J. Math. Soc. Japan
Vol. 60, No. 1 (2008) pp. 117–125
doi: 10.2969/jmsj/06010117

The relative cohomology of formal contact vector fields with respect to formal Poisson vector fields

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(Received Mar. 12, 2007)

Abstract. We review the method due to Gel'fand and Fuks to show the finite dimensionality of the cohomology ring of the Lie algebra of formal contact vector fields. We apply this method to prove the relative cohomology of it with respect to formal Poisson vector fields is trivial.

1. Introduction.

Let $\mathfrak{g}$ be the Lie algebra of formal contact vector fields in $\boldsymbol{R}^{2n+1}$ at the origin and $\mathfrak{h}$ that of formal Poisson vector fields in $\boldsymbol{R}^{2n}$ (for the precise definitions, see Section 2). We know the following facts about $H^*(\mathfrak{g})$. Gel'fand and Fuks showed its finite dimensionality in [**7**]. In [**8**], Guillemin and Shnider showed that $H^q(\mathfrak{g}) = 0$ for $0 < q \leq n$. In [**3**], using a spectral sequence, Feigin showed that it is isomorphic to the cohomology ring of the total space of the restriction of the standard universal $Sp(2n)$-bundle to the $(4n+2)$-skeleton of the base space.

In this paper we review the method due to Gel'fand and Fuks [**7**] and apply it to show that $H^q(\mathfrak{g}) = 0$ for $q > 2n^2 + 7n + 6$. This is of course a part of Feigin's results. We also apply this method to prove that the relative cohomology of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is trivial:

Theorem 1.1. *$H^q(\mathfrak{g}, \mathfrak{h}) = 0$ for $q > 0$.*

A $(2n+1)$-dimensional contact structure is considered to be a projectification of a $(2n+2)$-dimensional symplectic structure. It is also locally regarded as a (pre)quantization of $2n$-dimensional symplectic structure. In fact Feigin computed $H^q(\mathfrak{g})$ by looking at the linear part $\mathfrak{sp}(2n+2)$ of the Hamiltonian vector fields. This is the first point of view. However it does not seem that the second point of view has been well taken. As our problem is local, this point of view is one of the motivations of the present work. $2n$-dimensional Hamiltonian vector fields

2000 *Mathematics Subject Classification.* Primary 17B66; Secondary 17B56.
Key Words and Phrases. formal vector fields, cohomology of Lie algebras, contact structures.

do not naturally lift to $(2n+1)$-dimensional contact space but once they are enlarged to Poisson ones, they do.

The quotient $\mathfrak{h}' = \mathfrak{h}/\mathfrak{i}$ is naturally identified with the Lie algebra of formal Hamiltonian vector fields on $\boldsymbol{R}^{2n}$, where the ideal $\mathfrak{i}$ is the set of constant functions in the sense of Section 2 below. Therefore it seems that $\mathfrak{h}'$ and $\mathfrak{h}$ are very close to each other. There is one important problem asking whether $H^*(\mathfrak{h}')$ is infinite dimensional or not. So far its calculation is regarded as a difficult problem. For example $n=1$, its dimension is at least 112, among which we explicitly know only eight generators (six by Gel'fand, Kalinin and Fuks [**4**] and other two by Metoki [**10**]). Moreover practically nothing more is known about $H^*(\mathfrak{h}')$ for $n>1$. This problem also motivated this article.

The author would like to express his gratitude to Professor Yoshihiko Mitsumatsu for his encouragement and the members of Do Semi at TIT for many useful comments.

