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Stationary reflection and the club filter

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Abstract. Suppose that κ is a regular uncountable cardinal. It has been known that the club filter on $\mathscr{P}_{\omega_1}\kappa$ can be presaturated. In this paper we extend the result to the case of $\mathscr{P}_\mu\kappa$, where μ is a regular uncountable cardinal $\leq \kappa$. This involves suitably weakening the notion of presaturation. A new reflection principle for stationary sets in $\mathscr{P}_\kappa\lambda$ plays a key role.

1. Introduction.

In [**5**] Foreman, Magidor and Shelah established the following:

Theorem 1. *The club filter on ω_1 is presaturated in the extension by the Levy collapse of a supercompact cardinal to ω_2.*

Goldring [**10**] extended and refined Theorem 1 as follows:

Theorem 2. *Suppose that $\omega < \kappa < \nu$, κ is regular and ν is a Woodin cardinal. Then the club filter on $\mathscr{P}_{\omega_1}\kappa$ is presaturated in the extension by the Levy collapse* $\mathrm{Col}(\kappa, \nu)$.

Let us review the relevant notions quickly. See Section 2 for a more detailed exposition and [**13**] for background. Suppose that $\mu \leq \kappa < \nu$ are all regular uncountable cardinals. For a set A the set of all subsets of A of size less than μ is denoted by $\mathscr{P}_\mu A$. The club filter is the filter generated by the closed and unbounded sets. Supercompactness and Woodinness are two of the major large cardinal notions, with the former much stronger than the latter.

We abbreviate a partially ordered set as a poset. Each poset gives rise to generic extensions of the ground model (of ZFC set theory). The Levy collapse $\mathrm{Col}(\kappa, \nu)$ is the standard poset by which every $\alpha < \nu$ has size at most κ in the extension.

Suppose that F is a filter on $\mathscr{P}_\mu\kappa$. We say that a subset S of $\mathscr{P}_\mu\kappa$ is F-positive if $S \cap X \neq \emptyset$ for every $X \in F$. The set of all F-positive sets is denoted by F^+. We always view F^+ as the poset whose ordering is defined by:

$$T \leq S \text{ iff } T \cap X \subset S \text{ for some } X \in F.$$

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The setup enables us to weaken the notion of an ultrafilter:

Definition. F is presaturated if and only if every set of ordinals of size $|\kappa|$ in the extension by F^+ can be covered by some set of size κ in the ground model.

Originally Baumgartner and Taylor [**1**] defined presaturation as "precipitousness plus preservation of κ^+ as a cardinal". In general this is weaker than our "covering" property. However the new definition is more natural as a combinatorial property and is more common in the recent literature.

It is easy to see that the club filter is not an ultrafilter. In fact Shelah [**21**] proved that the club filter on $\mathscr{P}_{\omega_1}\kappa$ is not even 2^κ-saturated if $\kappa > \omega_1$ (see also [**26**]). In this case presaturation is the strongest property known to date that the club filter on $\mathscr{P}_{\omega_1}\kappa$ can have. In contrast Woodin [**29**] established that the club filter on ω_1 can be even ω_1-dense.

In this paper we extend Theorem 2 further to the case of $\mathscr{P}_\mu\kappa$. This involves suitably weakening the notion of presaturation. Indeed, extending a result of Shelah [**20**], Burke and Matsubara [**3**] showed that the club filter on $\mathscr{P}_\mu\kappa$ is not presaturated if $\omega_1 < \mu < \kappa$.

Definition. F is weakly presaturated if and only if every *countable* set of ordinals in the extension by F^+ can be covered by some set of size κ in the ground model.

It is easy to see that weak presaturation coincides with presaturation for a filter on $\mathscr{P}_{\omega_1}\kappa$. It turns out that weak presaturation has been called ω_1-presaturation [**9**] or ω-presaturation [**29**] as well.

Here is the main result of this paper:

Theorem 3. *Suppose that $\omega < \mu \leq \kappa < \nu$, μ and κ are both regular, and ν is 2^ν-supercompact. Then the club filter on $\mathscr{P}_\mu\kappa$ is weakly presaturated below the set $\{x \in \mathscr{P}_\mu\kappa : \operatorname{cf}\sup x = \omega\}$ in the extension by* $\operatorname{Col}(\kappa,\nu)$.

Here $\operatorname{cf}\gamma$ is the cofinality of an ordinal γ under the canonical well-ordering.

We prove Theorem 3 in Section 5. The key element of the proof is a new reflection principle for stationary sets in $\mathscr{P}_\kappa\lambda$. Here λ is a cardinal $> \kappa$. Recall from [**5**] the following:

Definition. Stationary Reflection in $\mathscr{P}_{\omega_1}\lambda$ holds if and only if for every stationary $S \subset \mathscr{P}_{\omega_1}\lambda$ there is $\omega_1 \subset A \subset \lambda$ of size ω_1 such that $S \cap \mathscr{P}_{\omega_1}A$ is stationary in $\mathscr{P}_{\omega_1}A$.

In what follows we write SR for Stationary Reflection.

In [**5**] it was shown that SR in $\mathscr{P}_{\omega_1}\lambda$ holds for every $\lambda > \omega_1$ in the model of Theorem 1. Todorčević proved in effect that SR in $\mathscr{P}_{\omega_1}\lambda$ with λ large enough implies the presaturation of the club filter on ω_1 (see [**2**]). Extending both results in Section 3, we reprove Theorem 2 under the stronger hypothesis that ν is supercompact.

Our proof of Theorem 3 follows the same pattern. It is more involved though. One reason is the lack of definitive "SR in $\mathscr{P}_\kappa\lambda$" in the case $\omega_1 < \kappa < \lambda$. Indeed it was shown in [**22**] that the most natural version of "SR in $\mathscr{P}_\kappa\lambda$" fails in this case. Consequently the new reflection principle is somewhat awkward. See Section 4 for the precise statement of the principle.

It is known by the results of [**7**], [**9**] that the club filter on an inaccessible cardinal or on the successor of a regular cardinal can be weakly presaturated. We do not know whether the club filter on $\mathscr{P}_\mu\kappa$ can be weakly presaturated in the case $\omega_1 < \mu < \kappa$. We do know, however, that Theorem 3 is optimal with respect to the size of the covering sets. This is proved in Section 6.

2. Preliminaries.

Throughout the paper κ is a regular uncountable cardinal and λ is a cardinal $\geq \kappa$. Suppose $f : [\lambda]^{<\omega} \to \mathscr{P}_{\omega_1}\lambda$. Here $[\lambda]^{<\omega}$ is the set of all finite subsets of λ. Define

$$C(f) = \{x \subset \lambda : f``[x]^{<\omega} \subset \mathscr{P}(x)\}.$$

For $x \subset \lambda$ the closure of x under f is denoted by $\mathrm{cl}_f\, x$. A set of the form $\mathscr{P}_\kappa\lambda \cap C(f)$ is called σ-club in $\mathscr{P}_\kappa\lambda$. The σ-club filter on $\mathscr{P}_\kappa\lambda$ is the filter generated by the σ-club sets in $\mathscr{P}_\kappa\lambda$. It is denoted by $\mathscr{C}^\sigma_{\kappa\lambda}$. It is easy to see that $\mathscr{C}^\sigma_{\kappa\lambda}$ is countably complete and is fine, i.e. $\{x \in \mathscr{P}_\kappa\lambda : \alpha \in x\} \in \mathscr{C}^\sigma_{\kappa\lambda}$ for every $\alpha < \lambda$. Moreover $\mathscr{C}^\sigma_{\kappa\lambda}$ is closed under diagonal intersections, i.e. if $\{X_\alpha : \alpha < \lambda\} \subset \mathscr{C}^\sigma_{\kappa\lambda}$, then $\triangle\{X_\alpha : \alpha < \lambda\} = \{x \in \mathscr{P}_\kappa\lambda : \forall\alpha \in x(x \in X_\alpha)\} \in \mathscr{C}^\sigma_{\kappa\lambda}$. We call such a filter normal. It is easy to see that $\mathscr{C}^\sigma_{\kappa\lambda}$ is the smallest filter on $\mathscr{P}_\kappa\lambda$ that is normal in this sense.

The club filter on $\mathscr{P}_\kappa\lambda$ is the filter generated by the σ-club filter on $\mathscr{P}_\kappa\lambda$ together with the set $\{x \in \mathscr{P}_\kappa\lambda : x \cap \kappa \in \kappa\}$. This is not the original definition of the club filter but is equivalent to it. For a proof see e.g. [**23**]. The club filter on $\mathscr{P}_\kappa\lambda$ is denoted by $\mathscr{C}_{\kappa\lambda}$. It is easy to see that $\mathscr{C}_{\kappa\lambda}$ is the smallest normal κ-complete filter on $\mathscr{P}_\kappa\lambda$. In particular $\mathscr{C}_{\omega_1\lambda} = \mathscr{C}^\sigma_{\omega_1\lambda}$.

The prototype of Lemma 4 can be found in the proof of Shelah [**19**] that Chang's conjecture holds in the model of Theorem 1. It has been exploited quite extensively in the subsequent works including [**2**], [**4**], [**5**], [**10**], [**11**], [**14**], [**17**], [**22**], [**26**], [**28**].

LEMMA 4. *Suppose that $\omega < \kappa < \nu < \lambda = 2^\nu$, κ is regular and D is σ-club in $\mathscr{P}_\kappa 2^\lambda$. List all functions from ν to $\mathscr{P}_{\omega_1}\lambda$ as $\{e_\alpha : \alpha < \lambda\}$. Then there is a map $d : [2^\lambda]^{<\omega} \to \mathscr{P}_{\omega_1}2^\lambda$ such that*

- *$\mathscr{P}_\kappa 2^\lambda \cap C(d) \subset D$ and*
- *if $z \in C(d)$ and $\xi < \nu$, then* $\mathrm{cl}_d(z \cup \{\xi\}) \cap \lambda = \bigcup\{e_\alpha(\xi) : \alpha \in z \cap \lambda\} = \mathrm{cl}_d((z \cap \lambda) \cup \{\xi\}) \cap \lambda.$

PROOF. By recursion on $n < \omega$ define

$$d_n : [2^\lambda]^{<\omega} \to \mathscr{P}_{\omega_1}2^\lambda \text{ and } \alpha_n : [2^\lambda]^{<\omega} \to \lambda$$

so that the following hold:

(1) $\mathscr{P}_\kappa 2^\lambda \cap C(d_0) \subset D \cap \{z \in \mathscr{P}_\kappa 2^\lambda : \forall\alpha \in z \cap \lambda(e_\alpha``(z \cap \nu) \subset \mathscr{P}(z))\}$,
(2) $e_{\alpha_n(a)} = \langle \mathrm{cl}_{d_n}(a \cup \{\xi\}) \cap \lambda : \xi < \nu\rangle$ and
(3) $d_{n+1}(a) = d_n(a) \cup \{\alpha_n(a)\}$ for every $a \in [2^\lambda]^{<\omega}$.

