Singular solutions of nonlinear partial differential equations with resonances

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Abstract. We present a Frobenius type theorem for a system of nonlinear partial differential equations. Typical application is the normal form theory of a singular vector field. The construction of a singular solution is closely related with a Riemann-Hilbert factorization.

1. Introduction.

In this paper we shall study singular solutions of a system of nonlinear singular partial differential equations with resonances. If a resonance occurs, then we can easily see that the equations are not solvable in general even in a class of formal power series unless a compatibility condition is assumed. Nevertheless, in view of the applications, it is important to show the solvability without assuming any compatibility conditions when a resonance occurs.

In order to show the solvability in such a case, we recall the Frobenius method in the theory of ordinary differential equations. Namely, one can construct a solution by introducing logarithmic singularities. We will extend the idea to systems of nonlinear singular partial differential equations with nontrivial nilpotent parts by introducing logarithmic singularities of several variables.

We will apply our method to the linearization of a singular vector field with a resonance. In fact, we will see that a Frobenius type singular solution may yield a finitely differentiable solution in the real domain under a certain condition on the resonances. This gives a new interpretation of well-known Hartmann's theorem from the complex-analytic point of view. (cf. Remark 2 and [**1**]).

This paper is organized as follows. In Section 2 we state the main theorem for a first order system and applications to the linearization problem of a singular vector field. In Section 3 we give the proof of the main theorem. In Section 4 we briefly state an extension to higher order systems.

2. First order systems.

Let $x = (x_1, x_2, \ldots, x_n) \in \boldsymbol{C}^n$ be the variable in $\boldsymbol{C}^n$ $(n \geq 2)$ and $\boldsymbol{Z}_+$ be the set of all nonnegative integers. We use the notation $\partial_{x_j} = \partial/\partial x_j$, $j = 1, 2, \ldots, n$. Let $A \subset \{1, 2, \ldots, n\}$, $A \neq \{1, 2, \ldots, n\}$ be given. We consider the system of semilinear first order partial differential equations

$$P_k u_k(x) = d_k(x) + f_k(x, u), \quad k = 1, 2, \ldots, N, \tag{2.1}$$

where $u = (u_1, \ldots, u_N)$, $N \geq 1$ and P_k is given by

$$P_k = \sum_{j=1}^n a_{j,k}(x) \partial_{x_j} + \sum_{j \in A} b_{j,k}(x) \partial_{x_j} + c_k(x). \tag{2.2}$$

Here the functions $a_{j,k}(x)$, $b_{j,k}(x)$, $c_k(x)$ and $d_k(x)$ are holomorphic in some neighborhood of the origin such that

$$a_{j,k}(x) = x_j\big(\lambda_{j,k} + \tilde{a}_{j,k}(x)\big), \quad \tilde{a}_{j,k}(0) = 0, \tag{2.3}$$

$$b_{j,k}(x) = x_{j-1}\varepsilon_{j,k} + \tilde{b}_{j,k}(x), \quad \tilde{b}_{j,k}(x) = O(|x|^2), \tag{2.4}$$

where $\tilde{a}_{j,k}(x)$ and $\tilde{b}_{j,k}(x)$ are holomorphic in some neighborhood of the origin, and $\lambda_{j,k}$ and $\varepsilon_{j,k}$ are complex constants. The functions $f_k(x, u)$ $(k = 1, 2, \ldots, N)$ are holomorphic in some neighborhood of the origin of $(x, u) = (0, 0) \in \boldsymbol{C}^n \times \boldsymbol{C}^N$ such that $f_k(x, u) = O(|u|^2)$ for $k = 1, \ldots, N$.

We define the indicial polynomial $p_k(\alpha)$ by

$$p_k(\alpha) = \sum_{j=1}^n \lambda_{j,k}\alpha_j + c_k(0) \quad \text{for} \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \boldsymbol{Z}_+^n. \tag{2.5}$$

Then we say that $\alpha \in \boldsymbol{Z}_+^n$ is a resonance if $p_k(\alpha) = 0$ for some k, $(1 \leq k \leq N)$.

We assume the following three conditions.

ASSUMPTION 1.   There exists a resonance $\alpha \in \boldsymbol{Z}_+^n$.

ASSUMPTION 2.   For every resonance $\alpha = (\alpha_1, \ldots, \alpha_n) \in \boldsymbol{Z}_+^n$, there exists an $\ell \in \{1, \ldots, n\} \setminus A$ such that $\alpha_\ell \neq 0$.

ASSUMPTION 3 (Poincaré condition).   For every k, $k = 1, 2, \ldots, N$, the convex hull of the set $\{\lambda_{j,k}; j = 1, \ldots, n\}$ in $\boldsymbol{C}$ does not contain the origin.

We can easily see that Assumption 3 implies that the set of all resonances is a finite set. We define $(\log x)^\beta = \prod_{j=1}^n (\log x_j)^{\beta_j}$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{Z}_+^n$.

THEOREM 1. *Assume that Assumptions 1, 2 and 3 are satisfied. Then the system of equations (2.1) has a solution $u = (u_1, u_2, \ldots, u_N)$ of the form*

$$u(x) = \sum_{|\alpha| \geq 1, \beta \in \mathbf{Z}_+^n} u_{\alpha, \beta} x^\alpha (\log x)^\beta, \qquad (2.6)$$

where the summation with respect to β in (2.6) is taken for β such that $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{Z}_+^n$, $\beta_j = 0$ if $j \in A$. Moreover, there exist an integer J and a constant $r > 0$ such that the series (2.6) is written as the power series of x and $X_{\nu,p} := x_\nu (\log x_\nu)^p$, $(p = 1, 2, \ldots, J, \nu \in \{1, \ldots, n\} \setminus A)$, which converges in the domain

$$\{x \in \mathbf{C}^n; |x| < r, \ |X_{\nu,p}| < r, \ p = 1, 2, \ldots, J, \quad \nu \in \{1, \ldots, n\} \setminus A\}.$$

REMARK 1. We note that Tahara [**13**] showed the existence of a singular solution with one singular variable for a class of Fuchsian partial differential equations. Our singular solution contains singularities of several variables and our equations admit nilpotent parts. In the several variable case, the construction of a singular solution essentially depends on the Riemann-Hilbert factorization problem of several variables. (cf. Step 3 of the proof of Theorem 1.) In fact, we will show more precise formula, from which we can show that the singularity of the solution is generated by a finite number of singular functions.

Our method can also be applied to the normal form theory of a singular vector field. We consider

$$\mathscr{X}(x) = \sum_{j=1}^n a_j(x) \frac{\partial}{\partial x_j}, \quad a_j(0) = 0 \quad (j = 1, \ldots, n), \qquad (2.7)$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbf{C}^n$, and $a_j(x)$ are holomorphic in some neighborhood of the origin $x = 0$. We set $X(x) = (a_1(x), a_2(x), \ldots, a_n(x))$ and write $X(x)$ in the form

$$X(x) = x\Lambda + R(x), \qquad (2.8)$$

where

$$R(x) = (R_1(x), \ldots, R_n(x)), \quad R(x) = O(|x|^2), \tag{2.9}$$

and Λ is an $n \times n$ constant matrix. We want to linearize $\mathscr{X}(x)$ by the change of the variables $x = y + v(y)$, $v = O(|y|^2)$. Noting that

$$X(x)\frac{\partial}{\partial x} = X(y + v(y))\frac{\partial y}{\partial x}\frac{\partial}{\partial y} = X(y + v(y))\left(\frac{\partial x}{\partial y}\right)^{-1}\frac{\partial}{\partial y},$$

the linearization condition is equivalent to $X(y+v)(1+\partial_y v)^{-1} = y\Lambda$, namely

$$\mathscr{L}v - v\Lambda \equiv y\Lambda\partial_y v - v\Lambda = R(y + v(y)), \quad v = (v_1, \ldots, v_n). \tag{2.10}$$

We study the solvability of (2.10). We assume that the matrix Λ is put in a Jordan normal form for the sake of simplicity. This implies that the differentiations in $\mathscr{L}$ have the forms $y_j\frac{\partial}{\partial y_j}$ or $y_{k-1}\frac{\partial}{\partial y_k}$ for some j or k. The former term comes from the semisimple part of Λ, and the latter one appears from the nilpotent part of Λ. In view of this we define $A \subset \{1, 2, \ldots, n\}$, $A \neq \{1, 2, \ldots, n\}$ as the set of all integers $k \geq 2$ such that $y_{k-1}\frac{\partial}{\partial y_k}$ appears in $\mathscr{L}$. If $v(y)$ has the form (2.6), then it follows from the condition $R(x) = O(|x|^2)$ that in the expansion of the right-hand side of (2.10) only the terms $y^\alpha(\log y)^\beta$ with $|\alpha| \geq 2$, $|\beta| \geq 0$ may appear in each component. On the other hand, in the left-hand side of (2.10) the term $\mathscr{L}v - v\Lambda$ may contain $y^\gamma(\log y)^\delta$ for some $|\gamma| = 1$, $|\delta| \geq 0$ in the components. Because we are interested in the solvability of (2.10), we will look for v of the form (2.6) such that $|\alpha| \geq 2$ in the summation. We denote the eigenvalues of Λ by λ_j, $j = 1, \ldots, n$ with multiplicity. Then we have

THEOREM 2. *Assume that the Poincaré condition is satisfied. Suppose that there exists a resonance $\alpha \in \mathbf{Z}_+^n$, $|\alpha| \geq 2$. Moreover, assume that for every resonance $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{Z}_+^n$ such that $|\alpha| \geq 2$, there exists an $\ell \in \{1, 2, \ldots, n\} \setminus A$ such that $\alpha_\ell \neq 0$. Then (2.10) has a solution v of the form*

$$v(y) = \sum_{|\alpha| \geq 2, \beta \in \mathbf{Z}_+^n} v_{\alpha\beta}\, y^\alpha(\log y)^\beta, \tag{2.11}$$

where $(\log y)^\beta = \prod_{j=1}^n (\log y_j)^{\beta_j}$, and the summation with respect to β in (2.11) is taken for β such that $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{Z}_+^n$, $\beta_j = 0$ if $j \in A$. There exist an integer J and a constant $\varepsilon > 0$ such that the series $v(y)$ can be written as the power series of y and $Y_{\nu,p} := y_\nu(\log y_\nu)^p$, $(p = 1, 2, \ldots, J; \nu \in \{1, \ldots, n\} \setminus A)$, which converges in the domain

$$\{y \in \boldsymbol{C}^n; |y| < \varepsilon,\ |Y_{\nu,p}| < \varepsilon,\ p = 1, 2, \ldots, J,\ \nu \in \{1, \ldots, n\} \setminus A\}.$$

REMARK 2. (a) If there is no resonance, then Theorem 2 is still valid. The solution (2.11) coincides with the classical solution constructed by Poincaré because no logarithmic term appears.