2. The Lie algebras of formal contact vector fields.

First we recall contact vector fields. We consider the standard contact 1-form $\alpha = dz + x_1 dy_1 + \cdots + x_n dy_n$ on $\boldsymbol{R}^{2n+1}$. A vector field X on $\boldsymbol{R}^{2n+1}$ is a contact vector field if it preserves the contact plane field $\operatorname{Ker}\alpha$. We associate a contact vector field X with a function $-\alpha(X)$. Conversely, for any function $f \in C^\infty(\boldsymbol{R}^{2n+1})$, there exists a unique contact vector field X_f such that $\alpha(X_f) = -f$. This correspondence determines a Lie algebra structure on $C^\infty(\boldsymbol{R}^{2n+1})$ (for detail, see [**1**]). The Lie bracket relation between two contact vector fields are translated in the following form between functions:

$$[f,g] = -\sum_{s=1}^{n}\left(\frac{\partial f}{\partial x_s}\frac{\partial g}{\partial y_s} - \frac{\partial f}{\partial y_s}\frac{\partial g}{\partial x_s}\right) - \frac{\partial f}{\partial z}\left(\sum_{s=1}^{n} x_s \frac{\partial g}{\partial x_s} - g\right) + \frac{\partial g}{\partial z}\left(\sum_{s=1}^{n} x_s \frac{\partial f}{\partial x_s} - f\right).$$

Here we formalize the functions. We define the Lie algebra $\mathfrak{g}$ of formal contact vector fields on $\boldsymbol{R}^{2n+1}$ as the vector space of formal power series $\boldsymbol{R}[[x_1,\dots,x_n,y_1,\dots,y_n,z]]$ with the bracket presented above. Since the subspace $\boldsymbol{R}[[x_1,\dots,x_n,y_1,\dots,y_n]]$ of $\mathfrak{g}$ is stable under this bracket and it reduces to the following form

$$[f,g] = -\sum_{s=1}^{n}\left(\frac{\partial f}{\partial x_s}\frac{\partial g}{\partial y_s} - \frac{\partial f}{\partial y_s}\frac{\partial g}{\partial x_s}\right),$$

this subalgebra is naturally identified with the Lie algebra $\mathfrak{h}$ of formal Poisson vector fields on $\boldsymbol{R}^{2n}$.

For convenience, we prepare the following notation on multi-indices. Let $\mathscr{N}$ denote the set of non-negative integers. For multi-indices $\alpha_s = (\alpha_{s,1}, \ldots, \alpha_{s,2n+1}) \in \mathscr{N}^{2n+1}$ $(s = 1, 2)$ and $a \in \mathscr{N}$, we set

$$\alpha_1 + \alpha_2 = (\alpha_{1,1} + \alpha_{2,1}, \ldots, \alpha_{1,2n+1} + \alpha_{2,2n+1}), \quad a\alpha_1 = (a\alpha_{1,1} + \cdots + a\alpha_{1,2n+1}).$$

We set the following multi-indices:

$$0 = (0, \ldots, 0), \quad \varepsilon_s = (\underbrace{0, \ldots, 0}_{s-1}, 1, 0, \ldots, 0) \ \ (s = 1, \ldots, 2n+1).$$

For a multi-index $\alpha_1 \in \mathscr{N}^{2n+1}$, we define the length by $|\alpha_1| = \alpha_{1,1} + \cdots + \alpha_{1,2n+1}$ and

$$\boldsymbol{x}^{\alpha_1} = x_1^{\alpha_{1,1}} \cdots x_n^{\alpha_{1,n}} y_1^{\alpha_{1,n+1}} \cdots y_n^{\alpha_{1,2n}} z^{\alpha_{1,2n+1}}.$$

Therefore $f \in \mathfrak{g}$ is expressed as an infinite sum

$$f = \sum_{\alpha_1 \in \mathscr{N}^{2n+1}} a_{\alpha_1} \boldsymbol{x}^{\alpha_1} \quad (a_{\alpha_1} \in \boldsymbol{R}).$$

We define the topology of $\mathfrak{g}$ by the family of semi-norms

$$f = \sum_{\alpha_1 \in \mathscr{N}^{2n+1}} a_{\alpha_1} \boldsymbol{x}^{\alpha_1} \mapsto \sup_{|\alpha_1| \le k} |a_{\alpha_1}| \quad (k = 0,\ 1,\ \ldots).$$

Therefore the topological dual of $\mathfrak{g}$ is isomorphic to the vector space of the polynomials on $\boldsymbol{R}^{2n+1}$.