Define $d : [2^\lambda]^{<\omega} \to \mathscr{P}_{\omega_1} 2^\lambda$ by

$$d(a) = \bigcup\nolimits_{n<\omega} d_n(a).$$

We claim that d is as desired.

By the definition of d and (1) we have $\mathscr{P}_\kappa 2^\lambda \cap C(d) \subset \mathscr{P}_\kappa 2^\lambda \cap C(d_0) \subset D$. Suppose next $z \in C(d)$ and $\xi < \nu$. It suffices to show that

$$\mathrm{cl}_d(z \cup \{\xi\}) \cap \lambda \subset \bigcup \{e_\alpha(\xi) : \alpha \in z \cap \lambda\} \subset \mathrm{cl}_d((z \cap \lambda) \cup \{\xi\}) \cap \lambda.$$

For the first inclusion it suffices to show that

$$\begin{aligned} \mathrm{cl}_d(z \cup \{\xi\}) \cap \lambda &= \bigcup \{\mathrm{cl}_{d_n}(z \cup \{\xi\}) \cap \lambda : n < \omega\} \\ &= \bigcup \{\mathrm{cl}_{d_n}(a \cup \{\xi\}) \cap \lambda : n < \omega \wedge a \in [z]^{<\omega}\} \\ &= \bigcup \{e_{\alpha_n(a)}(\xi) : n < \omega \wedge a \in [z]^{<\omega}\} \\ &\subset \bigcup \{e_\alpha(\xi) : \alpha \in z \cap \lambda\}. \end{aligned}$$

The first equality follows from the definition of d and (3). The third equality follows from (2). For the last inclusion, fix $n < \omega$ and $a \in [z]^{<\omega}$. It suffices to show that

$$\alpha_n(a) \in d_{n+1}(a) \subset d(a) \subset z.$$

These follow from (3), the definition of d and $z \in C(d)$ respectively.

For the second inclusion, fix $\alpha \in z \cap \lambda$. It suffices to show that

$$e_\alpha(\xi) \subset \mathrm{cl}_d((z \cap \lambda) \cup \{\xi\}).$$

Note that $\mathrm{cl}_d((z \cap \lambda) \cup \{\xi\}) \in C(d) \subset C(d_0)$ by the definition of d. Since $\alpha, \xi \in \mathrm{cl}_d((z \cap \lambda) \cup \{\xi\})$, we get the desired inclusion by (1). $\square$

Suppose that μ is a regular uncountable cardinal $\leq \kappa$. Recall that a $\mathscr{C}_{\mu\lambda}$-positive set is called stationary in $\mathscr{P}_\mu \lambda$. Likewise a $\mathscr{C}^\sigma_{\mu\lambda}$-positive set is called σ-stationary in $\mathscr{P}_\mu \lambda$. It is easy to see that a σ-stationary subset of $\mathscr{P}_\mu \lambda$ is σ-stationary in $\mathscr{P}_\kappa \lambda$. The following facts are invoked without further mention:

- If $T \subset \mathscr{P}_\mu \lambda$ is σ-stationary, then $\{x \cap \kappa : x \in T\}$ is σ-stationary in $\mathscr{P}_\mu \kappa$ and $\{\sup(x \cap \kappa) : x \in T\}$ is stationary in κ.
- If $S \subset \mathscr{P}_\mu \kappa$ is σ-stationary, then $\{x \in \mathscr{P}_\mu \lambda : x \cap \kappa \in S\}$ is σ-stationary.
- If $S \subset \{\alpha < \kappa : \mathrm{cf}\, \alpha < \mu\}$ is stationary, then $\{x \in \mathscr{P}_\mu \lambda : \sup(x \cap \kappa) \in S\}$ is stationary.

Suppose that F is a normal filter on $\mathscr{P}_\mu \kappa$. A subset A of F^+ is called an antichain in F^+ if $X \cap Y \notin F^+$ for every $X \neq Y$ from A. We say that an antichain is maximal if it is so with respect to inclusion.

Suppose that $\{X_\alpha : \alpha < \kappa\} \subset F^+$. The least upper bound of $\{X_\alpha : \alpha < \kappa\}$ in F^+ is denoted by $\sum\{X_\alpha : \alpha < \kappa\}$. It is well-known that $\sum\{X_\alpha : \alpha < \kappa\}$ is given by the diagonal union of $\{X_\alpha : \alpha < \kappa\}$ defined by:

$$\bigtriangledown\{X_\alpha : \alpha < \kappa\} = \{x \in \mathscr{P}_\mu\kappa : \exists\alpha \in x(x \in X_\alpha)\}.$$

The following characterization of weak presaturation from [**9**] is useful:

Proposition 5. *F is weakly presaturated if and only if the following holds: Suppose that $S \in F^+$ and $\{A_n : n < \omega\}$ is a set of maximal antichains in F^+. Then there is $S^* \leq S$ such that $|\{X \in A_n : S^* \cap X \in F^+\}| \leq \kappa$ for every $n < \omega$.*

With Proposition 5 one can see that if F is weakly presaturated, then the generic ultrapower by F^+ is wellfounded (i.e. F is precipitous) and is closed under countable sequences. For $S \in F^+$ we say that F is weakly presaturated below S if the filter generated by $F \cup \{S\}$ is weakly presaturated.

For a set A of ordinals $\mathrm{Col}(\kappa, A)$ denotes the Levy collapse adjoining a surjection from κ to each $\gamma \in A$. It is easy to see that $\mathrm{Col}(\kappa, A)$ is a κ-closed poset, i.e. a poset in which every descending sequence of length $< \kappa$ has a lower bound. If $\nu > \kappa$ is inaccessible, then $\nu = \kappa^+$ in the extension by $\mathrm{Col}(\kappa, \nu)$.

We say that ν is λ-supercompact if there is a normal ν-complete ultrafilter on $\mathscr{P}_\nu\lambda$. This is equivalent to saying that there are a transitive class M and an elementary embedding j from the universe V of all sets to M such that $j|\nu$ is the identity, $j(\nu) > \lambda$ and M is closed under sequences of length λ.

3. SR and the presaturation of the club filter.

As a warmup we prove in this section two special cases of Theorem 3 by adapting Todorčević's argument from [**2**] in the case of ω_1. More specifically we prove the (weak) presaturation of the club filter from the following extension of SR:

Definition. Stationary κ-Reflection in $\mathscr{P}_{\omega_1}\lambda$ holds if and only if for every stationary $S \subset \mathscr{P}_{\omega_1}\lambda$ there is $\kappa \subset A \subset \lambda$ of size κ such that $S \cap \mathscr{P}_{\omega_1}A$ is stationary in $\mathscr{P}_{\omega_1}A$.

In what follows we write κ-SR for Stationary κ-Reflection. Note that ω_1-SR is just the original SR from [**5**].

Proposition 6. *Suppose that $\kappa < \nu < \lambda$ and ν is λ-supercompact. Then κ-*SR *in $\mathscr{P}_{\omega_1}\lambda$ holds in the extension by* $\mathrm{Col}(\kappa, \nu)$.

Proposition 6 was proved in effect in [**5**]. The proof is obtained by replacing Lemma 9 in the proof of Proposition 11 by the following lemma of Shelah [**19**]:

Lemma 7. *Every stationary set in $\mathscr{P}_{\omega_1}\lambda$ remains stationary in the extension by a countably closed poset.*

Here is the main result of this section:

THEOREM 8. *Assume κ-SR in $\mathscr{P}_{\omega_1}\lambda$ with λ large enough. Then the following hold*:

(1) *The club filter on $\mathscr{P}_{\omega_1}\kappa$ is presaturated.*
(2) *The club filter on κ is weakly presaturated below the set $\{\alpha < \kappa : \operatorname{cf}\alpha = \omega\}$.*

PROOF. (1) Set $\nu = 2^{\kappa^\omega}$ and $\lambda = 2^\nu$. Fix a stationary $S \subset \mathscr{P}_{\omega_1}\kappa$ and for each $n < \omega$ a maximal antichain $\{X^n_\xi : \xi < \nu\}$ in $\mathscr{C}^+_{\omega_1\kappa}$. It suffices to give a stationary $S^* \subset S$ such that $|\{\xi < \nu : S^* \cap X^n_\xi \in \mathscr{C}^+_{\omega_1\kappa}\}| \leq \kappa$ for every $n < \omega$.

Define

$$\bar{S} = \{y \in \mathscr{P}_{\omega_1}\lambda : \forall n < \omega \exists \xi \in y \cap \nu (y \cap \kappa \in S \cap X^n_\xi)\}.$$

MAIN CLAIM. *$\bar{S}$ is stationary.*

PROOF. Fix $f : [\lambda]^{<\omega} \to \mathscr{P}_{\omega_1}\lambda$. It suffices to show that $\bar{S} \cap C(f) \neq \emptyset$.

List all functions from $[\lambda]^{<\omega}$ to $\mathscr{P}_{\omega_1}\lambda$ as $\{f_\beta : \beta < 2^\lambda\}$. Define

$$D = \{z \in \mathscr{P}_{\omega_1}2^\lambda : \forall \beta \in z(z \cap \lambda \in C(f) \cap C(f_\beta))\}.$$

It is easy to see that D is club. By Lemma 4 there is a map $d : [2^\lambda]^{<\omega} \to \mathscr{P}_{\omega_1}2^\lambda$ such that

- $\mathscr{P}_{\omega_1}2^\lambda \cap C(d) \subset D$ and
- if $z \in C(d)$ and $\xi < \nu$, then $\operatorname{cl}_d(z \cup \{\xi\}) \cap \lambda = \operatorname{cl}_d((z \cap \lambda) \cup \{\xi\}) \cap \lambda$.

For each $n < \omega$ define

$$C_n = \{y \in \mathscr{P}_{\omega_1}\lambda : \exists \xi < \nu(\operatorname{cl}_d(y \cup \{\xi\}) \cap \kappa = y \cap \kappa \in X^n_\xi)\}.$$

CLAIM. *C_n has a club subset.*

PROOF. Fix a stationary $T \subset \mathscr{P}_{\omega_1}\lambda$. It suffices to give $y \in T \cap C_n$.