(b) If Λ is semisimple, then $A = \emptyset$. Hence the existence of $\alpha_\ell \neq 0$ for every resonance α in Theorem 2 is trivially satisfied.

(c) If we restrict the solution v to the domain $\{y_j \geq 0; j \in \{1, \ldots, n\} \setminus A\}$ in some neighborhood of the origin of $\boldsymbol{R}^n$, then, we obtain a smooth solution of (2.10) which is continuous up to $y_j = 0$ $(j \in \{1, \ldots, n\} \setminus A)$. By solving (2.10) in each region $\pm y_j > 0$ after the change of variables $y_j \mapsto -y_j$ and by patching these solutions we obtain a finitely smooth solution in a real domain. The linearizability under a continuous (finitely smooth) transformation is known as Hartman's theorem. (See [**1**]).

EXAMPLE 1. Let $n = 2$ and let $m \geq 2$ be an integer. Let $\mathscr{L}$ be given by

$$\mathscr{L} = y_1 \partial_{y_1} + m y_2 \partial_{y_2}. \tag{2.12}$$

Then the resonance α such that $|\alpha| \geq 2$ is given by $\alpha = (\alpha_1, \alpha_2) = (m, 0)$.

Indeed, the resonance $\alpha = (\alpha_1, \alpha_2) \in \boldsymbol{Z}_+^2$ satisfies $\alpha_1 + \alpha_2 \geq 2$ and

$$\alpha_1 + m\alpha_2 - 1 = 0, \quad \text{or} \quad \alpha_1 + m\alpha_2 - m = 0.$$

Since $\alpha_1 + m\alpha_2 - 1 \neq 0$, by the conditions $\alpha_1 + \alpha_2 \geq 2$ and $m \geq 2$, we obtain $\alpha_1 + m\alpha_2 = m$ and $\alpha_1 + \alpha_2 \geq 2$. It follows that $(\alpha_1, \alpha_2) = (m, 0)$. Hence, by Theorem 2 we have a singular solution of (2.10) containing $\log y_1$.

EXAMPLE 2. We consider (2.10) for

$$\mathscr{L} = y_1 \partial_{y_1} + 2y_2 \partial_{y_2} + 4(y_3 \partial_{y_3} + y_4 \partial_{y_4}) + \varepsilon y_3 \partial_{y_4},$$

where ε is a constant. The resonance relations are given by $|\alpha| \geq 2$ and one of the equations, $\mathscr{C}(\alpha) = 1$, $\mathscr{C}(\alpha) = 2$, $\mathscr{C}(\alpha) = 4$, where $\mathscr{C}(\alpha) = \alpha_1 + 2\alpha_2 + 4(\alpha_3 + \alpha_4)$. By simple computations the resonances are given by $\alpha = (2, 0, 0, 0), (0, 2, 0, 0), (2, 1, 0, 0), (4, 0, 0, 0)$. Hence a singular solution contains singularities, either $y_1^2 \log y_1$ or $y_2^2 \log y_2$. We note that both singularities generally appear.

3. Proof of Theorems.

We will prove Theorem 1 in eight steps.

Step 1: Let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, $1 \le j \le n$ be the j-th unit vector. By substituting the expansion (2.6), $u = (u_1, \ldots, u_N)$, $u_k(x) = \sum_{|\alpha| \ge 1, \beta \in \mathbf{Z}_+^n} u_{k,\alpha,\beta} x^\alpha (\log x)^\beta$ $(1 \le k \le N)$ into (2.1), we have the following recurrence relation for $\{u_{k,\alpha,\beta}\}$

$$
\begin{aligned}
& p_k(\alpha) u_{k,\alpha,\beta} + \sum_{j \in A} (\alpha_j + 1)\varepsilon_{j,k} u_{k,\alpha+e_j-e_{j-1},\beta} + \sum_{j \notin A} \lambda_{j,k}(\beta_j + 1) u_{k,\alpha,\beta+e_j} \\
& = \tilde{d}_{k,\alpha,\beta} - \sum_{\eta+\xi=\alpha, |\eta| \ge 1} c_{k,\eta} u_{k,\xi,\beta} + F_k(\{u_{\zeta,\gamma}\}_{|\zeta|<|\alpha|,|\gamma|\le|\beta|}),
\end{aligned}
\tag{3.1}
$$

where $\tilde{d}_{k,\alpha,\beta} = d_{k,\alpha}$ (if $\beta = 0$), $= 0$ (if $\beta \ne 0$). Here $c_{k,\alpha}$ and $d_{k,\alpha}$ denote the coefficients of x^α in the Taylor expansions of $c_k(x)$ and $d_k(x)$ at the origin, and $F_k(\{Z_{\zeta,\gamma}\})$ is a polynomial of $\{Z_{\zeta,\gamma}\}$. We want to determine $u_{k,\alpha,\beta}$ from (3.1).

For a multi-integer $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbf{Z}_+^n$ we define the lexicographic order $\gamma \succ 0$ if there exists $j \ge 1$ such that $\gamma_1 = 0, \ldots, \gamma_{j-1} = 0, \gamma_j \ge 1$. Let $\nu > 0$ be an integer. Then we line up all $\alpha \in \mathbf{Z}_+^n$ such that $|\alpha| = \nu$, namely $\alpha^1 \succ \cdots \succ \alpha^r$ for the integer $r = \binom{\nu+n-1}{n-1}$. We note that $\alpha + e_j - e_{j-1} \prec \alpha$ for $j = 2, 3, \ldots, n$. Similarly, we line up all $\beta \in \mathbf{Z}_+^{n-\#A}$ in the lexicographic order, where $\#A$ is the cardinality of A. For every integer k, $(k = 1, 2, \ldots, N)$, we define the unknown vectors U_k by

$$
U_k = {}^t(U_{k,0}, U_{k,1}, \ldots, U_{k,\mu}, \ldots),
\tag{3.2}
$$

where $U_{k,\ell}$ are defined by $U_{k,\ell} = {}^t(U_k^\beta)_{|\beta|=\ell}$, $(\ell = 0, 1, 2, \ldots)$ and

$$
U_k^\beta := {}^t(u_{k,\alpha^1,\beta}, u_{k,\alpha^2,\beta}, \ldots, u_{k,\alpha^r,\beta}).
$$

Here we use the lexicographic order in the set of β such that $|\beta| = \ell$ when defining $U_{k,\ell}$.

In order to write (3.1) in the matrix form we denote the right-hand side of (3.1) by $g_{k,\alpha,\beta}$ for the sake of simplicity. We note that these terms are known terms in the recurrence relations. For every integer k, $1 \le k \le N$, we define G_k by the same way as U_k,

$$
G_k = {}^t(G_{k,0}, G_{k,1}, \ldots, G_{k,\mu}, \ldots),
\tag{3.3}
$$

where $G_{k,\ell} = {}^t(G_k^\beta)_{|\beta|=\ell}$, $(\ell = 0, 1, 2, \ldots)$, and

$$G_k^\beta := {}^t(g_{k,\alpha^1,\beta}, g_{k,\alpha^2,\beta}, \ldots, g_{k,\alpha^r,\beta}).$$

We write (3.1) in the infinite system of equations

$$\mathfrak{A}_k U_k = G_k, \quad k = 1, 2, \ldots, N, \tag{3.4}$$

where $\mathfrak{A}_k$ is given by

$$\mathfrak{A}_k = \begin{pmatrix} \mathfrak{B}_1 & \mathfrak{B}_{1,2} & & & \cdots & & 0 & & 0 \\ 0 & \mathfrak{B}_2 & \mathfrak{B}_{2,3} & & \cdots & & & & 0 \\ 0 & 0 & \mathfrak{B}_3 & \mathfrak{B}_{3,4} & & & & & 0 \\ & & & \ddots & \ddots & & & & \\ 0 & & & & \mathfrak{B}_{\mu-1} & \mathfrak{B}_{\mu-1,\mu} & & & 0 \\ 0 & & & & & \mathfrak{B}_\mu & \mathfrak{B}_{\mu,\mu+1} & & 0 \\ & & & & & & & & \\ 0 & & & & 0 & 0 & \ddots & & \ddots \end{pmatrix}. \tag{3.5}$$

Here $\mathfrak{B}_i$ $(i = 1, 2, \ldots)$ are diagonal matrices $\mathfrak{B}_i = A_0 \otimes I$, with A_0 given by

$$A_0 = \begin{pmatrix} p_k(\alpha^1) & & * \\ & \ddots & \\ & & p_k(\alpha^r) \end{pmatrix}, \tag{3.6}$$

where $*$ denotes terms which come from the second term of the left-hand side of (3.1). The size of the identity matrix in $\mathfrak{B}_i = A_0 \otimes I$ is equal to $\#\{\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{Z}_+^n; |\beta| = i - 1, \beta_j = 0 \text{ for } j \notin A\}$. Indeed, these matrices appear from the first and the second term of the left-hand side of (3.1) because the coefficients are independent of β. The matrix $\mathfrak{B}_{i,i+1}$ comes from the third term of the left-hand side of (3.1). The components are given by $\lambda_{\nu,k}(\beta_\nu + 1)I$ for some ν, where I is the identity matrix with the same size as A_0. If we write the third term of the left-hand side of (3.1) in the matrix form, then the lower diagonal part of $\mathfrak{B}_{i,i+1}$ vanishes.

Because the set of all resonances is a finite set, by Assumption 3, the set of all lengths $|\alpha|$ of resonances α is given by $\ell_j, (1 \leq j \leq r)$ for some integer $r \geq 1$. We may assume $1 \leq \ell_1 < \cdots < \ell_r$ without loss of generality. First we will determine $u_{k,\alpha,\beta}$ in the case where $1 \leq |\alpha| \leq \ell_1 - 1$. Because $p_k(\alpha) \neq 0$ for all $|\alpha| < \ell_1$, A_0 with

$|\alpha| = \nu$ is invertible for every $\nu < \ell_1$, and hence $\mathfrak{B}_1$ is invertible by the definition. We put $U_{k,j} = 0$ ($j \geq 1$), $k = 1, \ldots, N$. Then we can solve (3.4) recurrently for $|\alpha| < \ell_1$. Hence we can determine $\{u_{k,\alpha,0}\}_{|\alpha| \leq \ell_1 - 1}$.

