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Geometric Seifert 4-manifolds with hyperbolic bases

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Abstract. It is shown that Seifert 4-manifolds with hyperbolic bases are geometric in the sense of Thurston if and only if the monodromies are periodic. This result will be used to prove virtually geometric Seifert 4-manifolds with hyperbolic bases are geometric and thus give a classification of such manifolds in terms of finite covers.

Introduction.

Seifert fibred 3-manifolds were originally defined and classified by Seifert in [**6**]. Scott (in [**5**]) gives a survey of results connected with these classical Seifert spaces, in particular he shows they correspond to 3-manifolds having one of six of the eight 3-dimensional geometries (in the sense of Thurston). Thurston's geometrization conjecture asserts that any 3-manifold can be decomposed into such geometric pieces and so Seifert 3-manifolds are important building blocks. For manifolds of higher dimensions we can not expect similar results, however the class of geometric 4-manifolds is still an interesting class. In this paper we will generalise the concept of a Seifert manifold to four dimensions and find out if they similarly define geometric manifolds.

Essentially, a classical Seifert manifold is a circle bundle over a 2-orbifold. In general, we will define a Seifert manifold to be the total space of a bundle over a 2-orbifold with flat fibres (for Seifert 4 manifolds the fibre can either be a 2-torus or a Klein bottle). Ue has considered the geometries of orientable Seifert 4-manifolds (which have general fibre a torus) ([**9**], [**10**]). He proves that (with a finite number of exceptions) orientable manifolds of eight of the 4-dimensional geometries are Seifert fibred. However, Seifert manifolds with a hyperbolic base are not necessarily geometric. In this paper, we seek to extend Ue's work to the non-orientable case. Because Seifert manifolds with hyperbolic bases are not always geometric we will concentrate on that interesting case.

Ue proved that orientable Seifert 4-manifolds with hyperbolic bases are geometric if and only if the monodromies are periodic, and we will prove that this

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is also true for non-orientable Sefiert 4-manifolds. We will use this result to produce some corollaries concerning finite covers.

The proofs given here can be generalised to Seifert 4-manifolds with aspherical bases. The two types of Sefiert manifolds considered in this paper are the key types to understand the other types of Seifert 4-manifolds with aspherical bases. That is another reason to concentrate on hyperbolic bases.

We will prove this result by utilising the fundamental group. So in the first section we will clarify our definitions of orbfiolds and Seifert bundles and give a presentation for their fundamental group. We then will show how Vogt in [**11**] proves Seifert 4-manifolds with hyperbolic base are determined by their fundamental group. In the next two sections, we will introduce the two geometries that occur with such Seifert bundles: $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ and $\widetilde{SL}_2 \times \boldsymbol{E}$. In Section 4 we will give necessary and sufficient conditions for Seifert bundles with hyperbolic bases and 2-torus (T^2) fibres to be geometric (Theorem 4.7). We will prove this result in three parts. Firstly we will prove the result in the simpler case, the $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ case in order to show the general method, but we will defer proving the technical parts concerning reflector curves. Secondly we will show how the general method can be altered for the $\widetilde{SL}_2 \times \boldsymbol{E}$ case. Lastly in Section 5 we will prove the parts of the proof concerning reflector curves. In Section 6 we will give necessary and sufficient conditions for Seifert bundles with hyperbolic bases and Klein bottle (Kb) fibres to be geometric. To conclude, we will give two applications of our results to finite covers. Namely we will prove virtually geometric Seifert manifolds (= a Seifert manifold which is finitely covered by a geometric manifold) with hyperbolic bases are geometric and give a classification of such manifolds in terms of finite covers.

1. Seifert bundles.

In this section we will give some background results of orbifolds and Seifert bundles which are useful later on. Scott ([**5**]) gives a good survey of classical Seifert spaces (when the general fibre is the circle: S^1) and includes good background information on 2-orbifolds and their coverings. Bonahon and Seibenmann ([**1**]) considered classical Seifert spaces but also allowed for singularities in the total space. In their paper, they also give a good background of orbifolds. We will briefly introduce these ideas, but for more background information, see these papers.

An orbifold is a generalisation of a manifold. The quotient of manifold by a group which acts discretely and freely is another manifold. Orbifolds provide geometric models of quotients of manifolds by groups which act discretely but not necessarily freely. Recall for a n-manifold each point has a neighbourhood that is homeomorphic to an open subset of $\boldsymbol{R}^n$. We define an n-orbifold similarly: each point in a n-orbifold has a neighbourhood that is homeomorphic to a quotient of an open subset of $\boldsymbol{R}^n$ by a finite group. (See one of the above references for a more formal definition.)

Note that orbifolds do not need to be a quotient of a manifold by a discrete group - although if they are they are called good (as opposed to bad orbifolds). The orbifolds that we mostly consider in this paper are hyperbolic 2-orbifolds which are finitely covered by a hyperbolic manifold and so are good.

Orbifolds can be thought of as a topological space (called the underlying space) with marked singular points that have an associated finite group. We call a point in a orbifold regular if it has a neighbourhood homeomorphic to an open set of $\boldsymbol{R}^n$. Otherwise we call it a singular point. For 2-orbifolds there are three types of singular points. A cone point of order m is a singular point has a neighbourhood homeomorphic to a quotient of $\boldsymbol{R}^n$ by a cyclic group $\boldsymbol{Z}_m$ generated by rotation of order m. Cone points are always isolated. A reflector point is a singular point has a neighbourhood homeomorphic to a quotient of $\boldsymbol{R}^n$ by a group of order two $\boldsymbol{Z}_2$ generated by a reflection. A corner reflector of order m is a singular point has a neighbourhood homeomorphic to a quotient of $\boldsymbol{R}^n$ by a dihedral group D_{2m} generated by a reflection and a rotation of order m. A connected component of the set of reflector points and corner reflectors is an arc or closed loop and is called a reflector curve.

The idea of covering maps extends to orbifolds. A continuous map of orbifolds $f : X \to Y$ is an orbifold covering if every y in Y has an open neighbourhood U so that its preimage $f^{-1}(U)$ is a disjoint union, ie is $\bigcup_{\lambda\in\Lambda} V_\lambda$ for some indexing set Λ, where $f|_{V_\lambda} : V_\lambda \to U$ is a natural quotient map $\boldsymbol{R}^n/\Gamma \to \boldsymbol{R}^n/\Gamma'$, where $\Gamma \leq \Gamma'$ (both groups will be finite). Note the map between the underlying spaces is not necessarily a covering. (It may have branch points.)

We call an orbifold covering regular, if it is of the form $M/\Gamma \to M/\Gamma'$ where Γ is a normal subgroup of Γ' and which both act discretely on the orbifold M.

It can be proved that each orbifold X has a unique universal orbifold cover, and X is the quotient of its universal cover by some group G (see Proposition 13.2.4 of [**7**]). We call G the orbifold fundamental group of X and denote it as $\pi_1^{orb}(X)$. For a proof of existence of universal covers and construction of an orbifold fundamental group see [**5**] Section 2.

As we will define below, Seifert manifolds have a bundle structure with a 2-orbifold as a base. Since the base is in general an orbifold and not simply a manifold we will broaden the concept of a bundle map to orbifolds (following [**1**]).

Recall a bundle map is a continuous map $\eta : S \to B$ of spaces where there is a space F, such that for every point b of B there is an open neighbourhood, U_b and a commutative diagram:

$$\begin{array}{ccc} U_b \times F & \xrightarrow{\pi_{U_b}} & U_b \\ \phi_b \downarrow & & \| \\ \eta^{-1}U_b & \xrightarrow[\eta]{} & U_b \end{array}$$

in which π_{U_b} is a projection and ϕ_b is a homeomorphism. The fibre above b is defined to be the subspace $\eta^{-1}(b)$, which is homeomorphic to F. We call S the total space and say it has the structure of a bundle. We call B the base and call η the projection. (We shall always assume that F is connected.)

An orbifold bundle map is a continuous map $\eta : S \to B$ of orbifolds such that for every point b of B there is an open neighbourhood, U_b and a commutative diagram:

$$\begin{array}{ccc} \tilde{U}_b \times F_b & \xrightarrow{\pi_{\tilde{U}_b}} & \tilde{U}_b \\ \phi_b \downarrow & & \downarrow f_b \\ \eta^{-1}U_b & \xrightarrow[\eta]{} & U_b \end{array}$$

in which $\pi_{\tilde{U}_b}$ is a projection, ϕ_b and f_b are regular orbifold covering projections and F_b is some connected orbifold. The fibre above b is defined to be the suborbifold $\eta^{-1}(b)$, which will be covered by F_b. Without loss of generality, f_b can be a quotient map of $\tilde{U}_b$ under action by a finite group, ie $f_b : \tilde{U}_b \to \tilde{U}_b/\Gamma_b \cong U_b$. Similarly ϕ_b can be a quotient map of S, so long as F_b is good. We call S the total space and say it has the structure of an orbifold bundle. We call B the base and call η the projection.

If the fibres are all compact (for example if the total space is compact) then F_b can always be chosen to be a copy of a regular fibre F (see [**1**]), that is, the fibre above a regular point. The set of regular points is dense in B and so we commonly refer to F as the general fibre.

In this paper, we call orbifolds flat if they have a Euclidean geometry. The examples we consider are the circle (S^1), the 2-torus (T^2) and the Klein bottle (Kb).

We are now equipped to give the definition of a Seifert manifold that we will use in this paper:

DEFINITION 1.1 (Seifert bundle). A Seifert bundle is an orbifold bundle with base a 2-orbifold and compact flat fibres. A Seifert orbifold is the total space of a Seifert bundle.

Since the fibres are compact the bundle has a compact flat general fibre (see above). Explicitly, if S is a Seifert orbifold, then there is an associated (locally trivial) orbifold bundle map $\eta : S \to B$ (B is the 2-orbifold base) with general (compact flat) fibre, F. In this paper we will consider Seifert manifolds, that is when S has no singularities (and consequently neither has F). We will assume that all Seifert manifolds are compact (and hence the bases will be compact too).

We will concentrate our attention on Seifert 4-manifolds which have T^2 or Kb as fibres. Note when the manifold is orientable, it cannot have Kb general fibres.

When studying Seifert manifolds we will rely on the associated fundamental group sequence which the following lemma justifies.

LEMMA 1.2 (cf. Lemma 3.2 in [**5**]). *Let $\eta : S \to B$ be a orbifold bundle map with nonsingular total space and general fibre F. Then we have a short exact sequence*:

$$\pi_1(F) \to \pi_1(S) \to \pi_1^{orb}(B) \to 1,$$

where the maps are determined by the inclusion of fibre above the basepoint of B and η. Furthermore, if the base is good and aspherical then $\pi_1(F)$ maps injectively into $\pi_1(S)$.

DEFINITION 1.3 (Monodromies). If B is aspherical, then $\pi_1(F)$ is a normal subgroup of $\pi_1(S)$ from the previous lemma. Let $A : \pi_1(S) \to \mathrm{Aut}(\pi_1(F))$ be the homomorphism which sends ξ to the automorphism $h \mapsto \xi h \xi^{-1}$. If we quotient the domain of this map by $\pi_1(F)$ and the codomain by $\mathrm{Inn}(\pi_1(F))$, then we get a map $\bar{A} : \pi_1^{orb}(B) \to \mathrm{Out}(\pi_1(F))$. We call $\bar{A}$ the monodromy map, and we call $\mathrm{Im}(\bar{A})$ the group of monodromies. Also, for $\bar{\xi} \in \pi_1^{orb}(B)$ we call $\bar{A}(\bar{\xi})$ the monodromy above $\bar{\xi}$.

If $F = T^n$, then $\pi_1(F)$ is abelian, and so $\mathrm{Out}(\pi_1(F)) = \mathrm{Aut}(\pi_1(F)) = GL_n\boldsymbol{Z}$. If $\xi \in \pi_1(S)$ projects to $\bar{\xi} \in \pi_1^{orb}(B)$, then $\bar{A}(\bar{\xi}) = A(\xi) \in GL_n\boldsymbol{Z}$. In this case, the group of monodromies also equals $\mathrm{Im}\, A$.

The following proposition gives a presentation of the fundamental group for a Seifert manifold. Presentations for the fundamental group are mentioned by several authors, for example in Section 5 of [**9**] and Proposition 2.3 of [**11**].

PROPOSITION 1.4. *Suppose S is a Seifert manifold over a closed base B, such that B has k_0 cone points, l reflector curves and k_i corner reflector on the ith reflector curve. Let m_{0j} be the order of the jth cone point, and let m_{ij} be the order of the jth corner reflector on the ith reflector curve. Let B_a be the underlying manifold associated to B. Let $\pi_1(F)$ be generated by $h_1, \dots, h_q$, with set of relations W. Then $\pi_1(S)$ has the following presentation ($\tilde{e}_{ij}, \tilde{f}_i, \tilde{g}_{ij}, \tilde{b}_i$ and $\tilde{a}$ elements of $\pi_1(F)$).*

generators $h_1, \dots, h_q,$

$s_{ij}(i = 0, \dots, l; j = 1, \dots, k_i),$

$\partial_1, \dots, \partial_l,$

$r_1, \dots, r_l,$

$t_1, u_1, \dots, t_g, u_g$ *(if B_a is orientable, genus g)*

$v_1, \dots, v_g$ *(if B_a is non-orientable, g cross-caps)*

relations $W,$

$$\xi h_p \xi^{-1} = A(\xi) h_p \text{ for } \xi = s_{ij}, \partial_j, r_j, t_j, u_j$$

$$s_{ij}^{m_{ij}} \tilde{e}_{ij} = 1$$

$$r_i^2 \tilde{f}_i = 1$$

$$r_i s_{ij} r_i^{-1} = \left(\prod_{p=j+1}^{k_i} s_{ip} \right)^{-1} s_{ij}^{-1} \left(\prod_{p=j+1}^{k_i} s_{ip} \right) \tilde{g}_{ij} \quad (i \neq 0)$$

$$\partial_i^{-1} \left(\prod_{j=1}^{k_i} s_{ij} \right) r_i \partial_i r_i^{-1} = \tilde{b}_i \quad (i \neq 0)$$

$$\prod_{p=1}^{g} [t_p, u_p] \prod_{j=1}^{k_0} s_{0j} \prod_{i=1}^{l} \partial_i = \tilde{a} \text{ (\textit{if } } B_a \text{ \textit{is orientable})}$$

$$\prod_{p=1}^{g} v_p^2 \prod_{j=1}^{k_0} s_{0j} \prod_{i=1}^{l} \partial_i = \tilde{a} \text{ (\textit{if } } B_a \text{ \textit{is non-orientable})}$$

Note in the above presentation, A generates the monodromy map, s_{0j} corresponds to the jth cone point, s_{ij} $(i \neq 0)$ the jth corner reflector on the ith reflector curve, r_i the ith reflector curve and ∂_i the boundary of a neighbourhood of the ith reflector curve. Later we will also find it convenient to rewrite the generators s_{ij} $(i \neq 0)$ in terms of $\sigma_{ij} = \prod_{p=j}^{k_i} s_{ip}$ (for notational convenience we define $\sigma_{i,k_i+1} = 1$). [Note $s_{ij} = \sigma_{ij}\sigma_{i,j+1}^{-1}$.] [Other authors (for example in [**11**]) rewrite these generators in terms of $r_{ij} = \sigma_{ij} r_i$ for $j = 1, \dots k_i + 1$.]

