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On vanishing of L^2-Betti numbers for groups

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Abstract. We show that if a group G admits a finite dimensional contractible G-CW-complex X then the vanishing of the L^2-Betti numbers for all stabilizers G_σ of X determines that of the L^2-Betti numbers for G. We also give a relation among the L^2-Euler characteristics for X as a G-CW-complex and those for X as a G_σ-CW-complex under certain assumptions. Finally, we present a new class of groups satisfying the Chatterji-Mislin conjecture which amounts to putting Brown's formula within the framework of L^2-homology.

1. Introduction.

Let G be an arbitrary discrete group. In 1998, Lück [**9**], [**10**] defined the L^2-Betti numbers for an arbitrary G-space X using the extended von Neumann dimension of the $\mathscr{N}(G)$-modules $H_p^G(X, \mathscr{N}(G))$, motivated by Farber's work [**7**]. These L^2-Betti numbers extend the classical notion of L^2-Betti numbers for free G-CW-complexes of finite type. The L^2-Betti numbers for an arbitrary group G are defined as that of the classifying space EG. There are many results of the L^2-Betti numbers for an arbitrary G-spaces and groups. Notably, Cheeger and Gromov [**5**], [**11**] showed that the L^2-Betti numbers $b_p^{(2)}(G)$ for a group G which possess an infinite amenable normal subgroup vanish for all $p \geq 0$. Recently, Schafer [**12**, Corollary 3.11] extended this result for the case of groups G when G is the fundamental group of a graph of infinite amenable groups. The purpose of this paper is to give a vanishing result for a certain class of groups and its applications. Using the argument of Schafer, we show that if a group G admits a finite dimensional contractible G-CW-complex X for which the L^2-Betti numbers $b_p^{(2)}(G_\sigma)$ for all stabilizers G_σ of X vanish for $0 \leq p \leq n$ then so does the L^2-Betti numbers $b_p^{(2)}(G)$ for G, where n is a nonnegative integer or ∞ (Theorem 3.5). Also we obtain a relation among the L^2-Euler characteristics for X as a G-CW-complex and those for X as a G_σ-CW-complex under certain assumptions (Corollary 3.8). These results extend those of Schafer [**12**, Corollaries 3.11, 3.14] and Chatterji and Mislin [**4**, Lemma 2.4].

On the other hand, there is another interesting application of the above L^2-Euler characteristics formula between a group G acting on a finite-dimensional cocompact contractible G-CW-complex and stabilizers G_σ. In [**4**], Chatterji and Mislin gave a class of groups which satisfy their new conjecture [**4**, Conjecture 1] which is a generalization of Brown's formula [**3**]. Based on the idea due to Chatterji and Mislin, we give a new class of groups satisfying Chatterji-Mislin conjecture using Corollary 3.8 (Theorem 4.4, Remark 4.7).

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2. Preliminaries.

In this section, we briefly recall the notion of L^2-Betti numbers and L^2-Euler characteristics for arbitrary G-spaces and groups [**9**], [**10**], [**11**]. We also review the definition of $\mathscr{C}$-exact and weak $\mathscr{C}$-exact sequences of $\mathscr{N}(G)$-modules which was given by Schafer [**12**] and the Chatterji-Mislin conjecture and related facts [**4**]. For more details, we recommend each reference.

1. Let G be a discrete group. Let $l^2(G)$ be the Hilbert space of formal sums $\sum_{g\in G}\lambda_g\cdot g$ with complex coefficient λ_g such that $\sum_{g\in G}|\lambda_g|^2<\infty$. The group von Neumann algebra $\mathscr{N}(G)$ is the C^*-algebra $B(l^2(G))^G$ of G-equivariant bounded operators from $l^2(G)$ to $l^2(G)$.

Note that the von Neumann algebra $\mathscr{N}(G)$ is flat over $\boldsymbol{C}G$ if G is virtually cyclic, i.e., G is finite or contains $\boldsymbol{Z}$ as a normal subgroup of finite index and conjecturally these are the only ones [**11**, Conjecture 6.49]. Note also that if G is amenable, then the von Neumann algebra $\mathscr{N}(G)$ is dimension-flat over $\boldsymbol{C}G$, namely, for each $\boldsymbol{C}G$-module M

$$\dim_{\mathscr{N}(G)}(\mathrm{Tor}_p^{\boldsymbol{C}G}(\mathscr{N}(G),M))=0\quad\text{if } p\geq 1.$$

It is also conjectured whether the property of dimension-flat of the von Neumann algebra $\mathscr{N}(G)$ characterizes the amenability of G [**11**, Conjecture 6.48].

For any $\mathscr{N}(G)$-module M, there is the extended von Neumann dimension function $\dim_{\mathscr{N}(G)}(M)$ which takes values in $[0,\infty]$, which is a priori defined for finitely generated projective $\mathscr{N}(G)$-modules and is uniquely determined by Additivity, Cofinality and Continuity [**11**]. Due to this extended dimension function, the notion of L^2-Betti numbers for an arbitrary G-space X can be defined as follows. For an arbitrary G-space X, its p-th L^2-Betti number is defined by

$$b_p^{(2)}(X):=\dim_{\mathscr{N}(G)}(H_p^G(X,\mathscr{N}(G))),$$

where $H_p^G(X,\mathscr{N}(G))$ is the homology of the $\mathscr{N}(G)$-chain complex $\mathscr{N}(G)\otimes_{\boldsymbol{Z}G}C_*^{\mathrm{sing}}(X)$. The L^2-Euler characteristic of an arbitrary G-space X is defined by