Step 2: The case where $|\alpha| = \ell_1$. By the definition the resonance occurs for some $k = k_0$. If the resonance does not occur for the k-th equation, then the argument is similar to Step 1. Hence we consider the case $k = k_0$. Because there appears no logarithmic terms for $|\alpha| < \ell_1$, we have $G_{k,j} = 0$ for $j \geq 1$ and the quantities $G_{k,0}$ ($1 \leq k \leq N$) are already determined. For every resonance $\alpha = (\alpha_1, \ldots, \alpha_n)$, we take an integer $\ell = \ell(\alpha)$ such that $\alpha_\ell \neq 0$ by Assumption 2.

In order to solve (3.4), we will solve the Riemann-Hilbert factorization problem. The first step is to transform the infinite system (3.4) to a finite one by imposing additional conditions on $u_{k,\alpha,\beta}$. Namely we will look for the solution U_k with compact support in the sense that there exists s such that $U_{k,j} = 0$ for $j > s$ and $k = 1, \ldots, N$.

Let k, $1 \leq k \leq N$ be arbitrarily given. We line up all resonances of p_k satisfying $|\alpha| = \ell_1$ in the form

$$\alpha^1 \succ \alpha^2 \succ \cdots \succ \alpha^s, \tag{3.7}$$

where s is the number of resonances with $|\alpha| = \ell_1$.

We impose the additional conditions for $u_{k,\alpha,\beta}$

$$u_{k,\alpha,\beta} = 0 \quad \text{if} \quad \alpha \prec \alpha^{s-i+1}, \ |\beta| \geq i, \quad i = 1, 2, \ldots, s+1, \tag{3.8}$$

where we understand that $u_{k,\alpha,\beta} = 0$ if $|\beta| \geq s+1$, namely $U_{k,j} = 0$ for $j \geq s+1$. We will show that (3.8) is compatible with (3.4) with $G_k = 0$, namely

$$\mathfrak{B}_{j+1} U_{k,j} + \mathfrak{B}_{j+1,j+2} U_{k,j+1} = 0. \tag{3.9}$$

In order to see this, let us first consider (3.8) with $i = 1$

$$u_{k,\alpha,\beta} = 0 \quad \text{if} \quad \alpha \prec \alpha^s, \ |\beta| \geq 1. \tag{3.10}$$

We want to show that for every j, $j = 1, 2, \ldots$, the set of equations in the system (3.9) corresponding to $\alpha \prec \alpha^s$ is trivially satisfied. Indeed, if we write (3.9) in the recurrence relation, then we obtain the left-hand side of (3.1). Because $\alpha^s \succ \alpha \succ \alpha + e_j - e_{j-1}$, the left-hand side of (3.1) vanishes by the condition (3.10). Next we consider the α^s-component of (3.9) with $j = 1$. By the condition (3.8) with $i = 2$ the α^s-component of $U_{k,2}$ vanishes, and hence the corresponding

component of $\mathfrak{B}_{2,3}U_{k,2}$ also vanishes. Although the α^s-component of $U_{k,1}$ may not vanish, the resonance condition $p_k(\alpha^s) = 0$ and the above argument for the nilpotent part show that the α^s-component of $\mathfrak{B}_2 U_{k,1}$ also vanishes. Hence the condition (3.10) is compatible with (3.9) with $j \geq 1$. We make the same argument for the cases $2 \leq i < s$. In order to verify that (3.8) with $i = s$ is compatible with (3.9) we consider

$$\mathfrak{B}_{s+1}U_{k,s} + \mathfrak{B}_{s+1,s+2}U_{k,s+1} = 0. \qquad (3.11)$$

We note that $U_{k,s+1} = 0$ by (3.8) with $i = s + 1$. The condition (3.8) with $i = s$ and the above argument for the nilpotent part show that the α-component of the left-hand side of (3.11) vanishes for $\alpha^1 \succ \alpha$. The equation corresponding to α^1 in (3.11) is clearly satisfied by the resonance condition $p_k(\alpha^1) = 0$ and the above argument for the nilpotent part.

Then the system of equations (3.4) with the additional conditions (3.8) is equivalent to the following system of equations together with (3.8).

$$\mathfrak{A}'_k U'_k = G'_k, \quad k = 1, 2, \ldots, N, \qquad (3.12)$$

where $\mathfrak{A}'_k$ is given by

$$\mathfrak{A}'_k = \begin{pmatrix} \mathfrak{B}_1 & \mathfrak{B}_{1,2} & & & \cdots & & 0 \\ 0 & \mathfrak{B}_2 & \mathfrak{B}_{2,3} & & \cdots & & 0 \\ 0 & 0 & \mathfrak{B}_3 & \mathfrak{B}_{3,4} & & & 0 \\ & & & \ddots & \ddots & & 0 \\ 0 & & & & & \mathfrak{B}_{s+1} & \mathfrak{B}_{s+1,s+2} \\ 0 & & & & & & \mathfrak{B}_{s+2} \end{pmatrix}. \qquad (3.13)$$

The vectors U'_k and G'_k are given by

$$U'_k = {}^t(U_{k,0}, U_{k,1}, \ldots, U_{k,s+1}), \ G'_k = {}^t(G_{k,0}, G_{k,1}, \ldots, G_{k,s+1}), \ k = 1, \ldots, N. \quad (3.14)$$

Here we note that $U_{k,s+1} = 0$ by (3.8). The introduction of $U_{k,s+1}$ is necessary in the following argument.

First we note that we may assume that the off-diagonal components of the matrix A_0 in (3.6) can be made arbitrarily small. Indeed, this is possible if we introduce a weight in the unknown variable $U_k^\beta = (u_{k,\alpha,\beta})_{\alpha,|\alpha|=\nu}$ with respect to the order $\succ$. More precisely, let $D_\varepsilon = \mathrm{diag}(1, \varepsilon, \varepsilon^2, \ldots)$ be the diagonal matrix for a

nonzero small constant ε. We set $V_k^\beta = D_\varepsilon^{-1} U_k^\beta$ and $H_k^\beta = D_\varepsilon^{-1} G_k^\beta$, and we consider the equations for V_k^β and H_k^β instead of U_k^β and G_k^β. Then we can easily see that the off-diagonal elements can be made arbitrarily small if ε is sufficiently small. In the following we assume the property without loss of generality.

We will show that the matrix $\mathfrak{A}_k'$ is nonsingular. In order to show this we use the so-called Frobenius' deformation. Let α be a resonance and let $\ell = \ell(\alpha)$ be given by Assumption 2. We exchange the column in $\mathfrak{B}_1$ which corresponds to α with the one in $\mathfrak{B}_{1,2}$ corresponding to $u_{k,\alpha,\beta+e_{\ell(\alpha)}}$. In view of the relation (3.1), the coefficient of $u_{k,\alpha,\beta+e_{\ell(\alpha)}}$ is given by $\lambda_{\ell(\alpha),k}(\beta_{\ell(\alpha)} + 1)$, which does not vanish by the Poincaré condition. Then there appear new matrix components as a $(2,1)$-element of $\mathfrak{A}_k'$, which comes from the nilpotent part of A_0 in $\mathfrak{B}_2$. We denote this new element by $R_{2,1}$. Then, the matrices $\mathfrak{B}_1$ and $\mathfrak{B}_{1,2}$ are deformed to the ones $\tilde{\mathfrak{B}}_1$ and $\tilde{\mathfrak{B}}_{1,2}$, respectively. By the definition, we can easily see that $\tilde{\mathfrak{B}}_1$ is nonsingular.

Next we exchange the column in $\mathfrak{B}_2$ which corresponds to a resonance α with the one in $\mathfrak{B}_{2,3}$ in such a way that the diagonal component of $\mathfrak{B}_2$ corresponding to a resonance α is replaced by a certain nonzero term $\lambda_{\ell(\alpha),k}(\beta_{\ell(\alpha)} + 1)$. Then we can deform $\mathfrak{B}_2$ to $\tilde{\mathfrak{B}}_2$. We note that the invertibility of $\tilde{\mathfrak{B}}_2$ does not follow from the above transformation because there appears a new component in the off-diagonal components.

We continue these deformations until we exchange those columns in $\mathfrak{B}_{s+1}$ corresponding to resonances with the ones in $\mathfrak{B}_{s+1,s+2}$. Then we obtain a new matrix $\tilde{\mathfrak{B}}_{s+1}$. Therefore the new matrix $\mathfrak{A}_k''$ is given by

$$\mathfrak{A}_k'' = \begin{pmatrix} \tilde{\mathfrak{B}}_1 & \tilde{\mathfrak{B}}_{1,2} & & & \cdots & & \tilde{\mathfrak{B}}_{1,s+2} \\ R_{2,1} & \tilde{\mathfrak{B}}_2 & \tilde{\mathfrak{B}}_{2,3} & & \cdots & & \tilde{\mathfrak{B}}_{2,s+2} \\ 0 & R_{3,2} & \tilde{\mathfrak{B}}_3 & \tilde{\mathfrak{B}}_{3,4} & & & \tilde{\mathfrak{B}}_{3,s+2} \\ & & & \ddots & \ddots & & \\ 0 & & & & R_{s+1,s} & \tilde{\mathfrak{B}}_{s+1} & \tilde{\mathfrak{B}}_{s+1,s+2} \\ 0 & & & & & R_{s+2,s+1} & \tilde{\mathfrak{B}}_{s+2} \end{pmatrix}. \quad (3.15)$$

We also define $\tilde{U}_k'' = {}^t(\tilde{U}_{k,0}, \tilde{U}_{k,1}, \ldots, \tilde{U}_{k,s+1})$ and $\tilde{G}_k''$ by the same permutation in the components as the one for the columns of $\mathfrak{A}_k$.