3. Cohomology of Lie algebras.

Following [**9**] and [**11**], we define the cohomology of Lie algebras as follows. We set $A^0(\mathfrak{g}) = \boldsymbol{R}$. For each positive integer q, we set

$$A^q(\mathfrak{g}) = \{\varphi : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{q} \longrightarrow \boldsymbol{R}; \text{alternating } \boldsymbol{R}\text{-multilinear continuous map}\}.$$

The exterior derivation $d : A^q(\mathfrak{g}) \to A^{q+1}(\mathfrak{g})$ is defined by

$$d\varphi(f_1, \dots, f_{q+1}) = \sum_{s<t} (-1)^{s+t} \varphi([f_s, f_t], f_1, \dots \hat{f}_s \dots \hat{f}_t \dots, f_{q+1})$$

for $\varphi \in A^q(\mathfrak{g})$, $f_1, \dots, f_{q+1} \in \mathfrak{g}$. If $\varphi \in A^0(\mathfrak{g})$ then $d\varphi = 0$. Jacobi's identity implies $d^2 = 0$ and we obtain the cochain complex $\{A^*(\mathfrak{g}), d\}$. We call the cohomology of $\{A^*(\mathfrak{g}), d\}$ the (continuous) cohomology of $\mathfrak{g}$ and it is denoted by $H^*(\mathfrak{g})$.

For $f \in \mathfrak{g}$, we define the interior product $i(f) : A^q(\mathfrak{g}) \to A^{q-1}(\mathfrak{g})$ by

$$(i(f)\varphi)(f_1, \dots, f_{q-1}) = \varphi(f, f_1, \dots, f_{q-1})$$

for $\varphi \in A^q(\mathfrak{g})$, $f_1, \dots, f_{q-1} \in \mathfrak{g}$, and the Lie derivative $\mathscr{L}_f : A^q(\mathfrak{g}) \to A^q(\mathfrak{g})$ by Cartan's formula:

$$\mathscr{L}_f = di(f) + i(f)d.$$

If $\varphi \in A^0(\mathfrak{g})$ then $i(f)\varphi = 0$ for all $f \in \mathfrak{g}$.

We put

$$A^q(\mathfrak{g}, \mathfrak{h}) = \{\varphi \in A^q(\mathfrak{g}); i(f)\varphi = \mathscr{L}_f\varphi = 0 \text{ for any } f \in \mathfrak{h}\},$$

then $\{A^*(\mathfrak{g}, \mathfrak{h}), d\}$ is a subcomplex of $\{A^*(\mathfrak{g}), d\}$. We call the cohomology group of $\{A^*(\mathfrak{g}, \mathfrak{h}), d\}$ the relative cohomology of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and it is denoted by $H^*(\mathfrak{g}, \mathfrak{h})$.

Remark 3.1. By definition, $H^0(\mathfrak{g}) = \boldsymbol{R}$ and $H^0(\mathfrak{g}, \mathfrak{h}) = \boldsymbol{R}$.

4. Cohomology of formal contact vector fields.

We review the method of Gel'fand and Fuks [**7**] and show $H^*(\mathfrak{g}) = 0$ for $q > 2n^2 + 7n + 6$. For $\alpha_1 \in \mathscr{N}^{2n+1}$, we define the 1-cochain $\delta_{\alpha_1} \in A^1(\mathfrak{g})$ by

$$\delta_{\alpha_1}(f) = a_{\alpha_1} \quad \text{for } f = \sum_{\beta_1 \in \mathscr{N}^{2n+1}} a_{\beta_1} \boldsymbol{x}^{\beta_1} \in \mathfrak{g}.$$

By the continuity, a cochain $\varphi \in A^q(\mathfrak{g})$ is expressed as a finite sum of monomials $\delta_{\alpha_1} \wedge \cdots \wedge \delta_{\alpha_q}$:

$$\varphi = \sum_{\alpha_1, \dots, \alpha_q \in \mathscr{N}^{2n+1}} a^{\alpha_1 \dots \alpha_q} \delta_{\alpha_1} \wedge \cdots \wedge \delta_{\alpha_q} \quad (a^{\alpha_1 \dots \alpha_q} \in \boldsymbol{R}). \tag{4.1}$$

We define the multi-order $\kappa_1 \in \mathscr{N}^{2n+1}$ of a monomial $\delta_{\alpha_1} \wedge \cdots \wedge \delta_{\alpha_q} \in A^q(\mathfrak{g})$ as

$$\kappa_1 = \alpha_1 + \cdots + \alpha_q.$$

We put the suffix 1 to κ in order to emphasize that κ_1 is a multi-index.