$$\chi^{(2)}(X;G)=\chi^{(2)}(X):=\sum_{p\geq 0}(-1)^p\cdot b_p^{(2)}(X)$$

provided that $h^{(2)}(X):=\sum_{p\geq 0}b_p^{(2)}(X)<\infty$. A G-space X is called L^2-finite if $h^{(2)}(X)<\infty$. Thus the condition of being L^2-finite ensures that the L^2-Euler characteristic of a G-space X, $\chi^{(2)}(X)$ converges absolutely. We define for any discrete group G its p-th L^2-Betti number by $b_p^{(2)}(G):=b_p^{(2)}(EG)$. The L^2-Euler characteristic of G is defined by $\chi^{(2)}(G):=\chi^{(2)}(EG)$ provided that $h^{(2)}(G):=h^{(2)}(EG)<\infty$.

2. Let $\mathscr{C}$ denote the class of $\mathscr{N}(G)$-modules of dimension zero. Notice that the class of $\mathscr{N}(G)$-modules $\mathscr{C}$ contains the zero module and if $0\to A\to B\to C\to 0$ is exact, then $B\in\mathscr{C}$ if and only if $A\in\mathscr{C}$ and $C\in\mathscr{C}$. A sequence of $\mathscr{N}(G)$-modules $\cdots\overset{\partial_{n+2}}{\to}M_{n+1}\overset{\partial_{n+1}}{\to}M_n\overset{\partial_n}{\to}M_{n-1}\overset{\partial_{n-1}}{\to}\cdots$ is called $\mathscr{C}$-exact if for each pair of consecutive maps ∂_{n+1} and ∂_n, the following two conditions hold:

(a) $\partial_n \circ \partial_{n+1} = 0$.
(b) $\ker \partial_n / \mathrm{im}\partial_{n+1} \in \mathscr{C}$.

A sequence of $\mathscr{N}(G)$-modules $\cdots \xrightarrow{\partial_{n+2}} M_{n+1} \xrightarrow{\partial_{n+1}} M_n \xrightarrow{\partial_n} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots$ is called weak $\mathscr{C}$-exact if there exist $\mathscr{N}(G)$-modules M_n' such that (1) For all n, M_n is $\mathscr{C}$-isomorphic to M_n' and (2) There exists an exact sequence of $\mathscr{N}(G)$-modules $\cdots \to M_{n+1}' \to M_n' \to M_{n-1}' \to \cdots$.

Notice that if $\cdots \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to \cdots$ is a weak $\mathscr{C}$-exact sequence of $\mathscr{N}(G)$-modules then the following holds: $\dim_{\mathscr{N}(G)}(M_{i+1}) = \dim_{\mathscr{N}(G)}(M_{i-1}) = 0$ implies $\dim_{\mathscr{N}(G)}(M_i) = 0$.

3. For a group G, we denote the Hattori-Stallings Trace (cf. [**4**]) as follows:

$$\mathrm{HS} : K_0(\boldsymbol{C}G) \to HH_0(\boldsymbol{C}G) = \bigoplus_{[G]} \boldsymbol{C}.$$

Here $K_0(\boldsymbol{C}G)$ is the projective class group of $\boldsymbol{C}G$, $HH_0(\boldsymbol{C}G)$ is the Hochschild homology of $\boldsymbol{C}G$ and $[G]$ means the set of conjugacy classes of elements in G. If P is a finitely generated projective $\boldsymbol{C}G$-module and $[P] \in K_0(\boldsymbol{C}G)$ the corresponding element, then we write

$$\mathrm{HS}(P) := \mathrm{HS}([P]) = \sum_{[s]\in[G]} \mathrm{HS}(P)(s) \cdot [s] \in \bigoplus_{[G]} \boldsymbol{C},$$

where $\mathrm{HS}(P)(s)$ depending only on the conjugacy class $[s]$ of $s \in G$.

Let G be a group of type FP over $\boldsymbol{C}$, i.e., $\boldsymbol{C}$ admits a projective resolution over $\boldsymbol{C}G$

$$P_* : 0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to \boldsymbol{C},$$

where each P_i is finitely generated projective over $\boldsymbol{C}G$. For a group of type FP over $\boldsymbol{C}$, the complete Euler characteristic $E(G)$ of G is defined by an alternating sum of the Hattori-Stallings rank of finite generated projective modules. Thus $E(G)$ is a finite linear combination of the conjugacy classes $[s]$ of elements of G. Denote by $E(G)(s)$ the coefficient of the conjugacy class $[s]$ of an element $s \in G$. Notice that $E(G)(1) = e(G)$ is the Euler characteristic of G in the sense of Bass [**1**] and Chiswell [**6**]. The element $W(G) := \sum_i (-1)^i [P_i] \in K_0(\boldsymbol{C}G)$ depends only on G and is called the Wall element. Under the Hattori-Stallings Trace, the Wall element $W(G)$ is mapped to the complete Euler characteristic of G, $E(G) = \sum_{[s]\in[G]} E(G)(s) \cdot [s]$. In [**3**], Brown conjectured under suitable finiteness conditions for G the following formula:

$$E(G)(s) = \begin{cases} e(C_G(s)) & \text{if } s \text{ has finite order} \\ 0 & \text{otherwise,} \end{cases}$$

and proved it in many cases. Brown's assumptions require in particular $C_G(s)$ to be of type FP over $\boldsymbol{C}$. In [**4**], Chatterji and Mislin proposed the following conjecture which amounts to putting Brown's formula within the framework of L^2-homology and proved it for a class of groups containing all G admitting a cocompact $\underline{E}G$, the classifying space for proper action.