We consider the equation $R_{s+2,s+1}\tilde{U}_{k,s} + \tilde{\mathfrak{B}}_{s+2}\tilde{U}_{k,s+1} = 0$. We note that the α-th column of $R_{s+2,s+1}$ does not vanish only if α is a resonance. Because the α-th component of $\tilde{U}_{k,s}$ is the α-th component of $U_{k,s+1} = 0$, it follows that α-th component of $\tilde{U}_{k,s}$ vanish for any resonance α. Hence we have $R_{s+2,s+1}\tilde{U}_{k,s} = 0$. We define $\tilde{U}_{k,s+1} = 0$. Next we define $\tilde{U}_k'$ and $\tilde{G}_k'$ by deleting the last components. We

also define $\tilde{\mathfrak{A}}'_k$ by deleting the $(s+2)$-th row and the $(s+2)$-th column of $\mathfrak{A}''_k$. Then the equation $\mathfrak{A}'_k U'_k = G'_k$ can be written in

$$\tilde{\mathfrak{A}}'_k \tilde{U}'_k = \tilde{G}'_k. \tag{3.16}$$

We shall solve (3.16). We write $\tilde{\mathfrak{A}}'_k = T + R$, where R denotes the lower triangular part of $\tilde{\mathfrak{A}}'_k$ and $T := \tilde{\mathfrak{A}}'_k - R$. Because the components of R consist of the elements in the upper triangular part of A_0, it follows that the norm of R can be made arbitrarily small. Therefore if we can show that $\det \tilde{\mathfrak{B}}_j \neq 0$ for $j = 1, 2, \ldots, s+1$, then T is invertible and $T + R = \tilde{\mathfrak{A}}'_k$ is invertible.

Step 3: We will show that $\det \tilde{\mathfrak{B}}_j \neq 0$ for $j = 1, 2, \ldots, s+1$. By the definition we can easily see that $\det \tilde{\mathfrak{B}}_1 \neq 0$. Hence we will show that $\det \tilde{\mathfrak{B}}_{j+1} \neq 0$ for $j > 0$. For this purpose we use the so-called Riemann-Hilbert factorization of $\tilde{\mathfrak{B}}_{j+1}$. For the sake of simplicity we assume that the rows of $\tilde{\mathfrak{B}}_{j+1}$ are assigned by the order $\beta^1 \prec \beta^2 \prec \cdots \prec \beta^q$. Then we have

$$\tilde{\mathfrak{B}}_{j+1} = \begin{pmatrix} A_{\beta^1} & & & & \\ & A_{\beta^2} & & P_{\beta,\gamma} & \\ & & \ddots & & \\ & * & & A_{\beta^{q-1}} & \\ & & & & A_{\beta^q} \end{pmatrix}, \tag{3.17}$$

where the upper triangular matrix A_β is obtained by replacing the columns of A_0 corresponding to a resonance α with the one in $\lambda_{\ell(\alpha),k}(\beta_{\ell(\alpha)} + 1)I$. Because $\lambda_{\ell(\alpha),k}$ does not vanish by the Poincaré condition, it follows that $\det A_\beta \neq 0$.

The matrix $P_{\beta,\gamma}$ is the (β, γ)-component of $\tilde{\mathfrak{B}}_{j+1}$, $|\beta| = |\gamma| = j$. The (β, γ)-component appears if there exists a nonzero element which comes from $u_{k,\alpha,\beta+e_i}$ in (3.1) in the column of $\mathfrak{B}_{j+1,j+2}$ corresponding to $u_{k,\alpha,\gamma+e_\ell}$ $(\alpha;$ resonance$)$, where $\ell = \ell(\alpha)$ is the integer given by Assumption 2. This implies that $\beta + e_i = \gamma + e_\ell$. By definition, $P_{\beta,\gamma}$ is given by

$$P_{\beta,\gamma} = \sum_{\alpha,\text{resonance}} c_\alpha^{\beta,\gamma} E_\alpha, \tag{3.18}$$

where $c_\alpha^{\beta,\gamma}$ is a certain constant and E_α is the projection matrix to the α-th component, namely, the diagonal matrix with the α-th component equal to 1 and the rest being zero.

We will show that $\tilde{\mathfrak{B}}_{j+1}$ can be factorized in the following form

$$\tilde{\mathfrak{B}}_{j+1}Q = \mathfrak{D}_{j+1} = \begin{pmatrix} A_{\beta^1} & & & \\ & \ddots & & 0 \\ & & \ddots & \\ * & & & \\ & & & A_{\beta^q} \end{pmatrix}, \tag{3.19}$$

where the matrix Q is the product of the matrices of the form

$$Q_1 = \begin{pmatrix} I & & & & \\ & \ddots & -A_{\gamma}^{-1}P_{\beta,\gamma} & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \end{pmatrix} \tag{3.20}$$

for some A_γ and $P_{\beta,\gamma}$, and where the lower triangular elements of $\mathfrak{D}_{j+1}$ are identical with the corresponding elements of $\tilde{\mathfrak{B}}_{j+1}$. Especially, we have $\det \tilde{\mathfrak{B}}_{j+1} \neq 0$. We note that the invariance of the lower triangular part is crucial in the argument of Step 6, where we show a certain finiteness of the singularity of the solution.

First we show that, if either $\gamma' \preceq \gamma \prec \beta$ or $\beta \prec \gamma' \preceq \gamma$ holds, then $P_{\beta,\gamma'}$ and $P_{\gamma,\beta}$ do not contain the same projection E_α for any resonance α. Indeed, assume that $\beta \prec \gamma' \preceq \gamma$ and that $P_{\gamma,\beta}$ and $P_{\beta,\gamma'}$ contain $c_\alpha^{\gamma,\beta}E_\alpha$ and $c_\alpha^{\beta,\gamma'}E_\alpha$, respectively. If neither $c_\alpha^{\gamma,\beta}$ nor $c_\alpha^{\beta,\gamma'}$ vanishes, then it follows that there exist $j > \ell = \ell(\alpha)$ such that

$$\beta + e_\ell = \gamma + e_j \tag{3.21}$$

and $\nu < \ell$ such that

$$\gamma' + e_\ell = \beta + e_\nu. \tag{3.22}$$

The condition $j > \ell$ follows from $\beta - \gamma = e_j - e_\ell \prec 0$, and the condition $\nu < \ell$ follows from $\beta \prec \gamma'$. By adding these equalities, we obtain

$$\gamma' + 2e_\ell = \gamma + e_j + e_\nu, \quad \nu < \ell < j. \tag{3.23}$$

Hence we have $\gamma' - \gamma = e_\nu + e_j - 2e_\ell \succ 0$, because $\nu < \ell < j$. This contradicts to the assumption $\gamma' \preceq \gamma$. The other cases will be proved similarly. Hence either $c_\alpha^{\gamma,\beta}$ or $c_\alpha^{\beta,\gamma'}$ vanishes. This proves the assertion.

We will erase the off-diagonal elements of the first row of $\tilde{\mathfrak{B}}_{j+1}$, $P_{\beta^1,\gamma}$, $(\gamma \succ \beta^1)$ for some β^1. We multiply the first column of $\tilde{\mathfrak{B}}_{j+1}$ by $-A_{\beta^1}^{-1} P_{\beta^1,\gamma}$ $(\gamma \succ \beta^1)$ from the right, and we add it to the γ-th column. Then the β-th component of the γ-th column of the transformed matrix is given by

$$\tilde{P}_{\beta,\gamma} := P_{\beta,\gamma} - P_{\beta,\beta^1} A_{\beta^1}^{-1} P_{\beta^1,\gamma}. \tag{3.24}$$

We will show that if $\gamma \preceq \beta$, then $\tilde{P}_{\beta,\gamma} = P_{\beta,\gamma}$. Indeed, we will show that $P_{\beta,\beta^1} A_{\beta^1}^{-1} P_{\beta^1,\gamma} = 0$. We write $A_{\beta^1} = B + N$, where B is the diagonal part of A_{β^1} and $N = A_{\beta^1} - B$ is an upper triangular nilpotent matrix. We note that B is invertible. We have

$$A_{\beta^1}^{-1} = (I - B^{-1}N + (B^{-1}N)^2 - (B^{-1}N)^3 + \cdots)B^{-1}, \tag{3.25}$$

where the summation is a finite sum. By what we have proved in the above, we obtain $P_{\beta,\beta^1} B^{-1} P_{\beta^1,\gamma} = 0$ because B^{-1} is a diagonal matrix, and P_{β,β^1} and $P_{\beta^1,\gamma}$ do not contain the same projection E_α. We will show that $P_{\beta,\beta^1} B^{-1} N B^{-1} P_{\beta^1,\gamma} = 0$. By the definition, the column of the matrix N for a resonant α vanishes. It follows that the α-th column of the matrix $B^{-1}NB^{-1}$ for a resonant α also vanishes. In view of the definition of $P_{\beta^1,\gamma}$ we see that the matrix $B^{-1}NB^{-1}P_{\beta^1,\gamma}$ vanishes except for those columns corresponding to the resonances. For a resonance α, the column is identical with the one of $B^{-1}NB^{-1}$ except for a constant factor. Hence $P_{\beta,\beta^1} B^{-1} N B^{-1} P_{\beta^1,\gamma} = 0$. Similarly, we can show that

$$P_{\beta,\beta^1}(B^{-1}N)^\nu B^{-1} P_{\beta^1,\gamma} = 0 \quad \text{for } \nu = 1,2,\dots$$

because the α-th column of $(B^{-1}N)^\nu B^{-1}$ for a resonance α vanishes. Therefore we obtain $P_{\beta,\beta^1} A_{\beta^1}^{-1} P_{\beta^1,\gamma} = 0$. We note that the lower triangular part of $\tilde{\mathfrak{B}}_{j+1}$ is invariant under the above transformation. For the sake of simplicity we denote the transformed matrix with the same letter $\tilde{\mathfrak{B}}_{j+1}$.

We will erase the components in the second row in the upper triangular part of $\tilde{\mathfrak{B}}_{j+1}$. For this purpose we multiply the second column of $\tilde{\mathfrak{B}}_{j+1}$ by $-A_{\beta^2}^{-1} \tilde{P}_{\beta^2,\gamma}$ $(\gamma \succ \beta^2)$ from the right and add it to the γ-th column. Then the (β,γ)-component of the transformed matrix $\tilde{\mathfrak{B}}_{j+1}$ is given by

$$\tilde{P}_{\beta,\gamma} - P_{\beta,\beta^2} A_{\beta^2}^{-1} \tilde{P}_{\beta^2,\gamma}$$
$$= P_{\beta,\gamma} - P_{\beta,\beta^1} A_{\beta^1}^{-1} P_{\beta^1,\gamma} - P_{\beta,\beta^2} A_{\beta^2}^{-1} (P_{\beta^2,\gamma} - P_{\beta^2,\beta^1} A_{\beta^1}^{-1} P_{\beta^1,\gamma}) \qquad (3.26)$$
$$= P_{\beta,\gamma} - P_{\beta,\beta^1} A_{\beta^1}^{-1} P_{\beta^1,\gamma} - P_{\beta,\beta^2} A_{\beta^2}^{-1} P_{\beta^2,\gamma} + P_{\beta,\beta^2} A_{\beta^2}^{-1} P_{\beta^2,\beta^1} A_{\beta^1}^{-1} P_{\beta^1,\gamma}.$$

For the sake of simplicity we write the left-hand side of (3.26) by the same letter $\tilde{P}_{\beta,\gamma}$.