By κ-SR in $\mathscr{P}_{\omega_1}\lambda$ there is $\kappa \subset A \subset \lambda$ of size κ such that $T \cap \mathscr{P}_{\omega_1}A$ is stationary in $\mathscr{P}_{\omega_1}A$. Fix a bijection $\pi : \kappa \to A$. Since $\{y \in \mathscr{P}_{\omega_1}A : \pi``(y \cap \kappa) = y\}$ is club, $\{y \in T \cap \mathscr{P}_{\omega_1}A : \pi``(y \cap \kappa) = y\}$ is stationary in $\mathscr{P}_{\omega_1}A$. Hence

$$T^* = \{y \cap \kappa : \pi``(y \cap \kappa) = y \in T \cap \mathscr{P}_{\omega_1}A\}$$

is stationary in $\mathscr{P}_{\omega_1}\kappa$.

Since $\{X^n_\xi : \xi < \nu\}$ is a maximal antichain in $\mathscr{C}^+_{\omega_1\kappa}$, there is $\xi < \nu$ such that $T^* \cap X^n_\xi$ is stationary in $\mathscr{P}_{\omega_1}\kappa$. Hence $\{z \in \mathscr{P}_{\omega_1}2^\lambda : z \cap \kappa \in T^* \cap X^n_\xi\}$ is stationary. Thus there is $z \in \mathscr{P}_{\omega_1}2^\lambda \cap C(d)$ such that $\xi \in z$, $\pi``(z \cap \kappa) \subset z$ and $z \cap \kappa \in T^* \cap X^n_\xi$. Since $z \cap \kappa \in T^*$, there is $y \in T$ such that $\pi``(y \cap \kappa) = y$ and $y \cap \kappa = z \cap \kappa$. It remains to prove that $y \in C_n$.

Since $y \cap \kappa = z \cap \kappa \in X^n_\xi$, it suffices to show that

$$y \cap \kappa \subset \operatorname{cl}_d(y \cup \{\xi\}) \cap \kappa \subset z \cap \kappa.$$

For the second inclusion note that $y = \pi``(y \cap \kappa) = \pi``(z \cap \kappa) \subset z$. Hence $y \cup \{\xi\} \subset z$ by $\xi \in z$. Since $z \in C(d)$, we have $\mathrm{cl}_d(y \cup \{\xi\}) \subset z$, as desired. $\square$(Claim)

Therefore $\bigcap_{n<\omega} C_n$ has a club subset. Take $\beta < 2^\lambda$ so that

$$\mathscr{P}_{\omega_1}\lambda \cap C(f_\beta) \subset \bigcap_{n<\omega} C_n.$$

By recursion on $n < \omega$ we define

$$z_n \in \mathscr{P}_{\omega_1}2^\lambda \cap C(d)$$

so that

- $\beta \in z_0$, $z_0 \cap \kappa \in S$,
- $z_n \subset z_{n+1}$,
- $z_n \cap \kappa \in X^n_\xi$ for some $\xi \in z_{n+1} \cap \nu$ and
- $z_n \cap \kappa = z_{n+1} \cap \kappa$.

Since S is stationary in $\mathscr{P}_{\omega_1}\kappa$, $\{z \in \mathscr{P}_{\omega_1}2^\lambda : z \cap \kappa \in S\}$ is stationary. Hence there is $z_0 \in \mathscr{P}_{\omega_1}2^\lambda \cap C(d)$ such that $\beta \in z_0$ and $z_0 \cap \kappa \in S$. Suppose next we have defined z_n as above. Since $\beta \in z_0 \subset z_n \in \mathscr{P}_{\omega_1}2^\lambda \cap C(d) \subset D$, we have $z_n \cap \lambda \in \mathscr{P}_{\omega_1}\lambda \cap C(f_\beta) \subset C_n$. Hence there is $\xi < \nu$ such that

$$\mathrm{cl}_d((z_n \cap \lambda) \cup \{\xi\}) \cap \kappa = z_n \cap \kappa \in X^n_\xi.$$

Set

$$z_{n+1} = \mathrm{cl}_d(z_n \cup \{\xi\}).$$

Then $z_n \cup \{\xi\} \subset z_{n+1} \in \mathscr{P}_{\omega_1}2^\lambda \cap C(d)$. Since $z_n \in C(d)$ and $\xi < \nu$, we have

$$z_{n+1} \cap \lambda = \mathrm{cl}_d(z_n \cup \{\xi\}) \cap \lambda = \mathrm{cl}_d((z_n \cap \lambda) \cup \{\xi\}) \cap \lambda$$

by the choice of d. Hence $z_{n+1} \cap \kappa = \mathrm{cl}_d((z_n \cap \lambda) \cup \{\xi\}) \cap \kappa = z_n \cap \kappa$, as desired.

Set

$$z = \bigcup_{n<\omega} z_n.$$

We show that $z \cap \lambda \in \bar{S} \cap C(f)$, which completes the proof of Main Claim.

To see that $z \cap \lambda \in \bar{S}$, note that $z \cap \kappa = z_0 \cap \kappa \in S$ and that for every $n < \omega$ there is $\xi \in z_{n+1} \cap \nu$ such that $z \cap \kappa = z_n \cap \kappa \in X^n_\xi$. To see that $z \cap \lambda \in C(f)$, note that $\{z_n : n < \omega\} \subset D$ is increasing and that D is club in $\mathscr{P}_{\omega_1}2^\lambda$. Since $z \in D$, we get the desired result. $\square$(Main Claim)

By κ-SR in $\mathscr{P}_{\omega_1}\lambda$ there is $\kappa \subset B \subset \lambda$ of size κ such that $\bar{S} \cap \mathscr{P}_{\omega_1}B$ is stationary in $\mathscr{P}_{\omega_1}B$. Fix a bijection $\pi : \kappa \to B$. Since $\{y \in \mathscr{P}_{\omega_1}B : \pi``(y \cap \kappa) = y\}$ is club, $\{y \in \bar{S} \cap$

$\mathscr{P}_{\omega_1}B : \pi``(y \cap \kappa) = y\}$ is stationary in $\mathscr{P}_{\omega_1}B$. Hence

$$S^* = \{y \cap \kappa : \pi``(y \cap \kappa) = y \in \bar{S} \cap \mathscr{P}_{\omega_1}B\}$$

is stationary in $\mathscr{P}_{\omega_1}\kappa$. Note that $y \cap \kappa \in S$ for every $y \in \bar{S}$. Hence $S^* \subset S$. We claim that S^* is as required above.

Fix $n < \omega$. First we show that

$$S^* \subset \bigtriangledown\{X^n_{\pi(\zeta)} : \zeta \in B^*\} = \sum\{X^n_{\pi(\zeta)} : \zeta \in B^*\},$$

where $B^* \subset \kappa$ is defined by $\pi``B^* = B \cap \nu$.

Fix $x \in S^*$. It suffices to give $\zeta \in B^* \cap x$ such that $x \in X^n_{\pi(\zeta)}$. Since $x \in S^*$, there is $y \in \bar{S} \cap \mathscr{P}_{\omega_1}B$ such that $\pi``(y \cap \kappa) = y$ and $y \cap \kappa = x$. Since $y \in \bar{S}$, there is $\xi \in y \cap \nu$ such that $y \cap \kappa \in X^n_\xi$. Set $\zeta = \pi^{-1}(\xi)$. Since $\xi \in y \cap \nu \subset B \cap \nu$, we have

$$\zeta = \pi^{-1}(\xi) \in \pi^{-1}``(B \cap \nu) = B^* \text{ and } \zeta = \pi^{-1}(\xi) \in \pi^{-1}``y = y \cap \kappa = x.$$

Moreover we have $x = y \cap \kappa \in X^n_\xi = X^n_{\pi(\zeta)}$, as desired.

Since $\pi : \kappa \to B$ is a bijection, we have

$$S^* \leq \sum\{X^n_{\pi(\zeta)} : \zeta \in B^*\} = \sum\{X^n_\xi : \xi \in B \cap \nu\}.$$

Since $\{X^n_\xi : \xi < \nu\}$ is a maximal antichain in $\mathscr{C}^+_{\omega_1\kappa}$, we have

$$\{\xi < \nu : S^* \cap X^n_\xi \in \mathscr{C}^+_{\omega_1\kappa}\} \subset B \cap \nu.$$

Since $|B| = \kappa$, we have $|\{\xi < \nu : S^* \cap X^n_\xi \in \mathscr{C}^+_{\omega_1\kappa}\}| \leq \kappa$, as desired. This completes the proof of (1).

(2) Set $\nu = 2^\kappa$ and $\lambda = 2^\nu$. Fix a stationary $S \subset \{\alpha < \kappa : \operatorname{cf}\alpha = \omega\}$ and for each $n < \omega$ a maximal antichain $\{X^n_\xi : \xi < \nu\}$ in $\mathscr{C}^+_\kappa$. Here $\mathscr{C}_\kappa$ is the club filter on κ. It suffices to give a stationary $S^* \subset S$ such that $|\{\xi < \nu : S^* \cap X^n_\xi \in \mathscr{C}^+_\kappa\}| \leq \kappa$ for every $n < \omega$.

Define

$$\bar{S} = \{y \in \mathscr{P}_{\omega_1}\lambda : \forall n < \omega \exists \xi \in y \cap \nu(\sup(y \cap \kappa) \in S \cap X^n_\xi)\}.$$

MAIN CLAIM. *$\bar{S}$ is stationary.*

PROOF. Fix $f : [\lambda]^{<\omega} \to \mathscr{P}_{\omega_1}\lambda$. It suffices to show that $\bar{S} \cap C(f) \neq \emptyset$.

List all functions from $[\lambda]^{<\omega}$ to $\mathscr{P}_{\omega_1}\lambda$ as $\{f_\beta : \beta < 2^\lambda\}$. Define

$$D = \{z \in \mathscr{P}_{\omega_1}2^\lambda : \forall \beta \in z(z \cap \lambda \in C(f) \cap C(f_\beta))\}.$$

It is easy to see that D is club. By Lemma 4 there is a map $d : [2^\lambda]^{<\omega} \to \mathscr{P}_{\omega_1}2^\lambda$ such that

- $\mathscr{P}_{\omega_1} 2^\lambda \cap C(d) \subset D$ and
- if $z \in C(d)$ and $\xi < \nu$, then $\mathrm{cl}_d(z \cup \{\xi\}) \cap \lambda = \mathrm{cl}_d((z \cap \lambda) \cup \{\xi\}) \cap \lambda$.