Conjecture A (Chatterji and Mislin [**4**]). *Let G be a group of type FP over $\boldsymbol{C}$ such that the centralizer of every element of finite order in G has finite L^2-Betti numbers. Then for every $s \in G$,*

$$E(G)(s) = \chi^{(2)}(C_G(s)). \tag{$\star$}$$

Notice that if $C_G(s)$ is of type FP over $\boldsymbol{C}$, then Conjecture A implies Brown's formula [**4**]. Notice also that if G satisfies Conjecture A, then

$$\chi(G) = \sum_{[s]\in[G]} \chi^{(2)}(C_G(s)),$$

where $\chi(G) := \Sigma(-1)^i \dim_{\boldsymbol{C}} H_i(G, \boldsymbol{C})$ [**4**, Corollary 4.5]. Thus if $K(G,1)$ is a finite complex, then this formula implies the well-known result: $\chi(G) = \chi^{(2)}(G)$ (cf. [**11**]).

3. Vanishing results for L^2-Betti numbers for groups.

Throughout this paper, we employ the following conventions: G is a discrete group and $\boldsymbol{Z}G$ is its group ring. We denote tensor product over $\boldsymbol{Z}G$ by $-\otimes_G -$. The notation $-\otimes-$ means tensor product over $\boldsymbol{Z}$.

Definition 3.1 ([**11**, Definition 7.1]). Let n be a non-negative integer or ∞. Define $\mathscr{B}_n$ to be the class of groups G whose L^2-betti numbers $b_p^{(2)}(G)$ vanish for $0 \leq p \leq n$, i.e.,

$$\mathscr{B}_n := \{G \mid b_p^{(2)}(G) = 0,\ 0 \leq p \leq n\}.$$

Notice that $\mathscr{B}_0$ is the class of infinite groups and $\mathscr{B}_\infty$ contains the Thompson group and all groups which contain an infinite amenable normal subgroup [**11**].

Consider the class of groups G which admit a finite-dimensional contractible G-CW-complex for which all stabilizers belongs to the class of groups $\mathscr{B}_n$. It is clear that this class contains $\mathscr{B}_n$. In Theorem 3.5, we will show that in fact these two classes are equal. Until we prove that these two classes of groups coincide, we use the temporary notation $\mathscr{C}_n$ for the above class of groups, where $0 \leq n \leq \infty$.

Note that if G is the fundamental group of a graph of infinite amenable groups then there exists an 1-dimensional contractible G-CW-complex X with infinite amenable stabilizers [**13**]. Thus $\mathscr{C}_\infty$ contains the class of groups which are the fundamental group of a graph of infinite amenable groups.

Lemma 3.2. *Let G be an arbitrary group. For any $\mathscr{N}(G)$-module M and $\boldsymbol{Z}$-module N,*

$$\dim_{\mathscr{N}(G)}(M \otimes N) = \dim_{\mathscr{N}(G)}(M) \cdot \dim_{\boldsymbol{C}}(\boldsymbol{C} \otimes N)$$

with the convention $0 \cdot \infty = \infty \cdot 0 = 0$.

PROOF. It follows directly from a special case of [**11**, Theorem 6.104 (2)]. □

In what follows, we denote the cellular chain complex of a G-CW complex X as $C_*(X)$. Recall that $C_i(X) \cong \bigoplus_{\sigma\in\sum_i} \boldsymbol{Z}(G/G_\sigma) \cong \bigoplus_{\sigma\in\sum_i} \boldsymbol{Z}G \otimes_{\boldsymbol{Z}G_\sigma} \boldsymbol{Z}$, where G_σ is the stabilizer of σ and $\sum_i$ is a set of representatives for the G-orbits of i-cells.

PROPOSITION 3.3. *Suppose that a group G admits a finite-dimensional contractible G-CW-complex Y and X is a G-CW-complex for which all stabilizers are amenable. Then there exists a $\mathscr{C}$-exact sequence of $\mathscr{N}(G)$-modules*

$$\begin{aligned}0 &\to \bigoplus_{\sigma\in\sum_m} (\mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma))\otimes_G C_*(X) \to \bigoplus_{\sigma\in\sum_{m-1}} (\mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma))\otimes_G C_*(X)\\ &\to \cdots \to \bigoplus_{\sigma\in\sum_0} (\mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma))\otimes_G C_*(X) \to \mathscr{N}(G)\otimes_G C_*(X)\to 0,\end{aligned}$$

where $m=\dim Y$, G_σ is the stabilizer of $\sigma\in Y$ and $\sum_i$ is a set of representatives for the G-orbits of i-cells of Y.