In general, when we erase the components in the μ-th ($\mu \le s$) row in the upper triangular part of $\tilde{\mathfrak{B}}_{j+1}$, the rest components are given by

$$\tilde{P}_{\beta,\gamma} = P_{\beta,\gamma} + \sum_{k=1}^{\mu} \sum_{1 \le \nu_1 < \cdots < \nu_k \le \mu} (-1)^k P_{\beta,\beta^{\nu_k}} A_{\beta^{\nu_k}}^{-1} \cdots A_{\beta^{\nu_1}}^{-1} P_{\beta^{\nu_1},\gamma}. \qquad (3.27)$$

Because the argument is similar to the case $\mu = 2$ and $\mu = 3$ we omit the proof.

We will show that if $\gamma \preceq\!\!\!= \beta$, then $\tilde{P}_{\beta,\gamma} = P_{\beta,\gamma}$. For this purpose we will show that

$$P_{\beta,\beta^{\nu_k}} A_{\beta^{\nu_k}}^{-1} \cdots A_{\beta^{\nu_1}}^{-1} P_{\beta^{\nu_1},\gamma} = 0. \qquad (3.28)$$

In view of the argument in (3.24) it is sufficient to show that

$$P_{\beta,\beta^{\nu_k}} \cdots P_{\beta^{\nu_1},\gamma} = 0. \qquad (3.29)$$

Therefore we will show that $P_{\beta,\beta^{\nu_k}}, \cdots, P_{\beta^{\nu_1},\gamma}$ do not contain the same projection E_α for any resonance α. Suppose that such an α exists. Let $\ell = \ell(\alpha)$ be an integer given by Assumption 2. Then we have

$$\gamma + e_\ell = \beta^{\nu_1} + e_{j_1}, \beta^{\nu_1} + e_\ell = \beta^{\nu_2} + e_{j_2}, \cdots, \beta^{\nu_k} + e_\ell = \beta + e_{j_k}, \qquad (3.30)$$

for some integers $j_1, j_2, \cdots, j_k$. By the condition $\beta^{\nu_1} \prec \cdots \prec \beta^{\nu_k} \prec \gamma \preceq\!\!\!= \beta$ we obtain $\gamma - \beta^{\nu_1} = e_{j_1} - e_\ell \succ 0$. It follows that we have $j_1 < \ell$. Similarly, we have $\ell < j_2, \ldots, \ell < j_k$. By adding both sides of (3.30) we obtain $\gamma + k e_\ell = \beta + \sum_{p=1}^k e_{j_p}$. Thus we have $0 \preceq\!\!\!= \beta - \gamma = k e_\ell - \sum_{p=1}^k e_{j_p} \prec 0$. This is a contradiction. Therefore the lower triangular part of $\tilde{\mathfrak{B}}_{j+1}$ is invariant under the deformation. For the sake of simplicity we write the deformed matrix with the same letter $\tilde{\mathfrak{B}}_{j+1}$.

Step 4: We consider the case where $|\alpha| = \nu$, $\ell_1 < \nu < \ell_2$. By Assumption 2, we have $\ell_1 \ge 1$. We first consider the case $\nu = \ell_1 + 1$. Let the integer k, $1 \le k \le N$

be given and fixed. By the construction of the formal solution in Step 2 the term $x^\alpha(\log x)^\beta$ in the approximate solution $u_k = \sum_{|\alpha|\leq\ell_1} u_{k,\alpha,\beta}\, x^\alpha(\log x)^\beta$ in Step 2 satisfies $|\alpha| = \ell_1 \geq 1$ if $\beta \neq 0$. By substituting the formal power series u into $f_k(x,u)$ we have

$$f_k(x,u) = \sum_{\gamma\geq 0} g_{k,\gamma}(x)(\log x)^\gamma = \sum_{\nu=0}^{\infty}\sum_{|\gamma|=\nu} g_{k,\gamma}(x)(\log x)^\gamma. \qquad (3.31)$$

The vanishing order of $g_{k,\gamma}(x), |\gamma| = \nu$ tends to infinity when $\nu \to \infty$. It follows that the length $|\beta|$ of $x^\alpha(\log x)^\beta$ which appears from $f_k(x,u)$ is bounded if $|\alpha| \leq \ell_1 + 1$. We choose an integer s such that $G_{k,j}$ vanishes for $j > s$ and $G_{k,s} \neq 0$. Because there is no resonance, it follows that $\mathfrak{B}_j$ is invertible for all $j \geq 1$. We define $U_{k,j} = 0$ for $j > s$. Then we want to solve the system of equations

$$\mathfrak{B}_{j+1}U_{k,j} + \mathfrak{B}_{j+1,j+2}U_{k,j+1} = G_{k,j}, \quad j = 0,1,\ldots,s. \qquad (3.32)$$

Indeed, we solve (3.32) for $j = s$ and we determine $U_{k,s}$. Then we solve (3.32) for $j = s - 1$ and so on. Hence we can determine $u_{k,\alpha,\beta}$ for $|\alpha| = \ell_1 + 1$.

We consider the case $|\alpha| \geq \ell_1 + 2$. We substitute the modified formal power series $u = \sum_{|\alpha|\leq\ell_1+1} u_{\alpha,\beta}\, x^\alpha(\log x)^\beta$ into $f_k(x,u)$, and we make the same argument as in the above. Then we can inductively determine the approximate solution $u = \sum_{|\alpha|<\ell_2} u_{\alpha,\beta}\, x^\alpha(\log x)^\beta$.

Step 5: We consider the case $|\alpha| \geq \ell_2$. We continue to use the same notation as in Steps 2 and 3. Let k, $1 \leq k \leq N$ be arbitrarily given and fixed. Because the case $|\alpha| \geq \ell_3$ is the same as the case $|\alpha| = \ell_2$ and the nonresonant case, we may consider the case $\ell_2 \leq |\alpha| < \ell_3$. The case $\ell_2 < |\alpha| < \ell_3$ is similar to the case $\ell_1 < |\alpha| < \ell_2$, and we can determine the coefficients $u_{k,\alpha,\beta}$ of the formal solution. Hence we consider the case $|\alpha| = \ell_2$.

We take the smallest integer τ such that $G_{k,j}$ vanishes for $j > \tau$ and we consider the infinite system of equations

$$\mathfrak{B}_{j+1}U_{k,j} + \mathfrak{B}_{j+1,j+2}U_{k,j+1} = G_{k,j}, \quad j \geq 0. \qquad (3.33)$$

We will deform the equation (3.33) to a finite system of equations similar to (3.12) by the argument in Step 2. For this purpose we impose the following conditions

$$u_{k,\alpha,\beta} = 0 \quad \text{if} \quad \alpha \prec \alpha^{s-i+1},\ |\beta| \geq i + \tau, i = 1,2,\ldots,s+1, \qquad (3.34)$$

where s is the number of resonances with $|\alpha| = \ell_2$. It follows that we may consider the following modified system of equations

$$\mathfrak{A}'_{k,\tau} U'_{k,\tau} = G'_{k,\tau}, \quad k = 1, 2, \ldots, N, \tag{3.35}$$

where $\mathfrak{A}'_{k,\tau}$ is given by

$$\mathfrak{A}'_{k,\tau} = \begin{pmatrix} \mathfrak{B}_1 & \mathfrak{B}_{1,2} & & & \cdots & & & 0 \\ 0 & \mathfrak{B}_2 & \mathfrak{B}_{2,3} & & \cdots & & & 0 \\ & & \ddots & \ddots & & & & 0 \\ 0 & 0 & & \mathfrak{B}_\tau & \mathfrak{B}_{\tau,\tau+1} & & & 0 \\ 0 & 0 & & & \mathfrak{B}_{\tau+1} & \mathfrak{B}_{\tau+1,\tau+2} & & \\ 0 & 0 & & & & \mathfrak{B}_{\tau+2} & \mathfrak{B}_{\tau+2,\tau+3} & \\ & & & & & \ddots & & \ddots \\ 0 & & & & & & \mathfrak{B}_{s+\tau+1} & \mathfrak{B}_{s+\tau+1,s+\tau+2} \\ 0 & & & & & & & \mathfrak{B}_{s+\tau+2} \end{pmatrix}. \tag{3.36}$$

The vectors $U'_{k,\tau}$ $(k = 1, 2, \ldots, N)$ are given by

$$U'_{k,\tau} = {}^t(U_{k,0}, U_{k,1}, \ldots, U_{k,s+\tau+1}), \ \ k = 1, \ldots, N, \tag{3.37}$$

and $G'_{k,\tau}$ are defined similarly.