Reflector curves provide the biggest technical difficulties when proving our results. So we will consider the restriction of a Seifert bundle to the bundle above a neighbourhood of a reflector curve in order to isolate these technicalities. The following lemmas give the fundamental group of such a bundle and list some relationships between $\tilde{f}$ and $\tilde{e}_j$ etc.

LEMMA 1.5. *If $\bar{\mathscr{C}}$ is a reflector curve with annulus neighbourhood, $\bar{\mathscr{A}}$, with k corner reflectors so that the jth corner reflector has order m_j and if $\bar{\eta} : \bar{\mathscr{R}} \to \bar{\mathscr{A}}$ is a Seifert bundle with general fibre F, then $\pi_1(\bar{\mathscr{R}})$ has the following presentation*:

$$\begin{aligned}
\text{generators}\quad & h_1, h_2, \dots, h_q, \sigma_1, \dots, \sigma_k, \partial, r \\
\text{relations}\quad & W, \\
& \xi h_p \xi^{-1} = A(\xi) h_p \text{ for } \xi = \sigma_j, \partial, r \\
& \left(\sigma_j \sigma_{j+1}^{-1}\right)^{m_j} \tilde{e}_j = 1 \\
& r^2 \tilde{f} = 1 \\
& r \sigma_j r^{-1} = \sigma_j^{-1} \tilde{G}_j \\
& \partial^{-1} \sigma_1 r \partial r^{-1} = \tilde{b},
\end{aligned}$$

where $\tilde{G}_j = \prod_{i=j}^{k} A(\sigma_{i+1}) \tilde{g}_i$.

Note $(\sigma_j \sigma_{j+1}^{-1})^{m_j} \tilde{e}_j = 1$ can be more simply expressed as $s_j^{m_j} \tilde{e}_j = 1$.

LEMMA 1.6. *The following relationships exist between the various parts of the presentations of $\pi_1(\bar{\mathscr{R}})$*:

$$\tilde{e}_j = A(\sigma_j \sigma_{j+1}^{-1})\big(\tilde{e}_j\big) \tag{1}$$

$$\tilde{e}_j = A(s_j)\big(\tilde{e}_j\big) \tag{1$'$}$$

$$\left(\prod_{i=0}^{m_j - 1} A(\sigma_j \sigma_{j+1}^{-1})^{-i} \Big(\tilde{G}_j \big(\tilde{G}_{j+1}\big)^{-1} \Big) \right) \tilde{e}_j A(\sigma_j r)\big(\tilde{e}_j\big) = 1 \tag{2}$$

$$\tilde{f} = A(r)\big(\tilde{f}\big) \tag{3}$$

$$\big(\tilde{f}\big)^{-1} A(\sigma_j r)\big(\tilde{f}\big) = \big(\tilde{G}_j\big)^{-1} A(\sigma_j r)\big(\tilde{G}_j\big) \tag{4}$$

$$\tilde{b} A(r)\big(\tilde{b}\big)\big(\tilde{f}\big)^{-1} A(\partial^{-1})\big(\tilde{f}\big) = A(\partial^{-1})\big(\tilde{G}_1\big) \tag{5}$$

PROOF. To get the relation (1′) (and hence (1) as well) conjugate $s_j^{m_j} \tilde{e}_j$ by s_j. To get the other relationships requires conjugating relations by r and then simplifying (for instance by using the other relations or inverting) to get new relations involving just the h_i. For instance, take the relation $\partial^{-1}\sigma_1 r\partial r^{-1} = \tilde{b}$. Conjugate by r, simplify r^2 and $r\sigma_1 r^{-1}$ via the relations to $\left(\tilde{f}\right)^{-1}$ and $\sigma_1^{-1}\tilde{G}_1$ respectively, invert then use the original relation to get (5). The relations are ordered the same way as in the relations (in Lemma 1.5) they are based on, so that (2) corresponds to $\left(\sigma_j\sigma_{j+1}^{-1}\right)^{m_j}\tilde{e}_j = 1$ etc. □

When the general fibre is a torus, its fundamental group is abelian and the above relations may be expressed as follows.

COROLLARY 1.7. *Suppose $F = T^n$, then the relations* (1)–(5) *become*

$$\tilde{e}_j = A(\sigma_j\sigma_{j+1}^{-1})\tilde{e}_j \tag{6}$$

$$\tilde{e}_j = A(s_j)\tilde{e}_j \tag{6'}$$

$$\left(\sum_{i=0}^{m_j-1} A(\sigma_j\sigma_{j+1}^{-1})^{-i}\left(\tilde{G}_j - \tilde{G}_{j+1}\right)\right) + (A(\sigma_j r) + I)\tilde{e}_j = 0 \tag{7}$$

$$\tilde{f} = A(r)\tilde{f} \tag{8}$$

$$(A(\sigma_j r) - I)\tilde{f} = (A(\sigma_j r) - I)\tilde{G}_j \tag{9}$$

$$(A(r) + I)\tilde{b} + (A(\partial^{-1}) - I)\tilde{f} = A(\partial^{-1})\tilde{G}_1 \tag{10}$$

PROOF. Since $\pi_1(T^n) \cong \boldsymbol{Z}^n$ is abelian, A induces a group homomorphism $A : \pi_1^{orb}(\bar{\mathscr{A}}) \to \mathrm{Out}(\pi_1(T^n)) = \mathrm{Aut}(\pi_1(T^n)) = GL_n\boldsymbol{Z}$. The rest follows from the lemma. □

The following theorem due to Vogt shows that a Seifert manifold with hyperbolic base is determined topologically by its fundamental group.

THEOREM 1.8 (Vogt). *Suppose S and S' are Seifert* 4*-manifolds with hyperbolic bases. Then every isomorphism $\phi : \pi_1(S) \to \pi_1(S')$ is realised by a fibre preserving homeomorphism $\theta : S \to S'$.*

PROOF. In Section 7 of [**11**], Vogt proves this result (in Theorem 7.4) for a larger class of 4-manifolds, $\mathscr{M}(4)$. We will show that S and S' are in $\mathscr{M}(4)$ and hence the result.

Vogt defines $\mathscr{M}(4)$ to be the class of all sufficiently complicated closed 4-manifolds, S, which have the structure of an orbifold bundle with aspherical fibre, F and base B, such that rank $\pi_1(F) < J(B)$ for all integers $J > 1$. By sufficiently complicated, Vogt means B is hyperbolic (Definition 5.1 in [**11**]). The expression $J(B)$ (Definition 6.4 in [**11**]) is the minimum rank of a normal subgroup of $\pi_1^{orb}(B)$ with finite index $J > 1$. Such subgroups are fundamental groups of a hyperbolic surface so in particular, $J(B) > 2$. Since the rank of $\pi_1(F) = 2$ for $F = T^2$ and Kb, Seifert 4-bundles with hyperbolic bases (that is orbifold 4-bundles with flat 2-fibres) are in $\mathscr{M}(4)$. $\square$

2. The geometry $\boldsymbol{H}^2 \times \boldsymbol{E}^2$.

In this section we look at the geometry $\boldsymbol{H}^2 \times \boldsymbol{E}^2$. We will prove that closed manifolds with these geometries are Seifert fibred.

The model space for the geometry $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ is $\boldsymbol{H}^2 \times \boldsymbol{R}^2$. It is also useful to consider the model space as $\boldsymbol{H}^2 \times \boldsymbol{C} = \{(z, w) \in \boldsymbol{C}^2 | \Im z > 0\}$. The group of isometries is $\mathrm{Isom}(\boldsymbol{H}^2 \times \boldsymbol{E}^2) = \mathrm{Isom}(\boldsymbol{H}^2) \times \mathrm{Isom}(\boldsymbol{E}^2)$.

There is a natural fibration of $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ which is preserved by the group of isometries:

$$\boldsymbol{R}^2 \to \boldsymbol{H}^2 \times \boldsymbol{E}^2 \to \boldsymbol{H}^2.$$

Seifert fibrations descend from this fibration as the following proposition shows.

PROPOSITION 2.1. *If S is a compact quotient of $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ by a discrete group of isometries then S is a Seifert 4-manifold over a hyperbolic base.*

PROOF. Now $S = (\boldsymbol{H}^2 \times \boldsymbol{E}^2)/\Gamma$ for some discrete group of isometries of $\boldsymbol{H}^2 \times \boldsymbol{E}^2$. Let $\hat{\Gamma} = \Gamma \cap \mathrm{Isom}(\boldsymbol{E}^2)$. Then $\hat{\Gamma}$ is a discrete subgroup of $\mathrm{Isom}(\boldsymbol{E}^2)$ and is cocompact by Theorem 6.3 of [**12**]. So $\boldsymbol{R}^2/\hat{\Gamma}$ will be a (flat) closed orbifold. Now $\hat{\Gamma}$ is the kernel of the projection $p : \Gamma \to \mathrm{Isom}(\boldsymbol{H}^2)$. Let $\bar{\Gamma}$ be the image of this projection. Then the action of Γ gives a Seifert fibration: $\boldsymbol{R}^n/\hat{\Gamma} \to (\boldsymbol{H}^2 \times \boldsymbol{E}^2)/\Gamma \to \boldsymbol{H}^2/\bar{\Gamma}$. $\square$

For future reference, if ξ is an isometry of $\boldsymbol{H}^2 \times \boldsymbol{E}^2$, let $\bar{\xi}$ be its image in $\mathrm{Isom}(\boldsymbol{H}^2)$ and let $(\mathscr{O}(\xi), y(\xi))$ be its image in $\mathrm{Isom}(\boldsymbol{E}^2)$. Thus for $(z, w) \in \boldsymbol{H}^2 \times \boldsymbol{R}^2$, $\xi(z, w) = (\bar{\xi}(z), \mathscr{O}(\xi)w + y(\xi))$.

3. The geometry $\widetilde{\boldsymbol{SL}}_2 \times \boldsymbol{E}$.

In this section we look at the geometry $\widetilde{SL}_2 \times \boldsymbol{E}$. We will prove that closed manifolds with these geometries are Seifert fibred.

The model space for the geometry $\widetilde{SL}_2$ is the universal covering space of $U(\boldsymbol{H}^2)$, the unit tangent bundle of $\boldsymbol{H}^2$. The model space is a trivial line bundle over $\boldsymbol{H}^2$, however, $\widetilde{SL}_2$ is not geometrically a product, rather $\widetilde{SL}_2$ can be considered as a twisted product of the geometries $\boldsymbol{H}^2$ and $\boldsymbol{E}$. The group $\widetilde{SL}_2$ is the universal cover of $PSL_2\boldsymbol{R}$. The latter group acts simply transitively on $U(\boldsymbol{H}^2)$, so a choice of base point in $\widetilde{U(\boldsymbol{H}^2)}$ determines diffeomorphisms $\widetilde{SL}_2 \to \widetilde{U(\boldsymbol{H}^2)}$ and $PSL_2\boldsymbol{R} \to U(\boldsymbol{H}^2)$. The action of $\widetilde{SL}_2$ on itself via left multiplication projects to the action of $PSL_2\boldsymbol{R}$ on itself. The isometry group of $\widetilde{SL}_2$ has two components, both orientable. The identity component of $\mathrm{Isom}(\widetilde{SL}_2)$ is $\widetilde{SL}_2 \times_{\boldsymbol{Z}} \boldsymbol{R}$, the quotient of the product of $\widetilde{SL}_2$ and $\boldsymbol{R}$ modulo identification of the centre of $\widetilde{SL}_2$ with the group of integers $\boldsymbol{Z} \subset \boldsymbol{R}$. The isometries of $\widetilde{SL}_2$ preserve the fibration $\boldsymbol{R} \to \widetilde{SL}_2 \to \boldsymbol{H}^2$ with the $\boldsymbol{R}$ factor acting as translations of the fibres. Compact manifolds which are geometric of this type are precisely the Seifert 3-manifolds where the base has negative euler characteristic, and where the euler number is nonzero. See [**5**] pp. 462–467 for the details of this description of $\widetilde{SL}_2$.

The model space for $\widetilde{SL}_2 \times \boldsymbol{E}$ is the universal covering space of the nonzero tangent bundle of $\boldsymbol{H}^2$. The hyperbolic plane $\boldsymbol{H}^2$ can be considered as a subset of $\boldsymbol{C}$ whose tangent space can be naturally associated with $\boldsymbol{C} \times \boldsymbol{C}$. Therefore, we shall identify the tangent bundle of $\boldsymbol{H}^2$ with the set $\boldsymbol{H}^2 \times \boldsymbol{C}$. Let v be a unit tangent vector to the hyperbolic plane at z. Then its Euclidean length is given by $|v| = \Im z$ and so we may identify the unit tangent bundle of $\boldsymbol{H}^2$ with the subset $\{(z, v) \in \boldsymbol{H}^2 \times \boldsymbol{C} | |v| = \Im z\}$. Hence the nonzero tangent bundle of $\boldsymbol{H}^2$ is the subspace $\boldsymbol{H}^2 \times (\boldsymbol{C} - \{0\})$, and so we may identify the model space of $\widetilde{SL}_2 \times \boldsymbol{E}$ with $\boldsymbol{H}^2 \times \boldsymbol{C}$. The universal cover $\widetilde{SL}_2 \times \boldsymbol{E} \longrightarrow \boldsymbol{H}^2 \times (\boldsymbol{C} - \{0\})$ is given by $(z, w) \mapsto (z, e^w)$. The model space of $\widetilde{SL}_2$ is then identified with the subset $\{(z, w) \in \boldsymbol{H}^2 \times \boldsymbol{C} | \Re w = \ln \Im z\}$.

The isometry group of $\widetilde{SL}_2 \times \boldsymbol{E}$ is $\mathrm{Isom}(\widetilde{SL}_2) \times \mathrm{Isom}(\boldsymbol{E})$. By extension of the $\widetilde{SL}_2$ case, $\mathrm{Isom}(\widetilde{SL}_2 \times \boldsymbol{E})$ preserves the fibration $\boldsymbol{R}^2 \to \widetilde{SL}_2 \times \boldsymbol{E} \to \boldsymbol{H}^2$. We then have an associated exact sequence:

$$1 \to \boldsymbol{R} \times \mathrm{Isom}(\boldsymbol{E}) \to \mathrm{Isom}(\widetilde{SL}_2 \times \boldsymbol{E}) \xrightarrow{p} \mathrm{Isom}(\boldsymbol{H}^2) \to 1$$

Note the sequence does not split. See [**13**] Section 1 for some descriptions of $\widetilde{SL}_2 \times \boldsymbol{E}$.