PROOF. Let $C_*(Y)$ be the cellular chain complex of Y. Then there exists an exact sequence of $\boldsymbol{Z}G$-modules

$$0\to \bigoplus_{\sigma\in\sum_m} \boldsymbol{Z}(G/G_\sigma) \xrightarrow{\partial_m} \cdots \xrightarrow{\partial_1} \bigoplus_{\sigma\in\sum_0} \boldsymbol{Z}(G/G_\sigma) \to \boldsymbol{Z}\to 0,$$

and tensoring this with $\mathscr{N}(G)$ over $\boldsymbol{Z}$ we obtain the exact sequence of $\mathscr{N}(G)$-modules

$$0\to \bigoplus_{\sigma\in\sum_m} (\mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)) \xrightarrow{\partial_m} \cdots \xrightarrow{\partial_1} \bigoplus_{\sigma\in\sum_0} (\mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)) \xrightarrow{\varepsilon} \mathscr{N}(G)\to 0.$$

Here $1\otimes\partial_m$ $(1\otimes\varepsilon)$ is denoted by ∂_m (ε, respectively) by a slight abuse of notation. Consider first the exact sequence of $\mathscr{N}(G)$-modules

$$0\to \ker\varepsilon \to \bigoplus_{\sigma\in\sum_0} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma) \xrightarrow{\varepsilon} \mathscr{N}(G)\to 0.$$

Tensoring this exact sequence with $C_*(X)$ over $\boldsymbol{Z}G$ we have the exact sequence

$$\begin{aligned}\cdots &\to \mathrm{Tor}_2^G(\mathscr{N}(G), C_*(X)) \to \mathrm{Tor}_1^G(\ker\varepsilon, C_*(X))\\ &\to \mathrm{Tor}_1^G\left(\bigoplus_{\sigma\in\sum_0} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma), C_*(X)\right)\end{aligned} \tag{3.1}$$

$$\begin{aligned}&\to \operatorname{Tor}_1^G(\mathscr{N}(G), C_*(X)) \to \ker\varepsilon \otimes_G C_*(X)\\ &\to \bigoplus_{\sigma\in\sum_0} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)\otimes_G C_*(X) \to \mathscr{N}(G)\otimes_G C_*(X)\to 0.\end{aligned}$$

Since all stabilizers of X are amenable, we deduce from [**12**, Proposition 3.1] that

$$\dim_{\mathscr{N}(G)}(\operatorname{Tor}_2^G(\mathscr{N}(G), C_*(X))) = \dim_{\mathscr{N}(G)}(\operatorname{Tor}_1^G(\mathscr{N}(G), C_*(X))) = 0.$$

Thus the following sequence

$$\begin{aligned}0 \to \ker\varepsilon\otimes_G C_*(X) \to \bigoplus_{\sigma\in\sum_0} (\mathscr{N}(G)\otimes\boldsymbol{Z}(G/G_\sigma))\otimes_G C_*(X)\\ \to \mathscr{N}(G)\otimes_G C_*(X)\to 0\end{aligned} \tag{3.2}$$

is $\mathscr{C}$-exact. Since $\boldsymbol{Z}(G/G_\sigma)$ is $\boldsymbol{Z}$-flat, we have

$$\operatorname{Tor}_1^G(\mathscr{N}(G)\otimes\boldsymbol{Z}(G/G_\sigma), C_*(X)) \cong \operatorname{Tor}_1^G(\mathscr{N}(G), C_*(X))\otimes\boldsymbol{Z}(G/G_\sigma).$$

By Lemma 3.2, we have

$$\begin{aligned}&\dim_{\mathscr{N}(G)}(\operatorname{Tor}_1^G(\mathscr{N}(G)\otimes\boldsymbol{Z}(G/G_\sigma), C_*(X)))\\ &\quad= \dim_{\mathscr{N}(G)}(\operatorname{Tor}_1^G(\mathscr{N}(G), C_*(X))\otimes\boldsymbol{Z}(G/G_\sigma))\\ &\quad= \dim_{\mathscr{N}(G)}(\operatorname{Tor}_1^G(\mathscr{N}(G), C_*(X)))\cdot\dim_{\boldsymbol{C}}(\boldsymbol{C}\otimes\boldsymbol{Z}(G/G_\sigma))\\ &\quad= 0\end{aligned}$$

and thereby

$$\begin{aligned}&\dim_{\mathscr{N}(G)}\left(\operatorname{Tor}_1^G\left(\bigoplus_{\sigma\in\sum_0}\mathscr{N}(G)\otimes\boldsymbol{Z}(G/G_\sigma), C_*(X)\right)\right)\\ &\quad= \dim_{\mathscr{N}(G)}\left(\bigoplus_{\sigma\in\sum_0}\operatorname{Tor}_1^G(\mathscr{N}(G)\otimes\boldsymbol{Z}(G/G_\sigma), C_*(X))\right)\\ &\quad= \sum_{\sigma\in\sum_0}\dim_{\mathscr{N}(G)}(\operatorname{Tor}_1^G(\mathscr{N}(G)\otimes\boldsymbol{Z}(G/G_\sigma), C_*(X)))\\ &\quad= 0.\end{aligned}$$

Since $\dim_{\mathscr{N}(G)}(\operatorname{Tor}_2^G(\mathscr{N}(G), C_*(X))) = 0$, we deduce from the exact sequence (3.1) that

$$\dim_{\mathscr{N}(G)}(\mathrm{Tor}_1^G(\ker\varepsilon, C_*(X))) = 0.$$

Now consider the following short exact sequence

$$0 \to \ker\partial_1 \to \bigoplus_{\sigma\in\sum_1} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma) \to \ker\varepsilon \to 0.$$

Tensoring this exact sequence with $C_*(X)$ over $\boldsymbol{Z}G$ we have the exact sequence

$$\begin{aligned}\mathrm{Tor}_1^G(\ker\varepsilon, C_*(X)) &\to \ker\partial_1 \otimes_G C_*(X)\\ &\to \bigoplus_{\sigma\in\sum_1} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)\otimes_G C_*(X)\\ &\to \ker\varepsilon\otimes_G C_*(X) \to 0.\end{aligned}$$