For each $n < \omega$ define

$$C_n = \{y \in \mathscr{P}_{\omega_1}\lambda : \exists \xi < \nu(\sup(\mathrm{cl}_d(y \cup \{\xi\}) \cap \kappa) = \sup(y \cap \kappa) \in X^n_\xi)\}.$$

Claim. *C_n has a club subset.*

Proof. Fix a stationary $T \subset \mathscr{P}_{\omega_1}\lambda$. It suffices to give $y \in T \cap C_n$.

By κ-SR in $\mathscr{P}_{\omega_1}\lambda$ there is $\kappa \subset A \subset \lambda$ of size κ such that $T \cap \mathscr{P}_{\omega_1} A$ is stationary in $\mathscr{P}_{\omega_1} A$. Fix a bijection $\pi : \kappa \to A$. Since $\{y \in \mathscr{P}_{\omega_1} A : \pi``(y \cap \kappa) = y\}$ is club, $\{y \in T \cap \mathscr{P}_{\omega_1} A : \pi``(y \cap \kappa) = y\}$ is stationary in $\mathscr{P}_{\omega_1} A$. Hence

$$T^* = \{\sup(y \cap \kappa) : \pi``(y \cap \kappa) = y \in T \cap \mathscr{P}_{\omega_1} A\}$$

is stationary in κ.

Since $\{X^n_\xi : \xi < \nu\}$ is a maximal antichain in $\mathscr{C}^+_\kappa$, there is $\xi < \nu$ such that $T^* \cap X^n_\xi$ is stationary in κ. Hence $\{z \in \mathscr{P}_\kappa 2^\lambda : z \cap \kappa \in T^* \cap X^n_\xi\}$ is stationary. Thus there is $z \in \mathscr{P}_\kappa 2^\lambda \cap C(d)$ such that $\xi \in z$, $\pi``(z \cap \kappa) \subset z$ and $z \cap \kappa \in T^* \cap X^n_\xi$. Since $z \cap \kappa \in T^*$, there is $y \in T$ such that $\pi``(y \cap \kappa) = y$ and $\sup(y \cap \kappa) = z \cap \kappa$. It remains to prove that $y \in C_n$.

Since $\sup(y \cap \kappa) = z \cap \kappa \in X^n_\xi$, it suffices to show that

$$\sup(y \cap \kappa) \subset \sup(\mathrm{cl}_d(y \cup \{\xi\}) \cap \kappa) \subset z \cap \kappa.$$

For the second inclusion note that $y = \pi``(y \cap \kappa) \subset \pi`` \sup(y \cap \kappa) = \pi``(z \cap \kappa) \subset z$. Hence $y \cup \{\xi\} \subset z$ by $\xi \in z$. Thus $\mathrm{cl}_d(y \cup \{\xi\}) \subset z$ by $z \in C(d)$. Since $z \cap \kappa = \sup(y \cap \kappa)$ is a limit ordinal, we have $\sup(\mathrm{cl}_d(y \cup \{\xi\}) \cap \kappa) \subset \sup(z \cap \kappa) = z \cap \kappa$, as desired. □(Claim)

Therefore $\bigcap_{n<\omega} C_n$ has a club subset. Take $\beta < 2^\lambda$ so that

$$\mathscr{P}_{\omega_1}\lambda \cap C(f_\beta) \subset \bigcap\nolimits_{n<\omega} C_n.$$

By recursion on $n < \omega$ we define

$$z_n \in \mathscr{P}_{\omega_1} 2^\lambda \cap C(d)$$

so that

- $\beta \in z_0$, $\sup(z_0 \cap \kappa) \in S$,
- $z_n \subset z_{n+1}$,
- $\sup(z_n \cap \kappa) \in X^n_\xi$ for some $\xi \in z_{n+1} \cap \nu$ and
- $\sup(z_n \cap \kappa) = \sup(z_{n+1} \cap \kappa)$.

Since $S \subset \{\alpha < \kappa : \mathrm{cf}\,\alpha = \omega\}$ is stationary, $\{z \in \mathscr{P}_{\omega_1} 2^\lambda : \sup(z \cap \kappa) \in S\}$ is stationary. Hence there is $z_0 \in \mathscr{P}_{\omega_1} 2^\lambda \cap C(d)$ such that $\beta \in z_0$ and $\sup(z_0 \cap \kappa) \in S$. Suppose next we have defined z_n as above. Since $\beta \in z_0 \subset z_n \in \mathscr{P}_{\omega_1} 2^\lambda \cap C(d) \subset D$, we have $z_n \cap \lambda \in \mathscr{P}_{\omega_1}\lambda \cap C(f_\beta) \subset C_n$. Hence there is $\xi < \nu$ such that

$$\sup(\mathrm{cl}_d((z_n \cap \lambda) \cup \{\xi\}) \cap \kappa) = \sup(z_n \cap \kappa) \in X^n_\xi.$$

Set

$$z_{n+1} = \mathrm{cl}_d(z_n \cup \{\xi\}).$$

Then $z_n \cup \{\xi\} \subset z_{n+1} \in \mathscr{P}_{\omega_1} 2^\lambda \cap C(d)$. Since $z_n \in C(d)$ and $\xi < \nu$, we have

$$z_{n+1} \cap \lambda = \mathrm{cl}_d(z_n \cup \{\xi\}) \cap \lambda = \mathrm{cl}_d((z_n \cap \lambda) \cup \{\xi\}) \cap \lambda$$

by the choice of d. Hence $\sup(z_{n+1} \cap \kappa) = \sup(\mathrm{cl}_d((z_n \cap \lambda) \cup \{\xi\}) \cap \kappa) = \sup(z_n \cap \kappa)$, as desired.

Set

$$z = \bigcup_{n<\omega} z_n.$$

We show that $z \cap \lambda \in \bar{S} \cap C(f)$, which completes the proof of Main Claim.

To see that $z \cap \lambda \in \bar{S}$, note that $\sup(z \cap \kappa) = \sup(z_0 \cap \kappa) \in S$ and that for every $n < \omega$ there is $\xi \in z_{n+1} \cap \nu$ such that $\sup(z \cap \kappa) = \sup(z_n \cap \kappa) \in X^n_\xi$. To see that $z \cap \lambda \in C(f)$, note that $\{z_n : n < \omega\} \subset D$ is increasing and that D is club in $\mathscr{P}_{\omega_1} 2^\lambda$. Since $z \in D$, we get the desired result. □(Main Claim)

The rest of the proof is essentially the same as before. □

Remark. By Proposition 6 with $\lambda = 2^\nu$ the model of Theorem 3 satisfies κ-SR in $\mathscr{P}_{\omega_1} 2^\nu$ with $\nu = 2^{\kappa^\omega}$. Hence by Theorem 8 (1) the club filter on $\mathscr{P}_{\omega_1}\kappa$ is presaturated in the model. Thus we get a much simpler proof of Theorem 2 under the stronger hypothesis that ν is 2^ν-supercompact.

As one might notice, Main Claim of the proof of Theorem 8 could have been that the set $\{y \cap \nu : y \in \bar{S}\}$ is stationary in $\mathscr{P}_{\omega_1}\nu$. This modification allows us to reduce the strength of κ-SR that was applied to $\bar{S}$. This would not, however, reduce the strength of κ-SR that was applied to prove Main Claim. Even if the current Main Claim is modified, Claim of the proof should remain the same because the subsequent recursion uses Lemma 4 together with the current Claim.

In [**7**] Gitik established that the club filter on an inaccessible cardinal can be presaturated. It has been unknown, however, whether the club filter on a supercompact cardinal can be weakly presaturated. Theorem 8 (2) gives a partial answer. Indeed, starting from a model with two supercompact cardinals and forcing with the poset from [**15**] and then with the Levy collapse, we get one in which the club filter on a supercompact cardinal κ is weakly presaturated below the set $\{\alpha < \kappa : \mathrm{cf}\,\alpha = \omega\}$.

In [**16**] Matsubara conjectured that the club filter on ω_1 should be presaturated if for some large enough λ there is a normal filter F on $\mathscr{P}_{\omega_2}\lambda$ such that F^+ is strategically countably closed. Together with Theorem 3 of [**17**], Theorem 8 with $\kappa = \omega_1$ proves the conjecture. See [**26**] for further applications of κ-SR.

4. Reflecting $\boldsymbol{\sigma}$-stationary sets in $\mathscr{P}_{\kappa}\boldsymbol{\lambda}$.

This section introduces a new reflection principle for stationary sets in $\mathscr{P}_\kappa\lambda$. Suppose that θ is a regular cardinal $> \kappa$. Recall that $H(\theta)$ is the set of all sets hereditarily of size $< \theta$. In [**5**] the notion of internal approachability was introduced. The set of all internally approachable sets is denoted by IA. It was shown that IA is stationary in $\mathscr{P}_\kappa H(\theta)$ and a stationary subset of IA remains stationary in the extension by a κ-closed poset.

In [**5**] a superset of IA was also introduced, which was denoted by IA* in [**11**]. Here we introduce an analogue of IA* in the context of $\mathscr{P}_\kappa\lambda$. For the rest of this section we assume $\lambda^{<\kappa} = \lambda$ and fix a bijection $\varphi : {}^{<\kappa}\mathscr{P}_\kappa\lambda \to \lambda$. Define

$$S^\varphi(\kappa,\lambda) = \Big\{x \in \mathscr{P}_\kappa\lambda : \exists\delta < \kappa\exists t : \delta \to \mathscr{P}_\kappa\lambda \\ \Big(x \subset \bigcup \operatorname{ran} t \wedge \{\gamma < \delta : \varphi(t|\gamma) \in x\} \text{ is unbounded in } \delta\Big)\Big\}.$$

It is easy to see that $S^\varphi(\kappa,\lambda)$ is σ-stationary in $\mathscr{P}_\kappa\lambda$. If $\psi : {}^{<\kappa}\mathscr{P}_\kappa\lambda \to \lambda$ is another bijection, then $S^\varphi(\kappa,\lambda)$ and $S^\psi(\kappa,\lambda)$ agree on the set $\{x \in \mathscr{P}_\kappa\lambda : (\psi\circ\varphi^{-1})``x = x\}$, which is σ-club. In what follows we write $S(\kappa,\lambda)$ for $S^\varphi(\kappa,\lambda)$.