LEMMA 4.1. *We have the following*:

(i) $\mathscr{L}_z(\delta_{\alpha_1} \wedge \cdots \wedge \delta_{\alpha_q}) = (\kappa_{1,1} + \cdots + \kappa_{1,n} + \kappa_{1,2n+1} - q)\delta_{\alpha_1} \wedge \cdots \wedge \delta_{\alpha_q}$.

(ii) $\mathscr{L}_{x_s y_s}(\delta_{\alpha_1} \wedge \cdots \wedge \delta_{\alpha_q}) = (\kappa_{1,n+s} - \kappa_{1,s})\delta_{\alpha_1} \wedge \cdots \wedge \delta_{\alpha_q} \quad (s = 1, \ldots, n)$.

PROOF. A cochain $\mathscr{L}_z\delta_{\alpha_1}$ is expressed as a finite sum

$$\mathscr{L}_z\delta_{\alpha_1} = \sum_{\gamma_1 \in \mathscr{N}^{2n+1}} a^{\gamma_1}\delta_{\gamma_1} \quad (a^{\gamma_1} \in \boldsymbol{R}).$$

Since $[z, \boldsymbol{x}^{\beta_1}] = -(\beta_{1,1} + \cdots + \beta_{1,n} + \beta_{1,2n+1} - 1)\boldsymbol{x}^{\beta_1}$ for $\beta_1 \in \mathscr{N}^{2n+1}$, we have

$$\begin{aligned} a^{\beta_1} = (\mathscr{L}_z\delta_{\alpha_1})(\boldsymbol{x}^{\beta_1}) &= -\delta_{\alpha_1}([z, \boldsymbol{x}^{\beta_1}]) \\ &= (\beta_{1,1} + \cdots + \beta_{1,n} + \beta_{1,2n+1} - 1)\delta_{\alpha_1}(\boldsymbol{x}^{\beta_1}) \end{aligned}$$

Then we have $a^{\beta_1} = 0$ for $\beta_1 \neq \alpha_1$ and $a^{\alpha_1} = \alpha_{1,1} + \cdots + \alpha_{1,n} + \alpha_{1,2n+1} - 1$. Hence we obtain $\mathscr{L}_z\delta_{\alpha_1} = (\alpha_{1,1} + \cdots + \alpha_{1,n} + \alpha_{1,2n+1} - 1)\delta_{\alpha_1}$. Thus we have (i) from this and the formula $\mathscr{L}_z(\varphi \wedge \psi) = \mathscr{L}_z\varphi \wedge \psi + \varphi \wedge \mathscr{L}_z\psi$, and (ii) in the same way as (i). □

We decompose $A^q(\mathfrak{g})$ to the eigen spaces by Lie derivation. We put

$$A^q_{\mu,v}(\mathfrak{g}) = \{\varphi \in A^q(\mathfrak{g}); \mathscr{L}_{x_s y_s}\varphi = \mu_s\varphi \ (s = 1, \ldots, n),\ \mathscr{L}_z\varphi = v\varphi\}$$

for $\mu = (\mu_1, \ldots, \mu_n) \in \boldsymbol{Z}^n$, $v \in \boldsymbol{Z}$. By Lemma 4.1, we have the decomposition

$$A^q(\mathfrak{g}) = \bigoplus_{\mu,v} A^q_{\mu,v}(\mathfrak{g}).$$

Since $dA^q_{\mu,v}(\mathfrak{g}) \subset A^{q+1}_{\mu,v}(\mathfrak{g})$, $\{A^*_{\mu,v}(\mathfrak{g}), d\}$ is a subcomplex of $\{A^*(\mathfrak{g}), d\}$. The cohomology of $\{A^*_{\mu,v}(\mathfrak{g}), d\}$ is denoted by $H^*_{\mu,v}(\mathfrak{g})$. Hence we obtain