Since $\dim_{\mathscr{N}(G)}(\mathrm{Tor}_1^G(\ker\varepsilon, C_*(X))) = 0$, we have the $\mathscr{C}$-exact sequence

$$\begin{aligned}0 \to \ker\partial_1 \otimes_G C_*(X) \to &\bigoplus_{\sigma\in\sum_1} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)\otimes_G C_*(X)\\ &\to \ker\varepsilon\otimes_G C_*(X) \to 0.\end{aligned} \tag{3.3}$$

Splicing the two $\mathscr{C}$-exact sequences (3.2) and (3.3), we have the following $\mathscr{C}$-exact sequence

$$\begin{aligned}0 \to \ker\partial_1 \otimes_G C_*(X) &\to \bigoplus_{\sigma\in\sum_1} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)\otimes_G C_*(X)\\ \to \bigoplus_{\sigma\in\sum_0} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)&\otimes_G C_*(X) \to \mathscr{N}(G)\otimes_G C_*(X) \to 0\end{aligned}$$

Continuing this process, we have the desired result. □

LEMMA 3.4. *Let G be an arbitrary group and X be an arbitrary G-CW-complex. Then for any stabilizer G_σ of X,*

$$H_*^G(X, \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)) \cong \mathscr{N}(G)\otimes_{\mathscr{N}(G)s} H_*^{G_\sigma}(X, \mathscr{N}(G)s).$$

PROOF. It can be proved by the same method as used in the proof of [**12**, Theorem 3.10]. □

THEOREM 3.5. *Let n be a non-negative integer or ∞. If a group G belongs to the class $\mathscr{C}_n$ of groups, then $b_p^{(2)}(G) = 0$ for all $0 \le p \le n$. Hence the classes $\mathscr{B}_n$ and $\mathscr{C}_n$ of groups coincide.*

PROOF. Since $G \in \mathscr{C}_n$, there is a finite-dimensional contractible G-CW-complex Y for which all stabilizers of Y belong to the class B_n of groups. It follows from Proposition 3.3 that there exists a $\mathscr{C}$-exact sequence of $\mathscr{N}(G)$-modules

$$\begin{aligned} 0 \to & \bigoplus_{\sigma\in\sum_m} (\mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma))\otimes_G C_*(EG) \\ \xrightarrow{\partial_m} & \bigoplus_{\sigma\in\sum_{m-1}} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)\otimes_G C_*(EG) \xrightarrow{\partial_{m-1}} \cdots \\ \xrightarrow{\partial_1} & \bigoplus_{\sigma\in\sum_0} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)\otimes_G C_*(EG) \xrightarrow{\partial_0} \mathscr{N}(G)\otimes_G C_*(EG) \to 0, \end{aligned}$$

where $m = \dim Y$. Consider first the $\mathscr{C}$-exact sequence of $\mathscr{N}(G)$-modules

$$\begin{aligned} 0 \to & \bigoplus_{\sigma\in\sum_m} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)\otimes_G C_*(EG) \\ \xrightarrow{\partial_m} & \bigoplus_{\sigma\in\sum_{m-1}} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)\otimes_G C_*(EG) \to \mathrm{im}\partial_{m-1} \to 0. \end{aligned}$$

From Lemma 3.4 and [**12**, Proposition 3.6] it follows that there is the weak $\mathscr{C}$-exact sequence

$$\begin{aligned} \cdots \to & \bigoplus_{\sigma\in\sum_m} \mathscr{N}(G)\otimes_{\mathscr{N}(G_\sigma)} H_n^{G_\sigma}(EG, \mathscr{N}(G_\sigma)) \\ \to & \bigoplus_{\sigma\in\sum_{m-1}} \mathscr{N}(G)\otimes_{\mathscr{N}(G_\sigma)} H_n^{G_\sigma}(EG, \mathscr{N}(G_\sigma)) \to H_n(\mathrm{im}\partial_{m-1}) \\ \to & \bigoplus_{\sigma\in\sum_m} \mathscr{N}(G)\otimes_{\mathscr{N}(G_\sigma)} H_{n-1}^{G_\sigma}(EG, \mathscr{N}(G_\sigma)) \to \cdots \\ \to & \bigoplus_{\sigma\in\sum_{m-1}} \mathscr{N}(G)\otimes_{\mathscr{N}(G_\sigma)} H_0^{G_\sigma}(EG, \mathscr{N}(G_\sigma)) \to H_0(\mathrm{im}\partial_{m-1}) \to 0. \end{aligned} \tag{3.4}$$

Notice that for any stabilizer G_σ of Y, EG is a model for EG_σ. Since $b_p^{(2)}(G_\sigma) = 0$ for $0 \le p \le n$, we have $\dim_{\mathscr{N}(G)}(H_p(\mathrm{im}\partial_{m-1})) = 0$ for $0 \le p \le n$ by the weak $\mathscr{C}$-exact sequence (3.4).