Here is an analogue of Lemma 7 in the context of $\mathscr{P}_\kappa\lambda$:

Lemma 9. *Every σ-stationary subset of $S(\kappa,\lambda)$ remains σ-stationary in $\mathscr{P}_\kappa\lambda$ in the extension by a κ-closed poset.*

Proof. Let p be a condition forcing $\dot f : [\lambda]^{<\omega} \to \mathscr{P}_{\omega_1}\lambda$. It suffices to give a σ-club $D \subset \mathscr{P}_\kappa\lambda$ such that $S(\kappa,\lambda) \cap D \subset \{x \in \mathscr{P}_\kappa\lambda : \exists q \le p(q \Vdash x \in C(\dot f))\}$.

Since our poset is κ-closed, we can define by recursion on the length of $t \in {}^{<\kappa}\mathscr{P}_\kappa\lambda$

$$p_t \le p \text{ and } f_t : \Big[\bigcup \operatorname{ran} t\Big]^{<\omega} \to \mathscr{P}_{\omega_1}\lambda$$

so that

(1) $p_t \le p_s$ if $s \subset t$, and
(2) p_t forces $f_t = \dot f|[\bigcup \operatorname{ran} t]^{<\omega}$.

Define

$$D = \Big\{x \in \mathscr{P}_\kappa\lambda : \forall t \in {}^{<\kappa}\mathscr{P}_\kappa\lambda\Big(\varphi(t) \in x \to f_t``\Big[x \cap \bigcup \operatorname{ran} t\Big]^{<\omega} \subset \mathscr{P}(x)\Big)\Big\}.$$

It is easy to see that D is σ-club. We claim that D is as required above.

Fix $x \in S(\kappa,\lambda) \cap D$. Take $\delta < \kappa$ and $t : \delta \to \mathscr{P}_\kappa\lambda$ that witness $x \in S(\kappa,\lambda)$. It suffices to show that p_t forces $x \in C(\dot f)$.

Fix $a \in [x]^{<\omega}$. Since $x \subset \bigcup t``\delta$ and $\{\gamma < \delta : \varphi(t|\gamma) \in x\}$ is unbounded, there is $\gamma < \delta$ such that $a \subset \bigcup t``\gamma$ and $\varphi(t|\gamma) \in x$. Then $p_{t|\gamma}$ forces $\dot{f}(a) = f_{t|\gamma}(a) \subset x$ by (2) and $x \in D$. Since $p_t \leq p_{t|\gamma}$ by (1), p_t forces $\dot{f}(a) \subset x$, as desired. $\square$

In [**4**] it was shown that IA has a club subset in the extension by $\mathrm{Col}(\kappa, \{|H(\theta)|\})$. Let us prove an analogue of this fact in the context of $\mathscr{P}_\kappa\lambda$:

Proposition 10. *$S(\kappa, \lambda)$ has a σ-club subset in the extension by $\mathrm{Col}(\kappa, \{\lambda\})$.*

Proof. We work in the extension by $\mathrm{Col}(\kappa, \{\lambda\})$. Fix a bijection $\pi : \kappa \to \lambda$. Note that the set ${}^{<\kappa}\mathscr{P}_\kappa\lambda$ remains the same after forcing with $\mathrm{Col}(\kappa, \{\lambda\})$. Hence we can define $f : \kappa \to \lambda$ by $f(\gamma) = \varphi(\langle \pi``\alpha : \alpha < \gamma \rangle)$. Define

$$C = \{x \in \mathscr{P}_\kappa\lambda : f``(x \cap \kappa) \subset x = \pi``(x \cap \kappa)\}.$$

It is easy to see that C is σ-club. It suffices to show that $C \subset S(\kappa, \lambda)$.

Fix $x \in C$. Set $\delta = \sup(x \cap \kappa) < \kappa$. Define $t : \delta \to \mathscr{P}_\kappa\lambda$ by $t(\alpha) = \pi``\alpha$. Then $x = \pi``(x \cap \kappa) \subset \pi``\delta = \bigcup \mathrm{ran}\, t$. Since $x \in C$, we have $\varphi(t|\gamma) = f(\gamma) \in x$ for every $\gamma \in x \cap \kappa$. Hence $\{\gamma < \delta : \varphi(t|\gamma) \in x\}$ is unbounded in δ, as desired. $\square$

We are now ready to state our reflection principle:

Definition. σ-Stationary Reflection in $\mathscr{P}_\kappa\lambda$ holds if and only if for every σ-stationary $S \subset S(\kappa, \lambda)$ there is $\kappa \subset A \subset \lambda$ of size κ such that $S \cap \mathscr{P}_\kappa A$ is σ-stationary in $\mathscr{P}_\kappa A$.

In what follows we write σ-SR for σ-Stationary Reflection.

Proposition 11. *Suppose that $\kappa < \nu < \lambda$, $\lambda^{<\kappa} = \lambda$ and ν is λ-supercompact. Then* σ-SR *in $\mathscr{P}_\kappa\lambda$ holds in the extension by* $\mathrm{Col}(\kappa, \nu)$.

Proof. Let $j : V \to M$ witness that ν is λ-supercompact. Fix a V-generic $G \subset \mathrm{Col}^V(\kappa, \nu)$. In $V[G]$ fix a σ-stationary $S \subset S(\kappa, \lambda)$. It suffices to give $\kappa \subset A \subset \lambda$ of size κ such that $S \cap \mathscr{P}_\kappa A$ is σ-stationary in $\mathscr{P}_\kappa A$.

Fix a $V[G]$-generic $H \subset \mathrm{Col}^V(\kappa, j(\nu) - \nu)$. Then $G \times H$ is isomorphic to a V-generic filter over $\mathrm{Col}^V(\kappa, j(\nu)) = \mathrm{Col}^M(\kappa, j(\nu)) = j(\mathrm{Col}^V(\kappa, \nu))$, which is denoted by $j(G)$. Henceforth we work in $V[j(G)]$. Note that $j``G = G$ by $G \subset \mathrm{Col}^V(\kappa, \nu) \subset V_\nu$. Hence we can extend j to an elementary embedding from $V[G]$ to $M[j(G)]$, which is also denote by j. Thus it suffices to show that in $M[j(G)]$ there is $j(\kappa) \subset A \subset j(\lambda)$ of size $j(\kappa)$ such that $j(S) \cap \mathscr{P}_{j(\kappa)}A$ is σ-stationary in $\mathscr{P}_{j(\kappa)}A$.

Note that $j(\kappa) = \kappa \subset j``\lambda \subset j(\lambda)$, $|j``\lambda|^{M[j(G)]} = \kappa$ and $\mathscr{P}_\kappa^{M[j(G)]} j``\lambda = \mathscr{P}_\kappa j``\lambda$. Hence it suffices to show that $j(S) \cap \mathscr{P}_\kappa j``\lambda$ is $\sigma$-stationary in $\mathscr{P}_\kappa j``\lambda$.

Since $\mathrm{Col}^V(\kappa, \nu)$ is κ-closed in V, $\mathrm{Col}^V(\kappa, j(\nu) - \nu)$ remains κ-closed in $V[G]$. Hence S is σ-stationary in $\mathscr{P}_\kappa\lambda$ by Lemma 9. Since $j|\lambda : \lambda \to j``\lambda$ is a bijection, $\{ j``x : x \in S\}$ is σ-stationary in $\mathscr{P}_\kappa j``\lambda$. Note that $j``x = j(x)$ for every $x \in V[G]$ of size $< \kappa$. Hence $\{ j``x : x \in S\} = j``S \subset j(S)$. Thus we get the desired result. $\square$

Propositions 11 and 12 entail Proposition 6 in the case $\lambda^{<\kappa} = \lambda$. Note that even the weaker form of Proposition 6 suffices for proving Theorem 8.

PROPOSITION 12. *Assume* $\lambda^{<\kappa} = \lambda$. *Then* σ-SR *in* $\mathscr{P}_\kappa\lambda$ *implies* κ-SR *in* $\mathscr{P}_{\omega_1}\lambda$.

PROOF. Fix a bijection $\varphi : {}^{<\kappa}\mathscr{P}_\kappa\lambda \to \lambda$. Define

$$C = \{x \in \mathscr{P}_{\omega_1}\lambda : \varphi``^{<\omega}([x]^{<\omega}) \subset x\}.$$

It is easy to see that C is club. We claim that $C \subset S(\kappa,\lambda)$.

Fix $x \in C$. Since x is countable, there is $t : \omega \to [x]^{<\omega}$ such that $x = \bigcup \operatorname{ran} t$. Since $x \in C$, we have $\varphi(t|n) \in x$ for every $n < \omega$. Hence t witnesses $x \in S(\kappa,\lambda)$, as desired.

To see κ-SR in $\mathscr{P}_{\omega_1}\lambda$, fix a stationary $S \subset \mathscr{P}_{\omega_1}\lambda$. Since C is club in $\mathscr{P}_{\omega_1}\lambda$, we can assume $S \subset C$. Hence $S \subset S(\kappa,\lambda)$ by the previous paragraph. Since S is stationary in $\mathscr{P}_{\omega_1}\lambda$, it is σ-stationary in $\mathscr{P}_\kappa\lambda$. By σ-SR in $\mathscr{P}_\kappa\lambda$ there is $\kappa \subset A \subset \lambda$ of size κ such that $S \cap \mathscr{P}_\kappa A$ is σ-stationary in $\mathscr{P}_\kappa A$. Note that $S \cap \mathscr{P}_\kappa A = S \cap \mathscr{P}_{\omega_1} A$ by $S \subset \mathscr{P}_{\omega_1}\lambda$. Hence $S \cap \mathscr{P}_{\omega_1} A$ is stationary in $\mathscr{P}_{\omega_1} A$, as desired. □

5. Main Theorem.

In this section we prove Theorem 13 and deduce Theorem 3 as a corollary.

THEOREM 13. *Assume* σ-SR *in* $\mathscr{P}_\kappa 2^{2^{2^{<\kappa}}}$. *Then the* σ-*club filter on* $\mathscr{P}_\kappa\kappa$ *is weakly presaturated below the set* $\{x \in \mathscr{P}_\kappa\kappa : \operatorname{cf}\sup x = \omega\}$.

PROOF. Set $\nu = 2^{2^{<\kappa}}$ and $\lambda = 2^\nu$. Fix a σ-stationary $S \subset \{x \in \mathscr{P}_\kappa\kappa : \operatorname{cf}\sup x = \omega\}$ and for each $n < \omega$ a maximal antichain $\{X^n_\xi : \xi < \nu\}$ in $(\mathscr{C}^\sigma_{\kappa\kappa})^+$. It suffices to give a σ-stationary $S^* \subset S$ such that $|\{\xi < \nu : S^* \cap X^n_\xi \in (\mathscr{C}^\sigma_{\kappa\kappa})^+\}| \le \kappa$ for every $n < \omega$.