In order to solve (3.35) we study the invertibility of $\mathfrak{A}'_{k,\tau}$ in (3.36). We make Frobenius' deformation. Namely, we exchange those columns in $\mathfrak{B}_1$ which corresponds to the resonance α with the ones in $\mathfrak{B}_{1,2}$. Then there appear new matrix components as a $(2,1)$-element which comes from the nilpotent part of A_0 in $\mathfrak{B}_2$. We denote this new element $R_{2,1}$. Similarly, the matrix $\mathfrak{B}_1$ and $\mathfrak{B}_{1,2}$ becomes $\tilde{\mathfrak{B}}_1$ and $\tilde{\mathfrak{B}}_{1,2}$, respectively. Next we exchange those columns in $\mathfrak{B}_2$ corresponding to the resonance α with the ones in $\mathfrak{B}_{2,3}$. We continue to make these deformations until we exchange those columns in $\mathfrak{B}_{s+\tau+1}$ with the ones in $\mathfrak{B}_{s+\tau+1,s+\tau+2}$. Then we make similar arguments as in (3.13), (3.15) and (3.16). Therefore the new matrix $\tilde{\mathfrak{A}}'_{k,\tau}$ is given by

$$\tilde{\mathfrak{A}}'_{k,\tau} =$$

$$\begin{pmatrix} \tilde{\mathfrak{B}}_1 & \tilde{\mathfrak{B}}_{1,2} & & & \cdots & & & 0 \\ R_{2,1} & \tilde{\mathfrak{B}}_2 & \tilde{\mathfrak{B}}_{2,3} & & \cdots & & & \\ & & \ddots & & \ddots & & & \\ 0 & & R_{\tau,\tau-1} & \tilde{\mathfrak{B}}_\tau & \tilde{\mathfrak{B}}_{\tau,\tau+1} & & & \\ 0 & & & R_{\tau+1,\tau} & \tilde{\mathfrak{B}}_{\tau+1} & \tilde{\mathfrak{B}}_{\tau+1,\tau+2} & & \\ 0 & 0 & & & R_{\tau+2,\tau+1} & \tilde{\mathfrak{B}}_{\tau+2} & \tilde{\mathfrak{B}}_{\tau+2,\tau+3} & \\ & & & & & \ddots & & \ddots \\ 0 & & & & & R_{s+\tau,s+\tau-1} & \tilde{\mathfrak{B}}_{s+\tau} & \tilde{\mathfrak{B}}_{s+\tau,s+\tau+1} \\ 0 & & & & & & R_{s+\tau+1,s+\tau} & \tilde{\mathfrak{B}}_{s+\tau+1} \end{pmatrix}.$$

$$(3.38)$$

We define

$$\tilde{U}'_{k,\tau} = {}^t(\tilde{U}_{k,0}, \tilde{U}_{k,1}, \ldots, \tilde{U}_{k,s+\tau})$$

by the same permutations and deletions in the components as those for $\tilde{\mathfrak{A}}'_{k,\tau}$. Then the equation (3.35) can be written in $\tilde{\mathfrak{A}}'_{k,\tau}\tilde{U}'_{k,\tau} = \tilde{G}'_{k,\tau}$.

We write $\tilde{\mathfrak{A}}'_{k,\tau} = T + R$, where R denotes the lower triangular part of $\tilde{\mathfrak{A}}'_{k,\tau}$ and $T := \tilde{\mathfrak{A}}'_{k,\tau} - R$. Because the components of R consist of the elements in the upper triangular part of A_β, it follows that the norm of R can be made arbitrarily small. Because we have

$$\det \tilde{\mathfrak{B}}_j \neq 0 \quad \text{for } j = 1, 2, \ldots, s + \tau + 1,$$

by exactly the same argument as in Step 3, it follows that T and $T + R = \tilde{\mathfrak{A}}'_{k,\tau}$ are invertible.

Step 6: By the above arguments, we can determine the solution

$$v_k(x) = \sum_{1 \leq |\alpha| \leq \ell_r, \beta} u_{k,\alpha,\beta} x^\alpha (\log x)^\beta, \quad k = 1, 2, \ldots, N. \quad (3.39)$$

We shall show the finiteness property of the singularity. Namely, $u_{k,\alpha,\beta}$ ($k = 1, 2, \ldots, N$) satisfy

$$u_{k,\alpha,\beta} = 0, \quad \text{if } \alpha_j = 0,\ \beta_j > 0,\ \exists j \in \{1,\ldots,n\} \setminus A. \tag{3.40}$$

Clearly (3.40) holds for $|\alpha| < \ell_1$ because there appears no logarithmic term. We now study the case $|\alpha| = \ell_1$.

We will show that every component of the solution $\tilde{U}'_k$ of the equation (3.16) satisfies (3.40). We write $\tilde{\mathfrak{A}}'_k = T + R$, where T is an upper triangular part of $\tilde{\mathfrak{A}}'_k$, and $R := \tilde{\mathfrak{A}}'_k - T$. We note that every component of R can be made arbitrarily small. Then we have

$$\tilde{\mathfrak{A}}'^{-1}_k = (T + R)^{-1} = (I - T^{-1}R + (T^{-1}R)^2 - (T^{-1}R)^3 + \cdots)T^{-1}. \tag{3.41}$$

In order to show that $\tilde{\mathfrak{A}}'^{-1}_k \tilde{G}'_k$ satisfies (3.40) it is sufficient to show that T^{-1} and R preserve (3.40).

By the definition of T we have

$$T^{-1} = \begin{pmatrix} \tilde{\mathfrak{B}}_1^{-1} & C_{1,2} & \cdots & \cdots & C_{1,s+1} \\ 0 & \tilde{\mathfrak{B}}_2^{-1} & C_{2,3} & \cdots & \\ & & \ddots & \ddots & \vdots \\ 0 & & & \tilde{\mathfrak{B}}_s^{-1} & C_{s,s+1} \\ 0 & & & & \tilde{\mathfrak{B}}_{s+1}^{-1} \end{pmatrix}, \tag{3.42}$$

where $C_{i,j}$ $(i < j)$ are given by

$$C_{i,j} = \sum_{m=1}^{j-i} \sum_{i=i(1)<i(2)<\cdots<i(m)<j} (-1)^m \tilde{\mathfrak{B}}_i^{-1} \tilde{\mathfrak{B}}_{i,i(2)} \tilde{\mathfrak{B}}_{i(2)}^{-1} \tag{3.43}$$
$$\cdots \tilde{\mathfrak{B}}_{i(m-2),i(m-1)} \tilde{\mathfrak{B}}_{i(m-1)}^{-1} \tilde{\mathfrak{B}}_{i(m-1),i(m)} \tilde{\mathfrak{B}}_{i(m)}^{-1},$$

where the summation $\sum_{i=i(1)<i(2)<\cdots<i(m)<j}$ is taken over all combinations.

First we shall show that if every component of F satisfies the condition (3.40), then every component of $\tilde{\mathfrak{B}}_j^{-1}F$ satisfies (3.40). Let $j = 1$ and set $U = (u_{k,\alpha,\beta}) = \tilde{\mathfrak{B}}_1^{-1}F$. Then we have the following recurrence relation

$$p_k(\alpha)u_{k,\alpha,\beta} + \sum_{\nu \in A}(\alpha_\nu + 1)\varepsilon_{\nu,k}u_{k,\alpha+e_\nu-e_{\nu-1},\beta} = \text{a known quantity}, \tag{3.44}$$

where we replace $p_k(\alpha)$ with $\lambda_{\ell(\alpha),k}(\beta_{\ell(\alpha)} + 1)$ when $p_k(\alpha)$ vanishes. Let β be given and fixed. Let j be such that $\beta_j > 0$, $j \in \{1, \ldots, n\} \setminus A$. Let α^1 be the smallest index with respect to the order $\prec$ satisfying $|\alpha| = \ell_1$ and

$$\alpha_j = 0. \tag{3.45}$$

For every $\nu \in A$, the j-th component of the index $\alpha^1 + e_\nu - e_{\nu-1}$ vanishes and $\alpha^1 \succ \alpha^1 + e_\nu - e_{\nu-1}$. Hence, by the minimality of α^1, there appears no term in the second term of the left-hand side of (3.44) satisfying (3.45). Because the right-hand side terms in (3.44) satisfy (3.40) and $\beta_j > 0$, $\alpha_j = 0$, it follows that the right-hand side vanishes. Hence $u_{k,\alpha^1,\beta}$ vanishes.

Let α^2 be the smallest $\alpha \succ \alpha^1$ satisfying (3.45). Because $\alpha^2 \succ \alpha^2 + e_\nu - e_{\nu-1}$, and $\alpha^2 + e_\nu - e_{\nu-1}$ satisfies (3.45) for every $\nu \in A$, there appears only the term $u_{k,\alpha^1,\beta}$ in the second term of the left-hand side of (3.44). Because $u_{k,\alpha^1,\beta}$ vanishes by the above argument, it follows that the second term of the left-hand side of (3.44) vanishes. Because the right-hand side of (3.44) also vanishes by the assumption, it follows that $u_{k,\alpha^2,\beta}$ vanishes. By induction on those α's satisfying (3.45), we see that $u_{k,\alpha,\beta}$ satisfies (3.40). This proves that $\tilde{\mathfrak{B}}_1^{-1}F$ satisfy (3.40).

We consider $\tilde{\mathfrak{B}}_{j+1}^{-1}F$ for $j = 1, 2, \ldots$. By (3.19) we have

$$\tilde{\mathfrak{B}}_{j+1}^{-1}F = Q\mathfrak{D}_{j+1}^{-1}F. \tag{3.46}$$

We set $H = \mathfrak{D}_{j+1}^{-1}F$, and we write the equation $\mathfrak{D}_{j+1}H = F$ in the recurrence relation. We recall that the diagonal component of $\mathfrak{D}_{j+1}$ is given by A_β. By the definition the difference between the matrices A_0 and A_β is that the term $p_k(\alpha)$ for the resonance α is replaced by $\lambda_{\ell(\alpha),k}(\beta_{\ell(\alpha)} + 1)$ and the elements in the α-th column corresponding to the term $u_{k,\gamma+e_\nu-e_{\nu-1},\beta}$, $\gamma + e_\nu - e_{\nu-1} = \alpha$ are deleted from A_0. We can easily see that the same argument as for the case $\tilde{\mathfrak{B}}_1^{-1}F$ still works. It follows that if every component of G satisfies (3.40), then $A_\beta^{-1}G$ satisfies (3.40).

We recall that the lower triangular part of $\mathfrak{D}_{j+1}$ is identical with that of $\tilde{\mathfrak{B}}_{j+1}$. Because the off-diagonal element $P_{\beta,\gamma}$ in $\mathfrak{B}_{j+1}$ is a sum of projections, it preserves (3.40). It follows that every component of $P_{\beta,\gamma}F$ satisfies (3.40) if F satisfies (3.40). Therefore we see that every component of $H = \mathfrak{D}_{j+1}^{-1}F$ satisfies (3.40).

We next consider QH. Because Q is the product of terms of the form Q_1, one can think Q_1 instead of Q. We have shown that $A_\beta^{-1}P_{\beta,\gamma}F$ satisfies (3.40) if F satisfies (3.40). This proves that Q_1H satisfies (3.40). Hence QH satisfies (3.40).

The matrix $\tilde{\mathfrak{B}}_{\nu,\mu}$ preserves (3.40) because $\tilde{\mathfrak{B}}_{\nu,\mu}$ is given by replacing the columns in $\mathfrak{B}_{\nu,\mu}$ corresponding to the resonance with a zero vector. Therefore, by (3.43) the matrix $C_{i,j}$ preserves the condition (3.40). It follows that T^{-1} preserves the condition (3.40).