Now consider the $\mathscr{C}$-exact sequence of $\mathscr{N}(G)$-modules

$$0 \to \mathrm{im}\partial_{m-1} \to \bigoplus_{\sigma\in\sum_{m-2}} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)\otimes_{G_\sigma} C_*(EG) \to \mathrm{im}\partial_{m-2} \to 0.$$

Repeated application of Lemma 3.4 and [**12**, Proposition 3.6] deduces that there is the weak $\mathscr{C}$-exact sequence

$$\begin{aligned}\cdots \to H_n(\mathrm{im}\partial_{m-1}) \to & \bigoplus_{\sigma\in\sum_{m-2}} \mathscr{N}(G)\otimes_{\mathscr{N}(G_\sigma)} H_n^{G_\sigma}(EG,\mathscr{N}(G_\sigma)) \\ \to H_n(\mathrm{im}\partial_{m-2}) & \to H_{n-1}(\mathrm{im}\partial_{m-1}) \to \cdots \\ \to \bigoplus_{\sigma\in\sum_{m-2}} & \mathscr{N}(G)\otimes_{\mathscr{N}(G_\sigma)} H_0^{G_\sigma}(EG,\mathscr{N}(G_\sigma)) \to H_0(\mathrm{im}\partial_{m-2}) \to 0.\end{aligned} \tag{3.5}$$

Since $b_p^{(2)}(G_\sigma)=0$ and $\dim_{\mathscr{N}(G)}(H_p(\mathrm{im}\partial_{m-1}))=0$ for $0\le p\le n$, we have $\dim_{\mathscr{N}(G)}(H_p(\mathrm{im}\partial_{m-2}))=0$ for $0\le p\le n$. Continuing this process, we can conclude that $b_p^{(2)}(G)=0$ for all $0\le p\le n$. □

Recall that the Bredon cohomological dimension $\underline{\mathrm{cd}}\,G$ is the cohomological dimension of $\underline{\boldsymbol{Z}}$ in the category of functors $\mathrm{Mod}_{\mathscr{F}}G$. For more details, see [**2**], [**8**].

COROLLARY 3.6. *Let n be a non-negative integer or ∞ and let $1\to N\to G\to Q\to 1$ be a short exact sequence of groups. If N belongs to the class $\mathscr{B}_n$ of groups and $\underline{\mathrm{cd}}Q<\infty$, then G belongs to the class $\mathscr{B}_n$ of groups.*

PROOF. Suppose first that $N\in\mathscr{B}_n$ and $\underline{\mathrm{cd}}Q<\infty$. Since $\underline{\mathrm{cd}}Q<\infty$, there exists a finite-dimensional contractible proper Q-complex X. Let G act on X via the quotient map $G\to Q$. Then X is a finite-dimensional contractible G-complex whose all stabilizers are of the form of extension of N by F, where F is a finite subgroup of Q. It follows from [**11**, Exercise 7.7] that all stabilizers of X belong to the class of groups $\mathscr{B}_n$. Hence G belongs to the class $\mathscr{B}_n$ of groups by Theorem 3.5. □

REMARK 3.7. It was known from [**11**, Theorem 7.2] that Corollary 3.6 holds without any additional assumption on the quotient group Q. However under the assumption that $\underline{\mathrm{cd}}Q<\infty$, our proof is new.

COROLLARY 3.8. *Suppose that a group G admits an m-dimensional cocompact contractible G-CW-complex Y and X is a G-CW-complex for which all stabilizers are amenable. If X is L^2-finite with respect to all stabilizers G_σ of Y, then X is L^2-finite with respect to G and*

$$\chi^{(2)}(X;G)=\sum_{i=0}^{m}(-1)^i\sum_{\sigma\in\sum_i}\chi^{(2)}(X;G_\sigma).$$

PROOF. From Proposition 3.3, it follows that there is a weak $\mathscr{C}$-exact sequence of $\mathscr{N}(G)$-modules

$$0\to\bigoplus_{\sigma\in\sum_m}(\mathscr{N}(G)\otimes\boldsymbol{Z}(G/G_\sigma))\otimes_G C_*(X)$$

$$\overset{\partial_m}{\to} \bigoplus_{\sigma\in\sum_{m-1}} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)\otimes_G C_*(X) \overset{\partial_{m-1}}{\to} \cdots$$
$$\overset{\partial_1}{\to} \bigoplus_{\sigma\in\sum_{0}} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)\otimes_G C_*(X) \overset{\partial_0}{\to} \mathscr{N}(G)\otimes_G C_*(X)\to 0,$$

where $m = \dim Y$. Consider first the $\mathscr{C}$-exact sequence of $\mathscr{N}(G)$-modules

$$0\to \bigoplus_{\sigma\in\sum_{m}} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)\otimes_G C_*(X)$$
$$\overset{\partial_m}{\to} \bigoplus_{\sigma\in\sum_{m-1}} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)\otimes_G C_*(X)\to \mathrm{im}\partial_{m-1}\to 0.$$

From Lemma 3.4 and [**12**, Proposition 3.6] it follows that there is a weak $\mathscr{C}$-exact sequence

$$\cdots\to \bigoplus_{\sigma\in\sum_{m}} \mathscr{N}(G)\otimes_{\mathscr{N}(G_\sigma)} H_n^{G_\sigma}(X,\mathscr{N}(G_\sigma))$$
$$\to \bigoplus_{\sigma\in\sum_{m-1}} \mathscr{N}(G)\otimes_{\mathscr{N}(G_\sigma)} H_n^{G_\sigma}(X,\mathscr{N}(G_\sigma))\to H_n(\mathrm{im}\partial_{m-1})$$
$$\to \bigoplus_{\sigma\in\sum_{m}} \mathscr{N}(G)\otimes_{\mathscr{N}(G_\sigma)} H_{n-1}^{G_\sigma}(X,\mathscr{N}(G_\sigma))\to\cdots$$
$$\to \bigoplus_{\sigma\in\sum_{m-1}} \mathscr{N}(G)\otimes_{\mathscr{N}(G_\sigma)} H_0^{G_\sigma}(X,\mathscr{N}(G_\sigma))\to H_0(\mathrm{im}\partial_{m-1})\to 0.$$