Since $\lambda^{<\kappa} = \lambda$, there is a bijection $\varphi : {}^{<\kappa}\mathscr{P}_\kappa\lambda \to \lambda$. Define

$$\bar{S} = \{y \in S(\kappa,\lambda) : \forall n < \omega \exists \xi \in y \cap \nu (y \cap \kappa \in S \cap X^n_\xi)\}.$$

MAIN CLAIM. $\bar{S}$ *is* σ-*stationary in* $\mathscr{P}_\kappa\lambda$.

PROOF. Fix $f : [\lambda]^{<\omega} \to \mathscr{P}_{\omega_1}\lambda$. It suffices to show that $\bar{S} \cap C(f) \neq \emptyset$.

List all functions from ν to $\mathscr{P}_{\omega_1}\lambda$ as $\{e_\alpha : \alpha < \lambda\}$ and those from $[\lambda]^{<\omega}$ to $\mathscr{P}_{\omega_1}\lambda$ as $\{f_\beta : \beta < 2^\lambda\}$. Define $g : \lambda \times \nu \to \lambda$ so that for every $s \in {}^{<\kappa}\mathscr{P}_\kappa\lambda$ and $\xi < \nu$

$$g(\varphi(s),\xi) = \varphi\Big(\Big\langle \bigcup\{e_\alpha(\xi) : \alpha \in s(\gamma)\} : \gamma \in \operatorname{dom} s \Big\rangle\Big),$$

and $h : \lambda^2 \to \lambda$ so that for every $s, t \in {}^{<\kappa}\mathscr{P}_\kappa\lambda$

$$h(\varphi(s),\varphi(t)) = \varphi(s \cup (t|(\operatorname{dom} t - \operatorname{dom} s))).$$

Define

$$D = \{z \in \mathscr{P}_\kappa 2^\lambda : \forall \alpha \in z \cap \lambda (e_\alpha``(z \cap \nu) \subset \mathscr{P}(z)) \wedge \forall \beta \in z (z \cap \lambda \in C(f_\beta)) \wedge \\ z \cap \lambda \in C(f) \wedge g``((z \cap \lambda) \times (z \cap \nu)) \subset z \wedge h``(z \cap \lambda)^2 \subset z\}.$$

It is easy to see that D is σ-club. By Lemma 4 there is a map $d : [2^\lambda]^{<\omega} \to \mathscr{P}_{\omega_1} 2^\lambda$ such that

- $\mathscr{P}_\kappa 2^\lambda \cap C(d) \subset D$ and
- if $z \in C(d)$ and $\xi < \nu$, then
  $\mathrm{cl}_d(z \cup \{\xi\}) \cap \lambda = \bigcup\{e_\alpha(\xi) : \alpha \in z \cap \lambda\} = \mathrm{cl}_d((z \cap \lambda) \cup \{\xi\}) \cap \lambda$.

For each $n < \omega$ define

$$S_n = \{y \in \mathscr{P}_\kappa \lambda : \exists \xi < \nu(\mathrm{cl}_d(y \cup \{\xi\}) \cap \kappa = y \cap \kappa \in X^n_\xi)\}.$$

Claim 1. *There is a σ-club $C_n \subset \mathscr{P}_\kappa \lambda$ such that $S(\kappa, \lambda) \cap C_n \subset S_n$.*

Proof. Fix a σ-stationary $T \subset S(\kappa, \lambda)$. It suffices to give $y \in T \cap S_n$.

By σ-SR in $\mathscr{P}_\kappa \lambda$ there is $\kappa \subset A \subset \lambda$ of size κ such that $T \cap \mathscr{P}_\kappa A$ is σ-stationary in $\mathscr{P}_\kappa A$. Fix a bijection $\pi : \kappa \to A$. Since $\{y \in \mathscr{P}_\kappa A : \pi``(y \cap \kappa) = y\}$ is $\sigma$-club, $\{y \in T \cap \mathscr{P}_\kappa A : \pi``(y \cap \kappa) = y\}$ is σ-stationary in $\mathscr{P}_\kappa A$. Hence

$$T^* = \{y \cap \kappa : \pi``(y \cap \kappa) = y \in T \cap \mathscr{P}_\kappa A\}$$

is σ-stationary in $\mathscr{P}_\kappa \kappa$.

Since $\{X^n_\xi : \xi < \nu\}$ is a maximal antichain in $(\mathscr{C}^\sigma_{\kappa\kappa})^+$, there is $\xi < \nu$ such that $T^* \cap X^n_\xi$ is σ-stationary in $\mathscr{P}_\kappa \kappa$. Hence $\{z \in \mathscr{P}_\kappa 2^\lambda : z \cap \kappa \in T^* \cap X^n_\xi\}$ is σ-stationary. Thus there is $z \in \mathscr{P}_\kappa 2^\lambda \cap C(d)$ such that $\xi \in z$, $\pi``(z \cap \kappa) \subset z$ and $z \cap \kappa \in T^* \cap X^n_\xi$. Since $z \cap \kappa \in T^*$, there is $y \in T$ such that $\pi``(y \cap \kappa) = y$ and $y \cap \kappa = z \cap \kappa$. It remains to prove that $y \in S_n$.

Since $y \cap \kappa = z \cap \kappa \in X^n_\xi$, it suffices to show that

$$y \cap \kappa \subset \mathrm{cl}_d(y \cup \{\xi\}) \cap \kappa \subset z \cap \kappa.$$

For the second inclusion note that $y = \pi``(y \cap \kappa) = \pi``(z \cap \kappa) \subset z$. Hence $y \cup \{\xi\} \subset z$ by $\xi \in z$. Since $z \in C(d)$, we have $\mathrm{cl}_d(y \cup \{\xi\}) \subset z$, as desired. □(Claim 1)

Therefore $\bigcap_{n<\omega} C_n$ is σ-club in $\mathscr{P}_\kappa \lambda$. Take $\beta < 2^\lambda$ so that

$$\mathscr{P}_\kappa \lambda \cap C(f_\beta) \subset \bigcap\nolimits_{n<\omega} C_n.$$

Define

$$E = \Big\{z \in \mathscr{P}_\kappa 2^\lambda \cap C(d) : \beta \in z \wedge \exists t : \sup(z \cap \kappa) \to \mathscr{P}_\kappa 2^\lambda \cap C(d) \\ \Big(t \text{ is increasing} \wedge z \subset \bigcup \operatorname{ran} t \wedge \forall \delta \in z \cap \kappa(\varphi(\langle t(\gamma) \cap \lambda : \gamma < \delta\rangle) \in z)\Big)\Big\}.$$

For each $t : \delta \to \mathscr{P}_\kappa 2^\lambda$ with $\delta < \kappa$, define $t^* : \delta \to \mathscr{P}_\kappa \lambda$ by

$$t^*(\gamma) = t(\gamma) \cap \lambda.$$

It is easy to see that if t witnesses $z \in E$, then t^* witnesses $z \cap \lambda \in S(\kappa, \lambda)$.

Claim 2. *$\{z \cap \kappa : z \in E\}$ has a subset σ-club in $\mathscr{P}_\kappa \kappa$.*

Proof. Define an increasing map $t : \kappa \to \mathscr{P}_\kappa 2^\lambda \cap C(d)$ recursively so that

$$\{\beta, \gamma, \varphi((t|\gamma)^*)\} \subset t(\gamma).$$

Set

$$Y = \bigcup \operatorname{ran} t.$$

Then $\kappa \cup \{\varphi((t|\gamma)^*) : \gamma < \kappa\} \cup \{\beta\} \subset Y \subset 2^\lambda$. Since $t : \kappa \to \mathscr{P}_\kappa 2^\lambda \cap C(d)$ is increasing, we have $Y \in C(d)$ and $|Y| = \kappa$. Note that for every $\eta \in Y$ there is $\gamma < \kappa$ such that $\eta \in t(\gamma)$. Define

$$C = \{z \in \mathscr{P}_\kappa Y \cap C(d) : \beta \in z \wedge \forall \eta \in z \exists \gamma \in z \cap \kappa(\eta \in t(\gamma)) \wedge \forall \delta \in z \cap \kappa(\varphi((t|\delta)^*) \in z)\}.$$

It is easy to see that C is σ-club in $\mathscr{P}_\kappa Y$. Hence $\{z \cap \kappa : z \in C\}$ has a subset σ-club in $\mathscr{P}_\kappa \kappa$. Thus it suffices to show that $C \subset E$.

Fix $z \in C$. By definition $\beta \in z \in \mathscr{P}_\kappa 2^\lambda \cap C(d)$. It remains to show that $t|\sup(z \cap \kappa)$ witnesses $z \in E$.

By construction $t|\sup(z \cap \kappa) : \sup(z \cap \kappa) \to \mathscr{P}_\kappa 2^\lambda \cap C(d)$ is increasing. Since $z \in C$, we have $z \subset \bigcup\{t(\gamma) : \gamma \in z \cap \kappa\} \subset \bigcup\{t(\gamma) : \gamma \in \sup(z \cap \kappa)\} = \bigcup \operatorname{ran}(t|\sup(z \cap \kappa))$ and $\varphi(\langle t(\gamma) \cap \lambda : \gamma < \delta\rangle) = \varphi((t|\delta)^*) \in z$ for every $\delta \in z \cap \kappa$, as desired. □(Claim 2)

By recursion on $n < \omega$ we define

$$z_n \in E \text{ and } t_n : \sup(z_n \cap \kappa) \to \mathscr{P}_\kappa 2^\lambda \cap C(d)$$

so that

- $z_0 \cap \kappa \in S$,
- $z_n \subset z_{n+1}$,
- $z_n \cap \kappa \in X^n_\xi$ for some $\xi \in z_{n+1} \cap \nu$
- $z_n \cap \kappa = z_{n+1} \cap \kappa$
- $t_n(\gamma) \subset t_{n+1}(\gamma)$ for every $\gamma \in \operatorname{dom} t_n$ and
- t_n witnesses $z_n \in E$.