PROPOSITION 3.1. *If S is a compact quotient of $\widetilde{SL}_2 \times \boldsymbol{E}$ by a discrete group of isometries then S is a Seifert 4-manifold over a hyperbolic base.*

PROOF. Firstly, $S = (\widetilde{SL}_2 \times \boldsymbol{E})/\Gamma$ for Γ a discrete subgroup of $\mathrm{Isom}(\widetilde{SL}_2 \times \boldsymbol{E})$. Let $\hat{\Gamma} = \Gamma \cap \ker(p)$ and $\bar{\Gamma} = p(\Gamma)$. The group $\hat{\Gamma}$ is a discrete cocompact subgroup of $\mathrm{Isom}(\boldsymbol{E}^2)$ and $\bar{\Gamma}$ is a discrete cocompact subgroup of $\mathrm{Isom}(\boldsymbol{H}^2)$ (Theorem 6.3 in [**12**]). So $\boldsymbol{R}^2/\hat{\Gamma}$ will be a (flat) closed orbifold. Then we have the Seifert fibration

$$F = \boldsymbol{R}^2/\hat{\Gamma} \to S \to \boldsymbol{H}^2/\bar{\Gamma} = B.$$

□

Now we shall consider how the isometries of $\widetilde{SL}_2 \times \boldsymbol{E}$ act in more detail. The radical of the isometry group is $\boldsymbol{R}^2$. The first factor (corresponding to the radical of $\widetilde{SL}_2$) acts via purely imaginary translations (where the generator of the centre: $\boldsymbol{Z}$, acts by $(z,w) \mapsto (z, w+2\pi i)$), while the second factor acts via real translations. Imaginary translations of the w factor project to $\mathrm{Isom}(U(\boldsymbol{H}^2))$ as rotations of the tangents. Real translations of the w factor project to isometries of the nonzero tangent bundle which change the length of the tangent vectors. The quotient of the identity component of the isometry group by this radical is $PSL_2\boldsymbol{R} = \mathrm{Isom}^+(\boldsymbol{H}^2)$. Therefore to understand the rest of the isometries in the identity component of $\mathrm{Isom}(\widetilde{SL}_2 \times \boldsymbol{E})$ it is sufficient to construct a set-theoretic section $s : \mathrm{Isom}^+(\boldsymbol{H}^2) \to \mathrm{Isom}(\widetilde{SL}_2 \times \boldsymbol{E})$ (ie $ps = 1$). We will do this below (as well as extending this section to $\mathrm{Isom}(\boldsymbol{H}^2)$).

The isometry group of $\widetilde{SL}_2$ has two components, both orientation preserving, with representative for the non-identity component given by $(z,w) \mapsto (-\bar{z}, \bar{w})$. (As this corresponds to a simultaneous reflection of $\boldsymbol{H}^2$ and the tangent space, it is orientation preserving). The isometry group of $\boldsymbol{E}$ also has two components, with the non-identity component represented by reflection about 0. There is a corresponding reflection of $\boldsymbol{H}^2 \times \boldsymbol{C}$ which fixes the subspace $\widetilde{SL}_2$ and which is given by $(z,w) \mapsto (z, 2\ln(\Im z) - \bar{w})$. So $\mathrm{Isom}(\widetilde{SL}_2 \times \boldsymbol{E})/\mathrm{Isom}^0(\widetilde{SL}_2 \times \boldsymbol{E}) = \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$, with generators represented by $(z,w) \mapsto (z, 2\ln(\Im z) - \bar{w})$ and $(z,w) \mapsto (-\bar{z}, \bar{w})$. We will label each component by the corresponding element of $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ thought of as $\{\epsilon, \delta | \epsilon = \pm 1, \delta = \pm 1\}$ (or as the corresponding diagonal matrices, $\left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \delta \end{smallmatrix}\right)$).

Define an orientation function $\mathrm{w} : \mathrm{Isom}(\ ) \mapsto \{\pm 1\}$ by setting $\mathrm{w}(\alpha) = 1$ if α is orientation preserving and -1 if it reverses orientation.

We shall now define the section $s : \mathrm{Isom}^+(\boldsymbol{H}^2) \to \mathrm{Isom}(\widetilde{SL}_2 \times \boldsymbol{E})$. Suppose $\bar{\alpha}$ is in $PSL_2\boldsymbol{R}$ $(= \mathrm{Isom}^+(\boldsymbol{H}^2))$, ie $\bar{\alpha}(z) = (az+b)/(cz+d)$. Then $\bar{\alpha}$ acts on $PSL_2\boldsymbol{R} = \{z, v \in \boldsymbol{H}^2 \times \boldsymbol{C} | |v| = \Im z\}$ via the map $(z,v) \mapsto (az+b)/(cz+d)$, $v/((cz+d)^2)$. Let $\log : \boldsymbol{C} - \{0\} \to \boldsymbol{R} \times (-\pi, \pi]i \subset \boldsymbol{C}$ be the principal value of the inverse of exp, extended so that $\log(-k) = \ln(k) + \pi i$ for $k > 0$. Let $s(\bar{\alpha})$ be the map $(z,w) \mapsto (\bar{\alpha}(z), w - \log(cz+d)^2)$. Then $s(\bar{\alpha})$ is an isometry of $\widetilde{SL}_2 \times \boldsymbol{E}$ (see [**13**]) and $p(s(\bar{\alpha})) = \bar{\alpha}$, so s is a set theoretic section for p. Note $s(\bar{\alpha})$ restricts to an isometry of $\widetilde{SL}_2$ as well (since $\ln(\Im z) = \Re w$ implies $\ln(\Im \bar{\alpha}(z)) = \ln(\Im z) - \ln(|cz+d|^2) = \Re(w - \log(cz+d)^2)$). Therefore for $\alpha \in \mathrm{Isom}^0(\widetilde{SL}_2)$, $p(\alpha^{-1} sp(\alpha)) = 1$ and so $\alpha(z,w) = s(p(\alpha))(z,w) + (0, iy)$ for some $y \in \boldsymbol{R}$. From now on write $\bar{\alpha}$ to mean $p(\alpha)$. Let α be in $\mathrm{Isom}^0(\widetilde{SL}_2) \times \mathrm{Isom}^+(\boldsymbol{E})$. Then $\alpha(z,w) = s(\bar{\alpha})(z,w) + (0, y(\alpha))$, for some $y(\alpha) \in \boldsymbol{C}$.

We will modify s slightly to have a special form when $\bar{\alpha}$ is elliptic, which will be useful later on. From above, $s(\bar{\alpha})^m(z,w) = s(p(s(\bar{\alpha})^m))(z,w) + (0, iy) = s(\bar{\alpha}^m)(z,w) + (0, iy)$ for some $y \in \boldsymbol{R}$. Suppose $\bar{\alpha}$ is elliptic of order m. Then $\bar{\alpha}^m = 1$.

Let $s'(\bar{\alpha})(z,w) = s(\bar{\alpha})(z,w) + (0,-iy/m)$, then $s'(\bar{\alpha})^m(z,w) = s(\bar{\alpha})^m(z,w) + (0,-iy) = s(\bar{\alpha}^m)(z,w)$ since $s(\bar{\alpha})$ commutes with imaginary translations, and so $s'(\bar{\alpha})^m = 1$. Replace $s(\bar{\alpha})$ by $s'(\bar{\alpha})$ to get the desired modification. As remarked in [**10**], if $\bar{\alpha}$ is hyperbolic the imaginary part of the second factor is defined by the parallel translation of the unit tangent vector along the axis of $\bar{\alpha}$. We will not encounter parabolic elements, since the fundamental groups of compact hyperbolic orbifolds contain no parabolic elements ([**3**] Corollary 4.2.7), so we will not alter the action. Note s is not a group homomorphism; however $s(\bar{\alpha}\bar{\beta})s(\bar{\alpha})^{-1}s(\bar{\beta})^{-1}$ will be a rational translation of $2\pi i$ (ie the subgroup $\boldsymbol{Q}$ of $\boldsymbol{R}$ in $\widetilde{SL}_2 \times_{\boldsymbol{Z}} \boldsymbol{R}$).

We can extend our definition of s to $\mathrm{Isom}(\boldsymbol{H}^2)$. Take $\bar{\beta} \in \mathrm{Isom}(\boldsymbol{H}^2) \setminus \mathrm{Isom}^+(\boldsymbol{H}^2)$, then $\bar{\beta}(z) = \bar{\alpha}(-\bar{z})$ for some $\bar{\alpha} \in PSL_2\boldsymbol{R}$. A lift of $\bar{\beta}$ is then $s(\bar{\beta})(z,w) = s(\bar{\alpha})(-\bar{z},\bar{w})$. As before, if $\mathrm{w}(\beta) = 1$, then $\beta(z,w) = s(\bar{\beta})(z,w) + (0, y(\beta))$ for some $y(\beta) \in \boldsymbol{C}$. Note, $\beta^2(z,w) = s(\bar{\beta}^2)(z,w) + (0, y(\beta) + \bar{y}(\beta))$. If $\alpha \in \mathrm{Isom}(\widetilde{SL}_2 \times \boldsymbol{E})$ is orientation reversing, then $\alpha(z,w) = s(\bar{\alpha})(z, 2\ln(\Im z) - \bar{w}) + (0, y(\alpha))$ for some $y(\alpha) \in \boldsymbol{C}$.

In all this, if α is in the component labelled (ϵ,δ), then $\mathrm{w}(\alpha) = \epsilon$ and $\mathrm{w}(\bar{\alpha}) = \delta$. Furthermore, if τ_t is a translation, ie $\tau_t(z,w) = (z, w+t)$, then $\alpha\tau_t\alpha^{-1}(z,w) = (z, w + \left(\begin{smallmatrix}\epsilon & 0\\ 0 & \delta\end{smallmatrix}\right)t)$, considering $\boldsymbol{C} = \boldsymbol{R}^2$. We will label this matrix $\mathscr{O}(\alpha) := \left(\begin{smallmatrix}\epsilon & 0\\ 0 & \delta\end{smallmatrix}\right) = \left(\begin{smallmatrix}\mathrm{w}(\alpha) & 0\\ 0 & \mathrm{w}(\bar{\alpha})\end{smallmatrix}\right)$.

The lifts $s(\bar{\xi})$ are chosen so that if $\bar{\xi}$ is elliptic with order m, then $s(\bar{\xi})^m = 1$. In [**10**], it was also observed if $\bar{l}_1\bar{l}_2\ldots\bar{l}_n$ is trivial in $\mathrm{Isom}(\boldsymbol{H}^2)$ and the $\bar{l}_i$ are all elliptic or hyperbolic (for example a relation of a hyperbolic orbifold), then $s(\bar{l}_1)s(\bar{l}_2)\ldots s(\bar{l}_n)$ is an imaginary translation with magnitude the holonomy corresponding to the relation, that is the composition of the parallel translations of the $\bar{l}_i$. For example the standard global relation of an orientable hyperbolic orbifold without reflectors is $\prod_{p=1}^{g}[t_p,u_p]\prod_{j=1}^{k}s_{0j}$. This relation corresponds to traversing the entire orbifold and so the holonomy corresponding to this relation is the holonomy of the orbifold. This connection can be seen in [**3**] Theorem 4.3.2 when constructing fundamental regions.

Note $s(\bar{\alpha})$ and reflection in $\widetilde{SL}_2$ commute. Suppose $\alpha_1,\alpha_2,\ldots,\alpha_l$ are isometries of $\widetilde{SL}_2 \times \boldsymbol{E}$ such that $\alpha_1\alpha_2\ldots\alpha_l = 1$ and $p(\alpha_i)$ are not parabolic. Then $s(\bar{\alpha}_1)s(\bar{\alpha}_2)\ldots s(\bar{\alpha}_n)$ is also an imaginary translation, τ say, with magnitude the parallel translate corresponding to the relation. Also $(\mathscr{O}(\alpha_1), y(\alpha_1))\ldots(\mathscr{O}(\alpha_l), y(\alpha_l)) = \tau^{-1}$.

When considering the $\widetilde{SL}_2 \times \boldsymbol{E}$ geometry below, it is convenient to treat it as similarly to $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ as possible. So we will find a new representation of $\widetilde{SL}_2 \times \boldsymbol{E}$ so that isometries act via an element of $\mathrm{Isom}(\boldsymbol{H}^2) \times \mathrm{Isom}(\boldsymbol{E})^2$ plus a correction term. Let θ be the self-homeomorphism of $\boldsymbol{H}^2 \times \boldsymbol{C}$ defined by $\theta(z,w) = (z, w - \ln(\Im(z)))$. A new representation for the geometry $\widetilde{SL}_2 \times \boldsymbol{E}$ can then given by θ. Namely, points get sent to their image by θ and isometries get sent to their conjugate by θ. Note that $\widetilde{SL}_2$ now becomes identified with the subspace $\boldsymbol{H}^2 \times i\boldsymbol{R}$. Translations and the map $(z,w) \mapsto (-\bar{z}, \bar{w})$ are preserved. Reflection in $\widetilde{SL}_2$ now becomes the map $(z,w) \mapsto (z, -\bar{w})$. For $\bar{\alpha}$ in $PSL_2\boldsymbol{R}$ ($\bar{\alpha}(z) = (az+b)/(cz+d)$), $s(\bar{\alpha})$ now becomes the map $(z,w) \mapsto (\bar{\alpha}(z), w - 2\log((cz+d)/(|cz+d|)))$ (plus a purely imaginary translation if $\bar{\alpha}$ is elliptic). Note $2\log((cz+d)/(|cz+d|))$ (plus possibly the translation) is purely imaginary and (as before) corresponds to the parallel translate corresponding to $\bar{\alpha}$. Any isometry is a composition of these maps. Therefore we can say that any isometry, ξ acts (under the new representation) via an element of $\mathrm{Isom}(\boldsymbol{H}^2) \times \mathrm{Isom}(\boldsymbol{E})^2$ plus a purely imaginary correction term corresponding to the parallel translate of $\bar{\xi}$ ($\bar{\xi}$ is the image of ξ in $\mathrm{Isom}(\boldsymbol{H}^2)$). The $\mathrm{Isom}(\boldsymbol{E}) \times \mathrm{Isom}(\boldsymbol{E})$ part of the isometry is given by $(\mathscr{O}(\xi), y(\xi))$ since conjugation by θ preserves translations ($\mathscr{O}(\xi)$ was defined by action on translations by conjugation and $y(\xi)$ was defined as the translation difference from $s(\bar{\xi})$).

In summary, if ξ is an isometry of $\widetilde{SL}_2 \times \boldsymbol{E}$, then $\xi(z,w) = (\bar{\xi}(z), \mathscr{O}(\xi)w + y(\xi) + \left(\begin{smallmatrix} 0 \\ c(\xi) \end{smallmatrix}\right))$ where c is the correction term corresponding to the parallel translate of $\bar{\xi}$. Note $\mathscr{O}(\xi')\left(\begin{smallmatrix} 0 \\ c(\xi) \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ \mathrm{w}(\bar{\xi}')c(\xi) \end{smallmatrix}\right)$.

4. Geometric Seifert manifolds with hyperbolic bases.

In this section we will give necessary and sufficient conditions for Seifert 4-manifolds with hyperbolic base and T^2 fibre to be geometric.

Firstly, though we will give some results which are useful when describing the group of monodromies and the conditions.

Theorem 4.1. *A subgroup G of $GL_2\boldsymbol{Z}$, is finite if and only if there is a $P \in GL_2\boldsymbol{R}$ such that PGP^{-1} is a subgroup of $O_2\boldsymbol{R}$.*

Moreover, G is conjugate (in $GL_2\boldsymbol{Z}$) to a subgroup of

1. $O_2\boldsymbol{Z} = \langle \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle \cong D_8$.
   OR
2. $\langle \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle \cong D_{12}$

Proof. If G is finite, then by p. 85 of [**14**] it is conjugate in $GL_2\boldsymbol{Z}$ to a subgroup of $O_2\boldsymbol{Z}$ or $\langle \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle = \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right) \langle \left(\begin{smallmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)^{-1}$ (where $a = 2 + \sqrt{3}$ and $b = 1$). Therefore G is conjugate to a subgroup of $O_2\boldsymbol{R}$.

