \in \boldsymbol{Z}$ by Lemma 4.1. $\square$

Proposition 4.3. *Let f be a nice map with $f(l) + f(-l) = 0$ and $l \geq 2$ (assume that l is the smallest natural number for this property) and let W be an* *overring of V with $J(W) = b^{-1}W = Wb^{-1}$ for some $b^{-1} \in J(W)$, $A_1 = Wa = a\sigma(W)$ and $A_{-1} = J(W)\alpha_{-1}$. Suppose that the cardinality of $\{V_\lambda \mid W \supset V_\lambda \supset V$ and V_λ are overrings of $V\}$ is larger than $\aleph$ and that $C_{\lambda l} = V_\lambda b^{f(l)}\alpha_l$ is a right $\sigma^l(V)$-ideal for each λ. Then the cardinality of $\mathscr{S} = \{B = \oplus_{i\in\boldsymbol{Z}} B_i X^i \mid B$ is a graded extension of V in $K[X, X^{-1}; \sigma]$ with $B_1 = Wa$ and $B_{-1} = J(W)\alpha_{-1}\}$ is larger than $\aleph$.*

PROOF. For each V_λ, by Lemma 4.2, there is a graded extension $C_\lambda = \oplus_{j\in\boldsymbol{Z}} C^\lambda_{jl} X^{jl}$ of V in $K[X^l, X^{-l}; \sigma^l]$ with $C^\lambda_l = V_\lambda b^{f(l)}\alpha_l$ and $Wb^{f(jl)-1}\alpha_{jl} \subset C^\lambda_{jl} \subset Wb^{f(jl)}\alpha_{jl}$ for all $j \in \boldsymbol{Z}$. Set $B^\lambda_i = Wb^{f(i)}\alpha_i$ for all $i \notin l\boldsymbol{Z}$ and $B^\lambda_{jl} = C^\lambda_{jl}$ for all $j \in \boldsymbol{Z}$. Then $B_\lambda = \oplus_{i\in\boldsymbol{Z}} B^\lambda_i X^i$ is a graded extension of V in $K[X, X^{-1}; \sigma]$ with $B^\lambda_1 = Wa$ and $B^\lambda_{-1} = J(W)\alpha_{-1}$ by Theorem 2.11. Hence $|\mathscr{S}| > \aleph$ follows. $\square$

In the following we will give a concrete example of total valuation ring and a nice map satisfying the conditions in Proposition 4.3, by using the method in [**6**]:

Let Λ be a totally ordered group with $|\Lambda| > \aleph$ and $G = \boldsymbol{Z}_1 \oplus \boldsymbol{Z}_2 \oplus (\oplus_{\lambda\in\Lambda} \boldsymbol{Z}_\lambda)$ be a direct sum of $\boldsymbol{Z}_i$ and $\boldsymbol{Z}_\lambda$ $(i = 1, 2, \lambda \in \Lambda)$, where $\boldsymbol{Z}_i$ and $\boldsymbol{Z}_\lambda$ are copies of $\boldsymbol{Z}$, which is a totally ordered abelian group by lexicographic ordering. Furthermore, let F_0 be a field and $F = F_0(\{x_i, x_\lambda\})$ be the rational function field over F_0 in indeterminates x_i and x_λ $(i = 1, 2, \lambda \in \Lambda)$. We let σ be an automorphism defined by; $\sigma(a) = a$ for any $a \in F_0, \sigma(x_\lambda) = x_\lambda, \sigma(x_1) = x_2$ and $\sigma(x_2) = x_1$ so that $\sigma^2 = 1$. We also define a valuation v of F as follows; $v(a) = 0$ for any $a \in F_0$, $v(x_i) = g_i$ and $v(x_\lambda) = g_\lambda$, where g_i and g_λ are elements in G such that the i-th component and the λ-component are 1, and the other components are all zeros, respectively. Let V_0 be the valuation ring of F determined by v. Then it is easy to see that $\sigma(V_0) \nsubseteq V_0$ and $\sigma^2(V_0) = V_0$. Set

$$\wp = \cap_{n\in\boldsymbol{N}} x_2^n V_0 \cap_{\lambda\in\Lambda} (\cap_{n\in\boldsymbol{N}} x_\lambda^n V_0)$$

and

$$\wp_\lambda = \cap_{\lambda<\mu} (\cap_{n\in\boldsymbol{N}} x_\mu^n V_0)$$

for each $\lambda \in \Lambda$. Then $\wp$ and $\wp_\lambda$ are all prime ideals of V_0 with $\wp \subset \wp_\lambda \subset \wp_\mu$ if $\mu > \lambda$ (see [**6**, example 2.5]). Let $W_0 = V_{0\wp}$ and $V_{0\lambda} = V_{0\wp_\lambda}$, the localization of V_0 at $\wp$ and $\wp_\lambda$, respectively. So we have $W_0 \supset V_{0\lambda} \supset V_{0\mu}$ if $\mu > \lambda$.