$$H^*(\mathfrak{g}) = \bigoplus_{\mu,v} H^*_{\mu,v}(\mathfrak{g}).$$

Lemma 4.2. *We have $H^q_{\mu,v}(\mathfrak{g}) = 0$ for $(\mu, v) \neq (0,0)$.*

Proof. If $\varphi \in A^q_{\mu,v}(\mathfrak{g})$ is a cocycle, Cartan's formula implies

$$\mu_s \varphi = \mathscr{L}_{x_s y_s} \varphi = di(x_s y_s)\varphi \quad (s = 1, \dots, n) \quad \text{and} \quad v\varphi = \mathscr{L}_z \varphi = di(z)\varphi.$$

Therefore, if $(\mu, v) \neq (0,0)$ then φ is a coboundary. □

Lemma 4.3. *We have $A^q_{0,0}(\mathfrak{g}) = 0$ for $q > 2n^2 + 7n + 6$.*

Proof. Take a monomial $\delta_{\alpha_1} \wedge \cdots \wedge \delta_{\alpha_q} \in A^q_{0,0}(\mathfrak{g})$. By Lemma 4.1, we have

$$\kappa_{1,s} = \kappa_{1,n+s} \ (s = 1, \dots, n), \quad \kappa_{1,1} + \cdots + \kappa_{1,n} + \kappa_{1,2n+1} = q.$$

Hence we obtain

$$2q = \kappa_{1,1} + \cdots + \kappa_{1,2n} + 2\kappa_{1,2n+1} \geq |\kappa_1|. \tag{4.2}$$

On the other hand, in general the length of a multi-index is at least 3 except for the following three cases of multi-indices, in each of which the length is 0, 1 or 2:

$$0, \quad \varepsilon_s \ (s = 1, \dots, 2n+1), \quad \varepsilon_s + \varepsilon_t \ (1 \leq s \leq t \leq 2n+1).$$

Thus we have

$$\begin{aligned} |\kappa_1| &= |\alpha_1| + \cdots + |\alpha_q| \\ &\geq 3q - 3 \times 1 - 2 \times (2n+1) - 1 \times (2n+1)(n+1) \\ &= 3q - (2n^2 + 7n + 6). \end{aligned} \tag{4.3}$$

Hence we obtain $2n^2 + 7n + 6 \geq q$ from (4.2) and (4.3). □

We now conclude that $H^*(\mathfrak{g}) = 0$ for $q > 2n^2 + 7n + 6$.

5. Relative cohomology.

Let us look at properties of the interior product and the Lie derivation.

Lemma 5.1. *For any monomial $\lambda = \delta_{\alpha_1} \wedge \cdots \wedge \delta_{\alpha_q} \in A^q_{\mu,v}(\mathfrak{g})$ and any multi-index $\beta_1 \in \mathscr{N}^{2n+1}$, we have $i(\boldsymbol{x}^{\beta_1})\lambda \in A^{q-1}_{\bar{\mu},\bar{v}}(\mathfrak{g})$ and $\mathscr{L}_{\boldsymbol{x}^{\beta_1}}\lambda \in A^q_{\bar{\mu},\bar{v}}(\mathfrak{g})$, where $\bar{\mu} = (\bar{\mu}_1, \dots, \bar{\mu}_n) \in \boldsymbol{Z}^n$ and $\bar{v} \in \boldsymbol{Z}$ are defined by $\bar{\mu}_s = \mu_s + \beta_{1,s} - \beta_{1,n+s}$ and $\bar{v} = v - \beta_{1,1} - \cdots - \beta_{1,n} - \beta_{1,2n+1} + 1$ respectively.*