Conversely suppose G is conjugate to an orthogonal group. Since $G \subset GL_2\boldsymbol{Z}$, G is a discrete subgroup of $GL_2\boldsymbol{R}$. Since $PGP^{-1} \subset O_2\boldsymbol{R}$, PGP^{-1} is a discrete subgroup of a compact space and consequently is finite. Therefore G is finite. □

LEMMA 4.2. *If $A \in GL_n\boldsymbol{R}$ such that $A^m = I$, then*

$$\operatorname{Im}\left(\sum_{j=0}^{m-1} A^j\right) = \ker(A - I) \tag{11}$$

$$\operatorname{Im}(A - I) = \ker\left(\sum_{j=0}^{m-1} A^j\right) \tag{12}$$

PROOF. Suppose $(A - I)v = 0$, then $Av = v$ and consequently, $mv = \sum_{j=0}^{m-1} A^j v$. Therefore $\ker(A - I) \subseteq \operatorname{Im}\left(\sum_{j=0}^{m-1} A^j\right)$.

Next, $(A - I)\left(\sum_{j=0}^{m-1} A^j\right) = \left(\sum_{j=0}^{m-1} A^j\right)(A - I) = A^m - I = 0$. This shows $\ker(A - I) \supseteq \operatorname{Im}\left(\sum_{j=0}^{m-1} A^j\right)$ and $\ker\left(\sum_{j=0}^{m-1} A^j\right) \supseteq \operatorname{Im}(A - I)$, which proves the first equality.

By considering the dimensions of these spaces (and the rank-nullity theorem), we get the second equality. □

Recall that (when the base is aspherical) there is a homomorphism $A : \pi_1(S) \to \operatorname{Aut}(\pi_1(F))$, which induces the monodromy map, $\bar{A} : \pi_1^{orb}(B) \to \operatorname{Out}(\pi_1(F))$. When the fibre is T^n the group of monodromies is contained in $\operatorname{Out}(\pi_1(F)) = GL_n\boldsymbol{Z}$. In this case, the monodromy map gives a $\boldsymbol{Z}[\pi_1^{orb}(B)]$-module structure to $\boldsymbol{Z}^n$ and hence to $\boldsymbol{Q}^n$. Let $\mathscr{I}_{\mathrm{w}} = \langle \bar{\xi} - \mathrm{w}(\bar{\xi}) | \bar{\xi} \in \pi_1^{orb}(B) \rangle$ be the w-twisted augmentation ideal, and let $V = \mathscr{I}_{\mathrm{w}}\boldsymbol{Q}^n$ be the submodule generated by $(A(\bar{\xi}) - \mathrm{w}(\bar{\xi})I)z$ for all $\bar{\xi} \in \pi_1^{orb}(B)$ and $z \in \boldsymbol{Q}^n$. Thus sometimes we will write V as $\sum_{\bar{\xi} \in \pi_1^{orb}(B)} \operatorname{Im}_{\boldsymbol{Q}^n}(A(\bar{\xi}) - \mathrm{w}(\bar{\xi})I)$.

LEMMA 4.3. *Let G be a finitely generated group with generators $\{g_1, \ldots, g_m\}$. Then the augmentation ideal $\mathscr{I} = \langle g - 1 | g \in G \rangle$ in $\boldsymbol{Z}[G]$ is generated as a two-sided ideal by $\{g_1 - 1, \ldots, g_m - 1\}$.*

PROOF. For each g we need to prove $g - 1$ is in the ideal generated by $\{g_1 - 1, \ldots g_m - 1\}$. Since $gh - 1 = (g - 1)h + (h - 1) = g(h - 1) + (g - 1)$ and $g^{-1} - 1 = (g - 1)(-g^{-1}) = -g^{-1}(g - 1)$, the result follows by induction on the length of the shortest word that represents g in terms of the generators (and their inverses). □

COROLLARY 4.4. *Let G be a finitely generated group with generators $\{g_1, \ldots, g_m\}$. Let $R : G \to GL_n\boldsymbol{R}$ be a group homomorphism. Then,*

$$\sum_{g\in G} \mathrm{Im}(R(g)-I) = \sum_{i=1}^{m} \mathrm{Im}(R(g_i)-I).$$

PROOF. The group homomorphism, R, gives a $Z[G]$-module structure on $\boldsymbol{R}^n$. Therefore $\sum_{g\in G} \mathrm{Im}(R(g)-I) = \mathscr{I}\boldsymbol{R}^n$. By the lemma $\mathscr{I}\boldsymbol{R}^n = \{g_1-1,\dots, g_m-1\}\boldsymbol{R}^n = \sum_{i=1}^{m} \mathrm{Im}(R(g_i)-I)$, hence the corollary. □

COROLLARY 4.5. *Let B be an orbifold, so that $\pi_1^{orb}(B)$ is generated by $\{g_1,\dots,g_m\}$. Let $A:\pi_1^{orb}(B)\to GL_n\boldsymbol{Z}$ and $\mathrm{w}:\pi_1^{orb}(B)\to\{\pm1\}$ be group homomorphisms. Then*

$$\sum_{g\in\pi_1^{orb}(B)} \mathrm{Im}(A(g)-\mathrm{w}(g)I) = \sum_{i=1}^{m} \mathrm{Im}(A(g_i)-\mathrm{w}(g_i)I).$$

PROOF. Take $R(g)=\mathrm{w}(g)A(g)$ and apply the previous result. Note $\mathrm{Im}(A(g)-\mathrm{w}(g)I) = \mathrm{Im}(\mathrm{w}(g)A(g)-I)$. □

PROPOSITION 4.6. *If S is a geometric manifold of type $\boldsymbol{H}^2\times\boldsymbol{E}^2$ or $\widetilde{SL}_2\times\boldsymbol{E}$ then it is a Seifert manifold with a hyperbolic base. Furthermore if T^2 is the general fibre then the group of monodromies is a finite subgroup of $GL_2\boldsymbol{Z}$.*

PROOF. By Propositions 2.1 and 3.1 respectively, $\boldsymbol{H}^2\times\boldsymbol{E}^2$ and $\widetilde{SL}_2\times\boldsymbol{E}$ manifolds are Seifert fibred with hyperbolic bases.

Let X be the geometry. Then $S=X/\Gamma$ for some discrete subgroup Γ of $\mathrm{Isom}(X)$. In both cases there is a projection $p:\mathrm{Isom}(X)\to\mathrm{Isom}(\boldsymbol{H}^2)$. Now for the fibre F, $\pi_1(F)$ is isomorphic to $\Gamma\cap\ker(p)$. Suppose the fibre is T^2, then $\Gamma\cap\ker(p)$ is isomorphic to $\boldsymbol{Z}^2$. However $\ker(p)$ is contained in $\mathrm{Isom}(\boldsymbol{E}^2)$ ($\ker(p)=\mathrm{Isom}(\boldsymbol{E})\times\boldsymbol{E}$ when $X=\widetilde{SL}_2\times\boldsymbol{E}$) and so $\Gamma\cap\ker(p)$ must consist of translations.

Now Γ acts on $\Gamma\cap\ker(p)$ by conjugation. The group $\mathrm{Isom}(X)$ acts on translations via orthogonal matrices (when $X=\widetilde{SL}_2\times\boldsymbol{E}$ they will furthermore act via matrices of the form $\left(\begin{smallmatrix}\epsilon & 0\\ 0 & \delta\end{smallmatrix}\right)$ where ϵ and δ are ±1: see Section 3 for details), so Γ acts on $\Gamma\cap\ker(p)$ via a subgroup of $O_2\boldsymbol{R}$. However the group of monodromies (a subgroup of $GL_2\boldsymbol{Z}$) is the action of $\pi_1(S)=\Gamma$ on $\pi_1(F)$ (since $F=T^2$). Therefore the group of monodromies is a subgroup of $GL_2\boldsymbol{Z}$ conjugate to a subgroup of $O_2\boldsymbol{R}$ (the conjugation is via the isomorphism $\pi_1(F)\cong\Gamma\cap\ker(p)$), and so by Corollary 4.1 is finite. □

We will now show that the condition that group of monodromies is finite, is not only necessary for a Seifert manifold with T^2 fibres to be geometric, but is also sufficient.

THEOREM 4.7. *Let S be a Seifert manifold over a hyperbolic base, with general fibre T^2. Let the base, B, have k_0 cone points, so that m_{0j} is the order of the jth cone point, and l reflector curves, such that the ith reflector curve has k_i corner reflectors so that m_{ij} is the order of the jth corner reflector on the ith reflector curve. Let A be standard map which induces the monodromy map and let $\tilde{a}$, $\tilde{b}_i$, $\tilde{e}_{0j}$, $\tilde{f}_i$, and $\tilde{g}_{ij}$ be the standard parts of the presentation of $\pi_1(S)$.*

Then S is geometric if and only if the group of monodromies is finite, ie it is conjugate in $GL_2\boldsymbol{Z}$ to a subgroup of $O_2\boldsymbol{Z}$ or $\langle(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})\rangle$.

Let $e = \tilde{a} + \sum_{j=1}^{k_0} \tilde{e}_{0j}/m_{0j} + \frac{1}{2}\sum_{i=1}^{l}\left(\tilde{b}_i + \sum_{j=1}^{k_i} \tilde{e}_{ij}/m_{ij}\right)$ and $V = \mathscr{I}_{\mathrm{w}}\boldsymbol{Q}^2$.

More precisely, S is geometric of type $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ if and only if the group of monodromies is finite and

$$e \in V.$$

S is geometric of type $\widetilde{SL}_2 \times \boldsymbol{E}$ if and only if the group of monodromies is finite and

$$e \notin V,$$

which implies the group $\{\mathrm{w}(\bar{\xi})A(\xi) | \xi \in \pi_1^{orb}(B)\}$ is conjugate in $GL_2\boldsymbol{Z}$ to a subgroup of $\langle(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})\rangle$ or $\langle(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})\rangle$.

The proof of the complete result is technical in places. Therefore, we do this in stages. In this section we first prove the result for the $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ case of the theorem while deferring the details of how to handle the reflector curves. Then we explain how the proof differs in the $\widetilde{SL}_2 \times \boldsymbol{E}$ case. Lastly in the next section, we examine the details associated with reflector curves.

LEMMA 4.8 ($\boldsymbol{H}^2 \times \boldsymbol{E}^2$ CASE OF THEOREM 4.7). *Let S be a Seifert manifold over a hyperbolic base, with general fibre T^2. Let the base, B, have k_0 cone points, so that m_{0j} is the order of the jth cone point, and l reflector curves, such that the ith reflector curve has k_i corner reflectors so that m_{ij} is the order of the jth corner reflector on the ith reflector curve. Let A be standard map which induces the monodromy map and let $\tilde{a}$, $\tilde{b}_i$, $\tilde{e}_{0j}$, $\tilde{f}_i$, and $\tilde{g}_{ij}$ be the standard parts of the presentation of $\pi_1(S)$.*

Let $e = \tilde{a} + \sum_{j=1}^{k_0} \tilde{e}_{0j}/m_{0j} + \frac{1}{2}\sum_{i=1}^{l}\left(\tilde{b}_i + \sum_{j=1}^{k_i} \tilde{e}_{ij}/m_{ij}\right)$ and $V = \mathscr{I}_{\mathrm{w}}\boldsymbol{Q}^2$.

Then S is geometric of type $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ if and only if the group of monodromies is finite and $e \in V$.

PROOF. Note for $M \in \boldsymbol{Z}[GL_2\boldsymbol{Z}]$, $\mathrm{Im}_{\boldsymbol{Q}^2}(M) = \boldsymbol{Q}^2 \cap \mathrm{Im}_{\boldsymbol{R}^2}(M)$, hence $V = \boldsymbol{Q}^2 \cap \mathscr{I}_{\mathrm{w}}\boldsymbol{R}^2$. So to prove a rational number e is contained in V, it is sufficient to prove $e \in \mathscr{I}_{\mathrm{w}}\boldsymbol{R}^2$. Therefore for the purpose of this proof, we can replace V by $\mathscr{I}_{\mathrm{w}}\boldsymbol{R}^2 = \sum_{\bar{\xi}\in\pi_1^{orb}(B)} \mathrm{Im}_{\boldsymbol{R}^2}(A(\bar{\xi}) - \mathrm{w}(\bar{\xi})I)$ and write Im to mean $\mathrm{Im}_{\boldsymbol{R}^2}$.

Assume first S is geometric of type $\boldsymbol{H}^2 \times \boldsymbol{E}^2$, so $\pi_1(S)$ is a lattice in $\mathrm{Isom}(\boldsymbol{H}^2 \times \boldsymbol{E}^2)$. Since the fibre is T^2, $\pi_1(F) = \langle h_1, h_2 | [h_1, h_2] = 1\rangle \cong \boldsymbol{Z}^2$ and so the h_i act on $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ by translations: $(z, w) \mapsto (z, w + y(h_i))$, for some complex numbers $y(h_i)$. Let P be the matrix with $y(h_i)$ as the columns (considering $\boldsymbol{C} = \boldsymbol{R}^2$). The $y(h_i)$ are linearly independent over $\boldsymbol{R}$ (since $\pi_1(F)$ is a free abelian group), and P is invertible. The other generators, ξ say, project to isometries of $\boldsymbol{H}^2$ which are denoted as $\bar{\xi}$. They act on $\boldsymbol{C}$ as $(\mathscr{O}(\xi), y(\xi))$.

We shall consider the consequences of the defining relations of $\pi_1(S)$ for the actions of the generators of $\pi_1(S)$ on the "Euclidean factor", $\boldsymbol{C} = \boldsymbol{R}^2$. The monodromy relations $\xi h_i \xi^{-1} = A(\xi) h_i$ determine equations:

$$\mathscr{O}(\xi) = PA(\xi)P^{-1} \tag{13}$$

in $GL_2\boldsymbol{R}$, where P is the matrix with columns $y(h_i)$. This implies the subgroup $\{A(\xi) | \xi \in \pi_1(S)\}$ of $GL_2\boldsymbol{Z} = \mathrm{Aut}(T^2)$ is conjugate to a subgroup of $O_2\boldsymbol{R}$, and so we may apply Theorem 4.1. Note this provides a more explicit proof of Proposition 4.6.

The next relations are of the type $s_{0j}^{m_{0j}}\tilde{e}_{0j} = 1$, where s_{0j} corresponds to the jth cone point which has order m_{0j}. Now $\bar{s}_{0j}$ is elliptic of order m_{0j}, so $s_{0j}^{m_{0j}}$ acts trivially on the $\boldsymbol{H}^2$ factor and acts via the map $(\mathscr{O}(s_{0j}), y(s_{0j}))^{m_{0j}}$ on the $\boldsymbol{C}$ factor. By looking at just the $\boldsymbol{C}$ factor we see $(\mathscr{O}(s_{0j}), y(s_{0j}))^{m_{0j}} = (I, P\tilde{e}_{0j})^{-1}$ or equivalently $(A(s_{0j}), P^{-1}y(s_{0j}))^{m_{0j}} = (I, \tilde{e}_{0j})^{-1}$. By expanding, we see $A(s_{0j})^{m_{0j}} = I$ and $\left(\sum_{p=0}^{m_{0j}-1} A(s_{0j})^p\right) P^{-1}y(s_{0j}) = -\tilde{e}_{0j}$. Then $A(s_{0j})\tilde{e}_{0j} = \tilde{e}_{0j}$. So, $\left(\sum_{p=0}^{m_{0j}-1} A(s_{0j})^p\right)(P^{-1}y(s_{0j}) + \tilde{e}_{0j}/m_{0j}) = 0$. Therefore $P^{-1}y(s_{0j}) + \tilde{e}_{0j}/m_{0j}$ is in $\ker\left(\sum_{p=0}^{m_{0j}-1} A(s_{0j})^p\right)$ which equals $\mathrm{Im}(A(s_{0j}) - I)$ by Lemma 4.2. So

$$P^{-1}y(s_{0j}) + \tilde{e}_{0j}/m_{0j} = (A(s_{0j}) - I)z(s_{0j}), \text{ for some } z(s_{0j}) \in \boldsymbol{R}^2. \tag{14}$$

By considering the relations connected to the reflector curves we can obtain the following equations for $i = 1, \dots, l$ (the details of this are deferred to Lemma 5.1):

$$P^{-1}y(\partial_i) + \frac{1}{2}\left(\tilde{b}_i + \sum_{j=1}^{k_i} \tilde{e}_{ij}/m_{ij}\right) = \sum_{\xi}(A(\xi) - \mathrm{w}(\bar{\xi})I)z(\xi), \tag{15}$$

where the sum on the right is taken over the following generators: $\{\partial_i, r_i, s_{i1}, \ldots, s_{ik_i}\}$.