Next we shall show that R preserves (3.40). Let the components of $\mathscr{F}$ satisfy (3.40). By the definition, the column of R consists of that of A_β corredponding to some resonance α. In order to show that every component of $R\mathscr{F}$ satisfies (3.40) we suppose that α satisfies $\alpha_j = 0$, $\beta_j > 0$ for some $j \notin A$. Then the α-th component of $R\mathscr{F}$ can be written in $\sum_{\nu \in A} c_\nu f_{k,\alpha+e_\nu-e_{\nu-1},\beta}$ for some c_ν. Because the j-th component of $\alpha + e_\nu - e_{\nu-1}$ vanishes by the conditions $j \notin A$ and $\nu \in A$, it follows that $f_{k,\alpha+e_\nu-e_{\nu-1},\beta} = 0$. Hence the α-th component of $R\mathscr{F}$ vanishes. This proves that every component of $R\mathscr{F}$ satisfies (3.40). Therefore, by (3.41) the matrix $\tilde{\mathfrak{A}}_k'^{-1}$ preserves (3.40). This completes the proof.

We consider the case $\ell_1 < \nu < \ell_2$, $|\alpha| = \nu$. Let us assume that we have constructed the formal solution

$$u_0(x) = \sum_{1 \le |\alpha| < \nu, \beta \in \boldsymbol{Z}_+^n} u_{\alpha,\beta} x^\alpha (\log x)^\beta,$$

such that $u_{\alpha,\beta}$'s satisfy (3.40). We will construct the solution of the form

$$u(x) = u_0(x) + \sum_{|\alpha| = \nu, \beta \in \boldsymbol{Z}_+^n} u_{\alpha,\beta} x^\alpha (\log x)^\beta.$$

If we substitute u into $f_k(x,u)$ we have the expression (3.31). Because $f_k(x,u) = O(|u|^2)$ and $|\alpha| \ge 1$, it follows that the term $x^\alpha (\log x)^\beta$, $|\alpha| = \nu$ appears only from $f_k(x,u_0)$. Because every $u_{\alpha,\beta}$ in u_0 satisfies (3.40), it follows that the coefficient $g_{\alpha,\beta}$ of $x^\alpha (\log x)^\beta$ of the expansion of (3.31) satisfies (3.40) with $u_{\alpha,\beta} = g_{\alpha,\beta}$. Therefore, every component $g_{k,\alpha,\beta}$ of $G_{k,j}$ in the right-hand side of (3.32) satisfies (3.40) with $u_{k,\alpha,\beta} = g_{k,\alpha,\beta}$. Because the recurrence relation (3.32) has the same form as to the case $|\alpha| = \ell_1$, we can inductively show that every component of $U_{k,j}$ satisfies (3.40).

We next consider the case $|\alpha| \ge \ell_2$. The nonresonant case $|\alpha| \ne \ell_j$ is the same as to the case $\ell_1 < \nu < \ell_2$, $|\alpha| = \nu$. On the other hand, the case $|\alpha| = \ell_2$ is similar to the case $|\alpha| = \ell_1$. In fact we solve $\tilde{\mathfrak{A}}_{k,\tau}' \tilde{U}_{k,\tau}' = \tilde{G}_{k,\tau}'$, where $\tilde{\mathfrak{A}}_{k,\tau}'$ is given by (3.38). We note that the right-hand side term $\tilde{G}_{k,\tau}'$ satisfies (3.40) by the argument in the above. Then the argument is almost identical with the case $|\alpha| = \ell_1$. Therefore we have proved (3.40).

Step 7: We introduce the new variables $Y = (Y_{\nu,j})_{\nu,j}$ by the relation

$$Y_{\nu,j} = x_j (\log x_j)^\nu, \quad (\nu = 1, \dots, \nu_j, j \notin A),$$

where ν_j is the highest power of $\log x_j$ in v_k in (3.39). Then we see that v_k can be expressed as a polynomial of x and Y. For the sake of simplicity, we denote the

polynomial with the same letter $v_k(x, Y)$. We note that if we take the integer ν_j sufficiently large, then the variable Y can be taken independent of k, $1 \leq k \leq N$.

We put $u_k(x) = w_k(x, Y) + v_k(x, Y)$. By substituting u_k into the equation (2.1), we see that $w_k = w_k(x, Y)$ satisfies the following equation

$$P_k w_k = d_k(x) - P_k v_k(x, Y) + f_k(x, w + v), \quad k = 1, 2, \ldots, N, \qquad (3.47)$$

where $w = (w_1, \ldots, w_N)$ and $v = (v_1, \ldots, v_N)$. We rewrite the left-hand side of (3.47) with the new variables x and Y, and we shall determine w_k as a convergent power series of x and Y with degree greater than $\ell_r + 1$.

First we look for the expression of P_k with respect to the variables x and Y. Because the second term in the right-hand side of (2.2) does not contain the differentiations with respect to the variable x_j $(j \notin A)$, we have

$$P_k(x^\alpha Y^\beta) = P_k(x^\alpha)Y^\beta + x^\alpha \sum_{j \notin A}(\lambda_{j,k} + \tilde{a}_{j,k}(x))x_j \partial_{x_j}(Y^\beta)$$

$$= P_k(x^\alpha)Y^\beta + x^\alpha \sum_{j \notin A}(\lambda_{j,k} + \tilde{a}_{j,k}(x))\left(\sum_{\ell=1}^{\nu_j} Y_{\ell,j}\frac{\partial}{\partial Y_{\ell,j}} + \ell Y_{\ell-1,j}\frac{\partial}{\partial Y_{\ell,j}}\right)(Y^\beta),$$

where $Y_{0,j} = x_j$. We define the operator $P_{k,0}$ and $P_{k,1}$ by

$$
\begin{aligned}
P_{k,0} &:= \sum_{j=1}^{n} \lambda_{j,k} x_j \partial_{x_j} + \sum_{j \in A} \varepsilon_{j,k} x_{j-1} \partial_{x_j} + c_k(0) \\
&\quad + \sum_{j \notin A}\sum_{\ell=1}^{\nu_j} \lambda_{j,k} Y_{\ell,j}\frac{\partial}{\partial Y_{\ell,j}} + \sum_{j \notin A}\sum_{\ell=2}^{\nu_j} \lambda_{j,k}\ell Y_{\ell-1,j}\frac{\partial}{\partial Y_{\ell,j}}\,.
\end{aligned}
\qquad (3.48)
$$

$$P_{k,1} := \sum_{j \notin A} \lambda_{j,k} x_j \frac{\partial}{\partial Y_{1,j}}\,. \qquad (3.49)$$

Then the equation (3.47) is written in the following:

$$(P_{k,0} + P_{k,1} + \tilde{P}_k)w_k = b_k(x, Y) + F_k(x, Y, w), \quad k = 1, 2, \ldots, N, \qquad (3.50)$$

where $\tilde{P}_k$ is the operator which maps polynomials of (x, Y) with homogeneous degree r to the one with homogeneous degree greater than or equal to $r + 1$. The functions $b_k(x, Y)$ and $F_k(x, Y, w)$ are given by

$$b_k(x, Y) = d_k(x) - P_k v_k(x, Y) + f_k(x, v), \quad F_k(x, Y, w) = f_k(x, w + v) - f_k(x, v),$$

where we substitute x and Y into v.

Step 8: We want to construct the solution w_k of (3.50) in the following form

$$w_k(x, Y) = \sum_{|\gamma|+|B|\geq \ell_r+1} w_{k,\gamma,B} x^\gamma Y^B.$$

Indeed, by the construction of v_k in the preceeding section we have that $d_k(x) - P_k v_k(x) + f_k(x, v)$ consists of powers $x^\alpha (\log x)^\beta$ such that $|\alpha| \geq \ell_r + 1$. If we substitute Y into $d_k(x) - P_k v_k(x) + f_k(x, v)$, then we see that $d_k(x) - P_k v_k(x) + f_k(x, v)$ can be written in the power series of $x^\gamma Y^B$ with $|\gamma| + |B| = |\alpha| \geq \ell_r + 1$. Hence $b_k(x, Y)$ consists of powers of x and Y with degree greater than or equal to $\ell_r + 1$. On the other hand we can easily see that if we expand $F_k(x, Y, w)$ in the Taylor series of w, then every coefficient of the linear term of w vanishes when $x = 0$, $Y = 0$.

We note that the operator $P_{k,0} + P_{k,1}$ maps a homogenous polynomial to the one with the same degree. On the other hand, $\tilde{P}_k$ maps polynomials of homogeneous degree r to the ones with homogeneous degree greater than or equal to $r + 1$. Hence, in order to show that the formal power series solution exists it is sufficient to show the invertibility of $P_{k,0} + P_{k,1}$ on the set of homogeneous polynomials of x and Y. Here we regard x and Y as independent variables. We note that $P_{k,0}$ preserves homogeneous polynomials of the form $U_{\nu,\mu} = \sum_{|\gamma|=\nu, |B|=\mu} w_{\gamma,B} x^\gamma Y^B$. On the other hand, the operator $P_{k,1}$ raises the degree of x one and decreases the degree of Y by one. Hence if we show the invertibility of $P_{k,0}$, then we can recursively construct the solution $U = \sum_{\nu+\mu=\ell_r+1} U_{\nu,\mu}$ of the equation $(P_{k,0} + P_{k,1})U = F$ for a given F. Indeed, we determine U_{0,ℓ_r+1} first. Then we determine U_{1,ℓ_r}. This is possible by the property of $P_{k,0}$ and $P_{k,1}$. And we can determine $U_{\nu,\mu}$ recursively.

In order to show the invertibility of $P_{k,0}$ it is sufficient to verify the nonresonance condition. The nonresonance condition for $P_{k,0}$ is given by

$$\sum_{j=1}^{n} \lambda_{j,k}\gamma_j + c_k(0) + \sum_{j\notin A}\sum_{\ell=1}^{\nu_j} \lambda_{j,k}B_{\ell,j} \neq 0, \tag{3.51}$$

$$|\gamma| + |B| \geq \ell_r + 1, \quad \gamma = (\gamma_1, \ldots, \gamma_n), \ B = (B_{\ell,j}).$$

The condition (3.51) can be written in the form

$$\sum_{j\in A} \lambda_{j,k}\gamma_j + c_k(0) + \sum_{j\notin A} \lambda_{j,k}\left(\gamma_j + \sum_{\ell=1}^{\nu_j} B_{\ell,j}\right) \neq 0.$$

This condition is clearly satisfied, because $|\gamma| + |B| = |\gamma| + \sum_{\ell, j \notin A} B_{\ell, j} \geq \ell_r + 1$ and there is no resonance such that $\geq \ell_r + 1$. Therefore we can construct a formal power series solution $w_k(x, Y)$. The convergence of $w_k(x, Y)$ is well known because $P_{k,0} + P_{k,1}$ satisfies a Poincaré condition. This ends the proof.