PROOF. If $\alpha_s \neq \beta_1$ for $s = 1, \ldots, q$, then $i(\boldsymbol{x}^{\beta_1})\lambda = 0 \in A^{q-1}_{\bar{\mu},\bar{v}}(\mathfrak{g})$. If there exists s such that $\alpha_s = \beta_1$ then $i(\boldsymbol{x}^{\beta_1})\lambda = (-1)^{s+1}\delta_{\alpha_1} \wedge \cdots \wedge \hat{\delta}_{\alpha_s} \wedge \cdots \wedge \delta_{\alpha_q}$. Set $\bar{\kappa}_1$ the multi-order of $\delta_{\alpha_1} \wedge \cdots \wedge \hat{\delta}_{\alpha_s} \wedge \cdots \wedge \delta_{\alpha_q}$. Then we have

$$\bar{\kappa}_1 = \kappa_1 - \alpha_s = \kappa_1 - \beta_1$$

where κ_1 is the multi-order of λ. By Lemma 4.1, we obtain $i(\boldsymbol{x}^{\beta_1})\lambda \in A^{q-1}_{\bar{\mu},\bar{v}}(\mathfrak{g})$. Moreover we obtain $\mathscr{L}_{\boldsymbol{x}^{\beta_1}}\lambda \in A^{q}_{\bar{\mu},\bar{v}}(\mathfrak{g})$ because the exterior derivation d preserves $\bar{\mu}$ and $\bar{v}$. $\square$

Now we begin to calculate the relative cohomology of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We decompose a relative cochain $\varphi \in A^q(\mathfrak{g}, \mathfrak{h})$ as

$$\varphi = \sum_{\mu,v} \varphi_{\mu,v} \quad (\varphi_{\mu,v} \in A^q_{\mu,v}(\mathfrak{g})).$$

Since $\boldsymbol{x}^{\alpha_1} \in \mathfrak{h}$ for $\alpha_1 \in \mathscr{N}^{2n} \times \{0\}$, we have

$$i(\boldsymbol{x}^{\alpha_1})\varphi = \sum_{\mu,v} i(\boldsymbol{x}^{\alpha_1})\varphi_{\mu,v} = 0, \quad \mathscr{L}_{\boldsymbol{x}^{\alpha_1}}\varphi = \sum_{\mu,v} \mathscr{L}_{\boldsymbol{x}^{\alpha_1}}\varphi_{\mu,v} = 0.$$

By Lemma 5.1, we have $i(\boldsymbol{x}^{\alpha_1})\varphi_{\mu,v} = 0$ and $\mathscr{L}_{\boldsymbol{x}^{\alpha_1}}\varphi_{\mu,v} = 0$ for $\alpha_1 \in \mathscr{N}^{2n} \times \{0\}$. Therefore $\varphi_{\mu,v}$ is a relative cochain. Moreover we obtain $\mathscr{L}_{x_s y_s}\varphi_{\mu,v} = 0$ because $x_s y_s \in \mathfrak{h}$ for $s = 1, \ldots, n$. Hence we obtain $\mu = 0$. We put

$$A^q_{0,v}(\mathfrak{g}, \mathfrak{h}) = \{\varphi \in A^q(\mathfrak{g}, \mathfrak{h}); \mathscr{L}_z\varphi = v\varphi\}.$$

for $v \in \boldsymbol{Z}$. Then we have

$$A^q(\mathfrak{g}, \mathfrak{h}) = \bigoplus_v A^q_{0,v}(\mathfrak{g}, \mathfrak{h}).$$

The cohomology of $\{A^q_{0,v}(\mathfrak{g}, \mathfrak{h}), d\}$ is denoted by $H^q_{0,v}(\mathfrak{g}, \mathfrak{h})$. Thus we obtain

$$H^q(\mathfrak{g}, \mathfrak{h}) = \bigoplus_v H^q_{0,v}(\mathfrak{g}, \mathfrak{h}).$$