The last relations to consider are

$$\prod_{p=1}^{g}[t_p, u_p] \prod_{j=1}^{k_0} s_{0j} \prod_{i=1}^{l} \partial_i = \tilde{a}$$

when the base is orientable, and

$$\prod_{p=1}^{g} v_p^2 \prod_{j=1}^{k_0} s_{0j} \prod_{i=1}^{l} \partial_i = \tilde{a}$$

when the base is non-orientable. [To simplify expressions below, let $d = \prod_{p=1}^{g}[t_p, u_p]$ when the base is orientable and $d = \prod_{p=1}^{g} v_p^2$ when the base is non-orientable.]

By considering how these relations act on the $\boldsymbol{C}$ factor we obtain the following equations which respectively correspond to the orientable and non-orientable base cases:

$$\begin{aligned}\sum_{i=1}^{g}\left(\prod_{1\le j<i}[\mathscr{O}(t_j), \mathscr{O}(u_j)]\right)\big(y(t_i) + \mathscr{O}(t_i)y(u_i) - \mathscr{O}(t_i u_i t_i^{-1})y(t_i) - \mathscr{O}([t_i, u_i])y(u_i)\big) + \\ \mathscr{O}(d)\sum_{j=1}^{k_0}\left(\prod_{1\le p<j}\mathscr{O}(s_{0p})\right)y(s_{0j}) + \mathscr{O}(d)\prod_{j=1}^{k_0}\mathscr{O}(s_{0j})\sum_{i=1}^{l}\left(\prod_{1\le p<i}\mathscr{O}(\partial_p)\right)y(\partial_i) = P\tilde{a}\end{aligned} \tag{16}$$

$$\begin{aligned}\sum_{i=1}^{g}\left(\prod_{1\le j<i}\mathscr{O}(v_j)^2\right)(\mathscr{O}(v_i)y(v_i) + y(v_i)) + \\ \mathscr{O}(d)\sum_{j=1}^{k_0}\left(\prod_{1\le p<j}\mathscr{O}(s_{0p})\right)y(s_{0j}) + \mathscr{O}(d)\prod_{j=1}^{k_0}\mathscr{O}(s_{0j})\sum_{i=1}^{l}\left(\prod_{1\le p<i}\mathscr{O}(\partial_p)\right)y(\partial_i) = P\tilde{a}\end{aligned} \tag{17}$$

We can substitute for $y(s_{0j})$ and $y(\partial_i)$ using equations (14) and (15). To convert to monodromies, we premultiply the equations by P^{-1}. In order to simplify these expressions it is best to look at these results from the perspective of a different set of generators. In light of this, we make the following definitions:

$$t_i' = \left(\prod_{1\le j<i}[t_j,u_j]\right) t_i u_i t_i^{-1} u_i^{-1} t_i^{-1} \left(\prod_{1\le j<i}[t_j,u_j]\right)^{-1}$$

$$z(t_i') = -A\left(\prod_{1\le j<i}[t_j,u_j]\right) A(t_i) P^{-1} y(u_i)$$

$$u_i' = \left(\prod_{1\le j<i}[t_j,u_j]\right) t_i u_i t_i^{-1} \left(\prod_{1\le j<i}[t_j,u_j]\right)^{-1}$$

$$z(u_i') = -A\left(\prod_{1\le j<i}[t_j,u_j]\right) P^{-1} y(t_i)$$

$$v_i' = \left(\prod_{1\le j<i} v_j^2\right) v_i \left(\prod_{1\le j<i} v_j^2\right)^{-1}$$

$$z(v_i') = A\left(\prod_{1\le j<i} v_j^2\right) P^{-1} y(v_i)$$

$$s_{0j}' = d\left(\prod_{1\le p<j} s_{0p}\right) s_{0j} \left(\prod_{1\le p<j} s_{0p}\right)^{-1} d^{-1}$$

$$z(s_{0j}') = A(d) A\left(\prod_{1\le p<j} s_{0p}\right) z(s_{0j})$$

$$\xi_i' = d\left(\prod_{j=1}^{k_0} s_{0j}\right)\left(\prod_{1\le p<i} \partial_p\right) \xi_i \left(\prod_{1\le p<i} \partial_p\right)^{-1} \left(\prod_{j=1}^{k_0} s_{0j}\right)^{-1} d^{-1}$$

$$z(\xi_i') = A(d)\left(\prod_{j=1}^{k_0} A(s_{0j})\right)\left(\prod_{1\le p<i} A(\partial_p)\right) z(\xi_i) \quad \text{for } \xi_i \in \{\partial_i, r_i, s_{i1}, \dots, s_{ik_i}\}.$$

Notice that v_i and v_i' are orientation reversing and they are the only generators of either set that are so.

$$\text{Let } e' = \tilde{a} + A(d)\sum_{j=1}^{k}\left(\prod_{1\le p<j} A(s_{0p})\right)\tilde{e}_{0j}/m_{0j}$$
$$+ A(d)\left(\prod_{j=1}^{k_0} A(s_{0j})\right)\sum_{i=1}^{l}\left(\prod_{1\le p<i} A(\partial_p)\right)\frac{1}{2}\left(\tilde{b}_i + \sum_{j=1}^{k_i} \tilde{e}_{ij}/m_{ij}\right),$$

We can then rearrange equations (16) and (17) to both have the form:

$$e' = \sum (A(\xi) - \mathrm{w}(\bar{\xi})I)z(\xi) \tag{18}$$

where the sum is over these new generators, namely t'_i, u'_i, s'_{ij}, r'_i and ∂'_i when the base is orientable and v'_i, s'_{ij}, r'_i and ∂'_i when non-orientable. By Corollary 4.5 which precedes this proof, the sum can be taken over all elements of $\pi_1^{orb}(B)$ and so $e' \in V$. Note e' can be replaced by e (as defined in the statement of this theorem) because they are equal modulo $V = \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$. As stated earlier the group of monodromies is finite. Thus the necessity of this lemma.

To establish sufficiency of the conditions we will show that the groups can be realised by geometric Seifert manifolds and then invoke Theorem 1.8. Suppose (conversely) that the group of monodromies is finite and $e \in V$. Then by Theorem 4.1, $A(\xi) = P^{-1}\mathscr{O}(\xi)P$ for some $P \in GL_2\boldsymbol{R}$ and some group of orthogonal matrices $\{\mathscr{O}(\xi)|\xi \in \pi_1(S)\}$ and we then can choose some $z(\xi)$ to satisfy (18). Next, reverse the process to get a faithful representation of $\pi_1(S)$ as isometries of $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ defined (on the generators) by $\xi \mapsto (\bar{\xi}, (\mathscr{O}(\xi), y(\xi)))$ (by construction the map will be well-defined). Therefore $\pi_1(S)$ is isomorphic to the fundamental group of a geometric Seifert manifold. Then Theorem 1.8 implies S is homeomorphic to a Seifert geometric manifold of type $\boldsymbol{H}^2 \times \boldsymbol{E}^2$, hence the lemma. [Let us be more explicit how to choose the $z(\xi)$: If for some ξ', $(A(\xi') - \mathrm{w}(\bar{\xi}')I)$ is invertible then choose the $z(\xi)$ so that $z(\xi') = (A(\xi') - \mathrm{w}(\bar{\xi}')I)^{-1}e'$ while all the others are 0. Otherwise $\mathrm{w}(\bar{\xi})A(\xi)$ is I or conjugate in $GL_2\boldsymbol{R}$ to $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$. If for all generators $\mathrm{w}(\bar{\xi})A(\xi)$ is I or A for some A conjugate to $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ then let all $z(\xi) = 0$ except for one generator, ξ' (if there is one that is) such that $\mathrm{w}(\bar{\xi}')A(\xi') = A$, let $z(\xi') = -\mathrm{w}(\bar{\xi}')(1/2)e'$ (note in this case $e' \in \mathrm{Im}(A - I)$). If none of the previous cases arise, then there are two generators, ξ' and ξ'' say, such that $\mathrm{w}(\bar{\xi}')A(\xi')$ and $\mathrm{w}(\bar{\xi}'')A(\xi'')$ are not equal but both conjugate to $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ while for the rest of the generators $\mathrm{w}(\bar{\xi})A(\xi)$ is either I or conjugate to $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$. Let $z(\xi') = -\mathrm{w}(\bar{\xi}')(1/2)p$, $z(\xi'') = -\mathrm{w}(\bar{\xi}'')(1/2)(e' - p)$ where p is the intersection point of the lines $(e' + \mathrm{Im}(A(\xi'') - \mathrm{w}(\bar{\xi}'')I))$ and $\mathrm{Im}(A(\xi') - \mathrm{w}(\bar{\xi}')I)$ (therefore $p \in \mathrm{Im}(A(\xi') - \mathrm{w}(\bar{\xi}')I)$ and $(e' - p) \in \mathrm{Im}(A(\xi'') - \mathrm{w}(\bar{\xi}'')I))$. For the other generators let $z(\xi) = 0$.] □

LEMMA 4.9 ($\widetilde{SL}_2 \times \boldsymbol{E}$ case of Theorem 4.7). *Let S be a Seifert manifold over a hyperbolic base, with general fibre T^2. Let the base, B, have k_0 cone points, so that m_{0j} is the order of the jth cone point, and l reflector curves, such that the ith reflector curve has k_i corner reflectors so that m_{ij} is the order of the jth corner reflector on the ith reflector curve. Let A be standard map which induces the monodromy map and let $\tilde{a}$, $\tilde{b}_i$, $\tilde{e}_{0j}$, $\tilde{f}_i$, and $\tilde{g}_{ij}$ be the standard parts of the presentation of $\pi_1(S)$.*

Let $e = \tilde{a} + \sum_{j=1}^{k_0} \tilde{e}_{0j}/m_{0j} + \frac{1}{2}\sum_{i=1}^{l}\Big(\tilde{b}_i + \sum_{j=1}^{k_i} \tilde{e}_{ij}/m_{ij}\Big)$ *and* $V = \mathscr{I}_{\mathrm{w}}\boldsymbol{Q}^2$.

Then S *is geometric of type* $\widetilde{SL}_2 \times \boldsymbol{E}$ *if and only if the group of monodromies is finite and* $e \notin V$.

Proof. The proof of this lemma is very similar to the previous lemma. So instead of repeating the proof we will explain the differences. Assume first S is geometric of type $\widetilde{SL}_2 \times \boldsymbol{E}$. Then as in the $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ case, the model space may be identified with $\boldsymbol{H}^2 \times \boldsymbol{C}$ (see Section 3). The generators again act via translations, while the other generators ξ act on $\boldsymbol{C}$ as $(\mathscr{O}(\xi), y(\xi))$ plus a correction term which corresponds to the parallel translation of $\bar{\xi}$, as discussed in Section 3. As before we look at how each of the relations act on the $\boldsymbol{C}$ factor. The differences all arise from these correction terms. The monodromy and cone point relations again give equations (13) and (14).

Let $\bar{\mathscr{A}}_i$ be an annular neighbourhood of the ith reflector curve. By considering the relations connected to the reflector curves (or more precisely these neighbourhoods) we can obtain the following equations for $i = 1, \ldots, l$ which are similar to (15) (the details of this are also deferred to Lemma 5.1):

$$P^{-1}y(\partial_i) + \frac{1}{2}\left(\tilde{b}_i + \sum_{j=1}^{k_i} \tilde{e}_{ij}/m_{ij}\right) - P^{-1}\begin{pmatrix} 0 \\ c_i \end{pmatrix} = \sum_{\xi}(A(\xi) - \mathrm{w}(\bar{\xi})I)z(\xi), \qquad (19)$$

where c_i is 2π multiplied by the orbifold euler characteristic of $\bar{\mathscr{A}}_i$, ie $c_i = 2\pi\chi^{orb}(\bar{\mathscr{A}}_i)$, and (as before) the sum on the right is taken over the following generators: $\{\partial_i, r_i, s_{i1}, \ldots, s_{ik_i}\}$. The term involving c_i is the only difference and arises due to the correction terms and is connected to the holonomy of $\bar{\mathscr{A}}_i$ which we will explain in Lemma 5.1.

By considering how the last relations (the ones connected to the global information) act on the $\boldsymbol{C}$ factor, we can get equations similar to equations (16) and (17). The only difference is that we get an extra term $\left(\begin{smallmatrix} 0 \\ c_0 \end{smallmatrix}\right)$ on the left hand side. Suppose S has no reflector curves. Then c_0 corresponds to the parallel translation connected to this last relation, namely it is the parallel translation around a fundamental domain of the base. This is the total holonomy of the base which equals $2\pi\chi^{orb}(B)$. In general, c_0 is the total holonomy of the base with the neighbourhoods of the reflector curves taken out, which has value $2\pi\chi^{orb}\Big(B \setminus \Big(\bigcup_{i=1}^{l} \bar{\mathscr{A}}_i\Big)\Big)$. See Proof of Theorem B (1) for $X = \widetilde{SL}_2 \times \boldsymbol{E}$ in [**10**] (cases I and III in particular) for an explanation of this in the orientable case.

We can simplify all cases as before (using the same generators and same expression for e') to the following equation (analogous to (18)):

$$e' - P^{-1}\begin{pmatrix} 0 \\ c \end{pmatrix} = \sum (A(\xi) - \mathrm{w}(\bar{\xi})I)z(\xi) \tag{20}$$

where the sum is over all elements of $\pi_1^{orb}(B)$ (by Corollary 4.5) and $c = c_0 + \sum_{i=1}^{l} c_i = 2\pi\chi^{orb}(B)$ which is nonzero. Note e' (as before) can be replaced by e. Furthermore $\{\mathrm{w}(\bar{\xi})\mathscr{O}(\xi)\}$ is contained in the group $\langle(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})\rangle$ (from Section 3, $\mathscr{O}(\xi) = \begin{pmatrix} \mathrm{w}(\xi) & 0 \\ 0 & \mathrm{w}(\bar{\xi}) \end{pmatrix}$, so $\{\mathrm{w}(\bar{\xi})\mathscr{O}(\xi)\} = \left\{\begin{pmatrix} \mathrm{w}(\bar{\xi})\mathrm{w}(\xi) & 0 \\ 0 & 1 \end{pmatrix}\right\}$). Remember $\mathscr{O}(\xi) = PA(\xi)P^{-1}$. So the group $\{\mathrm{w}(\bar{\xi})A(\xi)\}$ is contained in a group $\langle A\rangle \subset GL_2\boldsymbol{Z}$ where $PAP^{-1} = (\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})$ for some $P \in GL_2\boldsymbol{R}$. There are two cases, either $\{\mathrm{w}(\bar{\xi})A(\xi)\}$ is $\{I\}$ or $\langle A\rangle$. In the former case, $e = P^{-1}(\begin{smallmatrix} 0 \\ c \end{smallmatrix})$, which is nonzero, and so $e \notin \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I) = 0$. In the latter case, we have $e - P^{-1}(\begin{smallmatrix} 0 \\ c \end{smallmatrix}) \in \mathrm{Im}(A - I)$. Now if $P^{-1}(\begin{smallmatrix} 0 \\ c \end{smallmatrix}) = (A - I)w$ for some $w \in \boldsymbol{R}^2$ then $(\begin{smallmatrix} 0 \\ c \end{smallmatrix}) = (PAP^{-1} - I)Pw = (\begin{smallmatrix} * \\ 0 \end{smallmatrix})$, which contradicts $c \neq 0$ and so in both cases $e \notin V = \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$. Hence the necessity of this lemma.