By taking the alternating sum of L^2-Betti numbers in the weak $\mathscr{C}$-exact sequence above, we have

$$\sum_{i\geq 0}(-1)^i \dim_{\mathscr{N}(G)}(H_i(\mathrm{im}\partial_{m-1}))$$
$$= \sum_{\sigma\in\sum_{m-1}} \chi^{(2)}(X;G_\sigma) - \sum_{\sigma\in\sum_{m}} \chi^{(2)}(X;G_\sigma).$$

Now consider the $\mathscr{C}$-exact sequence of $\mathscr{N}(G)$-modules

$$0\to \mathrm{im}\partial_{m-1}\to \bigoplus_{\sigma\in\sum_{m-2}} \mathscr{N}(G)\otimes \boldsymbol{Z}(G/G_\sigma)\otimes_{G_\sigma} C_*(X)\to \mathrm{im}\partial_{m-2}\to 0.$$

By the same method as used above, we obtain

$$\sum_{i\geq 0}(-1)^i \dim_{\mathcal{N}(G)}(H_i(\mathrm{im}\partial_{m-2}))$$
$$= \sum_{\sigma\in\sum_{m-2}} \chi^{(2)}(X;G_\sigma) - \sum_{i\geq 0}(-1)^i \dim_{\mathcal{N}(G)}(H_p(\mathrm{im}\partial_{m-1})).$$

Thus we have

$$\sum_{i\geq 0}(-1)^i \dim_{\mathcal{N}(G)}(H_i(\mathrm{im}\partial_{m-2}))$$
$$= \sum_{\sigma\in\sum_{m-2}} \chi^{(2)}(X;G_\sigma) - \sum_{\sigma\in\sum_{m-1}} \chi^{(2)}(X;G_\sigma) + \sum_{\sigma\in\sum_{m}} \chi^{(2)}(X;G_\sigma).$$

Iterating this process, we have the desired formula. □

4. New examples of groups satisfying the Chatterji-Mislin conjecture.

In this section, we will give a new class of groups satisfying the Chatterji-Mislin conjecture. We start with the following lemma.

LEMMA 4.1. *Suppose that a group G admits an m-dimensional contractible cocompact G-CW-complex X for which all stabilizers G_σ are amenable. If X is L^2-finite with respect to all stablizers G_σ, then X is L^2-finite with respect to G and*

$$\begin{aligned}\chi^{(2)}(G) &= \chi^{(2)}(X;G)\\ &= \sum_{i=0}^{m}(-1)^i \sum_{\sigma\in\sum_i} \chi^{(2)}(X;G_\sigma)\\ &= \sum_{i=0}^{m}(-1)^i \sum_{\sigma\in\sum_i} \chi^{(2)}(G_\sigma).\end{aligned}$$

PROOF. Notice that $EG\times X$ is a model for EG, since X is contractible. Notice also that if G_σ is infinite amenable, then $b_p^{(2)}(G_\sigma)=0$ for all $p\geq 0$ ([**5**] or Theorem 3.5). Thus by [**11**, Theorem 6.80 (4)], we have $\chi^{(2)}(G)=\chi^{(2)}(EG\times X)=\chi^{(2)}(X)$. The result now follows from Corollary 3.8. □

Notice that for a group G which admits a finite-dimensional contractible cocompact G-CW-complex X, if each stabilizer G_σ of X is of type FP over $\boldsymbol{C}$, then so is G ([**3**, Exercise VII.6.8]).

LEMMA 4.2. *Suppose that a group G admits an m-dimensional contractible cocompact G-CW-complex X all of whose stabilizers G_σ are of type FP over $\boldsymbol{C}$. Then the complete Euler characteristic of G is given by*

$$E(G) = \sum_{i=0}^{m} (-1)^i \sum_{\sigma \in \sum_i} j_*^\sigma E(G_\sigma),$$

where $j^\sigma : G_\sigma \hookrightarrow G$.

PROOF. Let $C_*(X; \boldsymbol{C})$ be the cellular chain complex of X over the complex numbers $\boldsymbol{C}$. Notice that

$$C_p(X; \boldsymbol{C}) \cong \bigoplus_{\sigma \in \sum_p} \boldsymbol{C}(G/G_\sigma).$$

Since X is contractible, we have an exact sequence of $\boldsymbol{C}G$-modules

$$0 \to \bigoplus_{\sigma \in \sum_m} \boldsymbol{C}(G/G_\sigma) \to \cdots \to \bigoplus_{\sigma \in \sum_0} \boldsymbol{C}(G/G_\sigma) \to \boldsymbol{C} \to 0.$$

For each stabilizer G_σ of σ, let $P_*^\sigma : 0 \to P_n^\sigma \to \cdots \to P_0^\sigma \to \boldsymbol{C} \to 0$ be a projective resolution of type FP over $\boldsymbol{C}$. Then there is a following projective resolution of type FP over $\boldsymbol{C}$ of an induced module $\boldsymbol{C}(G/G_\sigma)$:

$$\widetilde{P}_*^\sigma : 0 \to \widetilde{P}_n^\sigma \to \cdots \to \widetilde{P}_0^\sigma \to \boldsymbol{C}(G/G_\sigma) \to 0,$$

where $\widetilde{P}_k^\sigma = \boldsymbol{C}(G/G_\sigma) \otimes P_k^\sigma$ is projective over $\boldsymbol{C}G$. Thus the Wall element for G is given by

$$W(G) = \sum_{i=0}^{m} (-1)^i \sum_{\sigma \in \sum_i} [\boldsymbol{C}(G/G_\sigma)],$$

where $[\boldsymbol{C}(G/G_\sigma)] = \sum_{k=0}^{n} (-1)^k [\widetilde{P}_k^\sigma] = j_*^\sigma W(G_\sigma) \in K_0(\boldsymbol{C}G)$. Hence the complete Euler characteristic of G is given by

$$E(G) = \sum_{i=0}^{m} (-1)^i \sum_{\sigma \in \sum_i} j_*^\sigma E(G_\sigma).$$