LEMMA 5.2. *We have $H^q_{0,v}(\mathfrak{g}, \mathfrak{h}) = 0$ for $v \neq 0$.*

PROOF. If $\varphi_{0,v} \in A^q_{0,v}(\mathfrak{g}, \mathfrak{h})$ is a cocycle, Cartan's formula implies $di(z)\varphi_{0,v} = v\varphi_{0,v}$. We have $i(z)\varphi_{0,v} \in A^{q-1}_{0,v}(\mathfrak{g}, \mathfrak{h})$ because

$$i(\boldsymbol{x}^{\alpha_1})(i(z)\varphi_{0,v}) = -i(z)i(\boldsymbol{x}^{\alpha_1})\varphi_{0,v} = 0$$

and

$$\begin{aligned}\mathscr{L}_{\boldsymbol{x}^{\alpha_1}}(i(z)\varphi_{0,v}) &= (di(\boldsymbol{x}^{\alpha_1}) + i(\boldsymbol{x}^{\alpha_1})d)i(z)\varphi_{0,v} = i(\boldsymbol{x}^{\alpha_1})di(z)\varphi_{0,v} \\ &= i(\boldsymbol{x}^{\alpha_1})\mathscr{L}_z\varphi_{0,v} = vi(\boldsymbol{x}^{\alpha_1})\varphi_{0,v} = 0\end{aligned}$$

for any $\alpha_1 \in \mathscr{N}^{2n} \times \{0\}$. Therefore for $v \neq 0$, $\varphi_{0,v}$ is a coboundary. □

We recall a cochain $\varphi_{0,0} \in A^q_{0,0}(\mathfrak{g}, \mathfrak{h})$ is expressed as the equality (4.1). If there exists $\alpha_s \in \mathscr{N}^{2n} \times \{0\}$ which δ_{α_s} appears in this presentation of $\varphi_{0,0}$, then $i(\boldsymbol{x}^{\alpha_s})\varphi_{0,0} \neq 0$. Since $\boldsymbol{x}^{\alpha_s} \in \mathfrak{h}$, this contradicts that $\varphi_{0,0}$ is a relative cochain. Then $\varphi_{0,0} \in A^q_{0,0}(\mathfrak{g}, \mathfrak{h})$ is expressed as

$$\varphi_{0,0} = \sum_{\alpha_1,\dots,\alpha_q \in \mathscr{N}^{2n} \times \boldsymbol{N}} a^{\alpha_1,\dots,\alpha_q}\delta_{\alpha_1} \wedge \cdots \wedge \delta_{\alpha_q}.$$

Lemma 5.3. *We have $A^q_{0,0}(\mathfrak{g}, \mathfrak{h}) = 0$ for all positive integer q.*

Proof. Take a cochain $\varphi_{0,0} \in A^q_{0,0}(\mathfrak{g}, \mathfrak{h})$ and a monomial $\lambda = \delta_{\alpha_1} \wedge \cdots \wedge \delta_{\alpha_q}$ which appears in the above presentation of $\varphi_{0,0}$. Since $\mathscr{L}_z\lambda = 0$ and $\alpha_{s,2n+1} \geq 1$ ($s = 1, \dots, q$), we obtain

$$q = \kappa_{1,1} + \cdots + \kappa_{1,n} + \kappa_{1,2n+1} \geq \kappa_{1,2n+1} = \alpha_{1,2n+1} + \cdots + \alpha_{q,2n+1} \geq q.$$

Then we have $\kappa_{1,2n+1} = q$ and $\kappa_{1,1} = \cdots = \kappa_{1,2n} = 0$. Thus $\alpha_s = \varepsilon_{2n+1}$ for $s = 1, \dots, q$. For $q > 1$, we have $\lambda = 0$ and $\varphi_{0,0} = 0$. Hence we obtain $A^q_{0,0}(\mathfrak{g}, \mathfrak{h}) = 0$ for $q > 1$. For $q = 1$, $\delta_{\varepsilon_{2n+1}}$ dose not belong to $A^1_{0,0}(\mathfrak{g}, \mathfrak{h})$ because $\mathscr{L}_{\boldsymbol{x}^0}\delta_{\varepsilon_{2n+1}} = 2\delta_{2\varepsilon_{2n+1}}$. Therefore we obtain $A^1_{0,0}(\mathfrak{g}, \mathfrak{h}) = 0$. □

This completes the proof of Theorem 1.1.

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