We shall now show these conditions are not only necessary but sufficient. Suppose the group of monodromies is finite and $e \notin \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$. For this to happen $\dim(\sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)) < 2$. This implies $\mathrm{w}(\bar{\xi})A(\xi)$ all have 1 as an eigenvalue and so are either I or a reflection. Since $A(\xi) \in GL_2\boldsymbol{Z}$, all the reflections will be the same since $\mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$ will be the common -1-eigenspace. So $\{\mathrm{w}(\bar{\xi})A(\xi)\}$ will be conjugate in $GL_2\boldsymbol{R}$ to a subgroup of $\langle(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})\rangle$. Suppose $\{\mathrm{w}(\bar{\xi})A(\xi)\} = \{I\}$ and $e \neq 0$, then let $z(\xi) = 0$ for all generators and choose P so that $e - P^{-1}(\begin{smallmatrix} 0 \\ c \end{smallmatrix}) = 0$ (possible since e and $c = 2\pi\chi^{orb}(B)$ are nonzero). Then we can reverse the process (as in the previous lemma) to show $\pi_1(S)$ is isomorphic to the fundamental group of $\widetilde{SL}_2 \times \boldsymbol{E}$ Seifert manifold. Then by Theorem 1.8, S is geometric of type $\widetilde{SL}_2 \times \boldsymbol{E}$. Alternatively suppose $\{\mathrm{w}(\bar{\xi})A(\xi)\}$ is conjugate to $\langle(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})\rangle$ and $e \notin \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$. Choose a matrix Q such that $Q\{\mathrm{w}(\bar{\xi})A(\xi)\}Q^{-1} = \langle(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})\rangle$. Then $Qe \notin \sum \mathrm{Im}(QA(\xi)Q^{-1} - \mathrm{w}(\bar{\xi})I) = \{(\begin{smallmatrix} x \\ 0 \end{smallmatrix}) | x \in \boldsymbol{R}\}$. Therefore $Qe = (\begin{smallmatrix} * \\ a \end{smallmatrix})$ for some nonzero number a. Let $P = \frac{c}{a}Q$ ($c = 2\pi\chi^{orb}(B)$). Then $P\{\mathrm{w}(\bar{\xi})A(\xi)\}P^{-1} = \langle(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})\rangle$ and $e - P^{-1}(\begin{smallmatrix} 0 \\ c \end{smallmatrix}) \in \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$. Choose a generator ξ' such that $\mathrm{w}(\bar{\xi}')A(\xi') = A$. Then let $z(\xi') = -\mathrm{w}(\bar{\xi}')(1/2)(e - P^{-1}(\begin{smallmatrix} 0 \\ c \end{smallmatrix}))$ and let $z(\xi) = 0$ for all the other generators. Consequently, equation (20) is satisfied. Again we can reverse the process to show S is geometric of type $\widetilde{SL}_2 \times \boldsymbol{E}$. Hence the lemma. □

Remark 4.10. Let $\eta : S \to B$ be a Seifert fibration with fibre F and aspherical base. Let M be $\mathscr{Z}(\pi_1(F)) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ considered as a $\boldsymbol{Z}[\pi_1^{orb}(B)]$-module with the action determined by the monodromy map. Let $e^{\boldsymbol{Q}}(\eta) \in H^2(\pi_1^{orb}(B), M)$ be the class corresponding to $\pi_1(S)$ as an extension of $\pi_1(S)/\mathscr{Z}(\pi_1(F))$ by $\mathscr{Z}(\pi_1(F))$

[note $\pi_1(S)/\mathscr{Z}(\pi_1(F)) \cong \pi_1^{orb}(B)$ when the general fibre is a torus]. Then $e^{\boldsymbol{Q}}(\eta)$ is called the rational euler class of the fibration. When the general fibre is the torus, it can be shown that $H^2(\pi_1^{orb}(B), M) \cong M/I_{\mathrm{w}}M(= \boldsymbol{Q}^2/V)$ and via this isomorphism $e^{\boldsymbol{Q}}(\eta)$ gets mapped to $e \bmod V$.

At the end of Section 13.4 in [**7**] the rational euler class for a Seifert 3-manifold is defined as the obstruction to the existence of a rational section. Can the rational euler class be similarly defined for 4-manifolds? In [**4**], Neumann and Raymond construct the rational euler class for 3-manifolds more explicitly and prove a naturality result for finite covers (see especially their Theorem 1.2). The definition of the e used in the above theorem is derived from this latter construction.

5. Reflector curves.

In order to complete Theorem 4.7 we will consider what is happening in the neighbourhood of a reflector curve. Consider a Seifert bundle $\eta : \bar{\mathscr{R}} \to \bar{\mathscr{A}}$ with T^2 general fibre, above an annulus neighbourhood of a reflector curve. We are trying to determine if there is, and if so what is the nature of, any injective homomorphism $\pi_1(\bar{\mathscr{R}}) \to \mathrm{Isom}(X)$ which preserves the bundle structure (where X is $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ or $\widetilde{SL}_2 \times \boldsymbol{E}$). More precisely, given an injection $i : \pi_1^{orb}(\bar{\mathscr{A}}) \to \mathrm{Isom}(\boldsymbol{H}^2)$, we would like to determine all injections $\tilde{i} : \pi_1(\bar{\mathscr{R}}) \to \mathrm{Isom}(X)$ which makes the following diagram commute:

$$\begin{array}{ccc} \pi_1(\bar{\mathscr{R}}) & \xrightarrow{\ \tilde{i}\ } & \mathrm{Isom}(X) \\ {\scriptstyle \pi_1(\eta)}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \pi_1^{orb}(\bar{\mathscr{A}}) & \xrightarrow[\ i\ ]{} & \mathrm{Isom}(\boldsymbol{H}^2), \end{array}$$

Fix a presentation of $\pi_1(\bar{\mathscr{R}})$ as given in Lemma 1.5. Then as we saw in Lemmas 4.8 and 4.9, if $\tilde{i}$ exists, $\tilde{i}(h_j)(z, w) = (z, w + p_j)$ for some linearly independent $p_j \in \boldsymbol{R}^2$. As before let P be the matrix in $GL_2\boldsymbol{R}$ whose columns are these p_j. Also from before $\tilde{i}(\partial)(z, w) = (i(\bar{\partial})(z), (PA(\partial)P^{-1}, y(\partial))(w) + \binom{0}{c(\partial)})$ for some $y(\partial) \in \boldsymbol{R}^2$, where $A(\partial)$ is the monodromy and where $c(\partial) = 0$ if $X = \boldsymbol{H}^2 \times \boldsymbol{E}^2$ and $c(\partial)$ corresponds to the parallel translate of $i(\bar{\partial})$ if $X = \widetilde{SL}_2 \times \boldsymbol{E}$.

The following lemma determines necessary and sufficient conditions on P and $y(\partial)$ for $\tilde{i}$ to exist.

Lemma 5.1. *Let $\bar{\mathscr{A}}$ be an annulus neighbourhood of a reflector curve and suppose $\eta : \bar{\mathscr{R}} \to \bar{\mathscr{A}}$ is a Seifert bundle with general fibre T^2. Suppose the reflector curve has k corner reflectors, so that the jth corner reflector has order m_j. Then an injection $i : \pi_1^{orb}(\bar{\mathscr{A}}) \to \mathrm{Isom}(\boldsymbol{H}^2)$, lifts to an injection $\tilde{i} : \pi_1(\bar{\mathscr{R}}) \to \mathrm{Isom}(X)$ (where X is either $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ or $\widetilde{SL}_2 \times \boldsymbol{E}$) defined as above if and only if*

1. $P\{A(\xi)|\xi \in \pi_1(\bar{\mathscr{R}})\}P^{-1} \subset O_2\boldsymbol{R}$ *when* $X = \boldsymbol{H}^2 \times \boldsymbol{E}^2$ *and* $P\{\mathrm{w}(\bar{\xi})A(\xi)|\xi \in \pi_1(\bar{\mathscr{R}})\}P^{-1} \subset \langle\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\rangle$ *when* $X = \widetilde{SL}_2 \times \boldsymbol{E}$, *and*
2. $$P^{-1}y(\partial) + \frac{1}{2}\left(\tilde{b} + \sum_{j=1}^{k} \tilde{e}_j/m_j\right) - P^{-1}\begin{pmatrix}0\\ c\end{pmatrix} \in \sum_{\bar{\xi} \in \pi_1^{orb}(\bar{\mathscr{A}})} \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I),$$

where $c = 0$ when $X = \boldsymbol{H}^2 \times \boldsymbol{E}^2$ and $c = 2\pi\chi^{orb}(\bar{\mathscr{A}})$ when $X = \widetilde{SL}_2 \times \boldsymbol{E}$.

Proof. The first condition (connected to P) in the statement of this lemma was proved to be necessary in part of Lemmas 4.8 and 4.9. So we will first show that the above condition on $y(\partial)$ is necessary. Then we will show that all steps are reversible and hence show the condition is sufficient.

Thus firstly, suppose $i : \pi_1(\bar{\mathscr{A}}) \to \mathrm{Isom}(\boldsymbol{H}^2)$ does lift to a homomorphism $\tilde{i} : \pi_1(\bar{\mathscr{R}}) \to \mathrm{Isom}(X)$. By the way $\tilde{i}$ was defined above, the induced map $\pi_1(T^2) \to \ker p$ is injective (where p is the projection $\mathrm{Isom}(X) \to \mathrm{Isom}(\boldsymbol{H}^2)$). It follows that $\tilde{i}$ is injective.

Since $\tilde{i}(\xi)$ is an isometry of X for $\xi \in \pi_1(\bar{\mathscr{R}})$, we have $\tilde{i}(\xi)(z,w) = (i(\bar{\xi})(z), (PA(\xi)P^{-1}, y(\xi))(w) + \left(\begin{smallmatrix}0\\ c(\xi)\end{smallmatrix}\right))$ for some $y(\xi) \in \boldsymbol{R}^2$, where $\bar{\xi} = \pi_1(\eta)(\xi)$ and where $c(\xi) = 0$ if $X = \boldsymbol{H}^2 \times \boldsymbol{E}^2$ and $c(\xi)$ corresponds to the parallel translate of $i(\bar{\xi})$ if $X = \widetilde{SL}_2 \times \boldsymbol{E}$.

By considering the relations $(\sigma_j\sigma_{j+1}^{-1})^{m_j}\tilde{e}_j = 1$ which are more neatly expressed as $s_j^{m_j}\tilde{e}_j = 1$, as we did in Section 4, we see equation (14)) is satisfied, and so $P^{-1}y(s_j) + \tilde{e}_j/m_j = (A(s_j) - I)z(s_j)$ for some $z(s_j) \in \boldsymbol{R}^2$. Similarly, $P^{-1}y(r) + \tilde{f}/2 = (A(r) - I)z(r)$, for some $z(r) \in \boldsymbol{R}^2$.

Recall $\sigma_j = \prod_{p=j}^{k} s_p$. Let $E_j = \sum_{p=j}^{k} A(\sigma_p^{-1})\tilde{e}_p/m_p$. By combining the expressions for $P^{-1}y(s_j)$ it can be shown that $P^{-1}y(\sigma_j) + A(\sigma_j)E_j = -A(\sigma_j)x(\sigma_j)$ for some $x(\sigma_j) = -\sum_{p=j}^{k} A(\sigma_p^{-1})(A(s_p) - I)z(s_p) = -\sum_{p=j}^{k}(A(\sigma_{p+1}^{-1}\sigma_p) - I)A(\sigma_p^{-1})z(s_p) \in \sum_{p=j}^{k} \mathrm{Im}(A(\sigma_{p+1}^{-1}\sigma_p) - I)$. However the group generated by $\sigma_{p+1}^{-1}\sigma_p$, $p = j, \dots, k$ is also generated by σ_p, $p = j, \dots, k$ (since $\sigma_p = \left(\prod_{q=p}^{k}(\sigma_{q+1}^{-1}\sigma_q)^{-1}\right)^{-1}$ and $\sigma_{k+1} = 1$), so by Corollary 4.5, $x(\sigma_j) \in \sum \mathrm{Im}(A(\xi) - I)$ where the sum is over $\xi \in \langle\sigma_j, \dots, \sigma_k\rangle$.

Next by considering the relation $\partial^{-1}\sigma_1 r\partial r^{-1} = \tilde{b}$ in a similar way to getting equation (16), we get:

$$A(\sigma_1 r) = A(\partial)A(r)A(\partial^{-1}) \tag{21}$$

$$\begin{aligned}&(I - A(\sigma_1 r))P^{-1}y(\partial) + (I - A(\partial))A(r)P^{-1}y(r)\\ &\quad + A(\sigma_1 r)P^{-1}y(\sigma_1) + A(\sigma_1 r\partial)P^{-1}\begin{pmatrix}0\\ c\end{pmatrix} = A(\sigma_1 r\partial)\tilde{b}\end{aligned} \tag{22}$$

Here c is 0 when $X = \boldsymbol{H}^2 \times \boldsymbol{E}^2$ and c is a correction term corresponding to the holonomy of the loop projected to the base when $X = \widetilde{SL}_2 \times \boldsymbol{E}$. By considering the double cover of $\bar{\mathscr{R}}$ induced by doubling the base along the reflector curve, we see the above relation is preserved (except $r\partial r^{-1}$ is now a loop corresponding to a lift of the other boundary). Projecting the relation to the base, we see it corresponds to traversing the base, thus the correction is the holonomy of the base: $\mathscr{A}$, that is $c = 2\pi\chi^{orb}(\mathscr{A})$. By definition of the Euler characteristic, $c = 4\pi\chi^{orb}(\bar{\mathscr{A}})$.