□

DEFINITION 4.3. Let G be a discrete group which admits a finite-dimensional contractible cocompact G-CW-complex X such that for each finite subgroup H of G, the fixed point complex X^H is contractible. We say that G satisfies Condition (F) if for any element of finite order $s \in G$, all stabilizers $H \leq C_G(s)$ appearing on the fixed point complex $X^{\langle s \rangle}$ are amenable and L^2-finite, where $\langle s \rangle$ is the finite cyclic group generated by s.

THEOREM 4.4. *Let G be a discrete group which admits a finite-dimensional contractible cocompact G-CW-complex X such that for each finite subgroup H of G, the fixed point complex X^H is contractible. Suppose that G satisfies the Condition* (F). *Then the following hold.*

(1) *If all stabilizers G_σ satisfy formula ($\star$) at all elements of infinite order, then so does G.*
(2) *If all stabilizers G_σ are of type FP over $\boldsymbol{C}$ and satisfy formula ($\star$) at all elements of finite order, then so does G.*

PROOF.
(1) Let $s \in G$ be an element of infinite order. Then $\langle s \rangle$ is an infinite cyclic normal subgroup of $C_G(s)$ and thereby $\chi^{(2)}(C_G(s)) = 0$ by Cheeger-Gromov's result [**5**] or Theorem 3.5. The result now follows from Lemma 4.2 and the assumption on the G_σ's.

(2) Let σ be an i-dimensional cell of X. Denote $[s, G_\sigma]$ be the conjugacy classes of elements in G_σ which are G-conjugate to s. From Lemma 4.2, it follows that for an element of finite order $s \in G$, we have

$$\begin{aligned} E(G)(s) &= \sum_{i=0}^{m} (-1)^i \sum_{[x] \in [s, G_\sigma]} E(G_\sigma)(x) \\ &= \sum_{i=0}^{m} (-1)^i \sum_{[x] \in [s, G_\sigma]} \chi^{(2)}(C_{G_\sigma}(x)), \end{aligned} \tag{4.1}$$

where $m = \dim X$. Consider the G_σ's as representatives for the stabilizers of the G-action on X so that a general stabilizer will have the form $gG_\sigma g^{-1}$. Since $\langle s \rangle$ is a finite (cyclic) group, $X^{\langle s \rangle}$ is contractible. Note that $C_G(s)$ acts on $X^{\langle s \rangle}$ via the restriction of the G-action on X and the stabilizer of $\sigma \in X^{\langle s \rangle}$ is of the form $C_G(s) \cap gG_\sigma g^{-1}$, where $s \in gG_\sigma g^{-1}$ so that $C_G(s) \cap gG_\sigma g^{-1} \cong C_{G_\sigma}(g^{-1}sg)$. Since G satisfies the condition (F), $\chi^{(2)}(C_{G_\sigma}(g^{-1}sg))$ is well defined and so is $\chi^{(2)}(C_G(s))$ by Lemma 4.1. Moreover, by Lemma 4.1 again, we have

$$\chi^{(2)}(C_G(s)) = \sum_{p=0}^{m} (-1)^i \sum_{x \in \sum_i^s} \chi^{(2)}(C_{G_\sigma}(x)),$$

where $\sum_i^s$ is a set of representatives for the C_{G_σ}-orbits of i-cells of $X^{\langle s \rangle}$. Note that the index set $\sum_i^s$ corresponds bijectively to conjugacy classes of elements x in the $[G_\sigma]$'s which are G-conjugate to s. Hence the last line of the equation (4.1) is equal to $\chi^{(2)}(C_G(s))$. This completes the proof. □

COROLLARY 4.5. *Let G be a discrete group which admits a finite-dimensional contractible cocompact G-CW-complex X satisfying that for each finite subgroup H of G, the fixed point complex X^H is contractible. If each stabilizer G_σ satisfies formula ($\star$) at all elements of infinite order, then G satisfies formula ($\star$) at all elements of infinite order. Moreover, $E(G)(s) = 0 = \chi^{(2)}(C_G(s))$ for an element $s \in G$ of infinite order.*

Proof. From Cheeger-Gromov's result [**5**] or Theorem 3.5 it follows that $\chi^{(2)}(C_G(s)) = 0$. By Theorem 4.4, $E(G)(s) = 0$. □

Theorem 4.6. *Let G be a discrete group which admits a finite-dimensional contractible cocompact G-CW-complex X satisfying that for each finite subgroup H of G, the fixed point complex X^H is contractible. Suppose that G satisfies the Condition* (F). *If each stabilizer G_σ is of type FP over $\boldsymbol{C}$ and satisfies Conjecture A, then G is of type FP over $\boldsymbol{C}$ and satisfies Conjecture A.*

Proof. This follows from [**3**, Exercise VII.6.8] and Theorem 4.4. □

Remark 4.7. In [**4**, Theorem 4.4], Chatterji and Mislin constructed the class $\mathscr{B}$ of groups satisfying Conjecture A. One can obtain more examples coming from interesting closure properties of $\mathscr{B}$ and Theorem 4.6.

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