Since S is σ-stationary in $\mathscr{P}_\kappa \kappa$, there is $z_0 \in E$ such that $z_0 \cap \kappa \in S$ by Claim 2. Take t_0 that witnesses $z_0 \in E$. Suppose next we have defined z_n and t_n as above. Since $z_n \in E$, we have $z_n \cap \lambda \in S(\kappa, \lambda)$ and $\beta \in z_n \in \mathscr{P}_\kappa 2^\lambda \cap C(d) \subset D$. Hence $z_n \cap \lambda \in S(\kappa, \lambda) \cap C(f_\beta) \subset S(\kappa, \lambda) \cap C_n \subset S_n$. Thus there is $\xi < \nu$ such that

$$\operatorname{cl}_d((z_n \cap \lambda) \cup \{\xi\}) \cap \kappa = z_n \cap \kappa \in X^n_\xi.$$

Set

$$z_{n+1} = \mathrm{cl}_d(z_n \cup \{\xi\}).$$

Then $z_n \cup \{\xi\} \subset z_{n+1} \in \mathscr{P}_\kappa 2^\lambda \cap C(d)$. Hence $\beta \in z_{n+1}$ by $\beta \in z_n$. Since $z_n \in C(d)$ and $\xi < \nu$, we have

$$z_{n+1} \cap \lambda = \mathrm{cl}_d(z_n \cup \{\xi\}) \cap \lambda = \mathrm{cl}_d((z_n \cap \lambda) \cup \{\xi\}) \cap \lambda$$

by the choice of d. Hence $z_{n+1} \cap \kappa = \mathrm{cl}_d((z_n \cap \lambda) \cup \{\xi\}) \cap \kappa = z_n \cap \kappa$.

Define $t_{n+1} : \sup(z_{n+1} \cap \kappa) \to \mathscr{P}_\kappa 2^\lambda \cap C(d)$ by

$$t_{n+1}(\gamma) = \mathrm{cl}_d(t_n(\gamma) \cup \{\xi\}).$$

Note that $\mathrm{dom}\, t_{n+1} = \mathrm{dom}\, t_n$ by $z_{n+1} \cap \kappa = z_n \cap \kappa$. By definition $t_n(\gamma) \subset t_{n+1}(\gamma)$ for every $\gamma \in \mathrm{dom}\, t_n$. It remains to prove the following:

Claim 3. *t_{n+1} witnesses $z_{n+1} \in E$.*

Proof. Since t_n is increasing, so is t_{n+1}. Next we show that

$$\begin{aligned} z_{n+1} = \mathrm{cl}_d(z_n \cup \{\xi\}) &\subset \mathrm{cl}_d\Big(\bigcup \mathrm{ran}\, t_n \cup \{\xi\}\Big) \\ &= \bigcup \{\mathrm{cl}_d(t_n(\gamma) \cup \{\xi\}) : \gamma \in \mathrm{dom}\, t_n\} \\ &= \bigcup \mathrm{ran}\, t_{n+1}. \end{aligned}$$

The first inclusion follows from $z_n \subset \bigcup \mathrm{ran}\, t_n$. Since t_n is increasing, we get the second equality. For last equality, recall the definition of t_{n+1}.

Finally it suffices to show that for every $\delta \in z_{n+1} \cap \kappa$

$$\begin{aligned} \varphi(t^*_{n+1}|\delta) &= \varphi\Big(\Big\langle \bigcup \{e_\alpha(\xi) : \alpha \in t^*_n(\gamma)\} : \gamma < \delta \Big\rangle\Big) \\ &= g(\varphi(t^*_n|\delta), \xi) \\ &\in \mathrm{cl}_d(z_n \cup \{\xi\}) = z_{n+1}. \end{aligned}$$

For the first equality, it suffices to show that for every $\gamma < \delta$

$$\begin{aligned} t^*_{n+1}(\gamma) = t_{n+1}(\gamma) \cap \lambda &= \mathrm{cl}_d(t_n(\gamma) \cup \{\xi\}) \cap \lambda \\ &= \bigcup \{e_\alpha(\xi) : \alpha \in t_n(\gamma) \cap \lambda\} \\ &= \bigcup \{e_\alpha(\xi) : \alpha \in t^*_n(\gamma)\}. \end{aligned}$$

Since $t_n(\gamma) \in C(d)$ and $\xi < \nu$, we get the third equality by the choice of d.

For the second equality, recall the definition of g. For the membership, note that t_n witnesses $z_n \in E$. Hence $\varphi(t_n^*|\delta) \in z_n$ by $\delta \in z_{n+1} \cap \kappa = z_n \cap \kappa$. Since $\mathrm{cl}_d(z_n \cup \{\xi\}) \in \mathscr{P}_\kappa 2^\lambda \cap C(d) \subset D$, $\mathrm{cl}_d(z_n \cup \{\xi\}) \cap \lambda$ is closed under g. Hence we get the desired result. $\square$(Claim 3)

This completes the description of the recursion.

Set

$$z = \bigcup\nolimits_{n<\omega} z_n.$$

We show that $z \cap \lambda \in \bar{S} \cap C(f)$, which completes the proof of Main Claim.

To see that $z \cap \lambda \in \bar{S}$, note that $z \cap \kappa = z_0 \cap \kappa \in S$ and that for every $n < \omega$ there is $\xi \in z_{n+1} \cap \nu$ such that $z \cap \kappa = z_{n+1} \cap \kappa \in X_\xi^n$. It remains to show the following:

Claim 4. $z \cap \lambda \in S(\kappa, \lambda)$.

Proof. Set $\delta = \sup(z \cap \kappa) < \kappa$. Since $z \cap \kappa \in S \subset \{x \in \mathscr{P}_\kappa \kappa : \mathrm{cf}\sup x = \omega\}$, we have $\mathrm{cf}\,\delta = \omega$. Take an unbounded $\{\gamma_n : n < \omega\} \subset z \cap \kappa$ so that $\gamma_n < \gamma_{n+1}$. Recall that $\mathrm{dom}\,t_n = \sup(z_n \cap \kappa) = \sup(z \cap \kappa) = \delta$.

Define $t : \delta \to \mathscr{P}_\kappa 2^\lambda$ by

$$t(\gamma) = t_n(\gamma),$$

where $n = \min\{i < \omega : \gamma < \gamma_i\}$. It suffices to show that $t^* : \delta \to \mathscr{P}_\kappa \lambda$ witnesses $z \cap \lambda \in S(\kappa, \lambda)$.

To see that $z \cap \lambda \subset \bigcup \mathrm{ran}\, t^*$, it suffices to show that for every $n < \omega$

$$z_n \subset \bigcup \mathrm{ran}\, t_n = \bigcup t_n``(\delta - \gamma_n) \subset \bigcup t``(\delta - \gamma_n) \subset \bigcup \mathrm{ran}\, t.$$

For the first inclusion, note that t_n witnesses $z_n \in E$. Since t_n is increasing, we get the equality. For the second inclusion, note that $t_n(\gamma) \subset t_{m+1}(\gamma) = t(\gamma)$ if $n \leq m < \omega$ and $\gamma \in \gamma_{m+1} - \gamma_m$.

To see that $\{\gamma < \delta : \varphi(t^*|\gamma) \in z \cap \lambda\}$ is unbounded in δ, it suffices to show by induction on $n < \omega$ that $\varphi(t^*|\gamma_n) \in z_n$.

For $n = 0$, note that t_0 witnesses $z_0 \in E$. Since $\gamma_0 \in z \cap \kappa = z_0 \cap \kappa$, we have $\varphi(t_0^*|\gamma_0) \in z_0$, as desired. Next assume $\varphi(t^*|\gamma_n) \in z_n$. It suffices to show that

$$\begin{aligned} \varphi(t^*|\gamma_{n+1}) &= \varphi(t^*|\gamma_n \cup t_{n+1}^*|(\gamma_{n+1} - \gamma_n)) \\ &= h(\varphi(t^*|\gamma_n), \varphi(t_{n+1}^*|\gamma_{n+1})) \in z_{n+1}. \end{aligned}$$

The first equality follows from $t|\gamma_{n+1} = t|\gamma_n \cup t_{n+1}|(\gamma_{n+1} - \gamma_n)$. For the second equality, recall the definition of h. For the membership, note that t_{n+1} witnesses $z_{n+1} \in E$. Since $\gamma_{n+1} \in z \cap \kappa = z_{n+1} \cap \kappa$, we have $\varphi(t_{n+1}^*|\gamma_{n+1}) \in z_{n+1}$. Also $\varphi(t^*|\gamma_n) \in z_n \subset z_{n+1}$. Since $z_{n+1} \in \mathscr{P}_\kappa 2^\lambda \cap C(d) \subset D$, $z_{n+1} \cap \lambda$ is closed under h. Thus we get the desired result. $\square$(Claim 4)

To see that $z \cap \lambda \in C(f)$, note that $\{z_n : n < \omega\} \subset D$ is increasing and that D is σ-club in $\mathscr{P}_\kappa 2^\lambda$. Since $z \in D$, we get the desired result. □(Main Claim)

By σ-SR in $\mathscr{P}_\kappa\lambda$ there is $\kappa \subset B \subset \lambda$ of size κ such that $\bar{S} \cap \mathscr{P}_\kappa B$ is σ-stationary in $\mathscr{P}_\kappa B$. Fix a bijection $\pi : \kappa \to B$. Since $\{y \in \mathscr{P}_\kappa B : \pi``(y \cap \kappa) = y\}$ is $\sigma$-club, $\{y \in \bar{S} \cap \mathscr{P}_\kappa B : \pi``(y \cap \kappa) = y\}$ is σ-stationary in $\mathscr{P}_\kappa B$. Hence

$$S^* = \{y \cap \kappa : \pi``(y \cap \kappa) = y \in \bar{S} \cap \mathscr{P}_\kappa B\}$$

is σ-stationary in $\mathscr{P}_\kappa\lambda$. Note that $y \cap \kappa \in S$ for every $y \in \bar{S}$. Hence $S^* \subset S$. We claim that S^* is as required above.

Fix $n < \omega$. First we show that

$$S^* \subset \bigtriangledown\{X^n_{\pi(\zeta)} : \zeta \in B^*\} = \sum\{X^n_{\pi(\zeta)} : \zeta \in B^*\},$$

where $B^* \subset \kappa$ is defined by $\pi``B^* = B \cap \nu$.

Fix $x \in S^*$. It suffices to give $\zeta \in B^* \cap x$ such that $x \in X^n_{\pi(\zeta)}$. Since $x \in S^*$, there is $y \in \bar{S} \cap \mathscr{P}_\kappa B$ such that $\pi``(y \cap \kappa) = y$ and $y \cap \kappa = x$. Since $y \in \bar{S}$, there is $\xi \in y \cap \nu$ such that $y \cap \kappa \in X^n_\xi$. Set $\zeta = \pi^{-1}(\xi)$. Since $\xi \in y \cap \nu \subset B \cap \nu$, we have

$$\zeta = \pi^{-1}(\xi) \in \pi^{-1}``(B \cap \nu) = B^* \text{ and } \zeta = \pi^{-1}(\xi) \in \pi^{-1}``y = y \cap \kappa = x.$$

Moreover we have $x = y \cap \kappa \in X^n_\xi = X^n_{\pi(\zeta)}$, as desired.