We can rewrite (22) by using the expressions for $P^{-1}y(r)$, $P^{-1}y(\sigma_1)$, equations (8), (9) and (10) and the equation $A(\xi)P^{-1}\binom{0}{c} = P^{-1}\binom{0}{\mathrm{w}(\bar{\xi})c}$. Therefore, we get:

$$\begin{aligned}&(I - A(\sigma_1 r))\Bigg(P^{-1}y(\partial) + A(\partial)\tilde{b}/2 - A(r)E_1/2 - P^{-1}\begin{pmatrix}0\\ c/2\end{pmatrix}\\ &\qquad - \tilde{f}/4 - A(r)x(\sigma_1)/2 - A(\partial)z(r) - (A(r) - I)z(r)/2\Bigg)\\ &= (I + A(\sigma_1 r))\big(\tilde{G}_1/4 + A(r)E_1/2 + A(r)x(\sigma_1)/2 + (A(r) - I)z(r)/2\big)\end{aligned} \tag{23}$$

By multiplying both sides by $(I - A(\sigma_1 r))$ we see that both sides are equal to zero. Therefore

$$\begin{aligned}&\Bigg(P^{-1}y(\partial) + A(\partial)\tilde{b}/2 - A(r)E_1/2 - P^{-1}\begin{pmatrix}0\\ c/2\end{pmatrix}\\ &\qquad - \tilde{f}/4 - A(r)x(\sigma_1)/2 - A(\partial)z(r) - (A(r) - I)z(r)/2\Bigg) \in \ker(I - A(\sigma_1 r)),\end{aligned}$$

and

$$\big(\tilde{G}_1/4 + A(r)E_1/2 + A(r)x(\sigma_1)/2 + (A(r) - I)z(r)/2\big) \in \ker(I + A(\sigma_1 r)).$$

However, $\ker(I - A(\sigma_1 r)) = \mathrm{Im}(I + A(\sigma_1 r))$ and $\ker(I + A(\sigma_1 r)) = \mathrm{Im}(I - A(\sigma_1 r))$. So by rearranging and using equation (8), we get the following equations:

$$P^{-1}y(\partial) + A(\partial)\tilde{b}/2 - A(r)E_1/2 - P^{-1}\begin{pmatrix} 0 \\ c/2 \end{pmatrix}$$
$$= A(r)x(\sigma_1)/2 + (A(\partial) - I)z(r) + (A(r) + I)(z(r)/2 + \tilde{f}/8) + (A(\sigma_1 r) + I)z(\partial) \tag{24}$$

$$\tilde{G}_1/4 + A(r)E_1/2 + A(r)x(\sigma_1)/2 + (A(r) - I)z(r)/2 = (I - A(\sigma_1 r))z(d), \tag{25}$$

for some $z(\partial)$ and $z(d) \in \boldsymbol{R}^2$.

By writing $A(\partial)\tilde{b} - A(r)E_1$ as $\tilde{b} + E_1 + (A(\partial) - I)\tilde{b} - (A(r) + I)E_1$, $(A(r) + I)$ as $(A(\sigma_1 r) + I)A(\sigma_1) + (I - A(\sigma_1))$ and for all j, $A(r)(A(\sigma_j) - I)$ as $(I - A(\sigma_j))A(\sigma_j^{-1} r)$ we see (24) becomes

$$P^{-1}y(\partial) + \frac{1}{2}\left(\tilde{b} + E_1\right) - P^{-1}\begin{pmatrix} 0 \\ c/2 \end{pmatrix} \in \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I), \tag{26}$$

where the sum is over a set of generators: $\{\sigma_1, \ldots, \sigma_k, \partial, \sigma_1 r\}$, and so by Corollary 4.5 can be taken to be over $\pi_1^{orb}(\bar{\mathscr{A}})$. Observe, when $X = \widetilde{SL}_2 \times \boldsymbol{E}$, $\frac{1}{2}c = 2\pi\chi^{orb}(\bar{\mathscr{A}})$. Also E_1 and $\sum_{j=1}^{k} \tilde{e}_j/m_j$ are equal modulo $\sum_\xi (A(\xi) - \mathrm{w}(\bar{\xi})I)$.

This shows the necessity of the lemma. To show sufficiency we will show this method is reversible. The only step which does not immediately seem to be reversible is the last step. So to complete the lemma, we need to show that for any $y(\partial)$ which satisfies equation (26), there are some $x(\sigma_1) \in \sum_{j=1}^{k} \mathrm{Im}(A(\sigma_j) - I)$, $z(r)$, $z(\partial)$ and $z(d)$ which satisfy both equation (24) and equation (25). That is, for all $v \in \mathscr{S} = \sum \mathrm{Im}(A(\xi) - \mathrm{w}(\bar{\xi})I)$, there are some $x(\sigma_1)$, $z(r)$, $z(\partial)$ and $z(d)$ such that

$$A(r)x(\sigma_1)/2 + (A(\partial) - I)z(r) + (A(r) + I)z(r)/2 + (A(\sigma_1 r) + I)z(\partial) = v \tag{27}$$
$$A(r)x(\sigma_1)/2 + \tilde{G}_1/4 + A(r)E_1/2 + (A(r) - I)z(r)/2 + (A(\sigma_1 r) - I)z(d) = 0 \tag{28}$$

In $\pi_1(\bar{\mathscr{R}})$, $\partial^{-1}\sigma_1 r \partial r^{-1} = \tilde{b}$. By considering its image by the monodromy map we see $A(\sigma_1)$ is the commutator $[A(\partial), A(r)]$ (and we also derive equation (21)). Since $\{A(\xi) | \xi \in \pi_1(S)\}$ is a dihedral group (by Theorem 4.1), $A(\sigma_1) \neq I$ implies $(A(\sigma_1) - I)$ is invertible (Lemma 5.2).

Suppose $\sum_{j=1}^{k} \mathrm{Im}(A(\sigma_j) - I) = \boldsymbol{R}^2$. Let $z(r) = 0$, and $z(d) = z(\partial) = (1/2)\big(v + \tilde{G}_1/4 + A(r)E_1/2\big)$ and then choose $x(\sigma_1)$ to satisfy equation (28) (note $x(\sigma_1)$ can take any value). Equation (27) is then satisfied and so the result is proved in this case.

Instead suppose $\sum_{j=1}^{k} \mathrm{Im}(A(\sigma_j) - I) \neq \boldsymbol{R}^2$, so in particular $A(\sigma_1) = I$. Notice $z(r)$ is independent of (25) since we can absorb it into $z(d)$. If $\sum_{j=1}^{k} \mathrm{Im}(A(\sigma_j) - I) \subset \mathrm{Im}(A(\partial) - I) + \mathrm{Im}(A(r) + I)$, first we choose $x(\sigma_1)$ and $z(d)$ to satisfy (28), then choose $z(r)$ and $z(\partial)$ to get (27) and the result. [Note a choice for $x(\sigma_1)$ and $z(d)$ is always possible. To see this consider equation (7) divided by $4m_j$ and taken mod $\mathrm{Im}(A(r) - I) + \sum_{p=1}^{k} \mathrm{Im}(A(\sigma_p) - I)$. By summing the set of equations over j, we see $\tilde{G}_1/4 + A(r)E_1/2$ is in $\mathrm{Im}(A(r) - I) + \sum_{p=1}^{k} \mathrm{Im}(A(\sigma_p) - I)$.]

Alternatively suppose $\sum_{j=1}^{k} \mathrm{Im}(A(\sigma_j) - I) \not\subset \mathrm{Im}(A(\partial) - I) + \mathrm{Im}(A(r) + I)$. Then firstly $\mathrm{Im}(A(\partial) - I) + \mathrm{Im}(A(r) + I) \neq \boldsymbol{R}^2$. Therefore $A(r) \neq I$ and so $A(r) = Q\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)Q^{-1}$ for some $Q \in GL_2\boldsymbol{R}$ [Suppose (aiming for contradiction) that $A(r) - I = P^{-1}(\mathscr{O}(r) - I)P$ is invertible. Then $(\mathscr{O}(r), y(r))$ has a fixed point $(I - \mathscr{O}(r))^{-1}y(r)$. However $(\mathscr{O}(r), y(r))$ cannot have a fixed point since $\bar{\mathscr{R}}$ is a manifold and so $r : (z, w) \mapsto (i(\bar{r})z), (\mathscr{O}(r), y(r))w$ must act freely, hence the contradiction]. Also $\mathrm{Im}(A(\partial) - I) \subseteq \mathrm{Im}(A(r) + I) = Q\{\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) | x \in \boldsymbol{R}\}$. However since $A(\sigma_1) = I$, $A(\partial)$ commutes with $A(r)$ and so $A(\partial) = I$ or $-A(r)$. In either case, the expression $(A(\partial) - I)z(r) + (A(r) + I)z(r)/2 + (A(\sigma_1 r) + I)z(\partial)$ from equation (27) is contained in $\mathrm{Im}(A(r) + I)$ so we may as well take $z(r) = 0$. Also we have assumed $\sum_{j=1}^{k} \mathrm{Im}(A(\sigma_j) - I) \not\subset \mathrm{Im}(A(r) + I) = Q\{\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) | x \in \boldsymbol{R}\}$.

Suppose $\sum_{j=1}^{k} \mathrm{Im}(A(\sigma_j) - I) \supseteq \mathrm{Im}(A(r) - I)$. Hence $\mathscr{S} = \boldsymbol{R}^2$ and $\sum_{j=1}^{k} \mathrm{Im}(A(\sigma_j) - I) = \mathrm{Im}(A(r) - I)$ because of our earlier assumption that $\sum_{j=1}^{k} \mathrm{Im}(A(\sigma_j) - I) \neq \boldsymbol{R}^2$. We choose $x(\sigma_1)$ and $z(\partial)$ to satisfy equation (27) and then choose $z(d)$ to satisfy equation (28) [which is again possible since $\tilde{G}_1/4 + A(r)E_1/2$ is in $\mathrm{Im}(A(r) - I) + \sum_{p=1}^{k} \mathrm{Im}(A(\sigma_p) - I) = \mathrm{Im}(A(r) - I)$]. Lastly suppose (aiming for a contradiction) $\sum_{j=1}^{k} \mathrm{Im}(A(\sigma_j) - I)$ does not contain $\mathrm{Im}(A(r) - I)$ and is not contained in $\mathrm{Im}(A(r) + I)$. Then $\sum_{j=1}^{k} \mathrm{Im}(A(\sigma_j) - I)$ is 1-dimesional and does not equal $\mathrm{Im}(A(r) - I)$ or $\mathrm{Im}(A(r) + I)$. Therefore there is a p such that $A(\sigma_p) \neq I$. Choose p to be the largest such number. Then $A(\sigma_p) = A(s_p)A(\sigma_{p+1}) = A(s_p)$ which is conjugate in $GL_2\boldsymbol{R}$ to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ (for the same reason $A(r)$ is, in the previous paragraph). However $A(r)A(\sigma_p)A(r)^{-1} = A(\sigma_p)^{-1}$, and therefore $A(\sigma_p) = \pm A(r)$. Both cases give the desired contradiction.

To completely show the representation of $\tilde{i}$ it is desirable to show how to obtain $x(\sigma_j)$ for $j > 2$. From earlier, we defined $x(\sigma_j)$ via the equation:

$$x(\sigma_j) = -\sum_{p=j}^{k} (A(\sigma_{p+1}^{-1}\sigma_p) - I)A(\sigma_p^{-1})z(s_p). \tag{29}$$

In all the above cases we have $x(\sigma_1)$. Choose values for $z(s_p)$, $p = 1 \dots k$ to satisfy equation (29) (there are no restrictions on the $z(s_p)$ so a choice is always possible). To conclude, define $x(\sigma_j)$ for $j > 2$ using these values and equation (29). □

LEMMA 5.2. *Suppose G is a dihedral group of order $2n$, with presentation $\langle r, s | rsr^{-1} = s^{-1}, r^2 = 1 = s^n \rangle$. The commutator subgroup G' is then $\langle s^2 | s^n = 1 \rangle$ which is cyclic of order $n/2$ if n even, or cyclic of order n if n odd. Incidentally, G/G' is then $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ or $\boldsymbol{Z}_2$ when n is even or odd respectively.*

In particular, if $G = PO_2\boldsymbol{Z}P^{-1} \cong D_8$, then $G' = \{\pm I\}$, also if $G = P\langle \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle P^{-1} \cong D_{12}$, then $G' = P\langle \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right) \rangle P^{-1}$.

PROOF. Direct calculation will quickly get the result. Alternatively, the abelianisation of G can be seen to have the extra relation $s^2 = 1$, therefore G' is generated by s^2. □

6. Geometric Klein bottled fibred 4-manifolds.

In this section, we will consider the case when the fibres are Klein bottles. This means the base has no corner reflectors [a singular fibre above a corner reflector is covered by the general fibre with dihedral covering group, however the only manifold which Kb covers is itself and the covering group is cyclic].

In the following, we suppose $\pi_1(Kb) = \langle h_1, h_2 | h_1 h_2 h_1^{-1} h_2 = 1 \rangle$.

THEOREM 6.1. *Let S be a Seifert manifold over a hyperbolic base B, with general fibre Kb. Then S is geometric. Suppose that the base has l reflector curves and k cone points, so that m_i is the order of the ith cone point. Let $A(\xi)$ be the automorphisms from the presentation of $\pi_1(S)$ which send h_1 to $h_1^{\epsilon(\xi)} h_2^{c(\xi)}$ and h_2 to $h_2^{\delta(\xi)}$ and let $\tilde{e}_i = h_1^{e_{i1}} h_2^{e_{i2}}$, $\tilde{b}_i = h_1^{b_{i1}} h_2^{b_{i2}}$ and $\tilde{a} = h_1^{a_1} h_2^{a_2}$ be the standard parts of the presentation of $\pi_1(S)$.*

Let $e = a_1 + \sum_{i=1}^{k} e_{i1}/m_i + \frac{1}{2}\sum_{i=1}^{l} b_{i1}$ and let $V = \sum_{\bar{\xi} \in \pi_1^{orb}(B)} \mathrm{Im}(\epsilon(\xi) - \mathrm{w}(\bar{\xi}))$. More precisely, S is geometric of type $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ if and only if

$$e \in V.$$

S is geometric of type $\widetilde{SL}_2 \times \boldsymbol{E}$ if and only if

$$e \notin V,$$

ie $\epsilon(\xi) = \mathrm{w}(\bar{\xi})$ *and* $e \neq 0$.

This can be proved by repeating the method used in the T^2 fibre case (Theorem 4.7). However we will prove it a different way using the orientation cover.

Proof. The basic idea is to first show that the orientation cover is geometric. We will then add one more isometry to show that $\pi_1(S)$ is isomorphic to a group of isometries thus showing S is geometric, by Theorem 1.8.

Let S' be the orientation cover of S, which will also be a Seifert manifold. Since $D^2 \times Kb$ is non-orientable, S' cannot have Kb as the general fibre, and so has T^2 fibres. Let B' be the base of S'. Then the degree of the covering of $S' \to S$ equals the product of the degrees of the coverings $T^2 \to Kb$ and $B' \to B$. However S' is an orientation cover, thus the product is 2, so the first is 2 and the second is 1, which means $B' = B$.

For each $\xi \in \pi_1(S)$, let $A(\xi)_T$ be the restriction of $A(\xi)$ to the unique maximal abelian subgroup (which corresponds to the T^2 which covers Kb). Then $A(\xi)_T = \left(\begin{smallmatrix} \epsilon(\xi) & 0 \\ 0 & \delta(\xi) \end{smallmatrix}\right)$. Note $A(h_1)_T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $A(h_2)_T = I$.

Now if ξ in $\pi_1(S)$ preserves orientation, it must preserve orientation on both fibre and base, or it must reverse the orientation of both. In general, $\mathrm{w}(\xi) = \mathrm{w}(\bar{\xi})\mathrm{w}(A(\xi)) = \mathrm{w}(\bar{\xi}) \det(A(\xi)_T) = \mathrm{w}(\bar{\xi})\epsilon(\xi)\delta(\xi)$. Therefore $\pi_1(S') = \{\xi \in \pi_1(S) | \delta(\xi) = \mathrm{w}(\bar{\xi})\epsilon(\xi)\}$ and $\pi_1(S) = \langle \pi_1(S'), h_1 \rangle$. The group of monodromies of $\pi_1(S')$ equals the image of $A(-)_T$ which is contained in the finite group $\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), -I \rangle$. Therefore, by Theorem 4.7 S' is geometric and so $\pi_1(S')$ is isometric to a group of isometries. Notably $\pi_1(F)$ acts via the following isometries: $h_1^2(w, z) = (w, z + P\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right))$ and $h_2(w, z) = (w, z + P\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right))$ for some $P \in GL_2\boldsymbol{R}$ such that $P \operatorname{Im} A(-)_T P^{-1}$ is in $O_2\boldsymbol{R}$ (more precisely, when the geometry is $\widetilde{SL}_2 \times \boldsymbol{E}$, $PA(\xi)_T P^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{w}(\bar{\xi}) \end{smallmatrix}\right)$). We claim that by adjusting P as necessary, the map $(w, z) \mapsto (w, P\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)P^{-1}z + P\left(\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right))$ is an isometry whose square equals the action of h_1^2. Thus we can define the action of h_1 by this map and so $\pi_1(S)$ is isomorphic to a group of isometries. By Theorem 1.8 this means S is geometric.