In order to prove that $J(W_0) = x_1 W_0$, let $U = F_0(\{x_2, x_\lambda\})[x_1]$, which is contained in W_0. Since $U \backslash x_1 U \subseteq V_0 \backslash \wp$, it follows that $W_0 \supseteq U_{x_1 U}$, a discrete rank one valuation ring of F and so $W_0 = U_{x_1 U}$ follows. In particular, $J(W_0) = x_1 W_0$.

Let $S = F[y, \sigma]$ be the skew polynomial ring over F in the indeterminate y and $T = S_{yS}$, the localization of S at the maximal ideal yS. For any $t = f(y)g(y)^{-1} \in T$, where $f(y) = f_0 + f_1 y + \cdots + f_n y^n$ and $g(y) = g_0 + g_1 y + \cdots + g_m y^m$ with $g_0 \neq 0$, we define the map

$$\varphi : T \longrightarrow F$$

by $\varphi(t) = f_0 g_0^{-1}$. Then φ is a ring epimorphism with ker$\varphi = yT$ (see [**6**, Section 1]). Set $W = \varphi^{-1}(W_0) = W_0 + yT, V_\lambda = \varphi^{-1}(V_{0\lambda}) = V_{0\lambda} + yT$ and $V = \varphi^{-1}(V_0) = V_0 + yT$, the complete inverse images of $W_0, V_{0\lambda}$ and V_0 by φ, respectively. Then W, V_λ and V are all total valuation rings of $K = F(y, \sigma)$, the quotient ring of S which is a division ring, with $J(W) = x_1 W = W x_1$ and $W \supset V_\lambda \supset V$ for each $\lambda \in \Lambda$ by [**6**, (1.6)]. Note that σ is naturally extended to an automorphism of K which is the conjugation by y. We denote it by the same symbol σ. It is clear that $\sigma^2 = 1$. Now we set $y^{-1} = a$ and $b^{-1} = x_1$. Then we have the following properties:

(i) $Wa = a\sigma(W)$ and $J(W) = b^{-1}W = Wb^{-1}$.

(ii) $\vartheta = \{V_\lambda \mid W \supset V_\lambda \supset V, \lambda \in \Lambda\}$ and $|\vartheta| > \aleph$.

(iii) For each $\lambda \in \Lambda$, $V_\lambda b\alpha_2$ is a right V-ideal.

The statements (i) and (ii) are obvious. In order to prove (iii), note that $\alpha_2 = y^{-2}$. So we have

$$V_\lambda b\alpha_2 = V_\lambda b y^{-2} = y^{-2} V_\lambda b = y^{-2}(V_{0\lambda} + yT)x_1^{-1} = y^{-2}(x^{-1}V_{0\lambda} + yT),$$

which is a right V-ideal (note that $\sigma^2(V) = V$). Let $f = f_{1/2}$ be the nice map defined in Lemma 3.4. Then it is clear that $f(2) + f(-2) = 0$. Hence, by Proposition 4.3, the cardinality of $\mathscr{S} = \{B = \oplus_{i \in \boldsymbol{Z}} B_i X^i \mid B$ is a graded extension of V in $K[X, X^{-l}; \sigma]$ with $B_1 = Wa$ and $B_{-1} = J(W)\alpha_{-1}\}$ is larger than $\aleph$.

Finally, we give some simple examples of total valuation rings satisfying the conditions in Theorems 2.4 $\sim$ 2.6:

Let $W_0 \supseteq V_0$ be valuation rings of a field F with an automorphism σ and let $F[y, \sigma]$ be the skew polynomial ring over F in indeterminate y. As before, let

$$\varphi : T = S_{yS} \longrightarrow F$$

be the ring epimorphism, $W = \varphi^{-1}(W_0)$ and $V = \varphi^{-1}(V_0)$, which are all total valuation rings of $K = F(y, \sigma)$. Since $J(W) = J(W_0)W$, it follows that $J(W) = J(W)^2$ if and only if $J(W_0) = J(W_0)^2$, and that $\sigma(W) \subseteq W$ if and only if $\sigma(W_0) \subseteq W_0$. Furthermore, for any nonzero element $a \in F$, we have $aW = Wa$, because a

is a unit in T. Hence we have the following:

(i) Suppose that $W_0 \supset \sigma(W_0)$ and $V_0 = \sigma(V_0)$ ([**6**, (2.5)]). Let a be a nonzero element in F. Then $Wa \supset a\sigma(W)$ (Theorem 2.4).

(ii) Suppose that $W_0 \subset \sigma(W_0)$ and $V_0 = \sigma(V_0)$. Let a be a nonzero element in F. Then $a\sigma(W) \supset aW = Wa = Wa\sigma(V)$ (Theorem 2.5).

(iii) Let $a = y^{-1}$ and suppose that $J(W_0) = J(W_0)^2$. Then $Wa = a\sigma(W)$ and $J(W_0) = J(W_0)^2$ (Theorem 2.6).

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