Since $\pi : \kappa \to B$ is a bijection, we have

$$S^* \leq \sum\{X^n_{\pi(\zeta)} : \zeta \in B^*\} = \sum\{X^n_\xi : \xi \in B \cap \nu\}.$$

Since $\{X^n_\xi : \xi < \nu\}$ is a maximal antichain in $(\mathscr{C}^\sigma_{\kappa\kappa})^+$, we have

$$\{\xi < \nu : S^* \cap X^n_\xi \in (\mathscr{C}^\sigma_{\kappa\kappa})^+\} \subset B \cap \nu.$$

Since $|B| = \kappa$, we have $|\{\xi < \nu : S^* \cap X^n_\xi \in (\mathscr{C}^\sigma_{\kappa\kappa})^+\}| \leq \kappa$, as desired. This completes the proof of Theorem 13. □

Proof of Theorem 3. By Proposition 11 with $\lambda = 2^\nu$ the model of Theorem 3 satisfies σ-SR in $\mathscr{P}_\kappa 2^\nu$ with $\nu = 2^{2^{<\kappa}}$. Hence by Theorem 13 the σ-club filter on $\mathscr{P}_\kappa\kappa$ is weakly presaturated below the set $\{x \in \mathscr{P}_\kappa\kappa : \mathrm{cf}\sup x = \omega\}$ in the model.

Fix a regular uncountable cardinal $\mu \leq \kappa$. Note that the σ-club filter on $\mathscr{P}_\mu\kappa$ is identical to the σ-club filter on $\mathscr{P}_\kappa\kappa$ below the set $\mathscr{P}_\mu\kappa$. Hence the σ-club filter on $\mathscr{P}_\mu\kappa$ is weakly presaturated below the set $\{x \in \mathscr{P}_\mu\kappa : \mathrm{cf}\sup x = \omega\}$. Therefore the club filter on $\mathscr{P}_\mu\kappa$ is weakly presaturated below the set $\{x \in \mathscr{P}_\mu\kappa : \mathrm{cf}\sup x = \omega\}$. □

Remark. Work of Steel [**27**] strongly suggests that we need to assume in Theorem 3 that ν is a Woodin cardinal. It is likely that Theorem 3 can be proved directly under the weaker hypothesis. However the direct proof should be much more involved. (Compare the proof of Theorem 8 (1) with that of Theorem 2 from [**10**].)

In [**28**] Woodin introduced the stationary tower forcing $\mathbf{P}_{<\nu}$. He proved that if ν is a Woodin cardinal, then the generic ultrapower by $\mathbf{P}_{<\nu}$ is wellfounded and is closed under sequences of length $< \nu$ (see [**14**]). In view of [**4**] it is natural to wonder if $\mathbf{P}_{<\nu}$ parallels the σ-club filter on $\mathscr{P}_\kappa\kappa$ in the extension by $\mathrm{Col}(\kappa,\nu)$. More specifically one may ask whether the σ-club filter on $\mathscr{P}_\kappa\kappa$ is presaturated in the model of Theorem 3. The answer is positive just in the case $\kappa = \omega_1$. This follows from Theorem 1 and the results of [**3**], [**20**] mentioned in Section 1. We do not know, however, whether the σ-club filter on $\mathscr{P}_\kappa\kappa$ is weakly presaturated in the model if $\kappa > \omega_1$. For more on the problem see Section 6.

6. Concluding remarks.

In [**11**] Goldring established the following:

THEOREM 14. *Suppose that $\omega < \kappa < \nu$, κ is regular and ν is supercompact. Then the σ-club filter on $\mathscr{P}_\kappa\kappa$ is precipitous in the extension by* $\mathrm{Col}(\kappa,\nu)$.

In particular the club filter on $\mathscr{P}_\mu\kappa$ is precipitous for every regular uncountable cardinal $\mu \le \kappa$. The latter result had been proved in [**5**] below some stationary set. The stationary set is the projection of IA to $\mathscr{P}_\mu\kappa$. In the same paper IA* was introduced and was shown in effect to project to a club set in $\mathscr{P}_\mu\kappa$. Goldring [**11**] showed that the proof of [**5**] went through with IA replaced by IA*.

In Section 5 Theorem 3 was proved as a corollary to Proposition 11 and Theorem 13. Likewise Theorem 14 follows from Proposition 11 and the following:

THEOREM 15. *Assume* σ-SR *in* $\mathscr{P}_\kappa 2^{2^{2^{<\kappa}}}$. *Then the σ-club filter on $\mathscr{P}_\kappa\kappa$ is precipitous.*

Theorem 15 is proved in effect by the proof of Theorem 13, although the former is not literally a corollary of the latter. Let us see this in more detail. Recall from [**12**] that a filter F is precipitous if and only if the following holds: Suppose that $S \in F^+$ and $\{A_n : n < \omega\}$ is a set of maximal antichains below S such that A_{n+1} refines A_n for every $n < \omega$. Then there is a descending sequence $\langle X_n : n < \omega\rangle$ in F^+ such that $X_n \in A_n$ and $\bigcap_{n<\omega} X_n \neq \emptyset$. Hence in the notation of the proof of Theorem 13 the claim $\bar{S} \neq \emptyset$ entails in effect that the σ-club filter on $\mathscr{P}_\kappa\kappa$ is precipitous. In particular Theorem 15 is proved by the proofs of Claims 1–3. In other words, in the proof of Theorem 13 the restriction to the set $\{x \in \mathscr{P}_\kappa\kappa : \mathrm{cf}\sup x = \omega\}$ was invoked only to prove Claim 4.

Fix a regular cardinal ν with $\kappa \le \nu \le \lambda$. Set

$$S^\nu_{\kappa\lambda} = \{x \in \mathscr{P}_\kappa\lambda : \mathrm{cf}\sup(x \cap \nu) = \omega\}.$$

Suppose that F is a normal filter on $\mathscr{P}_\kappa\lambda$ and $S^\nu_{\kappa\lambda} \in F^+$. For each $x \in S^\nu_{\kappa\lambda}$ fix an unbounded $\{\gamma^x_i : i < \omega\} \subset x \cap \nu$. For each $i < \omega$ and $\beta < \nu$ set

$$S^\beta_i = \{x \in S^\nu_{\kappa\lambda} : \gamma^x_i = \beta\}.$$

The prototype of Lemma 16 can be found in [**6**]. See [**24**], [**25**] for further applications of these lemmas.

LEMMA 16. *Suppose that* $S \leq S^{\nu}_{\kappa\lambda}$ *in* F^+. *Then there is* $i < \omega$ *such that the set* $\{\beta < \nu : S \cap S^{\beta}_i \in F^+\}$ *is unbounded in* ν.

PROOF. If not, then for each $i < \omega$ there is $\alpha_i < \nu$ such that for every $\beta \in \nu - \alpha_i$ there is $C^{\beta}_i \in F$ with $S \cap S^{\beta}_i \cap C^{\beta}_i = \emptyset$. Set

$$\alpha = \sup\nolimits_{i<\omega} \alpha_i \text{ and } C = \bigcap\nolimits_{i<\omega} \triangle\{C^{\beta}_i : \beta \in \nu - \alpha_i\}.$$

Since $\nu > \omega$ is regular and F is normal, we have $\alpha < \nu$ and $C \in F$. Since $S \leq S^{\nu}_{\kappa\lambda}$, there is $x \in S \cap C$ such that $\alpha < \sup(x \cap \nu)$ and $\operatorname{cf}\sup(x \cap \nu) = \omega$. Since $\{\gamma^x_i : i < \omega\}$ is unbounded in $x \cap \nu$, there is $i < \omega$ such that $\alpha < \gamma^x_i$. Set $\beta = \gamma^x_i$. Then $x \in S^{\beta}_i$ by definition. Since $\alpha_i \leq \alpha < \beta = \gamma^x_i \in x \in C$, we have $x \in C^{\beta}_i$. This contradicts that $S \cap S^{\beta}_i \cap C^{\beta}_i = \emptyset$, as desired. □

By Proposition 17 with $\kappa = \nu = \lambda$ we get a countable set of ordinals in the extension by $(\mathscr{C}^{\sigma}_{\kappa\kappa})^+$ below the set $S^{\kappa}_{\kappa\kappa} = \{x \in \mathscr{P}_\kappa\kappa : \operatorname{cf}\sup x = \omega\}$ that cannot be covered by any set of size $< \kappa$ in the ground model. This shows that Theorem 13 is optimal with respect to the size of the covering sets.

PROPOSITION 17. $S^{\nu}_{\kappa\lambda} \in F^+$ *forces* $\operatorname{cf}\nu = \omega$.

PROOF. By definition $S^{\alpha}_i \cap S^{\beta}_i = \emptyset$ if $\alpha < \beta < \nu$ and $i < \omega$. Hence we can define a F^+-name $\dot{g}$ of a map from ω to ν so that S^{β}_i forces $\dot{g}(i) = \beta$ if $S^{\beta}_i \in F^+$. It suffices to show that $S^{\nu}_{\kappa\lambda}$ forces that $\dot{g}$ is cofinal.

Fix $S \leq S^{\nu}_{\kappa\lambda}$ from F^+ and $\alpha < \nu$. By Lemma 16 there are $i < \omega$ and $\beta \in \nu - \alpha$ such that $S \cap S^{\beta}_i \in F^+$. Then $S \cap S^{\beta}_i$ forces $\dot{g}(i) = \beta \geq \alpha$, as desired. □

In [**8**] Gitik and Shelah observed that F is precipitous if F^+ is proper. In fact F is weakly presaturated if F^+ is proper. This is because every countable set of ordinals in the extension by a proper poset can be covered by a countable set in the ground model. Matsubara and Shelah [**18**] proved that F^+ is not proper if F is a normal κ-complete filter on $\mathscr{P}_\kappa\lambda$ and $\{x \in \mathscr{P}_\kappa\lambda : \operatorname{cf}(x \cap \kappa) = \mu\} \in F^+$ for some μ with $\mu^+ < \kappa$. Proposition 17 shows that F^+ is not proper if F is a normal filter on $\mathscr{P}_\kappa\lambda$ and $S^{\nu}_{\kappa\lambda} \in F^+$ for some ν.

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