Before proving the claim, we will find conditions to separate the two geometries. The presentation of $\pi_1(S')$, in particular the standard relations, determine the geometry of S. To make these relations easier to find, we will consider a different presentation of $\pi_1(S)$. Now for each generator, ξ, of $\pi_1(S)$ which corresponds to a generator of $\pi_1^{orb}(B)$, either ξ or ξh_1 is in $\pi_1(S')$. By changing sections if necessary (and thus getting an isomorphic presentation), we can suppose $\xi \in \pi_1(S')$ for each of these generators. [Suppose $\epsilon(\xi) = \mathrm{w}(\bar{\xi})$ for all $\xi \in \pi_1(S)$. Note changing v_p to $v_p h_1$, t_p to $t_p h_1$, u_p to $u_p h_1$ or r_i to $r_i h_1$ does not change a_1, e_{i1} or b_{i1}. Changing s_i to $s_i h_1$ adds 1 to a_1 and subtracts m_i from e_{i1}. Changing ∂_i to $\partial_i h_1$ adds 1 to a_1 and subtracts 2 from b_i. So e and $e \bmod V$ are invariant by these changes of section. If $\epsilon(\xi) \neq \mathrm{w}(\bar{\xi})$ for some ξ, then $V = \boldsymbol{R}$ and so $e \bmod V$ is invariant by changes of section. Therefore changing the section, does not alter the conditions $e \in V$ and $e \notin V$.]

Next, consider a relation in $\pi_1^{orb}(B)$: $\bar{\xi}_1 \bar{\xi}_2 \ldots \bar{\xi}_p = 1$. This lifts to a relation in $\pi_1(S)$ of the form $\xi_1 \xi_2 \ldots \xi_p = h_1^{\alpha_1} h_2^{\alpha_2}$. Now the ξ_i were chosen to be in $\pi_1(S')$, therefore $\epsilon(\xi_1 \xi_2 \ldots \xi_p) = \delta(\xi_1 \xi_2 \ldots \xi_p)\mathrm{w}(\bar{\xi}_1 \bar{\xi}_2 \ldots \bar{\xi}_p) = \delta(\xi_1 \xi_2 \ldots \xi_p)$. This implies $1 = \epsilon(h_1^{\alpha_1} h_2^{\alpha_2}) = \delta(h_1^{\alpha_1} h_2^{\alpha_2}) = (-1)^{\alpha_1}$, and so α_1 is even. So relations of this type, are also relations of $\pi_1(S')$. Consequently, for each standard part of the presentation of $\pi_1(S)$: $\tilde{a}, \tilde{e}_i, \tilde{b}_i$ etc., $\tilde{\alpha} = \binom{\alpha_1}{\alpha_2}$ in general, the corresponding part of $\pi_1(S')$ is given by $\tilde{\alpha}' = \binom{\alpha_1/2}{\alpha_2}$. Let $e' = \binom{a_1/2}{a_2} + \sum_{i=1}^{k} \binom{e_{i1}/2m_i}{e_{i2}/m_i} + (1/2)\sum_{i=1}^{l} \binom{b_{i1}/2}{b_{i2}}$ and $V' = \sum_{\xi \in \pi_1(S')} \mathrm{Im}(A(\xi)_T - \mathrm{w}(\bar{\xi})I) = \sum_{\xi \in \pi_1(S')} \mathrm{Im}(\epsilon(\xi) - \mathrm{w}(\bar{\xi})) \times \sum_{\xi \in \pi_1(S')} \mathrm{Im}(\epsilon(\xi) - 1)$.

We will now show $e'_2 = a_2 + \sum_{i=1}^{k} e_{i2}/m_i + (1/2)\sum_{i=1}^{l} b_{i2} \in V'_2 = \sum_{\xi \in \pi_1(S')} \mathrm{Im}(\epsilon(\xi) - 1)$. It is sufficient to show $e'_2 = 0$ when $\epsilon(\xi) = 1$ for all ξ, so suppose $\epsilon(\xi) = 1$ and so for $\xi \in \pi_1(S')$, $\delta(\xi) = \mathrm{w}(\bar{\xi})$. Recall $A(\xi)h_1 = h_1^{\epsilon(\xi)} h_2^{c(\xi)}$. Note $c(h_1) = 0$, $c(h_2) = -2$, $c(\xi\xi') = c(\xi) + \delta(\xi)c(\xi')$ and $c(\xi^{-1}) = -\delta(\xi)c(\xi)$. By considering the image of c of the relation, $s_i^{m_i} h_1^{e_{i1}} h_2^{e_{i2}} = 1$, we see $m_i c(s_i) - 2e_{i2} = 0$, or equivalently

$$e_{i2}/m_i = c(s_i)/2.$$

By instead taking the relation $\partial_i^{-1} r_i \partial_i r_i^{-1} = h_1^{b_{i1}} h_2^{b_{i2}}$, we see $-c(\partial_i) + c(r_i) - c(\partial_i) - c(r_i) = -2b_{i2}$ (note $\delta(r_i) = \mathrm{w}(\bar{r}_i) = -1$) which reduces to

$$\frac{1}{2} b_{i2} = c(\partial_i)/2.$$

Lastly we will look at the image of c of the relation connected to the global information; $d \prod_{j=1}^{k} s_j \prod_{i=1}^{l} \partial_i = h_1^{a_1} h_2^{a_2}$, where $d = \prod_{p=1}^{g} [t_p, u_p]$ when the base is orientable and $d = \prod_{p=1}^{g} v_p^2$ when the base is non-orientable. Note $c(d) = 0$, so we get $\sum_{j=1}^{k} c(s_j) + \sum_{i=1}^{l} c(\partial_i) = -2a_2$ which can rearrange and then use the above equations to get:

$$0 = a_2 + \sum_{j=1}^{k} c(s_j)/2 + \sum_{i=1}^{l} c(\partial_i)/2$$

$$= a_2 + \sum_{j=1}^{k} e_{i2}/m_i + \frac{1}{2}\sum_{i=1}^{l} b_{i2}$$
$$= e'_2.$$

Theorem 4.7 states, S' is geometric of type $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ ($\widetilde{SL}_2 \times \boldsymbol{E}$ respectively) if and only if $e' \in V'$ ($e' \notin V'$) which is satisfied if and only if $e = 2e'_1 \in V'_1 = V$ ($e \notin V$), since $e'_2 \in V'_2$. From above S is geometric and it will have the same geometry as S', so S is geometric of type $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ ($\widetilde{SL}_2 \times \boldsymbol{E}$ respectively) if and only if $e \in V$ ($e \notin V$), hence the theorem.

We now return to prove the claim: by adjusting P as necessary, the map $(w,z) \mapsto (w, P\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)P^{-1}z + P\left(\begin{smallmatrix}1/2\\ 0\end{smallmatrix}\right))$ is an isometry whose square equals the action of h_1^2. To prove the claim we must consider a few cases. If the geometry is $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ then we need to prove $P\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)P^{-1}$ is in $O_2\boldsymbol{R}$. Suppose $A(\xi)_T = \pm\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ for some ξ, then $P\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)P^{-1} = \pm PA(\xi)_T P^{-1} \in O_2\boldsymbol{R}$. Therefore suppose $\operatorname{Im} A(-)_T \subseteq \{\pm I\}$. Then we can conjugate the isometries by $(P^{-1}, 0)$ to get an isomorphic group of isometries. That is, we can suppose $P = I$, hence the claim. If the geometry is $\widetilde{SL}_2 \times \boldsymbol{E}$, then we need to prove P has the form $\left(\begin{smallmatrix}0 & \lambda\\ \mu & 0\end{smallmatrix}\right)$ for some nonzero real numbers λ and μ. However for this geometry, $PA(\xi)_T P^{-1} = \left(\begin{smallmatrix}1 & 0\\ 0 & \mathrm{w}(\bar{\xi})\end{smallmatrix}\right)$ for $\xi \in \pi_1(S')$ (since ξ is orientable), and from above $A(\xi)_T = \left(\begin{smallmatrix}\epsilon(\xi) & 0\\ 0 & \epsilon(\xi)\mathrm{w}(\bar{\xi})\end{smallmatrix}\right)$. If $\epsilon(\xi) = -1$ and $\mathrm{w}(\bar{\xi}) = 1$, then $A(\xi)_T = -I$ but $PA(\xi)_T P^{-1} = I$ which is a contradiction. Therefore, if $\epsilon(\xi) = -1$, then $\mathrm{w}(\bar{\xi}) = -1$. In this case, $A(\xi)_T = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$ and $PA(\xi)_T P^{-1} = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ and so $P = \left(\begin{smallmatrix}0 & \lambda\\ \mu & 0\end{smallmatrix}\right)$ as desired. Instead, suppose $\epsilon(\xi) = 1$ for all $\xi \in \pi_1(S')$ (indeed since $\epsilon(h_1) = 1$, we are assuming this for all $\xi \in \pi_1(S)$). If $\mathrm{w}(\bar{\xi}) = -1$ for some $\xi \in \pi_1(S')$, then $\operatorname{Im}(A(\xi)_T - \mathrm{w}(\bar{\xi})I) = \{\left(\begin{smallmatrix}x\\ 0\end{smallmatrix}\right) | x \in \boldsymbol{R}\}$ and so $V' \supseteq \{\left(\begin{smallmatrix}x\\ 0\end{smallmatrix}\right) | x \in \boldsymbol{R}\}$ and $V = \boldsymbol{R}$. However this is impossible since from above S' being geometric of type $\widetilde{SL}_2 \times \boldsymbol{E}$ implies $e \notin V$ and so $V \neq \boldsymbol{R}$. Therefore $\mathrm{w}(\bar{\xi}) = 1$ for all ξ, which means B is orientable. As a result $A(\xi)_T = I$ for all $\xi \in \pi_1(S')$. In this situation, P may not have desired form, however by choosing an isomorphic group of isometries, we can change P to have the desired form. Let $\mu = (2\pi\chi^{orb}(B))/(e'_1)$. An isomorphic action of $\pi_1(S')$ is defined by $h_1^2(w,z) = \Big(w, z + \left(\begin{smallmatrix}0\\ \mu\end{smallmatrix}\right)\Big)$, $h_2(w,z) = \Big(w, z + \left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)\Big)$, $s_{0i}(w,z) = s(\bar{s}_{0i})\Big(w, z - \left(\begin{smallmatrix}e_{i2}/m_i\\ \mu e_{i1}/2m_i\end{smallmatrix}\right)\Big)$, $t_i(w,z) = s(\bar{t}_i)(w,z)$ and $u_i(w,z) = s(\bar{u}_i)(w,z)$, where $s : \operatorname{Isom}(\boldsymbol{H}^2) \to \operatorname{Isom}(\widetilde{SL}_2 \times \boldsymbol{E})$ is the section defined in Section 3. For this group of isometries, $P = \left(\begin{smallmatrix}0 & 1\\ \mu & 0\end{smallmatrix}\right)$ hence the claim. □

Remark 6.2. In Remark 4.10, we defined M to be $\mathscr{Z}(\pi_1(F)) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ and $e^{\boldsymbol{Q}}(\eta) \in H^2(\pi_1^{orb}(B), M)$ to be the rational euler class. When the general fibre is the Klein bottle, it can be shown that $H^2(\pi_1^{orb}(B), M) \cong M/I_{\mathrm{w}}M (= \boldsymbol{Q}/V)$ and via this isomorphism $e^{\boldsymbol{Q}}(\eta)$ gets mapped to $e \bmod V$.

Thus Proposition 4.6, and Theorems 4.7 and 6.1 can be summarised by the following:

Theorem 6.3. *A manifold is geometric of type* $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ *if and only if it is a Seifert* 4*-bundle with finite group of monodromies, zero rational euler class and hyperbolic base.*

A manifold is geometric of type $\widetilde{SL}_2 \times \boldsymbol{E}$ *if and only if it is a Seifert* 4*-bundle with finite group of monodromies, nonzero rational euler class and hyperbolic base.*

7. Virtually Geometric Seifert Manifolds.

It is easily shown that if a non-orientable manifold is geometric then its orientation cover is also geometric and of the same type. However, the converse is not so clear. We will show that in our case a much stronger result is true: that Seifert manifolds with hyperbolic base which are finitely covered by a geometric manifold are themselves geometric. We call manifolds which are finitely covered by a geometric manifold, virtually geometric.

Theorem 7.1. *Let* S *be a Seifert* 4*-manifold over a hyperbolic base* B*, and let* $\hat{S}$ *be a finite cover of* S*.*

Then S *is geometric if and only if* $\hat{S}$ *is.*

That is, virtually geometric Seifert 4*-manifolds over a hyperbolic base are geometric.*

Proof. Note $\hat{S}$ is also a Seifert 4 manifold. If F denotes the fibre of S, and $\hat{F}$ and $\hat{B}$ denote the fibre and base of $\hat{S}$ respectively, then $\hat{F}$ (finitely) covers F and $\hat{B}$ (finitely) covers B.

Suppose first that $F = T^2$ and that $\hat{S}$ is geometric. We may assume without loss of generality that $\pi_1(\hat{S})$ is normal in $\pi_1(S)$. Hence $\pi_1(\hat{F})$ is also normal in $\pi_1(S)$, since it is characteristic in $\pi_1(\hat{S})$. If $\xi \in \pi_1(S)$ acts trivially on $\pi_1(\hat{F})$ then it acts trivially on $\pi_1(F)$, since the action of an automorphism of $\pi_1(F)$ is determined by its action on any subgroup of finite index.

Since the group of monodromies of $\hat{S}$ is finite then so is the group of monodromies of S and therefore S is geometric by Theorem 4.7.

If $F = Kb$ there is nothing to prove, since $\mathrm{Out}(\pi_1(F))$ is finite and so the theorem follows from Theorem 6.1.

The necessity of the condition is clear. □

Corollary 7.2. *Let* S *be a non-orientable Seifert* 4*-manifold over a hyperbolic base* B *and let its orientation cover be* $\hat{S}$*.*

Then S *is geometric if and only if* $\hat{S}$ *is.*

COROLLARY 7.3. *A Seifert 4-manifold S over a hyperbolic base is geometric if and only if it has a finite cover diffeomorphic to $\tilde{B} \times T^2$ or $M^3 \times S^1$ where $\tilde{B}$ is a hyperbolic surface and M^3 is a $\widetilde{SL}_2$ 3-manifold.*

PROOF. The manifolds $\tilde{B} \times T^2$ are geometric of type $\boldsymbol{H}^2 \times \boldsymbol{E}^2$ and the manifolds $M^3 \times S^1$ are geometric of type $\widetilde{SL}_2 \times \boldsymbol{E}$. So by the theorem, if S is finitely covered by one of these it is geometric. Conversely suppose S is geometric. Then it is finitely covered by such a manifold by Theorem 9.3 in [**2**]. □

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