PROOF OF THEOREM 2. Because the proof of Theorem 2 is similar to that of Theorem 1, we give the sketch of the proof. We note that if Λ is semisimple, then (2.10) has the same form as (2.1). Hence we can make the same argument as in the proof of Theorem 1 in order to solve (2.10). If Λ is not semisimple, then we first consider the case where there is only one Jordan block for the sake of simplicity. The general case can be treated by applying the argument to each Jordan block.

By the method of indeterminate coefficients, we can see that there appears $-u_{k,\alpha,\beta}$ in the left-hand side of (3.1) except for the terms appearing from $\mathscr{L}$. This implies that in the right-hand side of (3.4), G_k is replaced by $G_k + U_{k-1}$, where $U_0 = 0$. Hence, in a nonresonant case we can inductively determine $U_1, U_2, \ldots, U_n$ by the arguments of Steps 2 and 4. On the other hand, in a resonant case, U_1 can be determined by the same argument as in Theorem 1 because $U_0 = 0$. As to the terms U_k, $k \geq 2$, we use the argument of Step 5 and we determine U_k, $k \geq 2$ recursively, because the right-hand side term G_k may not vanish in general. As to the finiteness property (3.40) we can prove it inductively in view of the argument in Step 6 of the proof of Theorem 1. As to the convergence of formal solutions, the proof is similar to that of corresponding Steps 7 and 8 of Theorem 1. This ends the proof of Theorem 2.

4. Higher order systems.

We will briefly mention the extension of Theorem 1 to higher order equations. We consider the following system of partial differential equations of order m $(m \geq 1)$ for $u = (u_1, \ldots, u_N)$ $(N \geq 1)$

$$a_k(x, \{x^\beta \partial_x^\alpha u\}_{|\alpha| = |\beta| \leq m}) = 0, \quad k = 1, \ldots, N, \tag{4.1}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{Z}_+^n$ are multi-indices, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $|\beta| = \beta_1 + \cdots + \beta_n$. We assume that $a_k(x, Z)$, $Z = (Z_{\alpha, \beta, \nu})$ is holomorphic in x and Z in some neighborhood of the origin $x = 0, Z = 0$. For simplicity we assume

$$a_k(0, 0) = 0, \quad k = 1, \ldots, N.$$

We now assume that a_k is a nilpotent type operator. Namely, for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$ and $\beta \in \mathbf{Z}_+^n$ in $a_k(x, \{x^\beta \partial_x^\alpha u\})$ we have that

$$\beta - \alpha \succeq 0, \ |\alpha| = |\beta| \leq m, \ \text{and} \ \alpha_j = 0 \text{ if } j \notin A, \alpha \neq \beta, \tag{4.2}$$

where $A \subset \{1, 2, \ldots, n\}$, $A \neq \{1, 2, \ldots, n\}$ is a given set.

Let $P = (P_{j,k})_{j,k=1,\ldots,N}$ be the linearized matrix operator of (4.1) at $Z = 0$

$$P_{j,k} = \sum_{|\alpha|=|\beta|\leq m} \left(\frac{\partial a_j}{\partial Z_{\alpha,\beta,k}} \right)(x,0) x^{\beta} \partial_x^{\alpha}, \tag{4.3}$$

where $k = 1, \ldots, N$. We assume the following condition for P:

ASSUMPTION 4. The operator P is a lower triangular matrix. Namely $P_{j,k} = 0$ if $j < k$.

REMARK 3. We can assume that P is an upper triangular matrix instead of Assumption 4. Indeed, in the case where P is a lower triangular matrix, we replace the unknown function $(u_1, u_2, \ldots, u_N)$ by a new unknown function $(v_1, v_2, \ldots, v_N) := (u_N, u_{n-1}, \ldots, u_1)$. Then the operator P is changed by an upper triangular matrix.

Let P_k be defined by $P_k := P_{k,k}$. We define the indicial polynomial $p_k(\zeta)$, $\zeta = (\zeta_1, \ldots, \zeta_n)$ of P_k by

$$p_k(\zeta) = \sum_{|\alpha|\leq m} \left(\frac{\partial a_k}{\partial z_{\alpha,\alpha,k}} \right)(0,0) \frac{\Gamma(\zeta + e)}{\Gamma(\zeta + e - \alpha)}, \quad k = 1, \ldots, N, \tag{4.4}$$

where $e = (1, 1, \ldots, 1) \in \mathbf{Z}_+^n$, $\Gamma(\zeta) = \prod_{j=1}^n \Gamma(\zeta_j)$ ($\Gamma(\zeta_j)$ is the Gamma function). We say that $\alpha \in \mathbf{Z}_+^n$ is a resonance if $p_k(\alpha) = 0$ for some $k, 1 \leq k \leq N$.

Corresponding to Assumptions 2 and 3 of Theorem 1 we assume

ASSUMPTION 5. For every resonance $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$, there exists $\ell \in \{1, \ldots, n\} \setminus A$ such that $\alpha_\ell \neq 0$ and

$$\frac{\partial p_k}{\partial \zeta_\ell}(\alpha) \neq 0. \tag{4.5}$$

Let $\tilde{p}_k(\zeta)$ be the m-th homogeneous part of $p_k(\zeta)$.

ASSUMPTION 6. There exist constants $C > 0$ and $K > 0$ independent of k and $\zeta \in \mathbf{Z}_+^n$, $|\zeta| \geq 1$ such that

$$|\tilde{\boldsymbol{p}}_k(\zeta)| \geq C|\zeta|^m \tag{4.6}$$

for all k and $\zeta \in \boldsymbol{Z}_+^n$, $|\zeta| \geq K$.

Then we have

THEOREM 3. *Assume that there exists a resonance. Suppose that the above conditions are satisfied. Then the equation* (4.1) *has a solution* $u(x)$ *of the form*

$$u(x) = \sum_{|\alpha|\geq 1; \beta \in \boldsymbol{Z}_+^n} u_{\alpha\beta} x^\alpha (\log x)^\beta, \quad (\log x)^\beta = \prod_{j=1}^{n} (\log x_j)^{\beta_j}, \tag{4.7}$$

where the summation with respect to β *in* (4.7) *is taken for* β *such that* $\beta = (\beta_1, \ldots, \beta_n) \in \boldsymbol{Z}_+^n$, $\beta_j = 0$ *if* $j \in A$. *There exist* $\varepsilon > 0$ *and an integer* $J \geq 1$ *such that the series* (4.7) *is expressed as the power series of* x *and* $X_{\nu,p} := x_\nu (\log x_\nu)^p$, $(p = 1, \ldots, J, \nu \in \{1, \ldots, n\} \setminus A)$, *which converges in the domain*

$$\{x \in \boldsymbol{C}^n; |x| < \varepsilon, \ |X_{\nu,p}| < \varepsilon, \ p = 1, \ldots, J, \ \nu \in \{1, \ldots, n\} \setminus A\}.$$

EXAMPLE 3. Let $x = (x_1, x_2)$, and define $M(u) = u_{x_1 x_1} u_{x_2 x_2} - u_{x_1 x_2}^2$, where $u_{x_1 x_1} = \partial_{x_1}^2 u$ and so on. Let $u_0(x_1, x_2)$ be holomorphic at the origin. Let k be a real number and let $g(x)$ be a holomorphic function near the origin. Then we consider the solvability of the equation

$$M(u_0 + w) + k x_1 x_2 (u_0 + w)_{x_1 x_2} = f_0(x) + g(x), \tag{4.8}$$

where $f_0(x) = k x_1 x_2 (u_0)_{x_1 x_2} + M(u_0)$. If we set $u_0(x) = x_1^2 x_2^2$, then we have $f_0(x) = 4(k-3)x_1^2 x_2^2$. If we define

$$P = 2x_1^2 \partial_{x_1}^2 + 2x_2^2 \partial_{x_2}^2 + (k-8)x_1 x_2 \partial_{x_1} \partial_{x_2}, \tag{4.9}$$

then the equation (4.8) can be written in the following form

$$Pw + M(w) = g(x). \tag{4.10}$$

If we set $g(x) = x^\alpha h(x)$, $w(x) = x^\alpha v(x)$ ($|\alpha| \geq 5$), then we can easily see that (4.10) can be written in the form (4.1). We denote the characteristic variable corresponding to $x_j \partial_{x_j}$ by ξ_j. Then the characteristic polynomial $p(\eta)$, $\eta = \alpha + \xi$ corresponding to the transformed equation is given by

$$p(\eta) = 2\eta_1(\eta_1 - 1) + 2\eta_2(\eta_2 - 1) + (k - 8)\eta_1\eta_2. \tag{4.11}$$

The Poincaré condition reads; $2 + (k-8)t_1 t_2 \neq 0$ for all $(t_1, t_2) \in \boldsymbol{R}_+^2$, $|t| = 1$. We can easily see that this is equivalent to $k > 4$.

On the other hand, the nonresonance condition is given by

$$2\eta_1(\eta_1 - 1) + 2\eta_2(\eta_2 - 1) + (k - 8)\eta_1\eta_2 \neq 0, \quad \eta_1 \in \boldsymbol{Z}_+, \eta_2 \in \boldsymbol{Z}_+. \tag{4.12}$$

Because we consider the perturbative problem to f_0, we may assume that the Taylor expansion of g at the origin vanishes up to fourth order. Hence we may assume $\eta_1 + \eta_2 \geq 5$ in (4.12). We can easily see that the resonance occurs in the cases, (1) $(\eta_1, \eta_2) = (3, 2), (2, 3)$ if $k = 16/3$, (2) $(\eta_1, \eta_2) = (2, 4), (4, 2)$ if $k = 9/2$, (3) $(\eta_1, \eta_2) = (3, 3)$ if $k = 16/3$, (4) $(\eta_1, \eta_2) = (3, 4), (4, 3)$ if $k = 5$.

We can easily see that $A = \emptyset$ and the condition (4.5) is clearly satisfied. Therefore the equation (4.10) has a singular solution when there is a resonance.

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